METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 7, No. 1, pp. 57–64, March 2000

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CENTRALLY SYMMETRIC ORTHOGONAL POLYNOMIALS AND SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS*

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Abstract. We classify completely, up to a real change of variables, all differential equations

$$L[u] := Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y = \lambda_n u,$$

which have centrally symmetric orthogonal polynomial solutions.

1. Introduction and Preliminaries. Consider a second order partial differential equations of the type

$$(1.1) \qquad L[u] := Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y = \lambda_n u, \qquad n = 0, 1, 2, \cdots,$$

where $A \sim E$ are polynomials in x and y. Krall and Sheffer[5] classified equations (1.1), up to a complex linear change of variables, which have orthogonal polynomials as solutions.

However, complex linear change of variables does not preserve the positive-definiteness of orthogonality and the type of the equation (1.1). In this respect, we classify completely, up to a real change of variables, the equations (1.1) which have centrally symmetric orthogonal polynomials as solutions together with explicit representations of orthogonal polynomial solutions.

For any integer $n \geq 0$, let $\mathcal{P}_n$ be the space of real polynomials in two variables of (total) degree $\leq n$ and $\mathcal{P} = \bigcup_{n\geq 0} \mathcal{P}_n$. By a polynomial system(PS), we mean a sequence of polynomials $\{\phi_{mn}\}_{m,n=0}^{\infty}$ such that $\deg(\phi_{mn}) = m + n$ for m and $n \geq 0$ and $\{\phi_{n-j,j}\}_{j=0}^{n}$ are linearly independent modulo $\mathcal{P}_{n-1}$ for $n \geq 0$ ($\mathcal{P}_{-1} = \{0\}$). A PS $\{P_{mn}\}_{m,n=0}^{\infty}$ is said to be monic if

$$P_{mn}(x, y) = x^m y^n \text{ modulo } \mathcal{P}_{m+n-1}, \qquad m \text{ and } n \geq 0.$$

A linear mapping $\sigma : \mathcal{P} \to \mathbb{R}$ is called a *moment functional*, whose action on a polynomial $\phi \in \mathcal{P}$ is denoted by $\langle \sigma, \phi \rangle$. For any moment functional σ, we define the partial derivatives σ_x and σ_y of σ by

$$\langle \sigma_x, \phi \rangle := -\langle \sigma, \phi_x \rangle, \quad \langle \sigma_y, \phi \rangle := -\langle \sigma, \phi_y \rangle \quad (\phi \in \mathcal{P}),$$

and the multiplication $\psi\sigma$ for $\psi \in \mathcal{P}$ by $\langle \psi\sigma, \phi \rangle := \langle \sigma, \psi\phi \rangle$.

DEFINITION 1.1. *([5]) A PS $\{\phi_{mn}\}_{m,n=0}^{\infty}$ is a weak orthogonal polynomial system (WOPS) if there is a non-zero moment functional σ such that $\langle \sigma, \phi_{mn}\phi_{kl} \rangle = 0$, if $m + n \neq k + l$.*

If furthermore

$$\langle \sigma, \phi_{mn}\phi_{kl} \rangle = K_{mn}\delta_{mk}\delta_{nl}$$

where K_{mn} are non-zero(resp., positive) constants, we call $\{\phi_{mn}\}_{m,n=0}^{\infty}$ an orthogonal polynomial system(OPS) (resp., a positive-definite OPS). In this case, we say that $\{\phi_{mn}\}_{m,n=0}^{\infty}$ is a WOPS or an OPS relative to σ.

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*Received April 21, 1999; revised Nov. 15, 1999.
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For any PS $\{\phi_{mn}\}_{m,n=0}^{\infty}$, there is a unique moment functional σ, called the canonical moment functional of $\{\phi_{mn}\}_{m,n=0}^{\infty}$, defined by the conditions

$$\langle\sigma, 1\rangle = 1 \text{ and } \langle\sigma, \phi_{mn}\rangle = 0,\ m+n \geq 1\ .$$

In the following, we write a PS $\{\phi_{mn}\}_{m,n=0}^{\infty}$ as $\{\Phi_n\}_{n=0}^{\infty}$ where $\Phi_n = [\phi_{n0}, \phi_{n-1,1}, \cdots, \phi_{0n}]^T$ and let $\mathbf{x}^n = [x^n, x^{n-1}y, \cdots, y^n]^T$, $n \geq 0$. When $\Phi_n = A_n\mathbf{x}^n$ modulo $\mathcal{P}_{n-1}$, we call the monic PS $\{\mathbb{P}_n\}_{n=0}^{\infty}$ the normalization of $\{\Phi_n\}_{n=0}^{\infty}$, where $\mathbb{P}_n := A_n^{-1}\Phi_n$.

DEFINITION 1.2. *A moment functional σ is quasi-definite (resp., positive-definite) if there is an OPS (resp., a positive-definite OPS) relative to σ.*

PROPOSITION 1.3. *([1, 5]) For a moment functional $\sigma \neq 0$, σ is quasi-definite(resp., positive-definite) if and only if D_n is nonsingular (resp., positive-definite), where*

$$D_n := \begin{bmatrix} \sigma_{00} & \sigma_{10} & \sigma_{01} & \cdots & \sigma_{n0} & \cdots & \sigma_{0n} \\ \sigma_{10} & \sigma_{20} & \sigma_{11} & \cdots & \sigma_{n+1,0} & \cdots & \sigma_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \sigma_{0n} & \sigma_{1n} & \sigma_{0,n+1} & \cdots & \sigma_{nn} & \cdots & \sigma_{0,2n} \end{bmatrix},\ n \geq 0,$$

and $\sigma_{m,n} = \langle\sigma, x^m y^n\rangle$, m and $n \geq 0$, are the moments of σ.

For any PS $\{\Phi_n\}_{n=0}^{\infty}$, there are matrices

$$\begin{array}{ll} A_{ni} : (n+1)\times(n+2), & B_{ni} : (n+1)\times(n+1), \\ C_{ni} : (n+1)\times n, & D_{ni}^k : (n+1)\times(k+1) \end{array}$$

for $i = 1, 2$ and $k = 0, 1, \cdots, n-2$ such that

$$\mathbf{x}\Phi_n := \begin{bmatrix} x\Phi_n \\ y\Phi_n \end{bmatrix} = A_n\Phi_{n+1} + B_n\Phi_n + C_n\Phi_{n-1} + \sum_{k=0}^{n-2} D_n^k\Phi_k \tag{1.2}$$

where $A_n = \begin{bmatrix} A_{n1} \\ A_{n2} \end{bmatrix}$, $B_n = \begin{bmatrix} B_{n1} \\ B_{n2} \end{bmatrix}$, $C_n = \begin{bmatrix} C_{n1} \\ C_{n2} \end{bmatrix}$, $D_n^k = \begin{bmatrix} D_{n1}^k \\ D_{n2}^k \end{bmatrix}$.
Note that since $\{\Phi_n\}_{n=0}^{\infty}$ is a PS, rank $A_n = n+2$, $n \geq 0$.

PROPOSITION 1.4. *(Favard's theorem) (cf. [3, 5, 9]) Let $\{\Phi_n\}_{n=0}^{\infty}$ be a PS. Then $\{\Phi_n\}_{n=0}^{\infty}$ is a WOPS relative to a quasi-definite moment functional σ if and only if $D_n^k = 0$ for $k = 0, 1, \cdots, n-2$ so that $\{\Phi_n\}_{n=0}^{\infty}$ satisfy a three term recurrence relation*

$$\mathbf{x}\Phi_n(\mathbf{x}) = A_n\Phi_{n+1}(\mathbf{x}) + B_n\Phi_n(\mathbf{x}) + C_n\Phi_{n-1}(\mathbf{x}),\ n \geq 0\ (\Phi_{-1}(\mathbf{x}) \equiv 0) \tag{1.3}$$

and

$$\operatorname{rank} \widetilde{C}_n = n+1,\ n \geq 1, \tag{1.4}$$

where $\widetilde{C}_n := [C_{n1}, C_{n2}]$ is an $(n+1)\times 2n$ matrix.

If the equation (1.1) has a PS $\{\Phi_n\}_{n=0}^{\infty}$ as solutions, then it must be of the form

$$\begin{aligned} L[u] &= (ax^2 + d_1x + e_1y + f_1)u_{xx} + (2axy + d_2x + e_2y + f_2)u_{xy} \\ &\quad + (ay^2 + d_3x + e_3y + f_3)u_{yy} + (gx + h_1)u_x + (gy + h_2)u_y = \lambda_n u \end{aligned} \tag{1.5}$$

where $\lambda_n := an(n-1) + gn$([5]).

We always assume that $|A|+|B|+|C| \not\equiv 0$ since otherwise the equation (1.5) cannot have any OPS as solutions(cf. [1]). Following Krall and Sheffer[5], we also assume that the equation (1.5) is admissible, that is, $\lambda_m \neq \lambda_n$ for $m \neq n$ (or equivalently $an + g \neq 0, n \geq 0$) so that the equation (1.5) has a unique monic PS as solutions.

LEMMA 1.5. *([1, Lemma 3.1]) If the equation* (1.5) *has a PS* $\{\Phi_n\}_{n=0}^{\infty}$ *as solutions, then the canonical moment functional* σ *of* $\{\Phi_n\}_{n=0}^{\infty}$ *satisfies*

$$L^*[\sigma] := (A\sigma)_{xx} + 2(B\sigma)_{xy} + (C\sigma)_{yy} - (D\sigma)_x - (E\sigma)_y = 0. \tag{1.6}$$

If we set $S_n := \langle \sigma, \mathbf{x}^n \rangle$, $n \geq 0$ ($S_{-1} = S_{-2} = 0$) then we may rewrite (1.6) as

(1.7)
$$\langle L^*[\sigma], \mathbf{x}^n \rangle = \langle \sigma, \lambda_n \mathbf{x}^n + B_n \mathbf{x}^{n-1} + C_n \mathbf{x}^{n-2} \rangle = \lambda_n S_n + B_n S_{n-1} + C_n S_{n-2} = 0,$$

where

$$\begin{aligned} B_k &= D_k^1 D_{k-1}^1 (d_1 M_{k-2}^1 + e_1 M_{k-2}^2) + D_k^1 D_{k-1}^2 (d_2 M_{k-2}^1 + e_2 M_{k-2}^2) \\ &\quad + D_k^2 D_{k-1}^2 (d_3 M_{k-2}^1 + e_3 M_{k-2}^2) + h_1 D_k^1 + h_2 D_k^2, \\ C_k &= f_1 D_k^1 D_{k-1}^1 + f_2 D_k^1 D_{k-1}^2 + f_3 D_k^2 D_{k-1}^2, \end{aligned}$$

and $x\mathbf{x}^n = M_n^1 \mathbf{x}^{n+1}$, $y\mathbf{x}^n = M_n^2 \mathbf{x}^{n+1}$, $\partial_x \mathbf{x}^n = D_n^1 \mathbf{x}^{n-1}$, $\partial_y \mathbf{x}^n = D_n^2 \mathbf{x}^{n-1}$. Here, I_n is the $n \times n$ identity matrix and

$$\begin{aligned} M_n^1 &= [I_{n+1} \mid 0], & M_n^2 &= [0 \mid I_{n+1}], \\ D_n^1 &= [\mathrm{Diag}(n, \cdots, 1) \mid 0]^T, & D_n^2 &= [0 \mid \mathrm{Diag}(1, \cdots, n)]^T. \end{aligned}$$

The equation (1.7) is a three term recurrence relation for vector moments $\{S_n\}_{n=0}^{\infty}$ of σ.

PROPOSITION 1.6. *([1, Theorem 3.7]) Let* $\{\Phi_n\}_{n=0}^{\infty}$ *be a PS satisfying an admissible equation* (1.5) *and* σ *the canonical moment functional of* $\{\Phi_n\}_{n=0}^{\infty}$. *Then the following statements are all equivalent:*

(i) $\{\Phi_n\}_{n=0}^{\infty}$ *is a WOPS relative to* σ*;*
(ii) $M_1[\sigma] := (A\sigma)_x + (B\sigma)_y - D\sigma = 0$*;*
(iii) $M_2[\sigma] := (B\sigma)_x + (C\sigma)_y - E\sigma = 0$.

Note that $L^*[\sigma] = (M_1[\sigma])_x + (M_2[\sigma])_y$. We call $M_1[\sigma] = 0$ and $M_2[\sigma] = 0$ the moment equations for the equation (1.5).

Using the moments σ_{mn} of σ, we may express $L^*[\sigma] = 0$, $M_1[\sigma] = 0$, and $M_2[\sigma] = 0$ as (cf. [1, 5])

$$\begin{aligned} A_{mn} :&= \langle L^*[\sigma], x^m y^n \rangle = \frac{1}{2}(mC_{m-1,n} + nB_{m,n-1}); \\ B_{mn} :&= -2\langle M_2[\sigma], x^m y^n \rangle = 2\{a(m+n)+g\}\sigma_{m,n+1} + e_2 m \sigma_{m-1,n+1} \\ &\quad + (d_2 m + 2e_3 n + 2h_2)\sigma_{mn} + f_2 m \sigma_{m-1,n} + 2f_3 n \sigma_{m,n-1} + 2d_3 n \sigma_{m+1,n-1} = 0; \\ C_{mn} :&= -2\langle M_1[\sigma], x^m y^n \rangle = 2\{a(m+n)+g\}\sigma_{m+1,n} + (2d_1 m + e_2 n + 2h_1)\sigma_{mn} \\ &\quad + d_2 n \sigma_{m+1,n-1} + 2f_1 m \sigma_{m-1,n} + f_2 n \sigma_{m,n-1} + 2e_1 m \sigma_{m-1,n+1} = 0, \quad m \text{ and } n \geq 0. \end{aligned}$$

2. Centrally symmetric OPS and Partial differential equations.

DEFINITION 2.1. *We call a PS* $\{\Phi_n\}_{n=0}^{\infty}$ *to be centrally symmetric if* $\Phi_n(-x,-y) = (-1)^n\Phi_n(x,y)$, $n \geq 0$. *Also, we call a moment functional* σ *to be centrally symmetric if* $\langle \sigma, x^m y^n \rangle = 0$ *for* $m+n$ *odd.*

LEMMA 2.2. (*[10, Theorem 2.2.1]*) *Let* $\{\Phi_n\}_{n=0}^{\infty}$ *be a WOPS relative to a quasi-definite moment functional* σ *so that* (1.3) *holds. Then the following statements are all equivalent:*

(i) $\{\Phi_n\}_{n=0}^{\infty}$ *is centrally symmetric;*
(ii) σ *is centrally symmetric;*
(iii) $B_n = 0$, $n \geq 0$.

Furthermore, we have:

PROPOSITION 2.3. *Assume that the equation* (1.5) *has a WOPS* $\{\Phi_n\}_{n=0}^{\infty}$ *relative to a quasi-definite moment functional* σ *as solutions. Then* σ *is centrally symmetric if and only if the equation* (1.5) *is of the form*

(2.1)
$$L[u] = (ax^2 + f_1)u_{xx} + (2axy + f_2)u_{xy} + (ay^2 + f_3)u_{yy} + g(xu_x + yu_y) = \lambda_n u.$$

Moreover, in this case, $\Delta := f_2^2 - 4f_1f_3 \neq 0$.

Proof. It follows from (1.6), Proposition 1.3, and Proposition 1.6. □

It's easy to see(cf. [5]) that under a real linear change of variables $T(x,y) = (\alpha x + \beta y, \gamma x + \delta y)$, $\alpha\delta - \beta\gamma \neq 0$, the equation (2.1) is transformed into

(2.2)
$$L[u] = (ax^2 + f_1^*)u_{xx} + (2axy + f_2^*)u_{xy} + (ay^2 + f_3^*)u_{yy} + g(xu_x + yu_y) = \lambda_n u,$$

where $f_1^* = \alpha^2 f_1 + \alpha\beta f_2 + \beta^2 f_3$, $f_2^* = 2\alpha\gamma f_1 + (\alpha\delta + \beta\gamma)f_2 + 2\beta\delta f_3$, and $f_3^* = \gamma^2 f_1 + \gamma\delta f_2 + \delta^2 f_3$.
Therefore, we may transform the equation (2.1) into either

$$L[u] = ax^2 u_{xx} + (2axy + f_2)u_{xy} + ay^2 u_{yy} + g(xu_x + yu_y) = \lambda_n u \tag{2.3}$$

if $\Delta > 0$ or

$$L[u] = (ax^2 + f_1)u_{xx} + 2axyu_{xy} + (ay^2 + f_1)u_{yy} + g(xu_x + yu_y) = \lambda_n u, \tag{2.4}$$

if $\Delta < 0$ where $a = 0$ or 1 and $f_1 \neq 0$, $f_2 \neq 0$ provided that $\Delta \neq 0$.

LEMMA 2.4. *Assume that* $\Delta := f_2^2 - 4f_1f_3 \neq 0$. *Then the (unique) monic PS* $\{\mathbb{P}_n\}_{n=0}^{\infty}$ *of solutions to the equation* (2.1) *is always a WOPS.*

Proof. Under the complex linear change of variables $T(x,y) = (x+iy, x-iy)$, the equation (2.4) is transformed into the equation (2.3). Since weak orthogonality is preserved under any linear change of variables, we may consider only the equation (2.3). Let σ be the canonical moment functional of the monic PS $\{\mathbb{P}_n\}_{n=0}^{\infty}$ of solutions to the equation (2.3). Then by Lemma 1.5, $A_{mn} = 0$, m and $n \geq 0$ so that $\sigma_{n0} = \sigma_{0n} = 0$, $n \geq 1$. Hence $\sigma_{mn} = 0$ for $m \neq n$ by induction. Then it's easy to see that $B_{mn} = 0$, i.e., $M_2[\sigma] = 0$ so that $\{\mathbb{P}_n\}_{n=0}^{\infty}$ is a WOPS relative to σ by Proposition 1.6. □

THEOREM 2.5. *The equation* (1.5) *has a centrally symmetric OPS as solutions if and only if the equation* (1.5) *is of the form* (2.1) *and* $a \neq g$, $\Delta \neq 0$.

In order to prove Theorem 2.5, we need to extend Favard's theorem for WOPS's.

PROPOSITION 2.6. *Let* $\{\Phi_n\}_{n=0}^{\infty}$ *be a WOPS relative to* σ. *Then* σ *is quasi-definite if and only if the rank condition* (1.4) *holds.*

Proof. See the proof of Theorem 2 in [9](see also [4]). □

Proof. [**Proof of Theorem 2.5**] Consider the equation (2.1) where $a \neq g$. We may assume that the equation (2.1) is of the form (2.3). Let $\{\mathbb{P}_n\}_{n=0}^{\infty}$ be the unique monic PS of solutions to the equation (2.3) and σ the canonical moment functional of $\{\mathbb{P}_n\}_{n=0}^{\infty}$. Then by Lemma 2.4, $\{\mathbb{P}_n\}_{n=0}^{\infty}$ is a WOPS relative to σ so that it suffices to show the rank condition (1.4) holds for $\{\mathbb{P}_n\}_{n=0}^{\infty}$. We set $\mathbb{P}_n(\mathbf{x}) = \sum_{k=0}^{n} A_k^n \mathbf{x}^k$, $n \geq 0$. Then we have $A_n^n = I_{n+1}$, $A_{n-1}^n = A_{n-3}^n = \cdots = 0$ and $(\lambda_n - \lambda_{n-2})A_{n-2}^n = f_2 D_n^1 D_{n-1}^2 = f_2[0|\text{Diag}(n-1, 2(n-2), \cdots, (n-2)2, n-1)|0]^T$, $n \geq 2$.
We also have

$$(2.5) \qquad \begin{cases} x\mathbb{P}_n = A_{n1}\mathbb{P}_{n+1} + C_{n1}\mathbb{P}_{n-1} \text{ (modulo } \mathcal{P}_{n-2}) \\ y\mathbb{P}_n = A_{n2}\mathbb{P}_{n+1} + C_{n2}\mathbb{P}_{n-1} \text{ (modulo } \mathcal{P}_{n-2}) \end{cases}$$

where $A_{nj} = M_n^j$, $C_{nj} = A_{n-2}^n M_{n-2}^j - M_n^j A_{n-1}^{n+1}$, $n \geq 1$ ($A_{-1}^1 = 0$) and $j = 1, 2$. Hence

$$(2.6) \quad \begin{aligned} t_n t_{n+1} f_2^{-1} C_{nj} &= t_{n+1} D_n^1 D_{n-1}^2 M_{n-2}^j - t_n M_n^j D_{n+1}^1 D_n^2 \\ &= \begin{cases} [0|\text{Diag}\{t_{n+1}k(n-k) - t_n k(n-k+1)\}_{k=1}^n]^T & \text{for } j = 1 \\ [\text{Diag}\{t_{n+1}k(n-k) - t_n(k+1)(n-k)\}_{k=1}^n|0]^T & \text{for } j = 2, \end{cases} \end{aligned}$$

where $t_n = \lambda_n - \lambda_{n-2}$, $n \geq 1$ ($\lambda_{-1} = 0$).

Note that

$$|\text{Diag}\{t_{n+1}k(n-k) - t_n k(n-k+1)\}_{k=1}^n| = \begin{cases} -\lambda_1 = -g & \text{if } n = 1 \\ \prod_{k=1}^n 2k[(3-2k)a - g] & \text{if } n \geq 2. \end{cases}$$

Hence, $\text{rank}\, C_{n1} = \text{rank}(t_n t_{n+1} f_2^{-1} C_{n1}) = n, n \geq 1$ since $ak + g \neq 0$, $k \geq -1$.

Similarly, $\text{rank}\, C_{n2} = n$, $n \geq 1$. In particular, $\text{rank}\, \widetilde{C}_n = \text{rank}(t_n t_{n+1} f_2^{-1} \widetilde{C}_n) = n+1, n \geq 1$ since the first $n+1$ columns of $t_n t_{n+1} f_2^{-1} \widetilde{C}_n$ are linearly independent.

Conversely, assume that the equation (1.5) has a centrally symmetric OPS $\{\Phi_n\}_{n=0}^{\infty}$ as solutions. Then the equation (1.5) must be of the form (2.1) and $\Delta \neq 0$. Furthermore, we may assume that the equation (2.1) is of the form (2.3). Let $\{\mathbb{P}_n\}_{n=0}^{\infty}$ be the normalization of $\{\Phi_n\}_{n=0}^{\infty}$ and σ the canonical moment functional of $\{\mathbb{P}_n\}_{n=0}^{\infty}$. Then $\{\mathbb{P}_n\}_{n=0}^{\infty}$ is a WOPS relative to σ, which is quasi-definite and we have (2.5) and (2.6). We now assume $a = g = 1$. Then

$$t_2 t_3 f_2^{-1} \widetilde{C}_2 = -8 \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

so that $\text{rank}\, \widetilde{C}_2 = 2$, which contradicts Favard's theorem. □

By a suitable real linear change of variables, the differential equations (2.3) and (2.4) can be transformed into:

$$\begin{aligned} L[u] &= (x^2+1)u_{xx} + 2xyu_{xy} + (y^2-1)u_{yy} + gxu_x + gyu_y \\ &= n(g+n-1)u \, (\Delta > 0, \, a \neq 0); \end{aligned} \tag{2.7}$$

$$L[u] = u_{xx} - u_{yy} + 2xu_x + 2yu_y = 2nu \, (\Delta > 0, a = 0, g > 0); \tag{2.8}$$

$$L[u] = u_{xx} - u_{yy} - 2xu_x - 2yu_y = -2nu \, (\Delta > 0, a = 0, g < 0); \tag{2.9}$$

$$\begin{aligned} L[u] &= (x^2+1)u_{xx} + 2xyu_{xy} + (y^2+1)u_{yy} + gxu_x + gyu_y \\ &= n(g+n-1)u \, (\Delta < 0, a \neq 0, af_1 > 0); \end{aligned} \tag{2.10}$$

$$\begin{aligned} L[u] &= (x^2-1)u_{xx} + 2xyu_{xy} + (y^2-1)u_{yy} + gxu_x + gyu_y \\ &= n(g+n-1)u \, (\Delta < 0, a \neq 0, af_1 < 0); \end{aligned} \tag{2.11}$$

$$L[u] = u_{xx} + u_{yy} + 2xu_x + 2yu_y = 2nu \, (\Delta < 0, a = 0, gf_1 > 0); \tag{2.12}$$

$$L[u] = u_{xx} + u_{yy} - 2xu_x - 2yu_y = -2nu \, (\Delta < 0, a = 0, gf_1 < 0). \tag{2.13}$$

By Theorem 2.5, above seven equations have centrally symmetric OPS's as solutions if and only if $g = 1, 0, -1, \cdots$.

PROPOSITION 2.7. (*cf. Proposition* 4.1 *and Theorem* 4.5 *in [1]*) *Let* $\{\mathbb{P}_n\}_{n=0}^{\infty}$ *be the monic PS of solutions to the equation* (1.5) *and* σ *the canonical moment functional of* $\{\mathbb{P}_n\}_{n=0}^{\infty}$. *If* $A_y = 0$ *(resp.,* $C_x = 0$*), then* $P_{n0}(x,y) = P_{n0}(x)$ *(resp.,* $P_{0n}(x,y) = P_{0n}(y)$ *),* $n \geq 0$, *and* $\{P_{n0}(x)\}_{n=0}^{\infty}$ *(resp.,* $\{P_{0n}(y)\}_{n=0}^{\infty}$ *) is a WOPS in one variable satisfying the equation*

$$Au_{xx} + Du_x = \lambda_n u \ (\textit{resp., } Cu_{yy} + Eu_y = \lambda_n u).$$

Moreover, if σ *is positive-definite, then* $\{P_{n0}(x)\}_{n=0}^{\infty}$ *(resp.,* $\{P_{0n}(y)\}_{n=0}^{\infty}$ *) is a positive-definite classical OPS in one variable. If* $A_y = C_x = B = 0$, *then*

$$P_{mn}(x,y) = P_{m0}(x)P_{0n}(y), \quad m \text{ and } n \geq 0.$$

PROPOSITION 2.8. (*cf. [6, 8]*) *A second order ordinary equation*

$$\alpha(x)y''(x) + \beta(x)y'(x) = \lambda_n y(x),$$

where $\alpha(x) = ax^2 + bx + c \, (\not\equiv 0)$, $\beta(x) = dx + e$, *and* $\lambda_n = an(n-1) + dn$, *has an OPS (resp., a positive-definite OPS) as solutions if and only if for each* $n \geq 0$

$$s_n := an + d \neq 0 \text{ and } \alpha\left(\frac{-t_n}{s_{2n}}\right) \neq 0 \, (\textit{resp., } \frac{s_{n-1}}{s_{2n-1}s_{2n+1}}\alpha\left(\frac{-t_n}{s_{2n}}\right) < 0),$$

where $t_n := bn + e$.

By Propositions 2.7 and 2.8, the equations (2.7) $\sim$ (2.10) and (2.12) cannot have positive-definite OPS's as solutions. The equation (2.13) has a positive-definite OPS $\{H_{n-k}(x)H_k(y)\}_{k=0,n=0}^{n \ \ \infty}$ as solutions, where $\{H_n(x)\}_{n=0}^{\infty}$ are Hermite polynomials. It is well known that the equation (2.11) has a positive-definite OPS, called the circle polynomials, as solutions for $g > 1$. We now claim that the equation (2.11) has a positive-definite OPS as solutions only when $g > 1$(but has a quasi-definite OPS as solutions for $g \neq 1, 0, -1, \cdots$). Assume that the equation (2.11) has a positive-definite OPS as solutions. Then, by Propositions 2.7 and 2.8, $g > 0$. We now let σ be the canonical moment functional of the monic PS $\{\mathbb{P}_n\}_{n=0}^{\infty}$ of solutions to the equation (2.11). Then, we have from $A_{mn} = 0$

$$\sigma_{10} = \sigma_{01} = \sigma_{11} = \sigma_{30} = \sigma_{21} = \sigma_{12} = \sigma_{03} = \sigma_{31} = \sigma_{13} = 0, \; \sigma_{20} = \sigma_{02} = 1/(g+1),$$
$$\sigma_{40} = \sigma_{04} = 3\sigma_{22} = 3/(g+1)(g+3)$$

so that $\Delta_2 := \det D_2 = \frac{4(g-1)}{(g+1)^6(g+1)^3}$. Thus $g > 1$ since $\Delta_2 > 0$.

In summary, we have proved:

COROLLARY 2.9. *The equation* (2.1) *has a positive-definite OPS as solutions if and only if either* $\Delta < 0$, $af_1 < 0$, *and* $ag > 1$ *or* $\Delta < 0$, $a = 0$, $gf_1 < 0$.

Allowing complex linear change of variables, Krall and Sheffer[5] found only the equations (2.11) and (2.13). We now give the explicit form of OPS $\{\Phi_n\}_{n=0}^{\infty}$ of solutions to each of the equations (2.7) $\sim$ (2.13)(see [2]).

• The equation (2.7):

$$\phi_{n-k,k}(x,y) = \check{P}_{n-k}^{(\frac{g}{2}+k-1,\frac{g}{2}+k-1)}(x)(1+x^2)^{\frac{k}{2}} P_k^{(\frac{g}{2}-\frac{3}{2},\frac{g}{2}-\frac{3}{2})}(\frac{y}{\sqrt{1+x^2}}), \; 0 \le k \le n;$$

• The equation (2.8): $\phi_{n-k,k}(x,y) = \check{H}_{n-k}(x)H_k(y), \; 0 \le k \le n;$

• The equation (2.9): $\phi_{n-k,k}(x,y) = H_{n-k}(x)\check{H}_k(y), \; 0 \le k \le n;$

• The equation (2.10):

$$\phi_{n-k,k}(x,y) = \check{P}_{n-k}^{(\frac{g}{2}+k-1,\frac{g}{2}+k-1)}(x)(1+x^2)^{\frac{k}{2}} \check{P}_k^{(\frac{g}{2}-\frac{3}{2},\frac{g}{2}-\frac{3}{2})}(\frac{y}{\sqrt{1+x^2}}), \; 0 \le k \le n;$$

• The equation (2.11):

$$\phi_{n-k,k}(x,y) = P_{n-k}^{(\frac{g}{2}+k-1,\frac{g}{2}+k-1)}(x)(1-x^2)^{\frac{k}{2}} P_k^{(\frac{g}{2}-\frac{3}{2},\frac{g}{2}-\frac{3}{2})}(\frac{y}{\sqrt{1-x^2}}), \; 0 \le k \le n;$$

• The equation (2.12): $\phi_{n-k,k}(x,y) = \check{H}_{n-k}(x)\check{H}_k(y), \; 0 \le k \le n;$

• The equation (2.13): $\phi_{n-k,k}(x,y) = H_{n-k}(x)H_k(y), \; 0 \le k \le n.$

Here, $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$, $\{\check{P}_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$, and $\{\check{H}_n(x)\}_{n=0}^{\infty}$, $n \ge 0$ are Jacobi, twisted Jacobi, and twisted Hermite polynomials(see [7]), respectively.

ACKNOWLEDGEMENTS. This work is partially supported by KOSEF(98-0701-03-01-5) and Korea Ministry of Education(BK21-project). Authors thank the referees for their valuable comments.

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