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REGULARITY OF THE INVERSION PROBLEM FOR A STURM-LIOUVILLE EQUATION IN $L_p(R)$*

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Abstract. We consider an equation

$$-(r(x)y'(x))' + q(x)y(x) = f(x), \qquad x \in R \tag{1}$$

where $r(x) > 0$, $q(x) \geq 0$ for $x \in R$, $\frac{1}{r(x)} \in L_1^{\text{loc}}(R)$, $q(x) \in L_1^{\text{loc}}(R)$. We study the inversion problem for (1) with $f(x) \in L_p(R)$, $p \in [1, \infty]$ $(L_\infty(R) := C(R)$ is the space of bounded continuous functions in R with norm $\|f\|_{C(R)} = \sup_{x\in R} |f(x)|)$ in the case when $q(x)$ is oscillating (for example, $q(x) = 1 + \sin(|x|^\theta)$, $\theta \in (0, \infty)$). We obtain necessary and sufficient conditions under which, for all $p \in [1, \infty]$, the following assertions hold:

(1) equation (1) has a unique solution $y(x) \in L_p(R)$ and

$$y(x) = (Gf)(x) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} G(x,t)f(t)dt, \qquad x \in R;$$

(2) $$\|G\|_{p\to p} < \infty.$$

Here $G(x,t)$ is the Green function corresponding to (1).

1. Introduction.

In this paper we consider an equation

$$-(r(x)y'(x))' + q(x)y(x) = f(x), \qquad x \in R. \tag{1.1}$$

Here and throughout the paper $f(x) \in L_p(R)$, $p \in [1, \infty]$ $(L_\infty(R) := C(R)$ is the space of bounded continuous functions in R with norm $\|f\|_{C(R)} = \sup_{x\in R} |f(x)|)$; and $r(x)$ and $q(x)$ satisfy the following conditions:

$$r(x) > 0, \quad q(x) \geq 0 \quad \text{for} \quad x \in R; \quad \frac{1}{r(x)} \in L_1^{\text{loc}}(R),\ q(x) \in L_1^{\text{loc}}(R) \tag{1.2}$$

$$\lim_{|d|\to\infty} \left(\int_{x-d}^{x} \frac{dt}{r(t)} \cdot \int_{x-d}^{x} q(t)dt \right) = \infty. \tag{1.3}$$

Our goal is to study the inversion problem for (1.1) in the case when $q(x)$ is oscillating (for example, $q(x) = 1 + \sin(|x|^\theta), \theta \in (0, \infty)$). To be more precise, we find necessary and sufficient conditions under which, for all $p \in [1, \infty]$, the following assertions

1) equation (1.1) has a unique solution $y(x) \in L_p(R)$ and

$$y(x) = (Gf)(x) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} G(x,t)f(t)dt, \quad x \in R \tag{1.4}$$

2) $$\|G\|_{p\to p} < \infty. \tag{1.5}$$

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hold simultaneously. In (1.5), $G : L_p(R) \to L_p(R)$ is the linear integral operator defined by equality (1.4). Its kernel $G(x,t)$ is the Green function corresponding to (1.1):

$$G(x,t) = \begin{cases} u(x)v(t), & x \geq t \\ u(t)v(x), & x \leq t \end{cases} \tag{1.6}$$

and $\{u(x), v(x)\}$ is a special fundamental system of solutions (FSS) of the equation:

$$(r(x)z'(x))' = q(x)z(x), \qquad x \in R. \tag{1.7}$$

(see §2.) Below we say that, for a given equation (1.1), the inversion problem (i.p.) is regular in $L_p(R)$ if for all $p \in [1,\infty]$, 1)-2) hold simultaneously. Let us formulate the main result of the paper (Theorem 1.1). Introduce three auxiliary functions. Consider equations in $d \geq 0$:

$$1 = \int_{x-d}^{x} \frac{dt}{r(t)} \cdot \int_{x-d}^{x} q(t)dt, \qquad 1 = \int_{x}^{x+d} \frac{dt}{r(t)} \cdot \int_{x}^{x+d} q(t)dt \tag{1.8}$$

where x is fixed. For $x \in R$, each of the equations (1.8) has a unique finite positive solution (see [1]). Denote them by $d_1(x), d_2(x)$, respectively. Set for $x \in R$

$$\begin{gathered} \varphi(x) = \int_{x-d_1(x)}^{x} \frac{dt}{r(t)}, \quad \psi(x) = \int_{x}^{x+d_2(x)} \frac{dt}{r(t)}, \\ h(x) = \frac{\varphi(x)\psi(x)}{\varphi(x)+\psi(x)} \equiv \left(\int_{x-d_1(x)}^{x+d_2(x)} q(\xi)d\xi\right)^{-1}. \end{gathered} \tag{1.9}$$

Furthermore, for any fixed $x \in R$ the equation in $d \geq 0$

$$1 = \int_{x-d}^{x+d} \frac{dt}{r(t)h(t)}. \tag{1.10}$$

has a unique continuous positive solution (see [1]). Denote it by $d(x)$.

THEOREM 1.1. *For the regularity of the i.p. for (1.1) in $L_p(R)$, it is necessary and, under the condition:*

$$\alpha^{-1}\varphi(x) \leq \psi(x) \leq \alpha\varphi(x), \qquad x \in R, \qquad \alpha = \text{const} \tag{1.11}$$

sufficient that $B < \infty$. Here $B \stackrel{\text{def}}{=} \sup_{x \in R}(h(x)d(x))$.

Since the hypotheses of the theorem do not comprise any direct requirements to $r(x)$ and $q(x)$, one needs some comments.

Let us first consider condition (1.11). The proof of Theorem 1.1 is based on properties of the Green function (1.6). In particular, it is known that for $x,t \in R$ one has the following representation (1.12) of Davies-Harrell ([3]) and estimates (1.13) – (1.14)

$$G(x,t) = \sqrt{\rho(x)\rho(t)} \exp\left(-\frac{1}{2}\left|\int_x^t \frac{d\xi}{r(\xi)\rho(\xi)}\right|\right), \quad \rho(x) \stackrel{\text{def}}{=} G(x,t)\Big|_{t=x} \tag{1.12}$$

$$r(x)\,|\rho'(x)| < 1, \qquad x \in R \tag{1.13}$$

$$2^{-1}h(x) \le \rho(x) \le 2h(x), \qquad x \in R \tag{1.14}$$

(see also [1] and Theorem 3.1 below). In (1.12) $\rho(x)$ is a basic element for the construction of $G(x,t)$, and this function and its derivative are bounded a priori inequalities (1.14) and (1.13), respectively. We show (§3) that (1.11) is equivalent to the conditions

$$m \stackrel{\text{def}}{=} \sup_{x\in R} \left(r(x)\,|\rho'(x)|\right) < 1. \tag{1.15}$$

Below we analyze this condition in detail because in this paper the estimates for $G(x,t)$, whose proof is based on application of (1.15), are crucial. First, note that m may be any number from $[0,1)$ (say, arbitrarily close to 1), and therefore (1.15) is a weak strengthening of the a priori property (1.13). On the other hand, (1.15), (1.12) and (1.14) allow one to estimate the decreasing of $G(x,t)$ which depends on the deviation of x from t, and we did not manage to settle this problem in terms of the function $\rho(\cdot)$ without introducing condition (1.15). In order to clarify our intentions in what concerns (1.15), below we present a proof of this estimate. Note that, on the one hand, the proof is very simple; and, on the other hand, while applying this inequality we show the main idea of the paper. So, let $m = 1-\varepsilon$, $\varepsilon \in (0,1]$. Then for $\xi \in R$ and $x \ge t$, we get

$$\frac{\rho'(\xi)}{\rho(\xi)} \le \frac{1-\varepsilon}{r(\xi)\rho(\xi)} \Rightarrow \frac{\rho(x)}{\rho(t)} \le \exp\left((1-\varepsilon)\int_t^x \frac{d\xi}{r(\xi)\rho(\xi)}\right). \tag{1.16}$$

Hence for $x \ge t$, in view of (1.12) and (1.16), we get

$$\begin{aligned} G(x,t) &= \rho(t)\sqrt{\frac{\rho(x)}{\rho(t)}}\exp\left(-\frac{1}{2}\int_t^x \frac{d\xi}{r(\xi)\rho(\xi)}\right) \\ &\le \rho(t)\exp\left(-\frac{\varepsilon}{2}\left|\int_x^t \frac{d\xi}{r(\xi)\rho(\xi)}\right|\right). \end{aligned} \tag{1.17}$$

For $x \le t$, inequality (1.17) can be proved in a similar way. Indeed, for $\xi \in R$ and $x \le t$, we get

$$\begin{aligned} &\frac{\rho'(\xi)}{\rho(\xi)} \ge -\frac{1-\varepsilon}{r(\xi)\rho(\xi)} \Rightarrow \frac{\rho(t)}{\rho(x)} \ge \exp\left(-(1-\varepsilon)\int_x^t \frac{d\xi}{r(\xi)\rho(\xi)}\right) \Rightarrow \\ &\rho(x) \le \rho(t)\exp\left((1-\varepsilon)\int_x^t \frac{d\xi}{r(\xi)\rho(\xi)}\right). \end{aligned}$$

Thus, now (1.17) for $x \le t$ follows from the last inequality using (1.12) in the same way as (1.17) for $x \ge t$. As to development of estimate (1.17) with the help of inequalities (1.14), we postpone it till §8, where we prove Theorem 1.1 using the final estimate for $G(x,t)$.

Continuing our analysis, we note that under condition (1.15), a FSS $\{u(x), v(x)\}$ of equation (1.7) has many common properties with the analogous FSS of the simplest equation $z''(x) = z(x)$ (see Lemma 3.1 of §3 for a list of such properties). Therefore in the sequel, equations (1.1) and (1.7) satisfying (1.15) are called standard (because of the standard behavior, in the sense of Lemma 3.1, of the FSS $\{u(x), v(x)\}$). We denote by S the class of all standard equations and write (1.1) $\in S$, (1.7) $\in S$. Such a classification of equations (1.1) may be useful since in the class S one can thoroughly investigate the properties of the solution of (1.1) (see, for example, [2]).

As to the condition $B < \infty$, we note that in class S it is equivalent to the following:

$$A \overset{\text{def}}{=} \inf_{x\in R}\left(\frac{1}{2\mu(x)}\int_{x-\mu(x)}^{x+\mu(x)} q(t)dt\right) > 0, \tag{1.18}$$

where $\mu(x)$ is an auxiliary function in $r(x)$ and $q(x)$ which is continuous and positive for $x \in R$ with

$$\overline{\lim_{x\to-\infty}}(x+\mu(x)) = -\infty, \qquad \underline{\lim}_{x\to\infty}(x-\mu(x)) = \infty. \tag{1.19}$$

(see §3). Hence, the condition $B < \infty$ means that some local average value (of the Steklov type [6]) of the function $q(x)$ must be separated from zero on R. (A similar condition related to 1) – 2) was used in [1].) To conclude, note that in order to solve the problem of the validity of assertions 1) – 2) for equations (1.1) $\notin S$, one needs methods and tools different from those used in this paper. We shall investigate non-standard equations (1.1) in our forthcoming papers.

2. Preliminaries. Throughout Section 2 we assume that (1.2) – (1.3) hold. We denote by c absolute positive constants which are not essential for the exposition, and may even differ within a single chain of calculations.

THEOREM 2.1. [1]. *There exists a FSS $\{u(x), v(x)\}$ of (1.7) such that*

$$\begin{aligned} &u(x) > 0,\ \ v(x) > 0,\ \ u'(x) \le 0,\ \ v'(x) \ge 0 && \textit{for}\ \ x \in R\\ &r(x)[v'(x)u(x) - u'(x)v(x)] = 1 && \textit{for}\ \ x \in R \end{aligned} \tag{2.1}$$

$$\lim_{x\to-\infty}\frac{v(x)}{u(x)} = \lim_{x\to\infty}\frac{u(x)}{v(x)} = 0.$$

COROLLARY 2.1.1. *If $z(x) \in L_p(R)$, $p \in [1,\infty)$ satisfies (1.7), then $z(x) \equiv 0$.*
A FSS $\{u(x), v(x)\}$ with properties (2.1) is called a principal FSS (PFSS) [1].

LEMMA 2.1. [1,3]. *For $x \in R$, a PFSS of (1.7) admits the representations*

$$u(x) = \sqrt{\rho(x)}\exp\left(-\frac{1}{2}\int_{x_1}^{x}\frac{d\xi}{r(\xi)\rho(\xi)}\right), \quad v(x) = \sqrt{\rho(x)}\exp\left(\frac{1}{2}\int_{x_1}^{x}\frac{d\xi}{r(\xi)\rho(\xi)}\right) \tag{2.2}$$

where $\rho(x) = u(x)v(x)$ and x_1 is the unique root of the equation $u(x) = v(x)$. Moreover, for $G(x,t)$ (see (1.6)) one has (1.12), and for $\rho(x)$ one has

$$\int_{-\infty}^{0}\frac{d\xi}{r(\xi)\rho(\xi)} = \int_{0}^{\infty}\frac{d\xi}{r(\xi)\rho(\xi)} = \infty. \tag{2.3}$$

LEMMA 2.2. [1]. *For a fixed $x \in R$ each of the equations (1.8), (1.10), has a unique finite positive solution. Denote them by $d_1(x), d_2(x), d(x)$ respectively. The function $d(x)$ is continuous for $x \in R$.*

THEOREM 2.2. [1]. *For $x \in R$ the following inequalities hold (see (1.9)):*

$$2^{-1}v(x) \le [r(x)v'(x)]\,\varphi(x) \le 2v(x), \quad 2^{-1}u(x) \le [r(x)|u'(x)|]\,\psi(x) \le 2u(x) \tag{2.4}$$

$$2^{-1}h(x) \le \rho(x) \le 2h(x). \tag{2.5}$$

In particular, for $x \in R$ one has inequalities $v'(x) > 0$, $u'(x) < 0$.

LEMMA 2.3. [1]. *Let $x \in R$, $t \in [x - d(x),\ x + d(x)]$. Then,*

$$\alpha^{-1} \leq \frac{v(t)}{v(x)}, \frac{u(t)}{u(x)}, \frac{\rho(t)}{\rho(x)} \leq \alpha, \quad \alpha = \exp(2) \tag{2.6}$$
$$(4\alpha)^{-1} h(x) \leq h(t) \leq 4\alpha h(x).$$

DEFINITION 2.1. Let $\varkappa(t)$ be positive and continuous function for $t \in R$. We say that segments $\{\Delta_n, n = \pm 1, \pm 2, \dots\}$ form an $R(x, \varkappa(\cdot))$ covering of R, if

1. $\Delta_n = [\Delta_n^-, \Delta_n^+] \stackrel{\text{def}}{=} [x_n - \varkappa(x_n), x_n + \varkappa(x_n)], \quad n = \pm 1, \pm 2, \dots$
2. $\Delta_{n+1}^- = \Delta_n^+$ if $n \geq 1$; $\Delta_{n-1}^+ = \Delta_n^-$ if $n \leq -1$
3. $\Delta_1^- = \Delta_{-1}^+ = x, \quad \bigcup_{n \neq 0} \Delta_n = R.$

LEMMA 2.4. [1]. *For every $x \in R$ there exists an $R(x, d(\cdot))$-covering.*

Let us define

$$H := \sup_{x \in R} \int_{-\infty}^{\infty} G(x,t)\ dt. \tag{2.7}$$

LEMMA 2.5. [1]. *Let $p \in [1, \infty], H < \infty$, and suppose that (1.7) has no solutions $z(x) \in L_p(R)$ apart from $z(x) \equiv 0$. Then assertions 1) - 2) of §1 hold, and $\|G\|_{p \to p} \leq H$.*

3. Statement of results. In addition to Theorem 1.1, our main result, the following assertions hold.

THEOREM 3.1. *For $x \in R$, (1.13) holds. Conditions (1.11) and (1.15) are equivalent.*

THEOREM 3.2. *Let $p \in [1, \infty]$. Then $\|G\|_{p \to p} \geq 81^{-1} B$. In addition, if (1.1) $\in S$, then*

$$81^{-1} B \leq \|G\|_{p \to p} \leq C(\alpha) B, \quad C(\alpha) = 72 \left(1 - \exp\left(-\frac{1}{2(4\alpha + 1)}\right)\right)^{-1}. \tag{3.1}$$

Here, G and B are defined in (1.4) and in Theorem 1.1, and α is a constant from (1.11).

LEMMA 3.1. *Let (1.7) $\in S$, $\varepsilon = 1 - m > 0$ (see (1.15)). Then*

$$r(x)\ \rho(x)\ v'(x) \geq \frac{\varepsilon}{2} v(x), \ r(x)\ \rho(x)|u'(x)| \geq \frac{\varepsilon}{2} u(x), \quad x \in R \tag{3.2}$$

$$\lim_{x \to -\infty} v(x) = \lim_{x \to -\infty} r(x) v'(x) = \lim_{x \to \infty} u(x) = \lim_{x \to \infty} r(x)\ u'(x) = 0 \tag{3.3}$$

$$\lim_{x \to \infty} v(x) = \lim_{x \to \infty} r(x) v'(x) = \lim_{x \to -\infty} u(x) = \lim_{x \to -\infty} r(x)\ |u'(x)| = \infty. \tag{3.4}$$

COROLLARY 3.1.1. *If $z(x) \in L_p(R), p \in [1, \infty]$ satisfies (1.7) $\in S$, then $z(x) \equiv 0$.*

Introduce a function $\mu(x)$. For a fixed $x \in R$ consider an equation in $\mu \geq 0$:

$$1 = \int_{x-\mu}^{x+\mu} q(\xi)\ h(\xi)\ d\xi. \tag{3.5}$$

LEMMA 3.2. *Let (1.11) hold. Then (3.5) has at least one positive solution. Let us define*

$$\mu(x) = \inf_{\mu \geq 0} \left\{ \mu : \int_{x-\mu}^{x+\mu} q(\xi) h(\xi) d\xi = 1 \right\}.$$

The function $\mu(x)$ is continuous for $x \in R$, and (1.19) *holds.*

THEOREM 3.3. *The i.p. for* (1.1) $\in S$ *is regular in $L_p(R)$ if and only if $A > 0$ (see* (1.18)*). Moreover, for $p \in [1,\infty]$ one has inequalities: $c^{-1}A^{-1} \le \|G\|_{p\to p} \le cA^{-1}$.*

COROLLARY 3.3.1. *Let* (1.1) $\in S$ *and $q(x) \to 0$ as $x \to -\infty$ or $x \to \infty$. Then the i.p. for* (1.1) *is not regular in $L_p(R)$. In particular, $\|G\|_{p\to p} = \infty$ for any $p \in [1,\infty]$.*

COROLLARY 3.3.2. *Under condition* (1.11), $B < \infty$ *if and only if* $A > 0$.

COROLLARY 3.3.3. *Let* (1.1) $\in S$. *If the i.p. for* (1.1) *is regular in $L_p(R)$, then*

$$\int_{-\infty}^{0} q(t)dt = \int_{0}^{\infty} q(t)dt = \infty.$$

The following theorem shows that, under an additional condition to (1.2) – (1.3),

$$\gamma^{-1} \le r(x) \le \gamma, \qquad x \in R, \qquad \gamma = \text{const}, \tag{3.6}$$

the regularity of the i.p. for (1.1) in $L_p(R)$ depends only on properties of $q(x)$. First, for a fixed $x \in R$ consider an equation in $d \ge 0$:

$$2 = d\int_{x-d}^{x+d} q(t)dt. \tag{3.7}$$

Under conditions (1.2), (1.3), and (3.6), equation (3.7) has a unique positive continuous solution [1]. Denote it by $\tilde d(x)$. Note an inequality [1]:

$$(2\gamma+2)^{-1}\tilde d(x) \le \rho(x) \le 2^{-1}(2\gamma+1)\,\tilde d(x), \qquad x \in R. \tag{3.8}$$

THEOREM 3.4. *Under conditions* (1.2), (1.3) *and* (3.6), *the i.p. for* (1.1) *is regular in $L_p(R)$ if and only if $K < \infty$. Here $K \stackrel{\text{def}}{=} \sup_{x\in R} \tilde d(x)$.*

Remark. The function $\tilde d(x)$ was introduced by M. Otelbaev (see [5]).

Remark. In the proofs below we mainly used the results of [1] (see §2). These assertions hold under condition (1.2) and condition (3.9) which is weaker than (1.3):

$$\inf_{x\in R}\lim_{|d|\to\infty}\left(\int_{x-d}^{x}\frac{dt}{r(t)}\cdot\int_{x-d}^{x} q(t)\,dt\right) > 0 \tag{3.9}$$

(see [2]). Thus the theorems proved in this paper hold under assumptions (1.2) and (3.9), and their proofs do not differ from the one given in §5-§8.

4. Conditions for regularity of the inversion problem expressed in terms of coefficients. Examples. The main result of the present paper is formulated in terms of the auxiliary functins $\varphi(x)$, $\psi(x)$, $h(x)$ and $d(x)$ (see Theorem 1.1). Therefore, to apply it, one has to extract the needed information on these functions from the properties of the coefficients $r(x)$ and $q(x)$. Theorem 1.1 allows one to use sharp by order two-sided estimates of $\varphi(x)$, $\psi(x)$, $h(x)$ and $d(x)$ instead of their values. This significantly simplifies the study of concrete equations (see Examples 1 and 2 below). In Theorem 4.1, we give inequalities for the auxiliary functions expressed in terms of $r(x)$ and $q(x)$. See §9 for the proof of this theorem.

THEOREM 4.1. *Suppose that* (1.2) *holds and there exist continuous positive functions $r_1(x), q_1(x)$ and functions $r_2(x) \in L_1^{\text{loc}}(R), q_2(x) \in L_1^{\text{loc}}(R)$ such that:*

(1) $r(x) = r_1(x) + r_2(x), \quad q(x) = q_1(x) + q_2(x), \quad x \in R,$ (4.1)

(2) *for some constants* $a, b (b \geq 3a > 0)$ *and* $|x| \gg 1$ *one has the inequalities*

$$\frac{1}{a} \leq \frac{r_1(t)}{r_1(x)}, \frac{q_1(t)}{q_1(x)} \leq a \quad \text{for } |t-x| \leq b\hat{d}(x), \ \hat{d}(x) = \sqrt{\frac{r_1(x)}{q_1(x)}}; \tag{4.2}$$

(3) *there exists* $\delta \in (0,1]$ *such that* $r(x) \geq \delta r_1(x), \quad x \in R$;

(4) $\varkappa_1(x) \to 0, \ \varkappa_2(x) \to 0$ *as* $|x| \to \infty$, *where*

$$\begin{aligned} \varkappa_1(x) &= \frac{1}{\sqrt{r_1(x) q_1(x)}} \sup_{|z| \leq b\hat{d}(x)} \left| \int_x^{x+z} q_2(t) dt \right|, \\ \varkappa_2(x) &= \sqrt{r_1(x) q_1(x)} \sup_{|z| \leq b\hat{d}(x)} \left| \int_x^{x+z} \frac{r_2(t)}{r_1^2(t)} dt \right|. \end{aligned} \tag{4.3}$$

Then the following assertions hold:

A) *Condition* (1.3) *is satisfied, equation* (1.7) $\in S$ *and*

$$c^{-1} \leq h(x) \sqrt{r_1(x) q_1(x)} \leq c \quad \text{for } x \in R.$$

B) *If, in addition,* $b \geq 160 a^3 \delta^{-2}$, *then*

$$c^{-1} \hat{d}(x) \leq d(x) \leq c\hat{d}(x) \quad \text{for } x \in R.$$

C) *The i.p. for* (1.1) *is regular in* $L_p(R)$ *if and only if* $\inf_{x \in R} q_1(x) > 0$.

Example 1. Consider equation (1.1) with the coefficients for $x \in R$

$$r(x) = 1 + x^2 + \frac{1+x^2}{2} \cos(e^{|x|}), \quad q(x) = e^{|x|} + e^{|x|} \sin(e^{|x|}).$$

According to Theorem 4.1, choose $r_1(x) = 1 + x^2$, $q_1(x) = e^{|x|}$. Then $\hat{d}(x) = (1+x^2)^{1/2} e^{-|x|/2} \to 0$ as $|x| \to \infty$. Let us verify that for any fixed $b > 0$ all the hypotheses of Theorem 4.1 are satisfied. One can assume $x \geq 0$ because all the above functions are even. Since $\hat{d}(x) \to 0$ as $|x| \to \infty$ and

$$\frac{r_1(t)}{r_1(x)} = \frac{1+t^2}{1+x^2} = 1 + (t-x) \frac{t+x}{1+x^2}, \quad \frac{q_1(t)}{q_1(x)} = e^{t-x},$$

one can see that (2) holds, say for $a = \frac{5}{4}$. Clearly in this case, $\delta = \frac{1}{2}$, and (3) also holds. Furthermore,

$$\varkappa_1(x) = \frac{1}{\sqrt{1+x^2} e^{x/2}} \sup_{|z| \leq b\hat{d}(x)} \left| \int_x^{x+z} e^t \sin(e^t) dt \right| \leq \frac{2}{\sqrt{1+x^2} e^{x/2}}.$$

To estimate $\varkappa_2(x)$, we use the second mean theorem [7, ch.12, §3], which can be applied here for $x \geq 0$:

$$\begin{aligned} \varkappa_2(x) &= \frac{1}{2} \sqrt{1+x^2} e^{x/2} \sup_{|z| \leq b\hat{d}(x)} \left| \int_x^{x+z} \frac{e^t \cos(e^t) dt}{(1+t^2) e^t} \right| \\ &\leq \frac{1}{2} \sqrt{1+x^2} e^{x/2} \sup_{|z| \leq b\hat{d}(x)} \frac{1}{(1+x^2) e^x} \left| \int_x^{x+z} e^t \cos(e^t) dt \right| \leq \frac{c}{\sqrt{1+x^2} e^{x/2}}. \end{aligned}$$

From the estimates for $\varkappa_1(x)$ and $\varkappa_2(x)$, it follows that all hypotheses of Theorem 4.1, including A), B) and C) are satisfied. In particular, for $x \in R$, one has inequalities:

$$\frac{c^{-1}}{\sqrt{1+x^2}}\,\frac{1}{e^{|x|/2}} \le h(x) \le \frac{c}{\sqrt{1+x^2}}\,\frac{1}{e^{|x|/2}}, \qquad \frac{c^{-1}\sqrt{1+x^2}}{e^{|x|/2}} \le d(x) \le \frac{c\sqrt{1+x^2}}{e^{|x|/2}};$$

and i.p. for (1.1) here is regular in $L_p(R)$ because $q_1 = e^{|x|}$ and $\inf_{x\in R} e^{|x|} = 1 > 0$.

Example 2. Consider (1.1) for which (3.6) holds and $q(x)$ is of the form

$$q(x) = 1 + \sin(|x|^\theta), \quad \theta \in (0,\infty) \tag{4.4}$$

The i.p. for such an equation is regular in $L_p(R)$ for $\theta \in [1,\infty)$ and irregular for $\theta \in (0,1)$. Using Theorem 3.4, one has to verify $K < \infty$.

Below we use the following simple fact:

LEMMA 4.1. *Let $\tilde d(x)$ be the solution of (3.7), and let $\eta > 0$. The inequality $\eta \ge \tilde d(x)$ ($\eta \le \tilde d(x)$) holds if and only if*

$$\frac{1}{\eta} \le \frac{1}{2}\int_{x-\eta}^{x+\eta} q(\xi)\,d\xi \quad \left(\frac{1}{\eta} \ge \frac{1}{2}\int_{x-\eta}^{x+\eta} q(t)\,dt\right). \tag{4.5}$$

Proof.
Necessity. Let $\eta \ge \tilde d(x)$. Then $[x-\tilde d(x), x+\tilde d(x)] \subseteq [x-\eta, x+\eta]$, and, therefore,

$$\frac{1}{\eta} \le \frac{1}{\tilde d(x)} = \frac{1}{2}\int_{x-\tilde d(x)}^{x+\tilde d(x)} q(t)\,dt \le \int_{x-\eta}^{x+\eta} q(t)\,dt.$$

Sufficiency. Suppose that (4.5) holds. Assume the contrary: $\eta < \tilde d(x)$. Then $[x-\eta, x+\eta] \subset [x-\tilde d(x), x+\tilde d(x)]$ and, thus we obtain a contradiction

$$\frac{1}{\eta} \le \frac{1}{2}\int_{x-\eta}^{x+\eta} q(\xi)\,d\xi \le \frac{1}{2}\int_{x-\tilde d(x)}^{x+\tilde d(x)} q(\xi)\,d\xi = \frac{1}{\tilde d(x)}.$$

□

Return to our example for $\theta \in (0,1)$. Let us show that $\tilde d(x_k) \to \infty$ for $k \to \infty$ if $x_k = \left(2k\pi + \frac{3}{2}\pi\right)^{1/\theta}$, $k = 1, 2, \dots$. To do that, let us estimate J_k

$$J_k = \int_{x_k-\eta_k}^{x_k+\eta_k} \left[1 + \sin\left(|t|^\theta\right)\right]\,dt,$$

where $\eta_k = x_k^{\frac{1-\theta}{2}}$ It is easy to see that

$$J_k = 2\eta_k + \int_0^{\eta_k} \left[\sin\,(x_k+z)^\theta + \sin\,(x_k-z)^\theta\right]\,dz.$$

Since $\eta_k \cdot x_k^{-1} \to 0$ for $k \to \infty$, we have for $z \in [0,\eta_k]$:

$$(x_k+z)^\theta = x_k^\theta\left[1+\frac{z}{x_k}\right]^\theta = x_k^\theta + \theta x_k^\theta\left(\frac{z}{x_k}\right) + \frac{\theta(\theta-1)}{2!}\,x_k^\theta\left(\frac{z}{x_k}\right)^2 + \dots \stackrel{\text{def}}{=} x_k^\theta + \alpha_k(z)$$

$$(x_k-z)^\theta = x_k^\theta\left[1-\frac{z}{x_k}\right]^\theta = x_k^\theta - \theta x_k^\theta\left(\frac{z}{x_k}\right) + \frac{\theta(\theta-1)}{2!}\,x_k^\theta\left(\frac{z}{x_k}\right)^2 - \dots \stackrel{\text{def}}{=} x_k^\theta + \beta_k(z).$$

Note that $\eta_k x_k^{\theta-1} \to 0$ for $k \to \infty$, and the inequalities

$$|\alpha_k(z)| \le 2\theta x_k^{\theta-1} z, \quad |\beta_k(z)| \le 2\theta x_k^{\theta-1} z$$

hold for all $k \gg 1$ uniformly in $z \in [0, \eta_k]$. Hence, for $k \gg 1$ we obtain

$$J_k = 2\eta_k + \int_0^{\eta_k} \left[\sin\left(x_k^\theta + \alpha_k(z)\right) + \sin\left(x_k^\theta + \beta_k(z)\right)\right] \, dz = 2\eta_k$$

$$- \int_0^{\eta_k} \left[\cos(\alpha_k(z)) + \cos(\beta_k(z))\right] \, dz$$

$$= 2\eta_k - \int_0^{\eta_k} \left[2 - \frac{\alpha_k(z)^2 + \beta_k(z)^2}{2!} + \frac{\alpha_k(z)^4 + \beta_k(z)^4}{4!} - \dots\right] \, dz$$

$$\le \int_0^{\eta_k} \frac{\alpha_k(z)^2 + \beta_k(z)^2}{2} \, dz \le 4\theta^2 \, x_k^{2\theta-2} \int_0^{\eta_k} z^2 dz = \frac{4}{3} \, \theta^2 \, x_k^{2\theta-2} \eta_k^3 \le \frac{2}{\eta_k}$$

Thus for $k \gg 1$ by Lemma 4.1, we conclude that $\tilde{d}(x_k) \ge \eta_k$. Since $\eta_k \to \infty$ for $k \to \infty$, by Theorem 3.4 we obtain our claim for $\theta \in (0, 1)$.

Let $\theta \in (1, \infty)$. For $x \gg 1$ we have

$$\frac{1}{2} \int_{x-2}^{x+2} \left(1 + \sin t^\theta\right) dt = 2 - (\theta - 1) \frac{\cos t^\theta}{t^{\theta-1}} \Bigg|_{x-2}^{x+2} + (\theta - 1) \int_{x-2}^{x+2} \frac{\cos t^\theta dt}{t^{\theta-1}} > 2 - \frac{c}{x^{\theta-1}} \ge \frac{1}{2}.$$

Hence, $\tilde{d}(x) \le 2$ by Lemma 4.1. Similarly, we obtain $\tilde{d}(x) \le 2$ for $x \ll -1$. Since the function $\tilde{d}(x)$ is continuous, then the above proves that it is bounded for $x \in R$ and and inequality $K < \infty$ holds.

Let $\theta = 1$. Then,

$$\frac{1}{2} \int_{x-4}^{x+4} (1 + \sin t) \, dt = 4 - \cos t \, \Big|_{x-4}^{x+4} > 2 > \frac{1}{4}.$$

Hence, $\tilde{d}(x) \le 4$ for $x \ge 4$, and, similarly, $\tilde{d}(x) \le 4$ for $x \le -4$. Thus, $\tilde{d}(x)$ is bounded for $x \in R$ because it is continuous, and inequality $K < \infty$ holds.

Remark. All the assertions of Theorem 4.1 follows from the two-sided estimates for the roots $d_1(x)$ and $d_2(x)$ of equations (1.8) which are obtained in the course of the proof, and from Theorem 1.1 (see (§3 and §9)). The proof of these estimates rests on technical tools whose application guarantees that conditions (1) – (4) hold (see above). To any other method of obtaining estimates for $d_1(x)$ and $d_2(x)$ there would correspond another variant of an assertion such as Theorem 4.1. Thus, one can find some concrete conditions for regularity of the inversion problem for equation (1.1) in $L_p(R)$; each of such conditions will follows from Theorem 1.1.

Note that such a relationship between the general criterion (Theorem 1.1) and "concrete" sufficient conditions (such as Theorem 4.1) is typical. For example, in the theory of number series one has the same relationship between the Cauchy convergence criterion and various particular conditions for convergence.

5. Auxiliary assertions. In this section we prove Lemmas 3.1, 3.2 and some technical assertions.

Proof of Lemma 3.1. For $x \in R$ from (2.2) it follows that

$$(5.1) \quad 2r(x)\rho(x)v'(x) = [1 + r(x)\rho'(x)]v(x), \quad 2r(x)\rho(x)u'(x) = -[1 - r(x)\rho'(x)]u(x).$$

From (5.1) and (2.1) one deduces (1.13). Let $m = 1 - \varepsilon$, $\varepsilon \in (0,1]$ (see (1.15)). Then for $x \in R$ from (1.15) we get (5.2) and (3.2):

$$\frac{\varepsilon}{2r(x)\rho(x)} \le \frac{v'(x)}{v(x)} \le \frac{1}{r(x)\rho(x)}, \quad \frac{\varepsilon}{2r(x)\rho(x)} \le \frac{|u'(x)|}{u(x)} \le \frac{1}{r(x)\rho(x)}, \quad x \in R. \tag{5.2}$$

From (5.2), for $x \ge t$, $x, t \in R$ we obtain:

$$\exp\left(\frac{\varepsilon}{2}\int_t^x \frac{d\xi}{r(\xi)\rho(\xi)}\right) \le \frac{v(x)}{v(t)} \le \exp\left(\int_t^x \frac{d\xi}{r(\xi)\rho(\xi)}\right). \tag{5.3}$$

From (5.3) and (2.3) we get (3.3) and (3.4) for $v(x)$ and, similarly, for $u(x)$. Since $v'(\xi) \ne 0$ for $\xi \in R$ (see (2.4)), one has

$$\frac{(r(\xi)v'(\xi))'}{r(\xi)v'(\xi)} = q(\xi)\,\frac{v(\xi)}{r(\xi)v'(\xi)}\,, \qquad \xi \in R. \tag{5.4}$$

Hence, for $c = r(0)v'(0)$ we have

$$r(x)v'(x) = c\,\exp\left(\int_0^x q(\xi)\,\frac{v(\xi)d\xi}{r(\xi)v'(\xi)}\right)\,, \qquad x \ge 0 \tag{5.5}$$

$$r(x)v'(x) = c\,\exp\left(-\int_x^0 q(\xi)\,\frac{v(\xi)d\xi}{r(\xi)v'(\xi)}\right), \qquad x \le 0 \tag{5.6}$$

Let us verify that if (1.7) $\in S$, then

$$\int_{-\infty}^0 q(\xi)\rho(\xi)\,d\xi = \infty, \qquad \int_0^\infty q(\xi)\rho(\xi)\,d\xi = \infty. \tag{5.7}$$

Indeed, for $x \ge 0$ from (5.2) and (5.5), one deduces (5.8) and (5.8) implies (5.9):

$$\frac{\varepsilon v(x)}{2} \le r(x)\rho(x)v'(x) = c\rho(x)\exp\left(\int_0^x \frac{q(\xi)v(\xi)d\xi}{r(\xi)v'(\xi)}\right) \le c\rho(x)\exp\left(\frac{2}{\varepsilon}\int_0^x q(\xi)\rho(\xi)d\xi\right) \tag{5.8}$$

$$\frac{\varepsilon}{2}\,\exp\left(-\frac{2}{\varepsilon}\int_0^x q(\xi)\rho(\xi)d\xi\right) \le cu(x), \quad x \ge 0. \tag{5.9}$$

From (5.9) and (3.3), for $u(x)$, we deduce the second equality of (5.7) and, similarly, the first one. Now (5.2), (5.5) – (5.7) imply (3.3) – (3.4) for $r(x)v'(x)$ and $r(x)u'(x)$. □

Proof of Corollary 3.1.1. For $p \in [1, \infty)$ the proof follows from Corollary 2.1.1. Because of (2.1) and (3.3) – (3.4), it is obvious for $p = \infty$. □

Proof of Lemma 3.2. The existence of a positive root of (3.5) follows from (5.7) and (2.5). Let us check the inequalities:

$$|\mu(x+s) - \mu(x)| \le |s| \qquad \text{for} \qquad |s| \le \mu(x), \quad x \in R. \tag{5.10}$$

Let $s \in [0, \mu(x)]$. Then,

$$1 = \int_{x-\mu(x)}^{x+\mu(x)} q(\xi)h(\xi)d\xi \le \int_{(x+s)-(s+\mu(x))}^{(x+s)+(s+\mu(x))} q(\xi)h(\xi)d\xi$$

$$1 = \int_{x-\mu(x)}^{x+\mu(x)} q(\xi)h(\xi)d\xi \ge \int_{(x+s)-(\mu(x)-s)}^{(x+s)+(\mu(x)-s)} q(\xi)h(\xi)d\xi.$$

From this we conclude that $\mu(x+s) \le \mu(x)+s$, $\mu(x+s) \ge \mu(x)-s$, i.e., (5.10) holds. The case $s \in [-\mu(x), 0]$ can be treated in a similar way. From (5.10) it follows that $\mu(x)$ is continuous for $x \in R$. Let us verify, for example, the second equality of (1.19). If it is not true, then there are x_0 and $\{x_n\}_{n=1}^{\infty}$ such that $x_n \to \infty$ as $n \to \infty$, and $x_n - \mu(x_n) \le x_0$, $n = 1, 2, \dots$. Then, by (5.7) and (2.5) we get a contradiction:

$$1 = \int_{x_n-\mu(x_n)}^{x_n+\mu(x_n)} q(\xi)h(\xi)d\xi \ge \int_{x_0}^{x_n} q(\xi)h(\xi)d\xi \ge \frac{1}{2}\int_{x_0}^{x_n} q(\xi)\rho(\xi)d\xi \to \infty \quad \text{as } n \to \infty.$$

□

LEMMA 5.1. *Let* (1.7) $\in S$, $x \in R$, $t \in \Delta(x) \overset{\text{def}}{=} [x-\mu(x), x+\mu(x)]$. *Then,*

$$c^{-1} \le \frac{u(t)}{u(x)}, \frac{v(t)}{v(x)}, \frac{\rho(t)}{\rho(x)}, \frac{h(t)}{h(x)} \le c. \tag{5.11}$$

Proof. From (5.4) we get (5.12) and (5.13):

$$\frac{r(t)v'(t)}{r(x)v'(x)} = \exp\left(\int_x^t \frac{q(\xi)v(\xi)d\xi}{r(\xi)v'(\xi)}\right), \qquad x \le t \tag{5.12}$$

$$\frac{r(t)v'(t)}{r(x)v'(x)} = \exp\left(\int_t^x \frac{q(\xi)v(\xi)d\xi}{r(\xi)v'(\xi)}\right), \qquad x \ge t. \tag{5.13}$$

From (5.2), (5.12) and (2.5), for $t \in [x, x+\mu(x)]$, it follows that

$$\frac{r(t)v'(t)}{r(x)v'(x)} \le \exp\left(\frac{2}{\varepsilon}\int_x^t q(\xi)\rho(\xi)d\xi\right) \le \exp\left(\frac{4}{\varepsilon}\int_{\Delta(x)} q(\xi)h(\xi)d\xi\right) = \exp\left(\frac{4}{\varepsilon}\right). \tag{5.14}$$

Since $r(t)v'(t)$ does not decrease, (5.14) also holds for $t \in [x-\mu(x), x]$. Similarly, for $t \in [x-\mu(x), x]$, from (5.2), (5.13) and (2.5), we get

$$\frac{r(x)v'(x)}{r(t)v'(t)} \le \exp\left(\frac{2}{\varepsilon}\int_t^x q(\xi)\rho(\xi)d\xi\right) \le \exp\left(\frac{4}{\varepsilon}\int_{\Delta(x)} q(\xi)h(\xi)d\xi\right) = \exp\left(\frac{4}{\varepsilon}\right) \tag{5.15}$$

and $r(t)v'(t)$ does not decrease, and (5.15) remains true for $t \in [x, x+\mu(x)]$. Thus, for $t \in \Delta(x)$, $x \in R$ one has the inequality

$$c^{-1}r(x)v'(x) \le r(t)v'(t) \le c\, r(x)v'(x). \tag{5.16}$$

Now, from (5.16) and (5.2), for $t \in \Delta(x)$, we get

$$\frac{u(t)}{u(x)} = \frac{\rho(t)}{v(t)} \cdot \frac{v(x)}{\rho(x)} \le \frac{2}{\varepsilon}\,\frac{r(x)v'(x)}{r(t)v'(t)}, \quad \frac{u(t)}{u(x)} = \frac{\rho(t)}{v(t)}\,\frac{v(x)}{\rho(x)} \ge \frac{\varepsilon}{2}\,\frac{r(x)v'(x)}{r(t)v'(t)}. \tag{5.17}$$

From (5.16) – (5.17) we deduce (5.11) for $u(x)$ and, similarly, for $v(x)$. This immediately implies (5.11) for $\rho(x)$ and, by (2.5), also for $h(x)$. □

LEMMA 5.2. *If* (1.11) *holds, then for* $x \in R$ *there exists an* $R(x, \mu(\cdot))$*-covering.*

Proof. Let $\nu(t) = t-\mu(t)-x$. By Lemma 3.2, $\nu(t)$ is continuous for $t \in R$. Clearly, $\nu(x) = -\mu(x) < 0$. By (1.19), one has $\nu(t_0) > 0$ for some $t_0 > x$. Then $\nu(t_1) = 0$ for some $t_1 \in (x, t_0)$, i.e., $x = t_1 - \mu(t_1), t_1 > x$. Set $\Delta_1 = [t_1 - \mu(t_1),\ \ t_1 + \mu(t_1)]$. Similarly, one can construct segments $\Delta_2, \Delta_3, \dots$. Let us verify that $\bigcup_{n=1}^{\infty} \Delta_n = [x, \infty)$. Assume the contrary. Then there is $x_0 > x$, such that $\Delta_n^+ < x_0$ for all $n = 1, 2, \dots$.

By construction, the sequence $\{t_n\}_{n=1}^{\infty}$ is increasing and bounded by x_0. Let ξ be its limit, $\xi \in (x, x_0]$. Since $\infty > x_0 - x \geq 2 \sum_{n=1}^{\infty} \mu(t_n)$, one has $\mu(t_n) \to 0$ as $n \to \infty$. Since $\mu(\cdot)$ is continuous, we get $\mu(\xi) = 0$, a contradiction to Lemma 3.2. The other assertions of the lemma can be checked similarly. □

LEMMA 5.3. *For the regularity of the i.p. for* (1.1) *in* $L_p(R)$, *it is necessary and, under the condition* (1.1) $\in S$, *sufficient that* $H < \infty$. *One has* $\|G\|_{p\to p} \leq H$.

Proof. Necessity. Let the i.p. for (1.1) be regular in $L_p(R)$. Then the operator $G\colon C(R) \to C(R)$ is bounded (see (1.4)). Hence, for $f(x) \equiv 1$, $x \in R$ one has

$$H = \sup_{x\in R} \left| \int_{-\infty}^{\infty} G(x,t) f(t) dt \right| = \|(Gf)(x)\|_{C(R)} \leq \|G\|_{C(R)\to C(R)} \|f\|_{C(R)} < \infty.$$

The proof of sufficiency follows from Lemma 2.5 and Corollary 3.1.1. □

6. A criterion for a homogeneous Sturm-Liouville equation to be standard. In this section we prove Theorem 3.1.

Proof. Inequality (1.13) follows from (2.1):

$$\begin{aligned}(r(x)\rho'(x))^2 &= r^2(x)\,[v'(x)u(x) + u'(x)v(x)]^2 = r^2(x)\,[v'(x)u(x) - u'(x)v(x)]^2 \\ &\quad - 4r^2(x)\,|u'(x)|\,v'(x)u(x)v(x) = 1 - 4r^2(x)\,|u'(x)|\,v'(x)\rho(x) < 1.\end{aligned}$$

Let us verify that (1.11) and (1.15) are equivalent. Let $m = 1 - \varepsilon$, $\varepsilon \in (0,1)$. Then, from (2.4), (5.1) and (2.5) one deduces relations leading to (1.11):

$$\begin{aligned}\frac{2}{\varphi(x)} &\geq \frac{r(x)v'(x)}{v(x)} = \frac{1 + r(x)\rho'(x)}{2\rho(x)} \geq \frac{\varepsilon}{2}\frac{1}{\rho(x)} \geq \frac{\varepsilon}{4}\frac{1}{h(x)} = \frac{\varepsilon}{4}\left(\frac{1}{\varphi(x)} + \frac{1}{\psi(x)}\right) \\ \frac{2}{\psi(x)} &\geq \frac{r(x)|u'(x)|}{u(x)} = \frac{1 - r(x)\rho'(x)}{2\rho(x)} \geq \frac{\varepsilon}{2}\frac{1}{\rho(x)} \geq \frac{\varepsilon}{4}\frac{1}{h(x)} = \frac{\varepsilon}{4}\left(\frac{1}{\varphi(x)} + \frac{1}{\psi(x)}\right).\end{aligned}$$

Let (1.11) hold. Then (5.1) and (2.4) imply (6.1), and (6.1) implies (6.2):

$$(6.1)\quad \frac{1}{4\alpha} \leq \frac{1}{4}\frac{\varphi(x)}{\psi(x)} \leq \frac{|u'(x)|}{u(x)}\frac{|v(x)|}{v'(x)} = \frac{1 - r(x)\rho'(x)}{1 + r(x)\rho'(x)} \leq 4\frac{\varphi(x)}{\psi(x)} \leq 4\alpha, \qquad x \in R$$

$$(6.2)\qquad r(x)|\rho'(x)| \leq (4\alpha - 1)\,(4\alpha + 1)^{-1}, \qquad x \in R.$$

□

7. Necessary and sufficient conditions for the Green operator to be bounded. In this section we study the properties of operator G (see (1.4)).

LEMMA 7.1. *Let* $\omega(x)$ *be a positive function, continuous for* $x \in R$, *such that*

$$(7.1)\quad c^{-1}v(x) \leq v(t) \leq cv(x), \quad c^{-1}u(x) \leq u(t) \leq cu(x) \quad \text{for } |t - x| \leq \omega(x), \quad x \in R.$$

If $\|G\|_{p\to p} < \infty$ *for some* $p \in [1, \infty]$, *then* $\sup_{x\in R}(h(x)\omega(x)) \leq c\|G\|_{p\to p}$.

Proof. Let $p \in [1, \infty)$, let $f_x(t)$ be the characteristic function of the segment $\Delta(x) = [x - \omega(x), x + \omega(x)]$. From(7.1) and (2.5), for $t \in \Delta(x)$, we get (7.2), and (7.2) $\Rightarrow$ (7.3):

$$(7.2)\quad (Gf_x)(t) = u(t)\int_{x-\omega(x)}^{t} v(\xi)d\xi + v(t)\int_{t}^{x+\omega(x)} u(\xi)d\xi \geq \frac{u(x)v(x)\omega(x)}{c} \geq \frac{h(x)\omega(x)}{c}$$

$$(7.3)\qquad ||G||_{p\to p}^p = \sup_{0\neq f\in L_p(R)} \frac{||Gf||_p^p}{||f||_p^p} \geq \sup_{x\in R} \frac{||Gf_x||_{L_p(\Delta(x))}^p}{2\omega(x)} \geq \sup_{x\in R}\left(\frac{h(x)\omega(x)}{c}\right)^p.$$

For $p=\infty$ one can take $f(x)\equiv 1$, $x\in R$ as a test function and repeat the same argument. □

LEMMA 7.2. *Let the hypotheses of Lemma* 7.1 *hold. If for every $x\in R$ there exists an $R(x,\omega(\cdot))$-covering (see Definition* 2.1*) and*

$$(7.4)\qquad \theta = \inf_{x\in R}\left(\frac{1}{2\omega(x)}\int_{x-\omega(x)}^{x+\omega(x)} q(t)dt\right) > 0,$$

then $H\leq c\theta^{-1}$ (see (2.7)*). In addition, if for any $p\in[1,\infty]$ equation* (1.7) *has no solution $z(x)\in L_p(R)$ apart from $z(x)\equiv 0$, then the i.p. for* (1.1) *is regular in $L_p(R)$, and $||G||_{p\to p}\leq c\theta^{-1}$, $p\in[1,\infty]$.*

Proof. From (2.1) it follows that for $x\in R$ one has (see [1]):

$$(7.5)\qquad 1\geq u(x)\int_{-\infty}^x q(t)v(t)dt + v(x)\int_x^\infty q(t)u(t)dt \overset{\text{def}}{=} (Gq)(x)$$

We use an $R(x,\omega(\cdot))$-covering, (7.5), (7.1) and the notation $\Delta(x)=[x-\omega(x),x+\omega(x)]$:

$$\begin{aligned}
1 &\geq (Gq)(x) = u(x)\sum_{n=-\infty}^{-1}\int_{\Delta(x_n)} q(t)v(t)dt + v(x)\sum_{n=1}^{\infty}\int_{\Delta(x_n)} q(t)u(t)dt\\
&\geq \frac{2}{c}\left\{u(x)\sum_{n=-\infty}^{-1} v(x_n)\omega(x_n)\left[\frac{1}{2\omega(x_n)}\int_{\Delta(x_n)} q(t)dt\right]\right.\\
&\left.+ v(x)\sum_{n=1}^{\infty} u(x_n)\omega(x_n)\left[\frac{1}{2\omega(x_n)}\int_{\Delta(x_n)} q(t)dt\right]\right\}\\
&\geq \frac{\theta}{c}\left\{u(x)\sum_{n=-\infty}^{-1}\int_{\Delta(x_n)}\left[\frac{v(x_n)}{v(t)}\right]v(t)dt + v(x)\sum_{n=1}^{\infty}\int_{\Delta(x_n)}\left[\frac{u(x_n)}{u(t)}\right]u(t)dt\right\}\\
&\geq \frac{\theta}{c^2}\left\{u(x)\sum_{n=-\infty}^{-1}\int_{\Delta(x_n)} v(t)dt + v(x)\sum_{n=1}^{\infty}\int_{\Delta(x_n)} u(t)dt\right\} = \frac{\theta}{c^2}\int_{-\infty}^{\infty} G(x,t)dt.
\end{aligned}$$

Thus $H\leq c^2\theta^{-1}$. It remains to apply Lemma 2.5. □

8. Proof of the main results. In this section we prove Theorems 1.1, 3.1 – 3.4 and their corollaries.

Proof of Theorems 1.1 and 3.2.

Necessity. By Lemmas 2.2 and 2.3, the function $d(x)$ satisfies the hypothesis of Lemma 7.1. By this lemma we get for $\omega(x)=d(x)$ and we obtain $81^{-1}B\leq ||G||_{p\to p}$, $p\in[1,\infty]$ (see Theorem 1.1 and (3.1)).

Sufficiency. Since (1.1) $\in S$, by Corollary 3.1.1 and Lemma 2.5 it is sufficient to prove that $H<\infty$ (see (2.7)). From (1.11), (6.2), (1.17) and (2.5) we get

$$(8.1)\qquad G(x,t)\leq \rho(t)\exp\left(-c_0\left|\int_x^t \frac{d\xi}{r(\xi)h(\xi)}\right|\right)\quad x,t\in R.$$

Here $2c_0 = (4\alpha+1)^{-1}$, and α is a constant from (1.11). Below we use (2.5), (8.1), Lemmas 2.3, 2.4 and the definition of $d(x)$ (see Lemma 2.2):

$$\begin{aligned}
\int_{-\infty}^{\infty} G(x,t)dt &\le \int_{-\infty}^{x} \rho(t)\exp\left(-c_0\int_t^x \frac{d\xi}{r(\xi)h(\xi)}\right)dt \\
&+ \int_x^{\infty} \rho(t)\exp\left(-c_0\int_x^t \frac{d\xi}{r(\xi)h(\xi)}\right)dt \\
&\le \sum_{n=-\infty}^{-1} \exp\left(-c_0\int_{\Delta_n^+}^{\Delta_{-1}^+} \frac{d\xi}{r(\xi)h(\xi)}\right)\int_{\Delta_n}\rho(t)dt \\
&+ \sum_{n=1}^{\infty} \exp\left(-c_0\int_{\Delta_1^-}^{\Delta_n^-} \frac{d\xi}{r(\xi)h(\xi)}\right)\int_{\Delta_n}\rho(t)dt \\
&\le 36\left\{\sum_{n=-\infty}^{-1} \exp(-|\,|n|-1|c_0)h(x_n)d(x_n) + \sum_{n=1}^{\infty}\exp(-|n-1|c_0)h(x_n)d(x_n)\right\} \\
&\le c(\alpha)B. \qquad (8.2)
\end{aligned}$$

From (8.2) we get $H \le c(\alpha)B$, which completes the proof of Theorem 1.1 and (3.1). □

Proof of Theorem 3.3.

Necessity. Since the i.p. for (1.1) is regular in $L_p(R)$, the operator $G\colon L_p(R) \to L_p(R)$ is bounded for $p \in [1,\infty]$. By Lemmas 3.2 and 5.1, the function $\mu(x)$ satisfies the hypothesis of Lemma 7.1 and, therefore, $\mu_0 \overset{\text{def}}{=} \sup_{x\in R}(h(x)\mu(x)) < \infty$. From (3.5) and (5.11) we get $\Delta(x) := [x-\mu(x), x+\mu(x)]$:

$$1 = \int_{\Delta(x)} q(t)h(t)dt \le ch(x)\int_{\Delta(x)} q(t)dt \le c\mu_0\left[\frac{1}{2\mu(x)}\int_{\Delta(x)} q(t)dt\right], \quad x \in R.$$

Hence, $A \ge (c\mu_0)^{-1}$ (see (1.18)).

Sufficiency. By Lemmas 3.2, 5.1 and 5.2, the function $\mu(x)$ satisfies Lemma 7.2. Therefore, $H \le cA^{-1}$. It remains to refer to Lemma 5.3.

In Lemma 7.1, set $\omega(x) = \mu(x)$. Inequalities (5.11) then imply:

$$\begin{aligned}
c\|G\|_{p\to p} &\ge \sup_{x\in R}(h(x)\mu(x)) = \sup_{x\in R}\frac{h(x)\mu(x)}{\int_{x-\mu(x)}^{x+\mu(x)} q(\xi)h(\xi)d\xi} \\
&\ge c^{-1}\sup_{x\in R}\frac{\mu(x)}{\int_{x-\mu(x)}^{x+\mu(x)} q(\xi)d\xi} = c^{-1}\sup_{x\in R}\left(\frac{1}{2\mu(x)}\int_{x-\mu(x)}^{x+\mu(x)} q(\xi)d\xi\right)^{-1} \\
&\ge c^{-1}\left(\inf_{x\in R}\frac{1}{2\mu(x)}\int_{x-\mu(x)}^{x+\mu(x)} q(\xi)d\xi\right)^{-1} = c^{-1}A^{-1}.
\end{aligned}$$

□

Proof of Corollary 3.3.1. Let, for example, $q(x) \to 0$ as $x \to \infty$. Then by (1.16), for any $\varepsilon > 0$ there exists $x_0(\varepsilon) \gg 1$ such that for all $x \geq x_0(\varepsilon)$ one has the inequalities $q(x) \leq \varepsilon$, and there exists $x_1(\varepsilon) \geq x_0(\varepsilon)$ such that for all $x \geq x_1(\varepsilon)$ one has inequalities $x - \mu(x) \geq x_0(\varepsilon)$. From this, by Lemma 5.1, we get

$$1 = \int_{x-\mu(x)}^{x+\mu(x)} q(t)h(t)dt \leq c\varepsilon h(x)\mu(x) \ , \quad x \geq x_1(\varepsilon).$$

Hence, $\sup_{x\in R}(h(x)\mu(x)) = \infty$. Then by Lemma 7.1, the operator $G\colon L_p(R) \to L_p(R)$ is not bounded for any $p \in [1,\infty]$. □

Proof of Corollary 3.3.2. The estimates for $\|G\|_{p\to p}$, obtained in Theorems 3.2 and 3.3, imply the inequalities $c^{-1} \leq AB \leq c$ which prove the corollary. □

Proof of Corollary 3.3.3. From the hypothesis of the lemma and Theorem 3.3, we get $A > 0$. Suppose that the segments $\{\Delta_n,\ n = \pm1, \pm2, \dots\}$ form an $R(x, \mu(\cdot))$-covering of R. Then

$$\begin{cases} \dfrac{1}{2\mu(x_n)} \displaystyle\int_{x_n-\mu(x_n)}^{x_n+\mu(x_n)} q(t)dt \geq A > 0 \\ n = \pm1, \pm2, \dots \end{cases} \Rightarrow \begin{cases} \displaystyle\int_{x_n-\mu(x_n)}^{x_n+\mu(x_n)} q(t)dt \geq 2A\mu(x_n) \\ n = \pm1, \pm2, \dots \end{cases}$$

These inequalities imply:

$$\int_0^\infty q(t)dt = \sum_{n=1}^{\infty} \int_{\Delta_n} q(t)dt \geq 2A \sum_{n=1}^{\infty} \mu(x_n) = \infty,$$

$$\int_{-\infty}^0 q(t)dt = \sum_{n=-\infty}^{-1} \int_{\Delta_n} q(t)dt \geq 2A \sum_{n=-\infty}^{-1} \mu(x_n) = \infty.$$

□

Proof of Theorem 3.4. From (2.5) and (3.8) it follows that

$$(4\gamma + 4)^{-1}\tilde{d}(x) \leq h(x) \leq (2\gamma + 1)\tilde{d}(x) \ , \quad x \in R. \tag{8.3}$$

From (3.6), (1.10), (2.6) and (2.5) it follows that

$$c^{-1}d(x) \leq \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)} \leq ch(x) \leq c\tilde{d}(x) \ , \qquad x \in R \tag{8.4}$$

$$c^{-1}\tilde{d}(x) \leq c^{-1}h(x) \leq c\int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)} \leq cd(x), \qquad x \in R. \tag{8.5}$$

Necessity. It follows from Theorem 1.1 and (8.3) – (8.5):

$$\infty > B = \sup_{x\in R}(h(x)d(x)) \geq c^{-2} \sup_{x\in R} \tilde{d}(x)^2 = c^{-2}K.$$

Sufficiency. From (2.2), (2.3) and (3.6), it follows that

$$\begin{aligned}\int_{x_1}^{\infty}\frac{d\xi}{v^2(\xi)}&=\int_{x_1}^{\infty}\frac{1}{\rho(t)}\exp\left(-\int_{x_1}^{t}\frac{d\xi}{r(\xi)\rho(\xi)}\right)dt\\ &\le\gamma\int_{x_1}^{\infty}\frac{1}{r(t)\rho(t)}\exp\left(-\int_{x_1}^{t}\frac{d\xi}{r(\xi)\rho(\xi)}\right)dt=\gamma.\end{aligned}\tag{8.6}$$

From (8.6) and (2.1) we conclude that $v(x)\to\infty$ as $x\to\infty$ and, similarly, $u(x)\to\infty$ as $x\to-\infty$. Since $K<\infty$, by (2.5) and (8.3) one has $\rho(x)=u(x)v(x)\le 2h(x)\le 2(2\gamma+1)\tilde d(x)\le cK$ for $x\in R$. From this it follows that $v(x)\to 0$ as $x\to-\infty$, $u(x)\to 0$ as $x\to\infty$. The latter means that for $p\in[1,\infty]$, equation (1.7) has no solutions $z(x)\in L_p(R)$ apart from $z(x)\equiv 0$. Furthermore, using (3.6), (8.3), (1.12) and the inequality $K<\infty$, we get

$$\begin{aligned}H&=\sup_{x\in R}\left[\int_{-\infty}^{\infty}\sqrt{\rho(x)\rho(t)}\exp\left(-\frac12\left|\int_x^t\frac{d\xi}{r(\xi)\rho(\xi)}\right|\right)dt\right]\\ &\le c\sup_{x\in R}\left[\int_{-\infty}^{\infty}\exp\left(-\frac{|x-t|}{c}\right)dt\right]=c<\infty.\end{aligned}$$

Now the proof of the theorem follows from Lemma 2.5. □

9. Proofs of the assertions concerning the estimates of the auxiliary functions. The following elementary assertion is given without proof.

LEMMA 9.1. *Let $\delta\in(0,1]$, $x\ge -1+\delta$. Then one has inequalities*

$$-\frac{1+x}{\delta^2}\le\frac{x}{1+x}\le x.\tag{9.1}$$

Proof of Theorem 4.1. Let $\gamma\in(0,3a)$. Then one has the following relations:

$$\begin{aligned}&\int_{x-\gamma\hat d(x)}^{x}q(t)dt\\ &=\int_{x-\gamma\hat d(x)}^{x}q_1(t)dt+\int_{x-\gamma\hat d(x)}^{x}q_2(t)dt\ge\frac{\gamma}{a}\,q_1(x)\hat d(x)-\sup_{|z|\le 3a\hat d(x)}\left|\int_x^{x+z}q_2(t)dt\right|\\ &=\frac{\gamma}{a}\sqrt{q_1(x)r_1(x)}-\sqrt{q_1(x)r_1(x)}\,\varkappa_1(x)=\sqrt{q_1(x)r_1(x)}\left(\frac{\gamma}{a}-\varkappa_1(x)\right).\end{aligned}\tag{9.2}$$

In the following relations we use (9.1):

$$\begin{aligned}&\int_{x-\gamma\hat d(x)}^{x}\frac{dt}{r(t)}=\int_{x-\gamma\hat d(x)}^{x}\frac{dt}{r_1(t)}+\int_{x-\gamma\hat d(x)}^{x}\left(\frac{1}{r_1(t)+r_2(t)}-\frac{1}{r_1(t)}\right)dt\ge\frac{\gamma}{a}\frac{\hat d(x)}{r_1(x)}\\ &-\int_{x-\gamma\hat d(x)}^{x}\frac{r_2(t)r_1(t)^{-1}}{1+r_2(t)r_1(t)^{-1}}\frac{dt}{r_1(t)}\ge\frac{\gamma}{a}\frac{1}{\sqrt{q_1(x)r_1(x)}}-\int_{x-\gamma\hat d(x)}^{x}\frac{r_2(t)dt}{r_1(t)^2}\\ &\ge\frac{1}{\sqrt{q_1(x)r_1(x)}}\left(\frac{\gamma}{a}-\varkappa_2(x)\right).\end{aligned}\tag{9.3}$$

Hence, since $\varkappa_1(x)$ and $\varkappa_2(x)$ are sufficiently small as $|x| \gg 1$, one has

$$(9.4)\quad T_1(x) \overset{def}{=} \int_{x-\gamma\hat{d}(x)}^{x} \frac{dt}{r(t)} \cdot \int_{x-\gamma\hat{d}(x)}^{x} q(t)dt \geq \left(\frac{\gamma}{a} - \varkappa_2(x)\right) \left(\frac{\gamma}{a} - \varkappa_1(x)\right), |x| \gg 1.$$

Choose

$$\frac{\gamma}{a} = \sqrt{3} + \varkappa_1(x) + \varkappa_2(x) \ , \quad |x| \gg 1.$$

Clearly,

$$(9.5)\qquad T_1(x) \geq \left(\sqrt{3} + \varkappa_2(x)\right) \left(\sqrt{3} + \varkappa_1(x)\right) \geq 3 \ , \quad |x| \gg 1.$$

Similarly, we check that with such choice of γ and $|x| \gg 1$, one has

$$(9.6)\qquad T_2(x) \overset{def}{=} \int_{x}^{x+\gamma\hat{d}(x)} \frac{dt}{r(t)} \cdot \int_{x}^{x+\gamma\hat{d}(x)} q(t)dt \geq 3.$$

Moreover, from (9.5) and (9.6) it follows that

$$(9.7)\qquad \int_{-\infty}^{x} \frac{dt}{r(t)} \cdot \int_{-\infty}^{x} q(t)dt \geq 3 \ , \ \int_{x}^{\infty} \frac{dt}{r(t)} \cdot \int_{x}^{\infty} q(t) \ dt \geq 3, \quad |x| \gg 1.$$

From (9.7) it follows that for any $x \in R$ one has inequalities

$$\int_{-\infty}^{x} q(t) \ dt > 0 \ , \quad \int_{x}^{\infty} q(t)dt > 0.$$

In addition, according to a well-known criterion for convergence of improper integrals ([7, §1.5]), in each of the products $T_1(x)$ and $T_2(x)$ at least one of the integrals diverges. This implies (1.3).

Note that the auxiliary functions $d_{1,2}(x)$ (and hence functions $\varphi(x)$, $\psi(x)$ and $h(x)$), which are systematically used in our arguments, are continuous for $x \in R$. This can be proved by applying standard theorems on the properties of implciit functions ([4], Ch. 5, §109).

Now let us give estimates for $d_1(x)$ (for $d_2(x)$ they have the same form and can be proved in a similar way). Setting in (9.4)

$$\gamma = \gamma_1 = a(1 + \varkappa_1(x) + \varkappa_2(x)) \ , \quad |x| \gg 1,$$

we obtain $T_1(x) \geq 1$ as $|x| \gg 1$. Hence, $d_1(x) \leq \gamma_1\tilde{d}(x)$.

Now let us obtain a lower estimate for $d_1(x)$. As before, let $\gamma \in (0, 3a)$. Then

$$\begin{aligned} \int_{x-\gamma\hat{d}(x)}^{x} q(t)dt &= \int_{x-\gamma\hat{d}(x)}^{x} q_1(t)dt + \int_{x-\gamma\hat{d}(x)}^{x} q_2(t)dt \leq a\gamma q_1(x)\hat{d}(x) \\ &+ \sup_{|z|\leq 3a\hat{d}(x)} \left| \int_{x}^{x+z} q_2(t)dt \right| = a\gamma\sqrt{q_1(x)r_1(x)} + \sqrt{q_1(x)r_1(x)} \ \varkappa_1(x) \\ (9.8)\qquad & \\ &= (a\gamma + \varkappa_1(x)) \ \sqrt{q_1(x)r_1(x)}. \end{aligned}$$

In the following relations we use (9.1):

$$\int_{x-\gamma\hat d(x)}^{x}\frac{dt}{r(t)}=\int_{x-\gamma\hat d(x)}^{x}\frac{dt}{r_1(t)}+\int_{x-\gamma\hat d(x)}^{x}\left(\frac{1}{r_1(t)+r_2(t)}-\frac{1}{r_1(t)}\right)dt$$

(9.9)

$$=\int_{x-\gamma\hat d(x)}^{x}\frac{dt}{r_1(t)}-\int_{x-\gamma\hat d(x)}^{x}\frac{r_2(t)r_1^{-1}(t)}{1+r_2(t)r_1^{-1}(t)}\,\frac{dt}{r_1(t)}\le\int_{x-\gamma\hat d(x)}^{x}\frac{dt}{r_1(t)}$$

$$+\frac{1}{\delta^2}\int_{x-\gamma\hat d(x)}^{x}\left(1+\frac{r_2(t)}{r_1(t)}\right)\frac{dt}{r_1(t)}=\left(1+\frac{1}{\delta^2}\right)\int_{x-\gamma\hat d(x)}^{x}\frac{dt}{r_1(t)}+\frac{1}{\delta^2}\int_{x-\gamma\hat d(x)}^{x}\frac{r_2(t)}{r_1(t)^2}\,dt$$

$$\le\left(1+\frac{1}{\delta^2}\right)a\gamma\frac{\hat d(x)}{r_1(x)}+\frac{1}{\delta^2}\,\frac{\varkappa_2(x)}{\sqrt{r_1(x)q_1(x)}}=\frac{1}{\sqrt{r_1(x)q_1(x)}}\left[a\gamma\left(1+\frac{1}{\delta^2}\right)+\frac{1}{\delta^2}\,\varkappa_2(x)\right].$$

From (9.8) and (9.9) we get

(9.10)

$$T_1(x)\le\left(1+\frac{1}{\delta^2}\right)(a\gamma+\varkappa_1(x))\left(a\gamma+\frac{\varkappa_2(x)}{1+\delta^2}\right)\le\left(1+\frac{1}{\delta^2}\right)(a\gamma+\varkappa_1(x))\,(a\gamma+\varkappa_2(x)).$$

For $|x|\gg 1$ we have $\varkappa_1(x),\varkappa_2(x)$ sufficiently small and,therefore, $\gamma_2\in(0,3a)$:

$$\gamma_2\overset{\text{def}}{=}\frac{1}{a}\left[\frac{\delta}{\sqrt{1+\delta^2}}-\varkappa_1(x)-\varkappa_2(x)\right].$$

Setting $\gamma=\gamma_2$ in (9.10), we get:

$$T_1(x)\le\left(1+\frac{1}{\delta^2}\right)\left(\frac{\delta}{\sqrt{1+\delta^2}}-\varkappa_1(x)\right)\left(\frac{\delta}{\sqrt{1+\delta^2}}-\varkappa_1(x)\right)\le 1.$$

Hence, $d_1(x)\ge\gamma_2\tilde d(x)$, as $|x|\gg 1$. Thus,

$$\gamma_2\hat d(x)\le d_1(x)\le\gamma_1\hat d(x)\ ,\quad |x|\gg 1. \tag{9.11}$$

From this, taking a bigger $|x|$, if necessary, we get

$$\frac{\delta}{\sqrt3\ a}\ \hat d(x)\le d_1(x)\le\sqrt2\ a\ \ \hat d(x)\ ,\quad |x|\gg 1, \tag{9.12}$$

$$\frac{\delta}{\sqrt3\ a}\ \hat d(x)\le d_2(x)\le\sqrt2\ a\ \ \hat d(x)\ ,\quad |x|\gg 1. \tag{9.12$'$}$$

Inequality (9.12′) can be proved similarly to (9.12).

Let us estimate $\varphi(x)$, $\psi(x)$, $h(x)$. From (9.3) and (9.11),for $\gamma=\frac{\delta}{\sqrt3}\,\frac{1}{a}$ and $|x|\gg 1$ we obtain:

$$\varphi(x)=\int_{x-d_1(x)}^{x}\frac{dt}{r(t)}\ge\int_{x-\gamma\hat d(x)}^{x}\frac{dt}{r(t)}\ge\left(\frac{\gamma}{a}-\varkappa_2(x)\right)\frac{1}{\sqrt{q_1(x)r_1(x)}}\ge\frac{\delta}{2a^2}\ \frac{1}{\sqrt{r_1(x)q_1(x)}}.$$

Similarly, from (9.9) and (9.11), for $|x|\gg 1$ it follows that

$$\varphi(x)=\int_{x-d_1(x)}^{x}\frac{dt}{r(t)}\le\int_{x-\sqrt2 a\hat d(x)}^{x}\frac{dt}{r(t)}\le\frac{1}{\sqrt{r_1(x)q_1(x)}}\left[\sqrt2\,a\left(1+\frac{1}{\delta^2}\right)+\frac{\varkappa_2(x)}{\delta^2}\right]$$

$$\le\frac{2\sqrt3\ a^2}{\delta^2}\ \frac{1}{\sqrt{r_1(x)q_1(x)}}.$$

The estimates for $\psi(x)$ have the same form and can be obtained in a similar manner. Thus,

$$\frac{\delta}{4a^2}\,\frac{1}{\sqrt{q_1(x)r_1(x)}} \le \varphi(x),\psi(x) \le \frac{2\sqrt{3}}{\delta^2}\,a^2\frac{1}{\sqrt{q_1(x)r_1(x)}}\ , \quad |x|\gg 1. \tag{9.13}$$

By (9.13) and the definition of $h(x)$, it immediately follows that

$$\frac{\delta}{4a^2}\,\frac{1}{\sqrt{q_1(x)r_1(x)}} \le h(x) \le \frac{\sqrt{3}}{\delta^2}\,a^2\frac{1}{\sqrt{r_1(x)q_1(x)}}\ , \quad |x|\gg 1. \tag{9.14}$$

Let us extend (9.13) – (9.14) to the whole number axis. For example, $h(x)\sqrt{q_1(x)r_1(x)}$ is a continuous function with no zeros in R. Then, in every finite segment $[-\alpha,\alpha]$ there is $c(\alpha)$ such that

$$c(\alpha)^{-1} \le h(x)\ \sqrt{q_1(x)r_1(x)} \le c(\alpha), \quad x\in[-\alpha,\alpha]. \tag{9.15}$$

Then, (9.15), (9.14) and similar estimates for $\varphi(x)$ and $\psi(x)$ imply:

$$c^{-1} \le h(x)\ \sqrt{q_1(x)r_1(x)} \le c\ , \quad x\in R, \tag{9.16}$$

$$\frac{c^{-1}}{\sqrt{q_1(x)r_1(x)}} \le \varphi(x),\psi(x) \le c\frac{1}{\sqrt{q_1(x)r_1(x)}}, \quad x\in R. \tag{9.17}$$

Estimate (9.16) concludes the proof of assertion A) of Theorem 4.1.

Let us check B). We start with estimating $d(x)$. From (2.6) it follows that

$$4^{-1}\exp(-2)\int_{x-d(x)}^{x+d(x)}\frac{dt}{r(t)} \le h(x) \le 4\exp(2)\int_{x-d(x)}^{x+d(x)}\frac{dt}{r(t)}\ , \quad x\in R. \tag{9.18}$$

From (9.18) and (9.14), for $|x|\gg 1$ we obtain:

$$\frac{1}{\sqrt{r_1(x)q_1(x)}} \le \frac{16a^2\exp(2)}{\delta}\int_{x-d(x)}^{x+d(x)}\frac{dt}{r(t)}\ , \quad |x|\gg 1 \tag{9.19}$$

$$\frac{\delta^2}{8\sqrt{3}\,a^2}\exp(-2)\int_{x-d(x)}^{x+d(x)}\frac{dt}{r(t)} \le \frac{1}{\sqrt{r_1(x)q_1(x)}}\ , \quad |x|\gg 1. \tag{9.20}$$

Let us apply (9.9) and (9.19). Assume that $d(x)\le\gamma_1\tilde d(x)$ where $\gamma_1=\left(\frac{\delta}{a}\right)^3\frac{1}{640}$. Then, taking a bigger $|x|$, if necessary, we get

$$\begin{aligned}
\frac{1}{\sqrt{r_1(x)q_1(x)}} &\le \frac{16a^2\exp(2)}{\delta}\int_{x-d(x)}^{x+d(x)}\frac{dt}{r(t)} \le \frac{16a^2\exp(2)}{\delta}\int_{x-\gamma\hat d(x)}^{x+\gamma\hat d(x)}\frac{dt}{r(t)}\\
&\le \frac{16a^2\exp(2)}{\delta}\left[a\gamma\left(1+\frac{1}{\delta^2}\right)+\frac{\varkappa_2(x)}{\delta^2}\right]\frac{1}{\sqrt{r_1(x)q_1(x)}}\\
&\le \frac{32a^2}{\delta^3}\ \exp(2)\left(1+\frac{\varkappa_2(x)}{a\gamma_1(1+\delta^2)}\right)\ \frac{\gamma_1}{\sqrt{q_1(x)r_1(x)}}\\
&\le \frac{320a^3}{\delta^3}\ \gamma_1\frac{1}{\sqrt{q_1(x)r_1(x)}} = \frac{1}{2}\ \frac{1}{\sqrt{q_1(x)r_1(x)}}.
\end{aligned}$$

This is a contradiction. Hence, $d(x) \geq \gamma_1 \tilde{d}(x)$ for $|x| \gg 1$. Now let us apply (9.20) and (9.3). Assume that $d(x) \geq \gamma_2 \tilde{d}(x)$ where $\gamma_2 = \frac{160a^3}{\delta^2}$. Then, taking a bigger $|x|$, if necessary, we get

$$\frac{1}{\sqrt{q_1(x)r_1(x)}} \geq \frac{\delta^2}{8\sqrt{3}\,a^2}\exp(-2)\int_{x-d(x)}^{x+d(x)}\frac{dt}{r(t)} \geq \frac{\delta^2\exp(-2)}{8\sqrt{3}\,a^2}\int_{x-\gamma\hat{d}(x)}^{x+\gamma\hat{d}(x)}\frac{dt}{r(t)}$$

$$\geq \frac{\delta^2\exp(-2)}{8\sqrt{3}\,a^2}\left(\frac{\gamma_2}{a}-\varkappa_2(x)\right)\frac{1}{\sqrt{q_1(x)r_1(x)}} = \frac{\delta^2\exp(-2)}{8\sqrt{3}\,a^3}\left(1-\frac{a}{\gamma_2}\,\varkappa_2(x)\right)\frac{1}{\sqrt{r_1(x)q_1(x)}}$$

$$\geq \frac{\delta^2}{80\sqrt{3}}\,\frac{\gamma_2}{a^3}\,\frac{1}{\sqrt{r_1(x)q_1(x)}} = \frac{2}{\sqrt{3}}\,\frac{1}{\sqrt{r_1(x)q_1(x)}}.$$

This is a contradiction. Hence, $d(x) \leq \gamma_2 \tilde{d}(x)$ for $|x| \gg 1$. Thus,

$$\frac{1}{640}\left(\frac{\delta}{a}\right)^3\hat{d}(x) \leq d(x) \leq \frac{160a^3}{\delta^2}\,\hat{d}(x)\ , \quad |x| \gg 1. \tag{9.21}$$

From (9.21), using the fact that $d(x)$ is continuous (see Lemma 2.2), we obtain

$$c^{-1}\sqrt{\frac{r_1(x)}{q_1(x)}} \leq d(x) \leq c\sqrt{\frac{r_1(x)}{q_1(x)}}\ , \quad x \in R. \tag{9.22}$$

To obtain (9.22) for $x \in R$, we use the same method as for (9.16).

Now, claim C) follows from claims A) and B). □

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