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THE LINEAR DIFFRACTIVE PULSE EQUATION*

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Abstract. The asymptotic description of short wavelength wavetrain solutions of constant coefficient linear hyperbolic partial differential equations leads, for times of order 1, to amplitudes which satisfy the linear transport equation along rays. For linear phases and times of order $1/\epsilon$ with wavelength $\approx O(\epsilon)$, diffractive effects become important and the amplitudes satisfy a linear Schrödinger equation. The asymptotic description of pulse solutions involves the same linear transport equation for time of order one (see [A], [AR2], [MR], [Y1], [Y2]). For times of order $1/\epsilon$, instead of a Schrödinger equation, the amplitudes satisfy a partial differential equation we call the Linear Diffractive Pulse Equation (LDPE) (see [A], [AR1]). Nonlinear analogues of all these results are known for solutions of the appropriate amplitudes. In this paper we examine in some detail the initial value problem for the LDPE.

Dedication. It is our pleasure to dedicate this paper to Cathleen Morawetz who has been an inspiration to us in our lives and our research. We hope that the tie to asymptotics recalls her profound work with Ludwig on the justification of geometric optics expansions in the exterior of convex obstacles [LM], that the slightly nonstandard multipliers recall her brilliant use of the *a,b,c* method for problems of mixed type for example in [M], and finally that the characteristic initial manifold recalls her deep contributions to characteristic initial value problems of Tricomi type. Her work on exponential decay in exterior domains includes the crucial Morawetz multiplier and the first proof of decay for nontrapping obstacles which remains one of our favorite results in modern analysis [MRS]. Speaking for the many mathematicians who have turned to her for advice and guidance both personal and scientific, we offer our heartfelt thanks.

1. Well posedness of the initial value problem. The linear diffractive pulse equation (LDPE) is the partial differential equation

$$2u_{tx} = \Delta_y u\,, \qquad \Delta := \sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2}\,. \tag{1}$$

We will show that it is a good evolution equation.

A first observation is that the plane $\{t = 0\}$ is characteristic so there are null solutions, that is smooth nonzero solutions which vanish for $t \leq 0$. In fact, $u = f(t)$ is such a solution for any smooth nonzero f supported in $[0, \infty[$.

On the other hand, let $d := n + 1$ and define variables $(z_0, z_1, \dots, z_d)$, by

$$z_0 := \frac{t+x}{\sqrt{2}}\,, \qquad z_1 := \frac{t-x}{\sqrt{2}}\,, \qquad z_2, \dots, z_{1+n} = y_1, \dots, y_n\,.$$

The inverse transformation is

$$t = \frac{z_0 + z_1}{\sqrt{2}}\,, \qquad x = \frac{z_0 - z_1}{\sqrt{2}}\,, \qquad y_1, \dots, y_n = z_2, \dots, z_{1+n}\,.$$

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Then

$$\frac{\partial}{\partial t} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial z_0} + \frac{\partial}{\partial z_1}\right), \qquad \frac{\partial}{\partial x} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial z_0} - \frac{\partial}{\partial z_1}\right),$$

and equation (1) becomes the classical wave equation

$$\Box_z u = 0, \qquad \Box := \frac{\partial^2}{\partial z_0^2} - \sum_{i=1}^{1+n} \frac{\partial^2}{\partial z_i^2}. \tag{2}$$

Thus, equation (1) is simply the wave equation in coordinates rotated 45°.

Plane wave solutions $u = e^{i(\tau t + \xi x + \eta . y)}$ are given by solutions τ, ξ, η of the characteristic equation

$$-2\tau\xi + |\eta|^2 = 0, \qquad \tau = \frac{|\eta|^2}{2\xi}. \tag{3}$$

The fact that this dispersion relation is real shows that though the plane $\{t = 0\}$ is characteristic, equation (1) defines a unitary group on $H^s(\mathbb{R}^{1+n}_{x,y})$ for all s.

Equivalently one can solve (1) by Fourier Transform in the x, y variables to obtain

$$-2i\xi\, \hat{u}_t(t,\xi,\eta) \;=\; -|\eta|^2 \hat{u}(t,\xi,\eta), \qquad \hat{u}(t,\xi,\eta) \;=\; e^{it\frac{|\eta|^2}{2\xi}}\, \hat{u}(0,\xi,\eta).$$

The multiplier $e^{it|\eta|^2/(2\xi)}$ has modulus equal to one which implies that the $H^s(\mathbb{R}^d)$ norm of the solution is independent of $t \in \mathbb{R}$.

Another point of view is to write the equation formally as

$$u_t = \frac{1}{2}\left(\frac{\partial}{\partial x}\right)^{-1} \Delta_y\, u.$$

The commuting operators Δ_y and ∂_x^{-1} are symmetric and antisymmetric respectively. Thus with its natural domain defined with the aid of the Fourier transform, $\partial_x^{-1}\Delta_y$ is antiselfadjoint on $H^s(\mathbb{R}^{1+n})$. It generates the unitary group $\mathcal{F}^*\, e^{it|\eta|^2/(2\xi)}\, \mathcal{F}$ on $H^s(\mathbb{R}^{1+n})$. The solutions are continuous with values in H^s but are not in general differentiable in time even as a function with values in the tempered distributions (see Remark 4 after Theorem 1). It is not hard to show that solutions $u \in C(\mathbb{R}_t\,;\, H^s(\mathbb{R}^{1+n}))$ are characterized by the following equivalent conditions.

1. *Equation (1) holds in the sense of distributions on* $\mathbb{R}^{2+n}_{t,x,y}$.

2. $\hat{u}(t,\xi,\eta) \;=\; e^{it|\eta|^2/2\xi}\, \hat{u}(0,\xi,\eta)$.

3. $2i\xi\, \hat{u}_t(t,\xi,\eta) \;=\; |\eta|^2\, \hat{u}(t,\xi,\eta)$ *in the sense of distributions on* $\mathbb{R}^{1+n}_{t,\xi,\eta}$.

THEOREM 1. *If* $f(x,y) \in H^s(\mathbb{R}^{1+n})$, *there is a unique function*

$$u \in C\big(\mathbb{R}_t\,;\, H^s(\mathbb{R}^{1+n})\big), \qquad u\big|_{t=0} = f \tag{4}$$

satisfying (1) in the sense of distributions. The solution is given by the formula

$$\hat{u}(t,\xi,\eta) \;=\; e^{it|\eta|^2/\xi}\, \hat{f}(\xi,\eta). \tag{5}$$

In particular one has the conservation law,

$$\|u(t)\|_{H^s(\mathbb{R}^{1+n})} \;=\; \|f\|_{H^s(\mathbb{R}^{1+n})} \tag{6}$$

for all t. There are regular solutions $u_\epsilon \in C^\infty(\mathbb{R}_t\,;\,\mathcal{S}(\mathbb{R}^{1+n}))$ *of (1) which converge to* u *as* $\epsilon \to 0$ *in the topology of* $C(\mathbb{R}_t\,;\,H^s(\mathbb{R}^{1+n}))$.

Proof. Only the last sentence needs comment. Choose $\chi \in C^\infty(\mathbb{R}^{1+n})$ with χ vanishing on a neighborhood of the origin, and $\psi \in C_0^\infty(\mathbb{R}^{1+n})$ equal to one on a neighborhood of the origin. Then the solutions with initial data

$$\hat{u}^\epsilon(0,\xi,\eta) \;:=\; \chi(\xi/\epsilon,\eta/\epsilon)\;\psi(\epsilon\xi,\epsilon\eta)\;\hat{u}(0,\xi,\eta)$$

have the desired properties. □

EXAMPLES/REMARKS. 1. The fundamental solution is the solution u whose initial value is $u(0,x,y) = \delta(x,y)$. Then u is a distribution in t, x, y homogeneous of degree $-n-1$. This is immediate from the observation that if u is the fundamental solution then for $\lambda > 0$,

$$\lambda^{n+1}\,u(\lambda t, \lambda x, \lambda y_1, \ldots, \lambda y_n)$$

is also a fundamental solution. The homogeneity follows from uniqueness of solutions of the initial value problem.

2. If u is a solution then so is $u(-t,-x,y)$. It follows that if you know how to solve the initial value problem for $t > 0$ then one solves for $t < 0$ by remarking that $u(-t,x,y) = v(t,x,y)$ where v is the solution with initial data $f(-x,y)$. For this reason it suffices to describe the fundamental solution for $t > 0$.

3. The initial data $u(0,x,y)$ is arbitrary so long as it is square integrable at infinity in the sense of membership in some H^s. This freedom in data must not be confused with arbitrary Cauchy data. The equation is second order so that the Cauchy data is the pair $u(0), u_t(0)$ and the theorem shows that the first determines u and thereby the second. The differential equation represents a relation between $u(0)$ and $u_t(0)$. This is because the initial manifold $\{t = 0\}$ is characteristic, which in turn is equivalent to the fact that the conormal $(\tau, 0, 0)$ belongs to the characteristic variety.

4. Though the solutions are continuous in time, they will not in general be differentiable in time, even for initial data in the Schwartz space $\mathcal{S}(\mathbb{R}^{1+n})$. This is easy to see from the formula

$$2\xi\,\widehat{u_t}(t,\xi,\eta) = |\eta|^2 \hat{f}(\xi,\eta)\,. \tag{7}$$

Even for $f \in \mathcal{S}$ one will not have $\widehat{u_t}$ square integrable unless $\hat{f}$ vanishes at $\xi = 0$. A necessary and sufficient condition for u_t to be a bounded function of t with values in $L^2(\mathbb{R}^{1+n})$ is that

$$\int_{\mathbb{R}^{1+n}} \left| \frac{|\eta|^2\,\hat{f}(\xi,\eta)}{\xi} \right|^2 d\xi\,d\eta \;<\; \infty\,.$$

2. Domains of dependence and propagation speeds. Implicit differentiation of (3) yields

$$-2\tau_\xi\,\xi + 2\tau = 0\,, \qquad -2\tau_\eta\xi + 2\eta = 0\,. \tag{8}$$

Together with (3), this yields the group velocities

$$(\tau_\xi, \tau_\eta) = \left(\frac{|\eta|^2}{2\xi^2}, \frac{\eta}{\xi}\right). \tag{9}$$

Note in particular that the x component of the group velocity is always nonnegative. The group lines $(t, t|\eta|^2/2\xi, t\eta/\xi)$ foliate the light cone $2tx = |y|^2$.

The conservation laws (6) can also be obtained by the method of multipliers. In the next computations suppose that $u \in C^\infty(\mathbb{R}_+ \,;\, \mathcal{S})$. Theorem 1 shows that general solutions are limits of such functions. Multiplying the equation (1) by u_x yields the conservation identity

$$\partial_t (u_x)^2 + \left(\frac{|\nabla_y u|^2}{2}\right)_x - \sum_{i=1}^{n} \left(u_x\, u_{y_i}\right)_{y_i} \;=\; 0\,.$$

Integrating with respect to x, y shows that

$$\partial_t \int_{\mathbb{R}^{1+n}} (u_x)^2 \; dx\, dy \;=\; 0\,. \tag{10}$$

In the same way multiplying by

$$(1-\Delta_{x,y})^s\,(\epsilon - \partial_x^2)^{-1} u_x$$

yields

$$\partial_t \int_{\mathbb{R}^{1+n}} \left((1-\Delta_{x,y})^{s/2}\,(\epsilon-\partial_x^2)^{-1/2} u_x\right)^2 \; dx\, dy \;=\; 0\,.$$

Letting $\epsilon > 0$ decrease to 0 recovers (6).

The previous argument shows the utility of the multiplier u_x in deriving estimates for the LDPE. One also has the standard identity in the z coordinates with multiplier u_{z_0}. There is also a simple identity from the multiplier u_t which is in fact a linear combination of the above two. It reads

$$u_t\,(2u_{tx} - \Delta_y u) \;=\; (u_t^2)_x - \sum_{j=1}^{n} \left(u_t u_{y_j}\right)_{y_j} + \sum_{j=1}^{n} \left(u_{y_j}^2/2\right)_t\,.$$

We next investigate what remains of the finite speed of propagation for the wave equation in $z_0, \ldots, z_{n+1}$. It is natural to expect that for our initial value problem, the domain of determination of a point $(\underline{t}, \underline{x}, \underline{y})$ is the backward light cone sketched in Figure 1. The intersection with the plane $\{t = 0\}$ is a parabolic region denoted P in the next theorem.

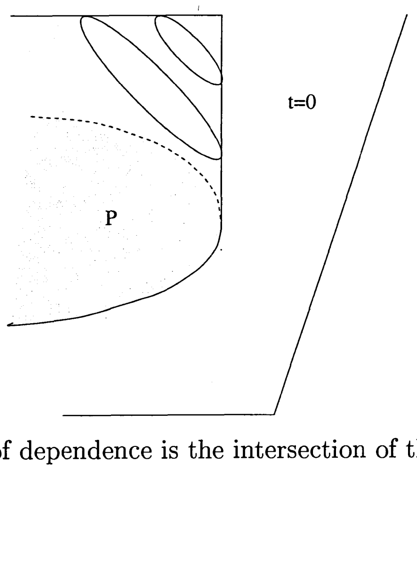

Figure 1. Domain of dependence is the intersection of the cone with $t = 0$.

The light cone in the figure has the equivalent equations

$$(11) \qquad 2(t-\underline{t})(x-\underline{x}) = \sum_{i=1}^{n}(y_i-\underline{y}_i)^2\,, \qquad (z_0-\underline{z}_0)^2 = \sum_{i=1}^{n+1}(z_i-\underline{z}_i)^2\,.$$

The assertion about the domain of dependence is the content of the next result.

THEOREM 2. *Suppose that $\underline{t} > 0$ and consider the intersection of the light cone (11) whose vertex is $\underline{t}, \underline{x}, \underline{y}$ with the plane $\{t = 0\}$. The interior of the cone cuts out the parabolic region*

$$(12) \qquad P := \Big\{(t,x,y)\ :\ t=0 \quad \text{and} \quad 2\underline{t}x < 2\underline{t}\,\underline{x} - \sum_{i=1}^{n}(y_i-\underline{y}_i)^2\Big\}\,.$$

If the initial data of a solution u from Theorem 1 vanishes in P then for $0 \le \underline{t}$, u vanishes inside the light cone (11) that is in the set

$$\Big\{(t,x,y)\ :\ 0<t<\underline{t} \quad \text{and} \quad 2(t-\underline{t})(x-\underline{x}) > \sum_{i=1}^{n}(y_i-\underline{y}_i)^2\Big\}\,.$$

In particular if the data vanish for $x \le a$ then for $t \ge 0$ the solution vanishes for $x \le a$. Influence propagates to the right. This is reasonable given the group velocities (9) with positive first component.

Proof. Regularizing the initial data with a smooth approximate delta function supported in an ϵ ball yields a solution with data in $\cap_s H^s$ supported in the parabola translated to the right by ϵ units. If the theorem were known for data in $\cap_s H^s$ one would conclude that the solutions u^ϵ vanished in the translated cone. Passing to the limit $\epsilon \to 0$ one obtains Theorem 2. Thus it suffices to prove the theorem for data in $\cap_s H^s$. Since both real and imaginary parts of a solution are themselves solutions, it suffices to treat real solutions with such smooth data.

Theorem 1 shows that

$$u \in C(\mathbb{R}_t\,;\,\cap_s H^s)$$

and the differential equation then shows that

$$u_x \in C^1(\mathbb{R}_t\,;\,\cap_s H^s)\,.$$

For $-\infty < \tilde{x} < \underline{x}$ denote by $C(\tilde{x})$ the part of the light cone in $t > 0$ and $\tilde{x} > x$,

$$C(\tilde{x}) := \Big\{t,x,y\ :\ 0<t<\underline{t}, \quad \tilde{x} > x\,, \quad 2(t-\underline{t})(x-\underline{x}) < |y|^2\Big\}\,.$$

Multiply the equation by u_x and integrate over C to find

$$0 = \int_{C(\tilde{x})} \partial_t(u_x)^2 + \Big(\frac{|\nabla_y u|^2}{2}\Big)_x - \sum_{i=1}^{n+1}\big(u_x\,u_{y_i}\big)_{y_i}\ dt\,dx\,dy\,.$$

Next integrate by parts obtaining three boundary terms

(13)

$$0 = \int_{\partial\text{Cone}} \nu_t(u_x)^2 + \nu_x|\nabla_y u|^2/2 - \sum \nu_{y_j}(u_x\,u_{y_j})d\sigma - \int_P (u_x)^2 dxdy + \int_{x=\tilde{x}} |\nabla_y u|^2/2dtdy,$$

where

$$\nu = (\nu_t, \nu_x, \nu_{y_1} \dots \nu_{y_n}) = \frac{(\underline{x} - x\,,\, \underline{t} - t\,,\, y - \underline{y})}{\|\underline{x} - x\,,\, \underline{t} - t\,,\, y - \underline{y}\|}$$

is the unit outward normal and $d\sigma$ is surface measure. The hypothesis on the initial data implies that the integral over P vanishes. The integrand in the ∂ Cone integral is nonnegative. One way to see this is that the surface is a limit of strictly spacelike surfaces on which the positivity of the integrand is nearly the definition of space like. Since all three integrals are nonegative it follows that they all vanish. In particular, the integral of $|\nabla_y u|^2$ over $x = \tilde{x}$ vanishes. Since $\tilde{x}$ is arbitrary we conclude that $\nabla_y u = 0$ at all points of $C(\tilde{x})$.

Knowing that $u_y = 0$, the surface integral yields $\int_{\partial\text{Cone}} \nu_t (u_x)^2 \, d\sigma = 0$. Since $\nu_t > 0$ this implies that $u_x = 0$ on the surface of the cone. The same argument holds for the cones with vertex lowered at any point between $(\underline{t}, \underline{x}, \underline{y})$ and $(0, \underline{x}, \underline{y})$. This proves that u_x and therefore $\nabla_{x,y} u$ vanishes at all points inside $C(\tilde{x})$.

By connectedness, it follows that for each $\underline{t} > t > 0$ the restriction of $u(t)$ to the interior of the light cone is constant. Since the time section has infinite measure, and $u(t)$ is square integrable, it follows that $u(t)\big|_{\text{light cone}} = 0$ and the proof is complete. □

This theorem has a dual form in terms of domain of influence.

COROLLARY 3. *For $t \geq 0$ the domain of influence of the origin for the LDPE, equivalently the support of the fundamental solution, is contained in the parabolic region*

$$\Big\{ \underline{t}, \underline{x}, \underline{y} \; : \; 2\underline{t}\,\underline{x} \;\geq\; \sum_{i=1}^{n} \underline{y}_i^2 \,.\Big\}.$$

Proof. Theorem 2 shows that the fundamental solution vanishes on a neighborhood of $\underline{t}, \underline{x}, \underline{y}$ if the origin is not contained in the region (12), that is if

$$0 \;\geq\; 2\underline{tx} - \sum_{i=1}^{n} \underline{y}_i^2 \,. \tag{14}$$

□

The fact that the support is contained in $x \geq 0$ is consistent with the fact that the group velocities have nonegative x component.

3. Relations with $\Box_z$. We next describe in more detail the relation between solutions $u(t, x, y)$ of the linear diffractive pulse equation and the corresponding solutions of the wave equation $\Box_z u = 0$. The solutions of the LDPE are given by the Fourier integral representation

$$u(t, x, y) \;=\; (2\pi)^{-(n+1)/2} \int e^{i(t|\eta|^2/(2\xi) \,+\, x\xi + y.\eta)} \,\hat{f}(\xi, \eta)\; d\xi\, d\eta\,, \tag{15}$$

while the solutions of $\Box_z v = 0$ are

(16) $$v(z_0, z_1, \dots, z_{n+1}) = v_+ + v_- =$$

$$(2\pi)^{-(n+1)/2} \cdot \sum_{\pm} \int e^{i((z_1,\dots,z_{n+1})\cdot(\zeta_1,\dots,\zeta_{n+1}) \pm z_0|\zeta_1,\dots,\zeta_{n+1}|)} a_\pm(\zeta_1,\dots,\zeta_{n+1}) d\zeta_1 \dots d\zeta_{n+1}.$$

To compare the formulas (15) and (16) use the relations between the t, x, y variables and the z variables given at the beginning of §1, and the dual relations

$$\zeta_0 = \frac{\tau+\xi}{\sqrt{2}}, \qquad \zeta_1 = \frac{\tau-\xi}{\sqrt{2}}, \qquad \zeta_2, \dots, \zeta_{n+1} = \eta_1, \dots, \eta_n .$$

$$\tau = \frac{\zeta_0+\zeta_1}{\sqrt{2}}, \qquad \xi = \frac{\zeta_0-\zeta_1}{\sqrt{2}}, \qquad \eta_1, \dots, \eta_n = \zeta_2, \dots, \zeta_{n+1} .$$

When these relations hold one has

(17) $$\tau t + \xi x + \eta . y = \frac{\tau+\xi}{\sqrt{2}} z_0 + \frac{\tau-\xi}{\sqrt{2}} z_1 + \eta . (z_2, \dots, z_{1+n}) := \zeta . z ,$$

and

$$\zeta_0^2 - \zeta_1^2 - \dots - \zeta_{n+1}^2 = 2\tau\xi - |\eta|^2 .$$

The last identity verifies that ζ belongs to the characteristic variety of $\Box_z$ if and only if τ, ξ, η is characteristic for the linear diffractive pulse equation. Denote this variety by Γ,

(18) $$\Gamma := \left\{ \zeta \neq 0 \; : \; \zeta_0^2 = \zeta_1^2 + \dots + \zeta_{n+1}^2 \right\} = \left\{ (\tau, \xi, \eta) \neq 0 \; : \; 2\tau\xi = |\eta|^2 \right\} .$$

The solution $u(t, x, y)$ of the LDPE correspond to the solution $v(z)$ of the wave equation by a change of independent variable,

(19) $$v(z) := (2\pi)^{-(n+1)/2} \int e^{i(\frac{z_0+z_1}{\sqrt{2}} \frac{|\eta|^2}{2\xi} + \frac{z_0-z_1}{\sqrt{2}}\xi + (z_2,\dots,z_{n+1}).\eta)} \hat{f}(\xi, \eta) \, d\xi \, d\eta_1 \dots d\eta_n .$$

The coefficient of z_0 in the exponent is

$$\frac{1}{\sqrt{2}} \left(\frac{|\eta|^2}{2\xi} + \xi \right) = \frac{|\eta|^2 + 2\xi^2}{2\sqrt{2}\,\xi} .$$

The sign of this quantity is the same as the sign of ξ. In formula (16) the solution of the wave equation is written as the sum of two terms each determined by the sign of the coefficient of z_0 in the exponent of the exponential. With this in mind let

(20)

$$v_\pm(z) := (2\pi)^{-(n+1)/2} \int_{\pm\xi>0} e^{i(\frac{z_0+z_1}{\sqrt{2}} \frac{|\eta|^2}{2\xi} + \frac{z_0-z_1}{\sqrt{2}}\xi + (z_2,\dots,z_{n+1}).\eta)} \hat{f}(\xi, \eta) \, d\xi \, d\eta_1 \dots d\eta_n .$$

To compare with (15) we change variables in the integral for $v_\pm$ noting that

$$e^{i((\zeta_1,\dots,\zeta_{n+1})\cdot(\frac{t-x}{\sqrt{2}}, y_1,\dots,y_n) \pm \frac{t+x}{\sqrt{2}} |\zeta_1,\dots,\zeta_{n+1}|)} = e^{i(\frac{|\eta|^2}{2\xi} t + x.\xi + y.\eta)}$$

where the map $(\zeta_1, \dots, \zeta_{n+1}) \to (\xi, \eta)$ is given by

(21) $$\xi = \frac{\pm|\zeta_1, \dots, \zeta_{n+1}| - \zeta_1}{\sqrt{2}}, \qquad \eta_1 \dots, \eta_n = \zeta_2, \dots, \zeta_{n+1} .$$

The inverse map is given by

$$(22)\qquad \zeta_1 = \frac{|\eta|^2 - 2\xi^2}{2\sqrt{2}\,\xi}\,, \qquad \zeta_2, \dots, \zeta_{n+1} = \eta_1 \dots, \eta_n\,.$$

The Jacobian determinants are

$$(23)\ \det \frac{\partial(\xi,\eta)}{\partial(\zeta_1 \dots, \zeta_{n+1})} = \frac{\partial \xi}{\partial \zeta_1} = \frac{1}{\sqrt{2}}\Big\{\frac{\pm\zeta_1}{|\zeta_1,\dots,\zeta_{n+1}|} - 1\Big\} = \frac{\pm\zeta_1 - |\zeta_1,\dots,\zeta_{n+1}|}{\sqrt{2}\,|\zeta_1,\dots,\zeta_{n+1}|}\,,$$

and

$$(24)\qquad \det \frac{\partial(\zeta_1 \dots, \zeta_{n+1})}{\partial(\xi,\eta)} = \frac{\partial \zeta_1}{\partial \xi} = \frac{-1}{2\sqrt{2}}\Big\{\frac{|\eta|^2 + 2\xi^2}{\xi^2}\Big\}.$$

Performing the change of variable in (20) with

$$\Big|\det \frac{\partial(\xi,\eta)}{\partial(\zeta_1 \dots, \zeta_{n+1})}\Big| = \frac{|\zeta_1,\dots,\zeta_{n+1}| \mp \zeta_1}{\sqrt{2}\,|\zeta_1,\dots,\zeta_{n+1}|}\,,$$

yields

$$(25)\qquad \begin{aligned} v_\pm = (2\pi)^{-(n+1)/2} \int & e^{i(\pm|\zeta_1,\dots,\zeta_{n+1}|z_0 + (\zeta_1,\dots,\zeta_{n+1})\cdot(z_1,\dots,z_{n+1}))} \times \\ & \times \hat{f}\Big(\frac{\pm|\zeta_1,\dots,\zeta_{n+1}| - \zeta_1}{\sqrt{2}}, \zeta_2, \dots, \zeta_{n+1}\Big) \frac{|\zeta_1,\dots,\zeta_{n+1}| \mp \zeta_1}{\sqrt{2}\,|\zeta_1,\dots,\zeta_{n+1}|}\, d\zeta_1 \dots d\zeta_{n+1}\,. \end{aligned}$$

To summarize the solution of the wave equation has amplitudes given by

$$(26)\ a_\pm(\zeta_1,\dots,\zeta_{n+1}) = \hat{f}\Big(\frac{\pm|\zeta_1,\dots,\zeta_{n+1}| - \zeta_1}{\sqrt{2}}, \zeta_2, \dots, \zeta_{n+1}\Big) \frac{|\zeta_1,\dots,\zeta_{n+1}| \mp \zeta_1}{\sqrt{2}\,|\zeta_1,\dots,\zeta_{n+1}|}\,.$$

Note that the restriction of $\hat{f}$ to $\pm\xi > 0$ determines $a_\pm$.

Example. Consider the fundamental solution of the LDPE defined by $u|_{t=0} = \delta(x,y)$. This solution is continuous in t with values in $H^s(\mathbb{R}^{1+n})$ provided that $s < -(1+n)/2$. The Fourier transform $\hat{f}$ is independent of ξ, η and therefore the amplitudes $a_\pm$ are functions homogeneous of degree zero and smooth except at the origin. It follows that $v_\pm$ and v are all $C(\mathbb{R}_{z_0}\,;\, H^s(\mathbb{R}^{n+1}_{z_1,\dots,z_{n+1}}))$ provided $s < -(n+1)/2$. In general the relation between the regularity of u and that of v is not quite this simple because of the nonlinear dependence on ζ of the argument of $\hat{f}$.

Continuing the computation using the shorthands $\zeta' := (\zeta_1,\dots,\zeta_{n+1})$, $z' := (z_1,\dots,z_{n+1})$ one has

$$\begin{aligned} v_\pm &= \frac{1}{(2\pi)^{n+1}} \int e^{i(\pm|\zeta'|z_0 + \zeta'.z')} \frac{|\zeta'| \mp \zeta_1}{\sqrt{2}\,|\zeta'|}\, d\zeta' \\ &= \frac{1}{i\,(2\pi)^{n+1}} \left(\pm\,\partial_{z_0} \mp \partial_{z_1}\right) \int e^{i(\pm|\zeta'|z_0 + \zeta'.z')} \frac{1}{\sqrt{2}\,|\zeta'|}\, d\zeta'\,. \end{aligned}$$

Therefore

$$v = v_+ + v_- = \frac{1}{i\,(2\pi)^{n+1}} \left(\partial_{z_0} - \partial_{z_1}\right) \int \left(e^{i(|\zeta'|z_0 + \zeta'.z')} - e^{i(-|\zeta'|z_0 + \zeta'.z')}\right) \frac{1}{\sqrt{2}\,|\zeta'|}\, d\zeta'\,.$$

So, up to a constant, v is equal to $\partial_{z_0} - \partial_{z_1}$ applied to the fundamental solution of $\Box_z$. We will recover this relation in the computation of the fundamental solution, u, in the next section.

There is a different strategy for transforming the initial value problem for the LDPE into a boundary value problem for the wave equation. Define

$$\tilde{u} := \begin{cases} u & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0. \end{cases}$$

Then

$$\left(2\partial_{tx}^2 - \Delta_y\right) \tilde{u} \;=\; 2\, u_x(0,x,y)\, \delta(t)\,.$$

In the variables z this reads

$$\begin{aligned} \Box_z \tilde{u} \;&=\; \frac{1}{\sqrt{2}} \left(\partial_{z_0} - \partial_{z_1}\right) f((z_0 - z_1)/\sqrt{2}, z_2, \ldots, z_{n+1})\, \delta((z_0 + z_1)/\sqrt{2}) \\ &=\; f_x((z_0 - z_1)/\sqrt{2}, z_2, \ldots, z_{n+1})\, \delta((z_0 + z_1)/\sqrt{2})/2\,. \end{aligned}$$

There are infinitely many solutions of this equation differing by solutions of the homogeneous wave equation. Since $\{t = 0\}$ is characteristic, there is even an infinite dimensional set which have support in $t \geq 0$. The correct solution is the one obtained by convolution with forward fundamental solution of $\Box_z$. The computation of fundamental solutions in the next section verify this assertion.

4. Explicit fundamental solutions. The fundamental solution is given by (15) with $\hat{f} = (2\pi)^{-(n+1)/2}$. The evaluation of the integral begins by performing the η integral using the identity

$$\int e^{-a|\eta|^2/2}\, e^{iy.\eta}\, d\eta \;=\; \left(\frac{2\pi}{a}\right)^{n/2} e^{-|y|^2/2a}\,, \qquad \operatorname{Re} a \geq 0, \quad a \neq 0\,. \tag{27}$$

The branch of the square root of $2\pi/a$ is the one with positive real part. With $a = -it/\xi$, $1/a = i\xi/t$, this yields

$$u \;=\; \frac{1}{(2\pi)^{1+n}} \int (2\pi i\xi/t)^{n/2}\, e^{-i\xi|y|^2/2t}\, e^{ix\xi}\, d\xi\,. \tag{28}$$

4.1. The case $n = 2k \in \mathbb{N}_{\text{even}}$. In this case, the wave equation $\Box_z$ is in a space of odd dimension equal to $2k+1$, so it is not surprising that the formulas are simpler and exhibit a Huyghens' principal. In the same vein, when $n = 2k$, there is no square root in formula (28) and one has

$$u \;=\; \frac{1}{(2\pi)^{1+n}} \int (2\pi i\xi/t)^k\, e^{-i\xi|y|^2/2t}\, e^{ix\xi}\, d\xi.$$

Differentiating the identity

$$\frac{1}{2\pi} \int e^{i\xi s}\, ds = \delta(s)$$

yields

$$\frac{1}{2\pi} \int (i\xi)^k\, e^{i\xi s}\, ds \;=\; \delta^{(k)}(s) \;:=\; \frac{d^k\delta}{ds^k}\,(s)\,.$$

Therefore

$$\begin{aligned} u &= \frac{(2\pi)^k}{(2\pi)^{n+1}\,t^k} \int (i\xi)^k\, e^{i\xi(x-|y|^2/2t)}\, d\xi \\ &= \frac{(2\pi)^{k+1}}{(2\pi)^{n+1}\,t^k}\, \delta^{(k)}(x-|y|^2/2t) \;=\; \frac{1}{(2\pi t)^k}\, \delta^{(k)}\big((2xt-|y|^2)/2t\big)\,. \end{aligned}$$

Since $\delta^{(k)}$ is positive homogeneous of degree $-k-1$, for $t>0$ this simplifies to

$$u \;=\; \frac{2t}{\pi^k}\, \delta^{(k)}(2xt-|y|^2)\,, \qquad t>0\,. \tag{29}$$

Since $\delta^{(k)}$ is homogeneous of degree $-k-1$ and $2xt-|y|^2$ is homogeneous of degree 2, it follows that $\delta^{(k)}(2xt-|y|^2)$ is homogeneous of degree $2(-k-1)$ and therefore that u is homogeneous of degree $-n-1$. Similarly, the local Sobolev regularity is $-n/2-1/2-\epsilon$. These verifications confirm properties already demonstrated about the fundamental solution. On the other hand, the formula (29) exhibits a Huyghens' principal. The support is contained entirely on the surface of the parabolic region P from Theorem 2. When $n=1$ and more generally for even dimensional space time, the support fills the solid light cone.

In the spirit of the last section, consider u as a solution of the wave equation in the z variables. Up to a constant factor, the function in (29) is equal to

$$\begin{aligned} t\,\delta^{(k)}\big(2xt-|y|^2\big) &= \frac{z_0+z_1}{\sqrt{2}}\, \delta^{(k)}\big(z_0^2-z_1^2-\cdots-z_{n+1}^2\big) \\ &= \frac{\partial_{z_0}-\partial_{z_1}}{2\sqrt{2}}\, \delta^{(k-1)}\big(z_0^2-z_1^2-\cdots-z_{n+1}^2\big)\,. \end{aligned}$$

The function $v := (2\pi)^{-k}\delta^{(k-1)}\big(z_0^2-z_1^2-\cdots-z_{n+1}^2\big)$ satisfies (see formula (41) of §VI.15.6 in [C])

$$\Box_z\, v \;=\; 0\,, \qquad v\big|_{z_0=0} = 0\,, \quad v_t\big|_{z_0=0} = \delta(z_1,\ldots,z_{n+1})\,.$$

Thus, the fundamental solution of the LDPE is equal to $c\,(\partial_{z_0}-\partial_{z_1})\,v$ where v is a fundamental solution of $\Box_z$.

4.2. The case $n=1$. When n is odd the branch of the square root of $2\pi i\xi/t$ in (28) is important. It is inherited from (27) and is given by

$$\begin{aligned} (2\pi i\xi/t)^{1/2} &= \big(2\pi i|\xi/t|\,\mathrm{sgn}\,(\xi/t)\big)^{1/2} \\ &= (2\pi|\xi/t|)^{1/2}\, \big(i\,\mathrm{sgn}\,(\xi/t)\big)^{1/2} \\ &= (2\pi|\xi/t|)^{1/2}\, \frac{1+i\,\mathrm{sgn}\,(\xi/t)}{\sqrt{2}}\,. \end{aligned} \tag{30}$$

Inserting (30) into (28) yields for $t>0$ and $n=1$,

$$u \;=\; \frac{1}{(2\pi)^{3/2}\sqrt{t}} \int \frac{1+i\,\mathrm{sgn}\,\xi}{\sqrt{2}}\, \sqrt{|\xi|}\; e^{i\xi(x-y^2/2t)}\, d\xi\,.$$

For $\xi>0$ the integrand includes the factor $(1+i)/\sqrt{2}$ while for $\xi<0$ this factor is replaced by

$$\frac{1-i}{\sqrt{2}} = -i\,\frac{1+i}{\sqrt{2}}\,.$$

Thus if we define the square root by

$$(\xi - i0)^{1/2} := \lim_{\epsilon\to 0}(\xi - i\epsilon)^{1/2} = \begin{cases} \sqrt{\xi} & \text{if } \xi \geq 0 \\ -i\sqrt{|\xi|} & \text{if } \xi \leq 0\,, \end{cases} \tag{31}$$

then for all real $\xi \neq 0$

$$\frac{1 + i\,\mathrm{sgn}\,\xi}{\sqrt{2}}\,\sqrt{|\xi|} = \frac{1+i}{\sqrt{2}}\,(\xi - i0)^{1/2} = e^{i\pi/4}\,(\xi - i0)^{1/2}\,.$$

Therefore

$$u = \frac{e^{i\pi/4}}{(2\pi)^{3/2}\sqrt{t}}\int(\xi - i0)^{1/2}\,e^{i\xi(x-y^2/2t)}\,d\xi\,.$$

Example 7.1.17 in [H] yields the identity

$$\frac{1}{\Gamma(-1/2)}\int_0^\infty x^{-3/2}\,e^{-ix\xi}\,d\xi = e^{i\pi/4}\,(\xi - i0)^{1/2}\,.$$

The Fourier Inversion Theorem yields

$$\frac{e^{i\pi/4}}{2\pi}\int e^{i\xi x}\,(\xi - i0)^{1/2}\,d\xi = \frac{x_+^{-3/2}}{\Gamma(-1/2)} := \chi_+^{-3/2}(x)$$

where we have introduced the notation (3.2.17) from [H]. Therefore

$$u = \frac{1}{\sqrt{2\pi t}}\,\chi_+^{-3/2}\big(x^2 - y^2/2t\big) = \frac{(2t)^{3/2}}{\sqrt{2\pi t}}\,\chi_+^{-3/2}(2xt - y^2)\,.$$

The composition has the standard interpretation as in [H, Theorem 6.1.2]. Simplifying yields the compact formula valid when $t > 0$ and $n = 1$

$$u = \frac{2t}{\sqrt{\pi}}\,\chi_+^{-3/2}(2xt - y^2)\,. \tag{32}$$

Note that the support of the distribution $u(t)$ agrees with that computed in Corollary 3 and also that the formula for u is homogeneous of degree -2 in t, x, y as it must be.

PROPOSITION 4. *The fundamental solution of the linear diffractive pulse equation is given for $t > 0$ by formula (29) when $n = 2k$ is even and by formula (32) when $n = 1$.*

The cases $n = 1$ and $n = 2$ are the most important for the applications. The case of n odd and greater than 1 can also be computed following the ideas used above. Numerical computations of some solutions of the LDPE can be found in [AR1].

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