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LARGE AMPLITUDE SOLUTIONS OF SPATIALLY NON-HOMOGENEOUS NON-LOCAL REACTION DIFFUSION EQUATIONS*

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Abstract. Existence and stability of a pulse solution of a spatially non-homogeneous non-local reaction diffusion equation are considered. Geometric singular perturbation theory is used to construct a large amplitude solution which lies in the transverse intersection of relevant invariant manifolds. The transverse intersection encodes a consistency condition of the non-local equation which determines the height of the pulse solution. An oscillation theorem for non-local eigenfunctions is used to prove the stability of the pulse. The results show that the spatial non-homogeneity in the non-linear and non-local terms is essential for stability of the pulse. Two different applications are considered, both of which are related to the microwave heating of ceramic materials.

1. Introduction. Non-local reaction diffusion equations arise in a variety of physical applications [4,6,11,18,19]. In the past few years, several analytic and numeric techniques have been developed for non-local equations to help determine the possible types of solutions and their ensuing stability properties [1,2,4,6,7,9-13,18-21]. There have been two standard approaches to these problems. One is to find a spatially homogeneous solution and determine whether any solutions bifurcate from it [4,9,10,20]. The second approach is to find a solution to a related local equation, and determine what impact the non-local term has on it [6,7,10-13,15,21]. A crucial observation that has been noted by various researchers is that non-local equations possess a number of asymptotically stable solutions that local equations do not. For example, it is well known that the only stable solution of a local scalar reaction diffusion equation, on a bounded domain, with Neumann boundary conditions, is spatially homogeneous. Using bifurcation arguments, Chafee [4] was the first to show that a non-local reaction diffusion equation with Neumann boundary conditions can support an asymptotically stable non-homogeneous solution. Fiedler and Poláčik [9] later showed that solutions of certain scalar non-local equations could have stable time periodic solutions with ω-limit sets of arbitrarily high dimension. Extensive work on the stability of solutions for non-local equations has been performed by Freitas [10-13]. Freitas notes that a major difficulty in determining the existence and stability of solutions to non-local equations is that the non-local equations rarely obey a maximum principle. Thus certain classical monotone methods are of no use.

With these restrictions in mind, we recently developed a number of techniques to analyze the existence and stability of large amplitude, non-homogeneous solutions of a non-local equation that arises in microwave heating applications [1,2,18]. The general form of the equation considered in those papers is

$$u_t = D^2 u_{xx} + G(u, \int_0^1 k(u)\, dx) \tag{1.1a}$$

$$u_x(0,t) = u_x(1,t) = 0, \tag{1.1b}$$

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where G and k are assumed to be sufficiently smooth, $D > 0$ is the diffusion constant and $0 < x < 1$. In [18], Kriegsmann employed asymptotic and numerical methods which suggested that (1.1) can support spatially localized, steady state, pulse or spike solutions for D sufficiently small. In [2], we proved that for certain choices of the non-linearities G and k, the 1-pulse solution is metastable and the n-pulse solutions are unstable. We also proved an Oscillation Theorem for the non-local eigenfunctions that is very much in the spirit of the oscillation results for standard Sturm-Liouville operators. In [1], Bose showed how to use geometric singular perturbation theory to construct steady-state pulse solutions to non-local equations and how to extend the stability results of [2]. The approach taken in these papers was to deal directly with the non-local equation as opposed to using bifurcation methods.

Note that in (1.1), there is no explicit dependence on the spatial variable x in the nonlinearities G or k. This has to to with the underlying geometry of the microwave heating apparatus. In the application discussed in [18], a thin ceramic cylinder is placed in a single mode cavity and is subjected to a uniform incident electrical field whose intensity is constant along the axis of the cylinder.

In this paper, we are motivated by a heating problem in which two ceramic cylinders are to be joined by heating the place at which these two cylinders meet. This problem is related to that considered in [1,2,18] by taking the cylinder in that application, and rotating it by ninety degrees. By doing so, the intensity of the incident electric field now has a spatial preference which is centered at the joining point. Kriegsmann [19] has recently modeled this situation and his work shows that the governing equation is still a non-local equation, but now one in which G and k explicitly depend on space. The form of the equation obtained in [19] is

$$u_t = D^2 u_{xx} + G(u, x, \int_0^1 k(u, x)\, dx) \tag{1.2a}$$

$$u_x(0, t) = u_x(1, t) = 0. \tag{1.2b}$$

For the applications we consider, existing monotone methods and bifurcation theory are again not suited to analyze (1.2). Thus it is of interest to determine whether any of the newer techniques can shed light on behavior of solutions of (1.2). In this paper, we show how the results described in [1,2,18] can be extended to analyze solutions of (1.2). In particular, we show that (1.2) supports an asymptotically stable, symmetric, spatially localized hot spot or 1-pulse solution. In contrast the 1-pulse solution of (1.1) is metastable. The main difference between the two is that the linearization of (1.1) about the 1-pulse yields an operator whose principle eigenvalue is exponentially small in D, whereas the operator obtained from (1.2) has an $O(1)$ stable principal eigenvalue. The hot spot forms at $x = 1/2$, which is the joining point of the two cylinders, and is thus symmetric about the midpoint of the computational domain $[0, 1]$.

A property that characterizes solutions of (1.1) and (1.2) is that they must satisfy a consistency condition. In particular, a solution $U^*(x)$ of (1.2) satisfies

$$U_t = D^2 U_{xx} + G(U, x, I^*) \tag{1.3a}$$

$$U_x(0, t) = U_x(1, t) = 0, \tag{1.3b}$$

where

$$I^* = \int_0^1 k(U, x)\, dx. \tag{1.3c}$$

Note that $U = U(x, t, I^*)$, where the value of I^* is determined by the consistency condition (1.3c). In section 2, we show how to construct a non-homogeneous, large amplitude steady state solution of (1.3) using geometric singular perturbation theory. In particular, we show that the symmetric 1-pulse solution lies in the transverse intersection of relevant invariant manifolds. These manifolds are constructed in a way that naturally incorporates the consistency condition (1.3c). In section 3, we discuss stability of solutions of non-local equations. Here we provide sufficient conditions under which eigenpairs of relevant non-local linear operators obey an Oscillation Theorem [1,2]. In section 4, the Oscillation Theorem is applied to assess the stability of the 1-pulse solution. Numerical results for the full time dependent problem are also presented which show that the basin of attraction of the 1-pulse solution is quite large. We also consider a second problem of joining two thin slabs in a microwave cavity. The solution of interest is a spatially localized hot stripe. We prove that this solution is unstable. These results are provided in section 5, along with more numerical simulations. A brief Discussion concludes the paper.

2. The Joining Problem. In this section, we consider the specific problem of joining two ceramic cylinders by microwave heating their interface as modeled in [19]. The equation of interest is of the form (1.2) and is given by

$$u_t = D^2 u_{xx} + \frac{p \sin^2(\pi x) f(u)}{(1 + \chi \int_0^1 \sin^2(\pi x) f(u)\, dx)^2} - h(u). \tag{2.1a}$$

$$u_x(0, t) = u_x(1, t) = 0, 0 \leq x \leq 1 \tag{2.1b}$$

The interface between the two cylinders is assumed to be at $x = 1/2$. The variable u denotes the temperature distribution along the axis of the cylinders, p is the non-dimensional power of the mode which excites the cylinders, χ is proportional to the product of the aspect ratio of the cylinders and the Q of the cavity and $D \ll 1$ is proportional to the aspect ratio of the cylinders. The functions $f(u)$ and $h(u)$ are assumed to be sufficiently smooth and satisfy the following properties.

a) $f(0) = 1$, $h(0) = 0$.

b) $f(u)$, $h(u) \to \infty$ as $u \to \infty$ and are strictly increasing for $u > 0$.

The function $f(u)$ represents the electrical conductivity of the cylinder, while $h(u)$ models the heat loss at the surface of the boundaries due to convection and radiation. We take

$$h(u) = 2(u + \beta[(u + 1)^4 - 1]),$$

as is the case in [1,2,18,19]. The parameter $\beta \ll 1$ is the ratio of radiative to convective heat loss at the ambient temperature. For $f(u)$ there are a few different choices. A commonly used form is $f(u) = e^{c_1 u}$, $c_1 > 0$, which is a good choice for low loss ceramic materials such as alumina [14]. Another choice is $f(u) = 1 + u^2$ which yields solutions of (2.1) with the same qualitative properties as $f(u) = e^{c_1 u}$. We first develop our theory with $f(u) = 1 + u^2$ and then discuss how it applies to $f(u) = e^{c_1 u}$.

REMARK. The equations considered in [1] are the same as (2.1) except that there is neither a $\sin^2 \pi x$ factor in the numerator of the second term, nor in the integrand of the denominator. See (4.4) below.

2.1 Existence of a spatially localized hot spot. We now prove the existence of a steady state pulse solution to (2.1). The main idea is to recast the right hand side of (2.1) into an autonomous system of first order equations in such a way as to obtain a *local* boundary value problem. To this end, introduce the auxiliary variable $v(x) = \int_0^x \sin^2(\pi x)(1+u^2)\,dx$, and let $I = \int_0^1 \sin^2(\pi x)(1+u^2)\,dx$. Thus a steady state solution of (2.1) will satisfy the following first order system of equations, where $' = d/dz$.

$$
\begin{aligned}
Du' &= w \\
Dw' &= h(u) - \frac{p\sin^2(\pi x)(1+u^2)}{(1+\chi I)^2} \\
v' &= \sin^2(\pi x)(1+u^2) \\
I' &= 0 \\
x' &= 1
\end{aligned}
\tag{2.2a}
$$

The boundary conditions that need to be satisfied are

$$
\begin{aligned}
(u,w,v,I,x) &= (u(0),0,0,I^*,0) \\
(u,w,v,I,x) &= (u(1),0,I^*,I^*,1).
\end{aligned}
\tag{2.2b}
$$

By phase plane considerations, it is easy to see that any pulse solution of (2.2) must be symmetric about the midpoint $x = 1/2$. This forces $u(0) = u(1)$. Equation (2.2) is a local boundary value problem in which the consistency condition (1.3c) has been embedded. A symmetric pulse solution of (2.2) with D sufficiently small will be close to $u = 0$ on most of $[0,1]$, except in a small neighborhood of $x = 1/2$ where the solution will make a large excursion in phase space away from and back to near $u = 0$. The large amplitude excursion corresponds to a homoclinic orbit of an appropriately scaled version of (2.2). More precisely, at $x = 1/2$, the maximum value of u and the value of I^* goes to infinity with D, so we need to rescale the independent variables. Let $\hat{u} = D^a u$, $\hat{w} = D^a w$, $\hat{v} = D^b v$, $\hat{I} = D^b I$ and $\xi = (z-1/2)/D$. Introducing these scalings into (2.2), letting $\dot{} = d/d\xi$ and noting that $\sin^2 \pi/2 = 1$, we obtain

$$
\begin{aligned}
\dot{\hat{u}} &= \hat{w} \\
\dot{\hat{w}} &= 2(\hat{u} + \beta D^a[(D^{-a}\hat{u}+1)^4 - 1]) - \frac{p(D^{a+2b} + D^{2b-a}\hat{u}^2)}{(D^b + \chi\hat{I})^2} \\
\dot{\hat{v}} &= D^{b+1} + D^{b+1-2a}\hat{u}^2 \\
\dot{\hat{I}} &= 0 \\
\dot{x} &= D.
\end{aligned}
\tag{2.3}
$$

To guarantee the existence of a homoclinic solution of (2.3), we require that the linear $(2u)$ and quadratic (u^2) terms are $O(1)$ with respect to D and that all others are $O(D^\gamma)$, $\gamma > 0$ or higher. This can be achieved by choosing $a = 2/3$, $b = 1/3$ and enforcing that $\beta = O(D^{2+\alpha})$, for $\alpha > 0$. Making these substitutions and rescaling back to the z variable we obtain

$$
\begin{aligned}
D\hat{u}' &= \hat{w} \\
D\hat{w}' &= 2(\hat{u} + D^{8/3+\alpha}[(D^{-2/3}\hat{u}+1)^4 - 1]) - \frac{p\sin^2 \pi x(D^{4/3}+\hat{u}^2)}{(D^{1/3}+\chi\hat{I})^2} \\
D\hat{v}' &= D^{4/3} + \sin^2(\pi x)\hat{u}^2 \\
\hat{I}' &= 0 \\
x' &= 1.
\end{aligned} \tag{2.4}
$$

The equations (2.3), with $a = 2/3$ and $b = 1/3$, are called the 'fast' equations whereas (2.4) are called the 'slow' equations.

The construction of a 1-pulse solution of (2.2) follows by showing that it lies in the transverse intersection of relevant invariant manifolds. In order to do this, the smallness of the parameter D is exploited to allow the use of techniques from geometric singular perturbation theory [8,16]. Our analysis parallels the approach in [1] and we refer the reader there for complete details.

We first construct a singular 1-pulse solution which is obtained by piecing together solutions of the 'reduced' fast and slow equations. The slow reduced equations, which are also referred to as the outer equations, are obtained by setting $D = 0$ in (2.4).

$$
\begin{aligned}
0 &= \hat{w} \\
0 &= 2\hat{u} - \frac{p\sin^2(\pi x)\hat{u}^2}{(\chi\hat{I})^2} \\
0 &= \sin^2(\pi x)\hat{u}^2 \\
\hat{I}' &= 0 \\
x' &= 1
\end{aligned} \tag{2.5}
$$

Notice in (2.5) that $\hat{u} = \hat{w} = 0$, and that $\hat{v}$ and $\hat{I}$ act as free parameters. The fast reduced equations, also called the inner equations, which are valid in a neighborhood of $x = 1/2$, are obtained by setting $D = 0$ in (2.3).

$$
\begin{aligned}
\dot{\hat{u}} &= \hat{w} \\
\dot{\hat{w}} &= 2\hat{u} - \frac{p\hat{u}^2}{(\chi\hat{I})^2} \\
\dot{\hat{v}} &= \hat{u}^2 \\
\dot{\hat{I}} &= 0 \\
\dot{x} &= 0.
\end{aligned} \tag{2.6}
$$

The equations (2.6) are the same set of inner equations that were obtained and analyzed in [1]. The $\hat{u}$ and $\hat{w}$ components of (2.6) form a Hamiltonian system which for each value of $\hat{I} > 0$ possesses a homoclinic orbit connecting the origin to itself. This homoclinic surrounds a second critical point at $\hat{u} = 2(\chi\hat{I})^2/p$. The consistency condition (1.3c), which had been cast into the boundary condition (2.2b), can now be reformulated as $\hat{v}(1) = \hat{I}^*$. Notice in (2.5) that the value of $\hat{v}$ does not change for the slow reduced equations. Thus in the singular limit, only the fast reduced equations are needed to check consistency.

Let

$$B_0 = \{(\hat{u}, \hat{w}, \hat{v}, \hat{I}, x) : \hat{u} = \hat{w} = \hat{v} = 0, x = 0\},$$

and

$$B_1 = \{(\hat{u}, \hat{w}, \hat{v}, \hat{I}, x) : \hat{u} = \hat{w} = 0, \hat{v} = \hat{I}, x = 1\}$$

denote the one-dimensional boundary manifolds associated with (2.2). Let M_0 denote the two-dimensional manifold obtained by flowing B_0 forward under (2.5). At $z = 1/2$, we define a jump off curve J_0, which is the one-dimensional restriction of M_0 to $z = 1/2$. Similarly, let M_1 denote the two-dimensional manifold obtained by flowing B_1 backward under (2.5). Restricting M_1 to $z = 1/2$ yields a one-dimensional consistency curve, T_c.

Next consider the inner equations (2.6) together with J_0 and T_c. We restrict to the $(\hat{u}, \hat{w}, \hat{I})$ phase space. Since the origin is a saddle point for the $(\hat{u}, \hat{w})$ components of the vector field, the jump off curve has a two-dimensional center-unstable manifold, $W^u(J_0)$. Similarly the consistency curve has a two-dimensional center-stable manifold $W^s(T_c)$.

PROPOSITION 2.1. *The manifold $W^u(J_0)$ transversely intersects $W^s(T_c)$ in $(\hat{u}, \hat{w}, \hat{I})$ space at some $\hat{I} = \hat{I}_0$.*

Proof. The $\hat{u}$ and $\hat{w}$ components of (2.6) form a version of Fischer's equation for which a closed form solution is available. It is easily checked that

$$\hat{u}(\xi) = \frac{3(\chi\hat{I})^2}{p} sech^2(\frac{\xi}{\sqrt{2}}), \quad \hat{w}(\xi) = \dot{\hat{u}} \tag{2.7}$$

satisfies the $\hat{u}$-$\hat{w}$ components of (2.6). Define $\hat{v}(\hat{I}) = \int_{-\infty}^{\infty} \hat{u}^2(\xi, \hat{I})\, d\xi$. A straightforward integration yields

$$\hat{v}(\hat{I}) = \frac{12\sqrt{2}(\chi\hat{I})^4}{p^2}. \tag{2.8}$$

The manifolds $W^u(J_0)$ and $W^s(T_c)$ will intersect transversely in $\mathbf{R}^3$ if the curve $\hat{v} = \hat{v}(\hat{I})$ transversely intersects $\hat{v} = \hat{I}$ in $\mathbf{R}^2$. Since $\hat{v}(\hat{I})$ is a quartic in $\hat{I}$, it is easy to see that the curves are transverse. By solving $\hat{v}(\hat{I}) = \hat{I}$, and using (2.8), we obtain the unique point of intersection as

$$\hat{I}_0 = \left(\frac{p^2}{12\sqrt{2}\chi^4}\right)^{1/3}. \tag{2.9}$$

We use the notation I_0 with subscript 0 to remind the reader that this is a $D = 0$ singular result. Substituting (2.9) into (2.7) and evaluating at $\xi = 0$ yields the maximum height of the singular solution.

REMARK. Because a closed from of the solution was available, we used it to obtain transversality. See [1] which describes a shooting method in $\hat{I}$ for obtaining transversality in the absence of a closed form solution.

Using the results of Proposition 2.1 and the outer flows, we can now obtain a unique consistent, singular 1-pulse solution. Namely, let q_0 denote the point obtained by restricting B_0 to the value $\hat{I}_0$ (See Figure 1). Flowing q_0 forward under (2.5) defines a one-dimensional trajectory, denoted by $q_0 \cdot z$. At $z = 1/2$, $q_0 \cdot 1/2$ intersects J_0. Now using the inner equations (2.6), Proposition 2.1 shows that $W^u(J_0)|_{q_0 \cdot 1/2}$ transversely intersects $W^s(T_c)$ along a one-dimensional curve. Denote by q_c the point on T_c which is connected to $q_0 \cdot 1/2$. Finally, flow q_c forward under (2.5) to $z = 1$, and define this point by $q_0 \cdot 1$. By construction, $q_0 \cdot 1$ intersects B_1. The trajectory which connects q_0 to $q_0 \cdot 1$ is the unique, consistent singular 1-pulse. Note that the solution as depicted in Figure 1 does not look like a pulse. Indeed in the three-dimensional $(\hat{u}, \hat{w}, \hat{v})$ phase space, the pulse is actually a heteroclinic orbit which connects $(0,0,0)$ to $(0,0,\hat{I}_0)$ since the $\hat{v}$ values at $\pm\xi$ are different. For this reason, the restriction of the inner pulse solution appears as a heteroclinic orbit in Figure 1.

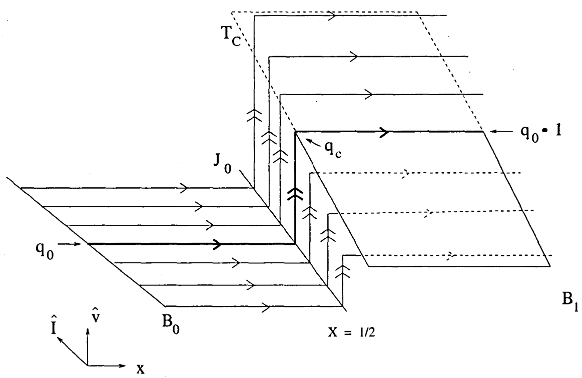

FIG 1. *The $(\hat{v}, \hat{I}, x)$ phase space. The bold curve represents the unique, consistent singular 1-pulse solution. Flows with single arrows represent solutions of the slow equations (2.5); double arrows represent solutions of the fast equations (2.6).*

We now show that the singular 1-pulse perturbs to an actual 1-pulse solution for D sufficiently small. To do this we invoke Fenichel's invariant manifold theory [8], together with the Exchange Lemma of Jones and Kopell [17]. See [16] for a complete description of these results and a number of applications. There are four criteria which need to be met in order to obtain a 1-pulse for D small. These criteria all have to do with the singular objects described above.

1) The manifolds B_0, B_1, J_0, and T_c are all normally hyperbolic.
2) The outer flow on M_0 transversely intersects the jump off curve J_0.
3) The outer flow on M_1 transversely intersects the consistency curve T_c.
4) The manifolds $W^u(J_0)$ transversely intersects $W^s(T_c)$.

Consider the first criterion. The manifolds in question are all defined at the origin of the $\hat{u} - \hat{w}$ subsystem. The origin is a saddle point, which guarantees the normally hyperbolicity. The second and third criteria are easily seen to hold since each of the manifolds J_0 and T_c are simply the restrictions of M_0 and M_1 at a particular value of x, and $x' \neq 0$ on these manifolds. The fourth criteria follows directly from Proposition 2.1.

Results of Fenichel guarantee that normally hyperbolic manifolds persist under perturbations. The perturbed manifolds are $O(D)$ close to the singular manifolds, and retain their hyperbolic structure with distinct attracting and repelling directions. Thus the manifolds discussed in the first criterion perturb to the one-dimensional manifolds B_0^D, B_1^D, J^D and T_c^D. Similarly, there exists perturbed manifolds $W^u(J^D)$

and $W^s(T_c^D)$, both of which are still two-dimensional. Moreover, the transversality result of Proposition 2.1 implies that $W^u(J^D)$ and $W^s(T_c^D)$ intersect transversely $O(D)$ close to $\hat{I}_0$.

The four criteria stated above are enough to guarantee the existence of an actual 1-pulse for D sufficiently small. To see this, note that $B_0^D \cdot z$ flowed forward under (2.4) and $B_1^D \cdot z$ flowed backward under (2.4) are $O(D)$ close to M_0 and M_1 respectively. A consequence of the Exchange Lemma of Jones and Kopell is that when these manifolds veer away from M_0 and M_1 they are $C^1 - O(D)$ close to $W^u(J^D)$ and $W^s(T_c^D)$, respectively. Since these latter two manifolds intersect transversely, independently of D, $B_0^D \cdot z$ and $B_1^D \cdot z$ also intersect transversely at some $\hat{I} = \hat{I}^*$, where $\hat{I}^*$ is $O(D)$ close to $\hat{I}_0$.

3. The Eigenvalue Problem. Denote the 1-pulse solution obtained above by $U^*(x)$. To determine asymptotic stability of U^*, proving linear stability suffices as Chafee [4] has shown that linear stability of solutions implies asymptotic stability for solutions of (1.2), in an appropriate function space. Linear stability of U^* is determined by inserting $u = U^*(x) + e^{-\lambda t}\phi(x)$ into (1.2) and linearizing about $U^*(x)$. This yields the following non-local eigenvalue problem

$$D^2\phi'' + (\lambda + \frac{\partial G}{\partial u}(U^*(x), x, I^*))\phi = -\frac{\partial G}{\partial I}(U^*(x), x, I^*) \int_0^1 \frac{\partial k}{\partial u}(U^*(x))\phi\, dx, \tag{3.1a}$$

$$\phi'(0) = \phi'(1) = 0. \tag{3.1b}$$

We rewrite this as

$$D^2\phi'' + (\lambda + A(x))\phi = B(x) \int_0^1 C(x)\phi\, dx, \tag{3.2a}$$

$$\phi'(0) = \phi'(1) = 0, \tag{3.2b}$$

where $A(x) = \frac{\partial G}{\partial u}(U^*(x), x, I^*)$, $B(x) = -\frac{\partial G}{\partial I}(U^*(x), x, I^*)$ and $C(x) = \frac{\partial k}{\partial u}(U^*(x))$. Denote the linear operator associated with (3.2) by

$$L_1\phi = -D^2\phi'' - A(x)\phi + B(x) \int_0^1 C(x)\phi\, dx, \tag{3.3}$$

and the spectrum of L_1 by $\sigma(L_1)$. Since we work on the bounded interval $x \in [0, 1]$, $\sigma(L_1)$ consists only of discrete eigenvalues of finite multiplicity [5]. Thus U^* will be an asymptotically stable solution of (1.2) if $Re\ \sigma(L_1) > 0$.

In general, L_1 is not a self-adjoint operator, thus its spectrum need not be strictly real. There are several circumstances, however, where $\sigma(L_1) \subset \mathbf{R}$. Using results in [2] and of Freitas [11], Bose [1] showed that the symmetry of the underlying solution U^* is sufficient to guarantee that $\sigma(L_1) \subset R$. He also showed that an Oscillation Theorem that first appeared in [2], where the choice of nonlinearities G and k implied that L_1 is self-adjoint, holds for general non-linearities G and k.

We need to establish some notation. Let

$$L_0\psi = -D^2\psi'' - A(x)\psi. \tag{3.4}$$

The operator L_0 is a standard Sturm-Liouville operator associated to solutions of local reaction diffusion equations. The associated local eigenvalue problem is

$$D^2\psi'' + (\nu + A(x))\psi = 0 \tag{3.5a}$$

$$\psi'(0) = \psi'(1) = 0. \tag{3.5b}$$

It is well known [5] that there exists a countable infinity of eigenpairs $(\psi_n(x), \nu_n)$ of (3.5), where the eigenvalues can be ordered $\nu_i < \nu_{i+1}$ for $i = 0, 1, 2, \ldots$ Moreover, the eigenfunctions $\psi_n(x)$ each possess exactly n interior zeros. In particular, the eigenfunction $\psi_0(x)$ associated with the principal eigenvalue ν_0 is strictly of one sign.

Using the symmetry of U^*, it can be shown that $A(x)$, $B(x)$ and $C(x)$ are all even about $x = 1/2$. This implies that the local eigenfunctions $\{\psi_n\}$ break up into two subsets: $\{\psi_{2n}\}$ which are even about $x = 1/2$, and $\{\psi_{2n+1}\}$ which are odd about $x = 1/2$. This causes

$$\int_0^1 C(x)\psi_{2n+1}(x)\, dx = 0.$$

As a result, the odd local eigenpairs are also eigenpairs of the non-local eigenvalue problem (3.1). A further consequence of the symmetry is that the eigenvalues of L_1 are real. These results provide the basis for the following Oscillation Theorem.

OSCILLATION THEOREM. *Let λ be a non-local eigenvalue with corresponding eigenfunction ϕ. For $n \geq 1$,*

a) $\lambda = \nu_{2n-1}$ if and only if $\phi = \psi_{2n-1}$ has $2n - 1$ interior zeroes.

b) $\nu_{2n-1} < \lambda < \nu_{2n+1}$ if and only if ϕ has $2n$ interior zeroes.

c) Every interval (ν_{2n-1}, ν_{2n+1}) contains exactly one non-local eigenvalue except possibly one such interval which may contain at most two non-local eigenvalues.

REMARKS. 1. See [2] for a proof of the Oscillation Theorem in the case L_1 is self-adjoint and [1] in the case of general L_1.

2. The Oscillation Theorem makes no explicit comment about the case $n = 0$. However, the Theorem implies that $\lambda < \nu_1$ if and only if the associated eigenfunction ϕ is strictly of one sign. The theorem does not guarantee the existence of such an eigenpair, and this must be resolved on a case by case basis.

3. The potential lack of existence of a non-local eigenfunction of strictly one sign is the precise reason why a spatially non-homogeneous solution may be stable.

4. The eigenvalues $\{\nu_{2n+1}\}$ are said to be fixed, as they are eigenvalues of both the local and non-local operators. This definition is due to Freitas [11] who gives a thorough description of how the spectrum of a non-local operator can be obtained by considering the fixed and 'moving' spectra of a related local operator.

4. Stability of hot spots. We now show that the 1-pulse is an asymptotically stable solution of (2.1). The stability of the 1-pulse follows by using our Oscillation Theorem, as well as the classical comparison results of Sturm-Liouville Theory.

The non-local eigenvalue problem of interest is

$$D^2\phi'' + (\lambda + A(x))\phi = B(x)\int_0^1 C(x)\phi\, dx, \tag{4.1a}$$

$$\phi'(0) = \phi'(1) = 0, \tag{4.1b}$$

where

$$A(x) = -2 - 4\beta(U^* + 1)^3 + \frac{2p\sin^2(\pi x)U^*}{(1+\chi I^*)^2}$$
$$B(x) = \frac{2p\chi\sin^2(\pi x)(1+(U^*)^2)}{(1+\chi I^*)^3} \tag{4.1c}$$
$$C(x) = 2\sin^2(\pi x)U^*$$

From the results of section 3, we know the eigenvalues are all real. The corresponding local eigenvalue problem is

$$D^2\psi'' + (\nu + A(x))\psi = 0 \tag{4.2a}$$

$$\psi'(0) = \psi'(1) = 0 \tag{4.2b}$$

The symmetry of the 1-pulse forces $(\psi_{2n+1}, \nu_{2n+1})$, the odd eigenpairs of the local problem, to be eigenpairs of the non-local equation. If there is a non-local eigenfunction of one sign, then the corresponding eigenvalue λ_0 will be the principal eigenvalue of the non-local problem, and its sign will determine stability of the solution. If there is no such eigenfunction, then ν_1 will be the principal eigenvalue and will determine stability.

Let us assume that there exists an eigenfunction ϕ_0 of strictly one sign. Let $J = \int_0^1 2\sin^2(\pi x)U^*(x)\phi_0(x)\,dx$ and $\Gamma = p/(1+\chi I^*)^2$. Then by integrating (4.1) from zero to one, we obtain

$$(\lambda_0 - 2)\int_0^1 \phi_0\,dx + 4\beta\int_0^1 (U^*+1)^3\phi_0\,dx + \Gamma J = \frac{2\Gamma\chi I^* J}{(1+\chi I^*)^3}. \tag{4.3a}$$

Since $\beta > 0$, we obtain the following estimate.

$$(\lambda_0 - 2)\int_0^1 \phi_0\,dx > \frac{\Gamma J}{(1+\chi I^*)^3}[\chi I^* - 1]. \tag{4.3b}$$

By definition the integral on the left hand side of (4.3b) is positive, as are J and Γ. Recall that $I^* \to \infty$ as $D \to 0$. This implies that for D sufficiently small, the right hand side of (4.3b) is positive, which implies that $\lambda_0 > 2$. Thus from (4.3b), we see that if a strictly positive eigenfunction exists, then it corresponds to a stable eigenvalue. Moreover, in this case, the Oscillation Theorem guarantees that the other eigenvalues of (4.1) are such that $\lambda > \lambda_0 > 2$ and are thus all stable eigenvalues.

Alternatively, let us now assume that there does not exist a non-local eigenfunction of one sign. This forces ν_1 to be the principal eigenvalue of (4.1). We shall determine the sign of ν_1 by comparing the spectrum of the operator associated with (4.2) with that of an operator whose spectrum we determined in [1]. In [1], we considered the related heating problem

$$u_t = D^2u_{xx} + \frac{p(1+u^2)}{(1+\chi\int_0^1 1+u^2\,dx)^2} - 2(u + \beta[(u+1)^4 - 1]) \tag{4.4a}$$

$$u_x(0,t) = u_x(1,t) = 0. \tag{4.4b}$$

We showed (4.4) has a stationary 1-pulse solution U_b, with $I = I_b$ which is symmetric about $x = 1/2$. Upon linearizing (4.4) about U_B, one again obtains a non-local eigenvalue problem. The associated local eigenvalue problem is

$$D^2\Psi'' + (\mu + E(x))\Psi = 0 \tag{4.5a}$$

$$\Psi'0 = \Psi'(1) = 0, \tag{4.5b}$$

where $E(x) = -2+4\beta(U_b+1)^3+\frac{2pU_b}{(1+\chi I_b)^2}$. In [1], Bose showed that μ_1 is exponentially small in D and negative. Recall from the existence proof in section 3.1, that the inner equations (2.6) are exactly the inner equations obtained for (4.4). Thus the values of $U^*(x)$ and I^* obtained above are $O(D)$ close to U_b and I_b, respectively. As a result, it is not hard to see that $E(x) > A(x)$. Thus the comparison theorem for Sturm-Liouville operators implies that $\nu_n > \mu_n$ for $n > 0$. In particular, $\nu_1 > \mu_1$, which shows that ν_1 is bounded below by an exponentially small term. To obtain a better idea on the size of ν_1, we employed a shooting method described in [19] to numerically calculate ν_1. Relevant parameter values were chosen, and in all cases $\nu_1 > 0$. In particular, with the parameters $D = 0.1$, $\chi = 0.01$, $\beta = 0$ and $p = 1.0$, we find that $\nu_1 = 2.82$.

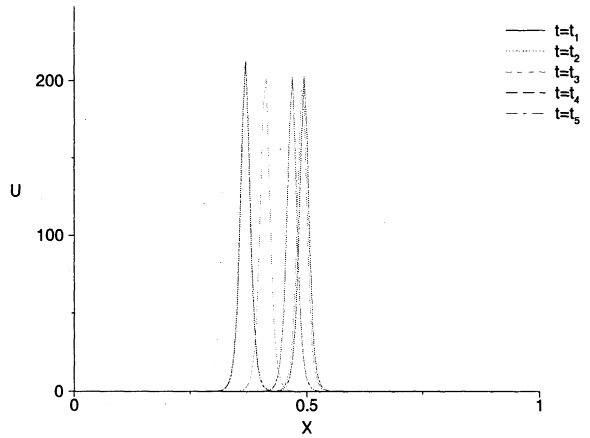

FIG 2. *The stable 1-pulse solution. An initial condition with a small local maxima at $x = 0.25$ was evolved. The pulse initially forms near $x = 0.35$ at $t = t_1$ and moves to the right towards $x = 0.5$ as time proceeds. The times $t_1 < t_2 < t_3 < t_4 < t_5$, where t_5 represents infinity.*

While the above analysis shows that perturbations of the 1-pulse decay to the 1-pulse, it says little about the basin of attraction of this solution. To address this issue, we also solved the full time dependent problem (2.1), using a semi-implicit Crank-Nicholson scheme (see [19]), for various different initial conditions. In Figure 2, it is seen how the 1-pulse attracts a solution that initially has a small local maxima (not shown) at $x = 0.25$. Here $D = 0.01, p = 1.0, \beta = 0$ and $\chi = 0.01$. By contrast, similar initial conditions when evolved under (4.4) yield a metastable spike that effectively remains fixed at $x = 0.25$ (see Figure 6 and Table 5.1 in [1]). Additional simulations using initial conditions that have a local maxima nearer to either of the $x = 0$ or $x = 1$ boundaries produce solutions that are also attracted to the 1-pulse. These results indicate that the basin of attraction for the 1-pulse is quite large.

In closing this section, we note that in principle, the above stability analysis also carries over for the case where $f(u) = e^{c_1 u}$. The Oscillation Theorem holds as stated. What is more difficult for this choice of non-linearity is an analytic determination of the sign of ν_1. Above we used the Sturm Comparison Theorem to obtain this information for the case $f(u) = 1+u^2$, which inherently relied on knowing some explicit information about the underlying solutions U^* and U_b. With the use of the exponential non-linearity, however, the existence results and in particular, the rescaling arguments for the inner equations are much more complicated due to the interplay of exponential and algebraic terms. As a result we are currently unable to make a similar comparison argument. Nonetheless, the eigenvalue problem can still be solved numerically, with the result that for D sufficiently small, the 1-pulse solution is asymptotically stable (see [19] for the details).

5. Heating a ceramic slab. In this section, we consider the problem of heating a thin ceramic slab. In particular, we ask the question as to whether or not two ceramic slabs can be joined in the same manner as the ceramic cylinders. The slab in question can be obtained by extending the cylinder of section 2 in the y direction, which is assumed to be parallel to the incident electric field. Kriegsmann derived the underlying equation of interest in [19], which is

$$u_t = D^2(u_{xx} + u_{yy}) + \frac{p \sin^2(\pi x) f(u)}{(1 + \frac{\chi}{H}\int_0^H \int_0^1 \sin^2(\pi x) f(u)\ dx\ dy)^2} - h(u) \tag{5.1a}$$

Here H denotes the height of the slab. Letting $\Omega = \{(x, y) : 0 < x < 1, 0 < y < H\}$, the boundary conditions are again of Neumann type

$$\frac{\partial}{\partial n} u = 0, (x, y) \in \partial\Omega \tag{5.1b}$$

where n is the outward normal and $\partial\Omega$ is the lateral boundary of the slab. We note that the incident electric field still has a spatial preference at $x = 1/2$, but is uniform in the y direction. Thus it is reasonable to expect that a spatially localized hot stripe extending from $y = 0$ to $y = H$ and symmetric about $x = 1/2$ should exist.

The microwave heating of a slab was consider by Brodwin and Johnson [3]. In their experiment, they did not observe a localized hot stripe. Instead, they observed an oval shaped hot spot which was centered about $x = 1/2$ and $y = H/2$. At first glance, this results seems somewhat surprising given that the stripe is nearly a superposition of the solution given above, which is stable, and the solution of (4.4) found in [1], which is metastable. We show here that (5.1) does indeed support a stripe solution, but that it is unstable for D sufficiently small. Furthermore, we show via simulation that the stripe decays to the aforementioned oval spot.

5.1 Existence and instability of a localized hot stripe. It is quite easy to obtain the stripe solution. Note that $U(x, y, t) = U^*(x)$, where $U^*(x)$ is the solution of (3.1) obtained above also solves (5.1). To determine the stability of $U^*(x)$ relative to (5.1), we linearize about the stripe. Using the ansatz $U(x, y, t) = U^*(x) + e^{-\Lambda t} V(x, y)$, we obtain the following eigenvalue problem,

$$D^2(V_{xx} + V_{yy}) + (\Lambda + A(x))V = B(x)\frac{1}{H}\int_0^H \int_0^1 C(x)V(x, y)\ dx\ dy, \tag{5.2a}$$

(5.2b) $$\frac{\partial}{\partial n}V = 0, (x, y) \in \partial\Omega,$$

where $A(x)$, $B(x)$ and $C(x)$ are defined in (4.1c). The partial differential equation (5.2) can be recast as an infinite system of ordinary differential equations by letting

(5.3) $$V(x, y) = \Sigma_{n=0}^{\infty} Q_n(x)\cos(\frac{n\pi y}{H}).$$

Taking derivatives and substituting into (5.2), we obtain the following equation for each n

(5.4a) $$D^2 Q_{n_{xx}}(x) + (\Lambda - \frac{D^2 n^2 \pi^2}{H^2} + A(x))Q_n(x) = B(x)\delta_{n0}(x)\int_0^1 C(x)Q_n(x)\,dx,$$

(5.4b) $$Q_{n_x}(0) = Q_{n_x}(1) = 0$$

where $\delta_{n0}(x)$ if $n = 0$ and 0 otherwise.

Consider (5.4) with $n \geq 1$, in which case the right hand side of the first equation vanishes. The equation then reduces to an infinite copy of the local Sturm-Liouville problem (4.2). Indexing the eigenvalues of (4.2) by the letter j, it follows that for $n \geq 1$,

(5.5) $$\Lambda_{nj} = \nu_j + \frac{D^2 n^2 \pi^2}{H^2}.$$

Note that ν_j depends on the diffusion constant D, but not the height of the slab H. In section 4, we determined that $\nu_1 > 0$. We now demonstrate that $\nu_0 < 0$.

Consider

(5.6a) $$D^2 U^*_{xx} + \frac{p\sin^2(\pi x)(1 + (U^*)^2)}{(1 + \chi I^*)^2} - 2(U^* + \beta[(U^* + 1)^4 - 1]) = 0,$$

(5.6b) $$D^2 \psi_{0_{xx}} + (\nu_0 + \frac{2p\sin^2(\pi x)U^*}{(1 + \chi I^*)^2} - 2(1 + 4\beta(U^* + 1)^3))\psi_0 = 0.$$

The eigenvalue ν_0 depends continuously on β, so we first restrict to $\beta = 0$. Multiplying (5.6a) by ψ_0 and (5.6b) by U^*, subtracting and integrating by parts, we obtain

(5.7) $$\nu_0 \int_0^1 U^*\psi_0\,dx = \frac{p}{(1 + \chi I^*)^2}\int_0^1 \sin^2(\pi x)(1 - (U^*)^2)\psi_0\,dx.$$

The solution U^* is positive, as by definition is ψ_0. Thus the sign of ν_0 is the same as the sign of the right hand side of (5.7). Recall that $I^* \to \infty$ as $D \to 0$. Furthermore the integral on the right hand side of (5.7) can be rewritten as $1 - I^*$. Thus for D sufficiently small the right hand side of (5.7) is clearly negative, which implies that $\nu_0 < 0$. By continuous dependence on parameters, the same must be true for β sufficiently small. Therefore, for fixed H and for D sufficiently small, $\Lambda_{n0} < 0$ for a finite number of n. This implies that the hot stripe is unstable.

Note that for $n = 0$, (5.4) reduces to (4.1), the non-local problem of section 4. In this case, the eigenvalues of (5.4) and (4.1) share the relationship

$$\Lambda_{0j} = \lambda_j + \frac{D^2 n^2 \pi^2}{H^2}.$$

In section 4, we showed that if λ_0 exists, then $\lambda_0 > 0$, and if not then $\lambda_1 > 0$. In either case $\lambda_{0j} > 0$ and thus $n = 0$ modes do not cause instabilities for the slab.

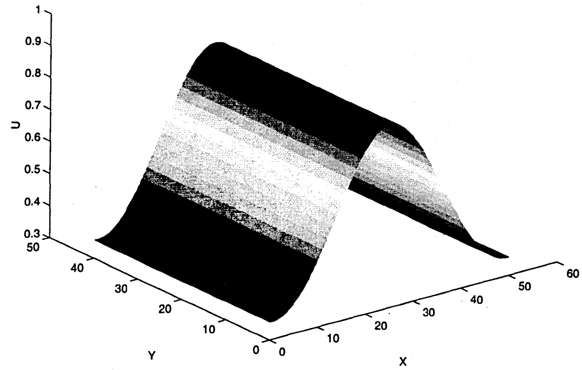

FIG 3. *The low amplitude stable hot stripe for large D.*

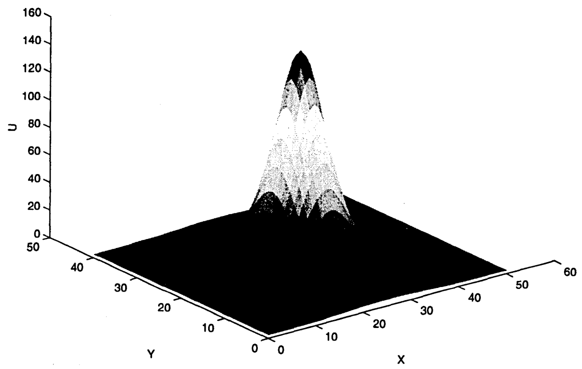

FIG 4. *The large amplitude stable hot spot for D small.*

If D is large enough, Λ_{n0} may be positive for $n \geq 1$. There are two terms on the right-hand side of (5.5) with the second term dominating the first if D is large. Indeed, we numerically solved (5.1) using the numerical scheme from [19] with $D = 0.1$, $\chi = 0.01$, $\beta = 0$, $H = 0.8$, and $p = 1.6$. A low amplitude stable hot stripe was observed (Figure 3). We then resolved the problem after decreasing $D = 0.01$. The stripe is no longer stable and the ensuing stable hot spot is shown in Figure 4. This hot spot corresponds to the oval shaped spot observed in experiments by Brodwin and Johnson [3].

6. Discussion. In this paper, we have shown that the techniques established in [1,2,18] also apply to cases where the non-linearity and non-local terms explicitly depend on the spatial variable. In particular, we presented techniques based on singular perturbation theory to construct a pulse solution of (2.1). We also used the non-local Oscillation Theorem, developed in [1,2], to show that the pulse is an asymptotically stable solution of (2.1), while the stripe is an unstable solution of (5.1).

The results presented here stand in contrast to those obtained for the cylinder problem considered in [1,2]. In those applications, the electric field has no spatial preference along the cylinder, and the ensuing 1-pulse is metastable; the linearization around the 1-pulse in those applications yielded an exponentially small principle eigenvalue. A consequence of the metastability is the 1-pulse could form at any point along the cylinder and move exponentially slowly towards one of the end points of the cylinder. In the present application, the electric field has a spatial preference at $x = 1/2$. This causes the 1-pulse to be asymptotically stable, with a large basin of attraction; the principle eigenvalue here is bounded away from the origin as $D \to 0$. In principle, symmetric n-pulse solutions for the current application may exist, but any such solution will necessarily be unstable [1,2,7].

The method with which we proved existence of solutions differs from much of the prior work in the area which often relies on bifurcation theory. Our approach is to recast the non-local problem into a higher dimensional local problem in order to construct pulse solutions. A primary advantage of this method is that it provides information about the structure of the solution in phase space, such as the amplitude and its dependence on various parameters in the equations. Our method also provides local uniqueness of solutions as a direct consequence of the transversality of intersections.

The stability results presented here relied in a non-trivial way in information obtained from the existence proof. In particular, the value of I^* as $D \to \infty$ was used to determine the sign of the principle eigenvalue of (4.1). It is this interplay between the existence and stability results which sets our work apart from some of the recent work on the Gray-Scott model and related problems [6,7] and on shadow systems [15,21]. In the former case, a system of two local reaction diffusion equations is considered. Existence of pulse solutions is obtained by geometric singular perturbation theory. Stability of these solutions is then analyzed by reducing the two second order linearized eigenvalue equations to a single, second order non-local eigenvalue problem. Thus the non-locality of the stability problem is an artifact of the method of solution as opposed to being inherent to the solution itself. Similarly, shadow systems which involve non-local terms arise as a method of solution for a higher dimensional system. These can be obtained again from a system of two local reaction diffusion equations in the limit where the diffusion constant of one of the equations goes to infinity. This yields a scalar non-local reaction diffusion equation. The existence and stability methods of the present work are well suited to address shadow systems. We note however, that the shadow system is used as a technical step to avoid the difficulties in dealing with a higher order system. As before, the actual equations of interest, are purely local.

An importance in recognizing these differences is that they point to a gap in the techniques for handling bona-fide non-local equations in both higher dimensional spaces, and for systems of non-local equations. In particular, in this paper, we have numerically shown that the hot stripe solution for the slab decays to a two-dimensional hot spot. It would be of interest to develop techniques to prove the existence and stability of this spot, and of other possible solutions of (5.1). We note that many of the techniques of Freitas and collaborators [10-13] are already suitable to address the stability of solutions in higher space dimensions.

Non-local equations with spatially dependent non-local terms have also been shown to exhibit stable periodic solutions [8,20]. In the current context, we only studied the existence and stability of stationary solutions. Linearization about the symmetric 1-pulse solution yields an operator whose spectrum is strictly real. This would appear to preclude the possibility of any Hopf bifurcations and periodic motion arising through this mechanism. Our analysis, however, in no way rules out the possibility of periodic solutions. Thus it would be of interest to develop a geometric method which does not rely on bifurcation arguments to construct periodic solutions of non-local equations.

We worked primarily with the quadratic non-linearity for $f(u)$ as opposed to the exponential non-linearity. The quadratic function yields qualitatively similar results to the physically more relevant exponential function. The main technical difference is that the quadratic yields maximum heating rates that are above the melting point of the ceramic. Also, the non-local equation with the quadratic terms is overly sensitive to β, the ratio of radiative to convective heat loss. In practice, this quantity is small, but in our quadratic model it must be smaller than $O(D^2)$. There is no such restriction when using the exponential function since as $u \to \infty$, the exponential dominates the quartic polynomial multiplying β. These technical details aside, the purpose of this paper was to provide a unified and systematic approach to analyzing non-homogeneous, non-local equations.

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