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TOTAL VARIATION DECAY OF SOLUTIONS TO THE NAVIER-STOKES EQUATIONS*

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Abstract. In this paper we study the decay of solutions to Navier-Stokes equations in the total variation norm of the solution and the Hardy H^1 norm of higher derivatives. We show that the solutions decays to zero at an algebraic rate.

1. Introduction. We consider the decay in the total variation norm of the solutions and derivatives to the Navier-Stokes equations in n spatial dimensions with $2 \leq n \leq 7$,

$$\begin{aligned} u_t - \Delta u + u \cdot \nabla u + \nabla p &= 0, \\ \text{div} u &= 0, \\ u(\cdot, 0) &= u_0 \in \mathbf{X}. \end{aligned} \tag{1.1}$$

Here $\mathbf{X}$, to be described below, will be chosen as an appropriate space which will insure the boundedness of the solution and derivatives in the total variation norm. We note that we will be work in a setting were the solutions are regular. Specifically we will either suppose that we start at a time sufficiently large so that our solutions are regular or the data is sufficiently small to insure global regularity.
The paper consists of an introduction and three sections. In the first section we introduce notation and give some preliminaries. In the second some observations on regularity are made. We show that solutions such that the product of the L^2 norms of the data and of the gradient is less than the inverse of the "Prodi constant" will have bounded L^2 norm of the gradient uniformly in time and hence be globally regular. This result is included to give one more setting under which we can obtain our decay results. In the last section we study the decay of the total variation norm of the solutions and of higher derivatives. We show that for the total variation we have the following behavior

$$||\partial_{x_i} u(t) - \partial_{x_i} e^{-At/2} u(t/2)||_1 \leq C_1 (t+1)^{-n/2-1/2}, i = 1, \cdots, n \tag{1.2}$$

$$||\partial_{x_i} u||_1 = C_o t^{-1/2} + C_1 (t+1)^{-n/2-1/2}, i = 1, \cdots, n \tag{1.3}$$

where e^{-At} as usual represents the Stokes semi-group. The decay obtained in the total variation norm is the same for the solution itself as the decay of the solution towards the solution of the heat equation.
For $p > 1$ we establish

$$||D^p u(t) - D^p e^{-At/2} u(t/2)||_{\mathcal{H}^1} \leq C_1 (t+1)^{-p/2-n/2} \tag{1.4}$$

$$||D^p u||_{\mathcal{H}^1} \leq C_o (t+1)^{-p/2} \tag{1.5}$$

2. Preliminaries and Notation. We shall use the multi-index notation $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$ with $\alpha_i \geq 0$, and write

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

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as usual

$$D^p u = \sum_{|\alpha|=p} D^\alpha u(x).$$

Further we define

$$C_o^m = C_o^m(\mathbb{R}^n) = \{u \in C^m(\mathbb{R}^n) : \lim_{|x|\to\infty} D^\alpha u(x) = 0 \ , \ |\alpha| \le m\}.$$
$$C_{o,\sigma}^m = C_o^m(\mathbb{R}^n) = \{u \in C^m(\mathbb{R}^n) : \lim_{|x|\to\infty} D^\alpha u(x) = 0 \ , \ |\alpha| \le m, \ \mathrm{div} = 0\}.$$

The L^2 norm (or energy norm) will be denoted by

$$\|u\| = \left(\int_{\mathbb{R}^n} |u(x)|^2 \, \mathrm{d}x\right)^{1/2},$$

where $x = (x_1, x_2, \ldots, x_n)$, $\mathrm{d}x = \mathrm{d}x_1 \, \mathrm{d}x_2 \ldots \mathrm{d}x_n$.

$$L_\sigma^2 = \{u : u \in L^2, \mathrm{div} u = 0\}$$

More generally we consider the L^p norm, for $1 \le p < \infty$, denoted by

$$\|u\|_p = \left(\int_{\mathbb{R}^n} |u(x)|^p \, \mathrm{d}x\right)^{1/p},$$

and the L^∞ norm,

$$\|u\|_\infty = \text{ess sup}_{x \in \mathbb{R}^n} |u(x)|.$$

The Sobolev H^m norm is defined by

$$\|u\|_{H^m} = \left(\int_{\mathbb{R}^n} \sum_{|\alpha|\le m} |D^\alpha u(x)|^2 \, \mathrm{d}x\right)^{1/2}.$$

We recall the definition of Hardy spaces on $\mathbb{R}^n$, denoted $\mathcal{H}^p$ for $0 < p < \infty$ (see, [8], [9]). Let $\mathcal{S}$ denote the Schwarz space of rapidly decreasing functions on $\mathbb{R}^n$. Let $\phi \in \mathcal{S}$ satisfy $\int_{\mathbb{R}^n} \phi(x) \, \mathrm{d}x = 1$. A tempered distribution f belongs to $\mathcal{H}^p$ whenever

$$f^+(x) = \sup_{t>0} |(\phi_t * f)| \in L^p(\mathbb{R}^n),$$

where $\phi_t(x) = t^{-n}\phi(x/t)$. The Hardy norm of f is defined by

$$\|f\|_{\mathcal{H}^p} = \|f^+\|_p \quad \text{for} \quad p \ge 1.$$

We recall that $\mathcal{H}^1$ is a Banach space strictly contained in L^1 and that $L^p \sim \mathcal{H}^p$ for $p > 1$.

We denote

$$\mathcal{H}_\sigma^1 = \{f : f \in \mathcal{H}^1 \ , \ \mathrm{div} f = 0\}$$

We need to recall the definition of **VMO**, the dual space to $\mathcal{H}^1$. We start by the definition of the space **BMO** of Bounded Mean Oscillation

DEFINITION 2.1. *[5] A measurable function f on $I\!R^n$ is in the space* **BMO** *if*

$$[f]_{\mathbf{BMO}} \equiv \sup_B \frac{1}{|B|}\int_B |f - f_B| \ dx < +\infty.$$

where the supremum is taken over all open balls B, and f_B is the average over B:

$$f_B = \frac{1}{|B|}\int_B f \ dx.$$

Where we have denoted by $|B|$ the Lebesgue measure of a measurable set $B \subset I\!R^n$.

Since $[f]_{\mathbf{BMO}} = 0$ if $f = constant$ we have only a semi-norm and not a norm, therefore it is natural to consider the quotient space $\mathbf{BMO}/I\!R$ with the norm induced by **BMO**. This quotient space is a Banach space and for simpli city we will also denote it by **BMO**. It is well known [3], [15] that $\mathbf{BMO} = \mathcal{H}^\infty(I\!R^n)^*$

DEFINITION 2.2. *[2] A function $f \in$* **BMO** *belongs to* **VMO** *if it satisfies*

$$\lim_{|B|\to 0}\frac{1}{|B|}\int_B |f - f_B| \ dx = 0, \quad lim_{|B|\to\infty}\frac{1}{|B|}\int_B |f - f_B| \ dx = 0,$$

uniformly for all open balls B.

We recall that **VMO** is the closure of **BMO** in $C_o(I\!R^n)$ of continuous functions vanishing at infinity and its dual space is $\mathbf{VMO}^* = \mathcal{H}^1(I\!R^n)$. For more details see [2], [8].
The following decay results for solutions to the Navier-Stokes equations in higher norms will be needed. We will work either with solutions that are globally smooth or we will suppose that our time is large enough to insure regularity see [4].

THEOREM 2.1. *Let $2 \le n \le 7$ Suppose $||u||_2^2 \le C_o(t+1)^{-2\mu}$ for $t \ge 0$, with some $\mu \ge 0$. Let T_o be given by Kato [4] such that*

$$||u||_\infty \le C_o(t - To)^{\frac{1}{2}} \quad \textit{for some} \quad C_o > 0 \tag{2.1}$$

Then for $m \in \mathbf{N}$ there is some $C = C_m(\mu, C_o)$ independent of T_o

$$||D^m u||_2^2 \le C_m(t - T_o - 1)^{-m-2\mu} \quad \textit{for} \quad t \ge T_o + 1 - 2^{-m} \tag{2.2}$$

Proof. See [12], [14],[17].

3. Regularity. It is well known that solutions of the Navier-Stokes equations are regular for a short period of time if we start with sufficiently smooth data and eventually become smooth for large time. The time period for $n \ge 3$ will depend on the norms of the data. For $n = 2$ there is global regularity. We recall that provided that the L^2 norm of the gradient of the solution is bounded it follows that the solutions are regular [7]. Here we give for completeness a very short proof which shows that if the data is such that the product of the L^2 norms of the data and gradient of the solution is smaller then the inverse of the constant appearing in the Prodi inequality [10] then the gradient of the solution remains bounded for all time. This result is presented for its simplicity. We give the proof in n dimensions and then show that for 2 and 3 dimensions the hypothesis reduce to the Prodi inequality [10]. We will suppose that we are working with solutions satisfying the energy inequality

$$\text{(3.1)}\quad \int_{I\!R^n} |u|^2 \ dx + \int_o^t \int_{I\!R^n} |u|^2 \ dx \ ds \leq \int_{I\!R^n} |\nabla u_o|^2 \ dx \ \text{ for all time } t > 0$$

It is known that for $2 \leq n \leq 5$ such solutions exists. In the next theorems we will use the notation

$$\phi = ||\nabla u||^2,$$
$$\phi_o = ||\nabla u_o||_2^2$$

THEOREM 3.1. *Let* $u_0 \in H^1(I\!R^n)$ *div* $u = 0$. *Let* $u(x,t)$ *be a solution to the Navier-Stokes equations satisfying the energy inequality (3.1) with data* u_0. *Suppose*

$$\text{(3.2)}\qquad a.\ \frac{d\phi}{dt} \leq C_n \phi^n \quad b.\ ||u_o||_2^2 (||\nabla u_o||_2^2)^{n-2} < (C_n)^{-1},$$

then the L^2 *of the gradient is bounded for all time.*

Proof. From (3.2) it follows that

$$\text{(3.3)}\qquad \frac{d\phi}{\phi^{n-1}} \leq C_n \phi \ dt$$

Integrating the last equation yields for $n \geq 3$

$$\text{(3.4)}\qquad \phi^{n-2} \leq [{\phi_o}^{2-n} - C_n(n-2) \int_o^t \phi \ \ ds]^{-1}$$

The energy inequality (3.1) yields

$$\int_o^t \phi \leq \int_{I\!R^n} |u_o|^2 \ dx$$

Thus combining the two last inequalities yields

$$\text{(3.5)}\qquad \phi^{n-2} \leq \frac{{\phi_o}^{n-2}}{1 - C_n(n-2) \int_o^t \phi(t)\, \mathrm{d}s (\phi_o)^{n-2}}$$
$$= \frac{(\int_{I\!R^n} |\nabla u_o|^2 \, \mathrm{d}x)^{n-2}}{1 - C_n(n-2) \int_{I\!R^n} |u_o|^2 \, \mathrm{d}x (\int_{I\!R^n} |\nabla u_o|^2 \, \mathrm{d}x)^{n-2}}.$$

Thus the gradient is bounded provided that

$$C_n(n-2) \int_{I\!R^n} |u_o|^2 \, \mathrm{d}x (\int_{I\!R^n} |\nabla u_o|^2)^{n-2} < 1.$$

Hence the theorem follows for $n \geq 3$. For $n = 2$ the same procedure leads to

$$\text{(3.6)}\qquad \log(\phi) \leq \log(\phi_o) + C_2 \int_o^t \phi dt$$

Thus in this case the L^2 norm of the gradient is bounded independently from the size of the data.

COROLLARY 3.1. *Let $u_o \in H^1(I\!R^n)$ $div\ u = 0$, with $n \geq 3$. Let $u(x,t)$ be a solution to the Navier-Stokes equations satisfying the energy inequality (3.1) with data u_o. Then $u(.,t) \in H^1(I\!R^n)$ for all $t \leq T$ where T is such that*

$$(n-2)C_n \int_{I\!R^n} ||\nabla u_o||_2^2 \ dx \int_o^T ||\nabla u||_2^2 \ dx \ \ ds \leq 1$$

where as before C_n is the constant that appears in (3.2)

Proof. Follows by (3.5).

REMARK 3.1. *For $n = 2,3$ the constant C_n in (3.2) a. is given by Prodi's inequality.*

REMARK 3.2. *We recall that Heywood using Prodi's inequality combined with a more geometric argument has gotten some other estimates that show for how long the 3D solution of (1.1) remains regular [6].*

4. Total variation decay for solutions to the Navier-Stokes equations. In this section we consider the decay of the total variation to solutions and higher derivatives to the Navier-Stokes equations in n dimensions, $2 \leq n \leq 7$. We obtain the decay for strong solutions. It is well known such solutions exist for small data as for example was shown in the last section. See also [7]. Our decay proofs apply also for large data, the smallness of the data is only used to insure regularity so that we can use the decay estimates on the derivatives given by Theorem (2.1). Since for sufficiently large time the solutions will become regular we can always modify the statement of the theorems to read for time larger that T_o , where T_o depends only on norms of the data. That such a time exists is a consequence of decay results of the L^2 and H^1 norms [12],[14] and [17].
The proof of the following theorem is done via Hardy estimates. That is we will obtain the decay of the L^1 norm of gradient via an estimate of the Hardy norm of the gradient. The ideas used are based on results of Miyakawa [8].

THEOREM 4.1. *Let $u_o \in L^2_\sigma \cap H^1 \cap \mathcal{H}^1_\sigma(I\!R^n)$. Suppose also that $\int_{I\!R^n} u_o = 0$ and let $n \geq 2$ Then if u is a weak solution to the Navier-Stokes equation , there exist T_o depending on the H^1 norms of the data such that for $t \geq T_o$*

$$||\nabla u(t) - \nabla e^{-At/2}u(t/2)||_1 \leq C(t+1)^{-n/2-1/2}, \tag{4.1}$$

where e^{-At} is the Stokes semi-group and C depends on the norms of the data. The L^1 norm of the gradient of the solution will decay as follows

$$\int_{I\!R^n} |\nabla u| dx \leq C_1(t+1)^{-n/2-1/2}, \tag{4.2}$$

with C_1 depending on the norms of the data.

Proof. We note that the condition imposed on the total mass, namely $\int_{I\!R^n} u_o = 0$, is essential since that will insure the decay of the homogeneous part of our equation. Recall that for positive data the L^1 norm of the solution to the Heat equation is independent of time.
We remark that if we take the data with the additional condition (3.2) from Theorem 3.1 then T_o can be taken equal to zero. Otherwise we work with large time. In both

situations we will work with regular solutions. Thus we can take a deriv ative of the equations (1.1)

$$u_{tx_i} + (u \cdot \nabla u)_{x_i} + (\nabla p)_{x_i} = (\Delta u)_{x_i}.$$

Let P be the projection onto divergence free fields hence the gradient of the solution can be expressed in the integral form

$$\partial u_{x_i} = e^{-At/2} u_o - \int_{t/2}^{t} \partial_{x_i} u[e^{A(t-s)}] P(u \cdot \nabla u) ds. \tag{4.3}$$

Note that we have put the derivative on the heat kernel in the last integral. Hence

$$||\partial_{x_i} u||_1 \leq ||\partial_{x_i} e^{-At/2} u_o||_1 + \int_{t/2}^{t} ||\partial_{x_i} [e^{A(t-s)}] P(u \cdot \nabla u)||_1 ds. \tag{4.4}$$

Thus we need to bound the two terms on the right side. The approach we use is based on ideas of Miyakawa [8], [9]. We need to show the following estimates

$$||\nabla u(t) - \nabla e^{-At/2} u(t/2)||_1 \leq C(t+1)^{-n/2-1}, \tag{4.5}$$

and

$$||\nabla e^{-At/2} u(t/2)||_1 \leq C t^{-1/2}. \tag{4.6}$$

The proof of (4.5) is as follows. Let $u \in \mathcal{H}^1$, for $t/2 \geq T_o$

$$\begin{aligned} < \partial_{x_i} u(t), \psi > = &< \partial_{x_i} e^{-At/2} u(t/2), \psi > - \\ &\int_{t/2}^{t} < \partial_{x_i} e^{-A(t-s)} P(u \cdot \nabla u)(s), \psi > ds. \end{aligned} \tag{4.7}$$

Here $\psi \in C^{\infty}_{o,\sigma}$, $0 \leq s \leq t$. Recalling that $[(\mathbf{VMO})_\sigma]^* = \mathcal{H}^1_\sigma$ it follows by (4.7) that

$$\begin{aligned} &< \partial_{x_i} u(t) - \partial_{x_i} e^{-At/2} u(t/2), \psi > \\ &\leq C_o \int_{t/2}^{t} \frac{1}{s^{1/2+n/2}} ||(u \cdot \nabla u)(s)||_{\mathcal{H}^1} ds [\psi]_{\mathbf{VMO}}. \end{aligned} \tag{4.8}$$

Here we used that

$$|\partial_{x_i} e^{-A(t-s)}| \leq C_o \frac{1}{s^{1/2+n/2}}.$$

Since $\mathcal{H}^1$ is strictly contained in L^1 we have

$$\begin{aligned} ||\partial_{x_i} u(t) - \partial_{x_i} e^{-At/2} u(t/2)||_1 &\leq ||\partial_{x_i} u(t) - \partial_{x_i} e^{-At/2} u(t/2)||_{\mathcal{H}^1} \\ &= \sup_{[\psi]_{\mathbf{VMO}=1}} < \partial_{x_i} u(t) - \partial_{x_i} e^{-At/2} u(t/2), \psi > \\ &\leq C_o \int_{t/2}^{t} \frac{1}{(t-s)^{1/2+n/2}} ||(u \cdot \nabla u)(s)||_{\mathcal{H}^1} ds \end{aligned} \tag{4.9}$$

We need the following estimate for the convective term.

THEOREM 4.2. *[1]. Let $n \geq 2$. If $u \in L^2_\sigma(\mathbb{R}^n)$ and $\nabla v \in L^2(\mathbb{R}^n)$, then $u \cdot \nabla v \in \mathcal{H}^1(\mathbb{R}^n)$, and we have the estimate*

$$\|u \cdot \nabla v\|_{\mathcal{H}^1} \leq C\|u\|_2\|\nabla v\|_2, \tag{4.10}$$

with C independent of u and v.

Proof. See [1].

From the last theorem and inequality (4.9) it follows that

$$\begin{aligned} \|\partial_{x_i}u(t) - \partial_{x_i}e^{-At/2}u(t/2)\|_1 &\leq \|\partial_{x_i}u(t) - \partial_{x_i}e^{-At/2}u(t/2)\|_{\mathcal{H}^1} \\ &\leq C_o\int_{t/2}^t \frac{1}{(t-s)^{1/2+n/2}}\|u\|_2\|\nabla u\|_2 ds \end{aligned} \tag{4.11}$$

We remark that $\int_{\mathbb{R}^n}|u_o|^2dx = 0$ yields that the solution to the underlying Heat equation with data u_o decays in the L^2 norm at a rate of $(t+1)^{-n/4-1/2}$. Thus under the conditions in the hypothesis it follows from theorem (2.1)

$$\|u(t)\|_2^2 \leq C_o(t+1)^{-n/2-1}.$$
$$\|\nabla u(t)\|_2^2 \leq C_o(t+1)^{-(n/2+2)}.$$

Replacing these last two inequalities in (4.11) yields after integrating

$$\|\partial_{x_i}u(t) - \partial_{x_i}e^{-At/2}u(t/2)\|_1 \leq C_1(t+1)^{-n/2-1/2} \tag{4.12}$$

Next we need a decay estimate on

$$\|\partial_{x_i}e^{-At/2}u_o\|_1.$$

An easy computation shows

$$\|\partial_{x_i}e^{-At/2}u_o\|_1 \leq C_o t^{-(n/2+1/2)}\|u(t/2)\|_1 \tag{4.13}$$

We are working with data such that the corresponding solution is bounded in $\mathcal{H}^1$. For a proof of former statement see [8].
Combining the inequalities from (4.4) and (4.13) yields the desired decay

$$\|\partial_{x_i}u\|_1 = C_1(t+1)^{-n/2-1/2} \tag{4.14}$$

This completes the proof.

The next theorem establishes the decay in $\mathcal{H}^1$ of higher order derivatives. The proof is similar to the last theorem, following ideas used by Miyakawa in [8]. The variant comes in the estimates of the non-homogeneous term

THEOREM 4.3. *Let $u_o \in L^2_\sigma \cap H^p \cap \mathcal{H}^1_\sigma(\mathbb{R}^n)$, with $p > 1$. Suppose also that $\int_{\mathbb{R}^n} u_o = 0$ and let $n \geq 2$. Then if u is a weak solution to the Navier-Stokes equation, there exist T_o depending on the H^1 norms of the data such that for $t \geq T_o$*

$$\|D^p u(t) - D^p e^{-At/2}u(t/2)\|_{\mathcal{H}^1} \leq C(t+1)^{-p/2-n/2}, \tag{4.15}$$

where as usual e^{-At} is the Stokes semi-group and C depends on the norms of the data. The p-derivative of the solution will decay as follows

$$\int_{\mathbb{R}^n}|D^p u|dx \leq C_o t^{-p/2} + C_1 t^{-n/2-p/2} \tag{4.16}$$

with C_o and C_1 depending on the norms of the data.

Proof. As in the last theorem we remark that if we take the data with the additional condition from (3.1) then T_o can be taken equal to zero. Otherwise we work with large time. In both situations we will work with regular solutions. Thus we can take "p" derivatives of the equations (1.1)

$$D^p u + D^p(u \cdot \nabla u) + D^P(\nabla p) = D^p(\Delta u).$$

Let P be the projection onto divergence free fields hence the D^p derivative of the solution can be expressed in the integral form

$$D^p u = e^{-At/2} D^p u_o - \int_{t/2}^{t} D^p[e^{A(t-s)}] P(u \cdot \nabla u) ds. \tag{4.17}$$

Note that we have put "p" derivatives on the heat kernel in the last integral. Hence

$$||D^p u||_{\mathcal{H}^1} \leq ||D^p e^{-At/2} u_o||_{\mathcal{H}^1} + \int_{t/2}^{t} ||D^p[e^{-As}] P(u \cdot \nabla u)(t-s)||_{\mathcal{H}^1} ds. \tag{4.18}$$

Thus we need to bound the two terms on the right side and show the following estimates

$$||D^p u(t) - D^p e^{-At/2} u(t/2)||_{\mathcal{H}^1} \leq C(t+1)^{-p/2}, \tag{4.19}$$

and

$$||D^p e^{-At/2} u(t/2)||_{\mathcal{H}^1} \leq C t^{-p/2}. \tag{4.20}$$

The proof of (4.20) is as follows. Let $u \in \mathcal{H}^1$, for $t/2 \geq T_o$

$$\begin{aligned} < D^p u(t), \psi > &= < D^p e^{-At/2} u(t/2), \psi > \\ &- \int_{t/2}^{t} < D^p e^{-As} P(u \cdot \nabla u)(t-s), \psi > ds. \end{aligned} \tag{4.21}$$

Here $\psi \in C^{\infty}_{o,\sigma}$, $0 \leq s \leq t$. Since $[(\mathbf{VMO})_\sigma]^* = \mathcal{H}^1_\sigma$ we have by (4.21) that

$$\begin{aligned} &< D^p u(t) - D^p e^{-At/2} u(t/2), \psi > \\ &\leq C_o \int_{t/2}^{t} ||D^p e^{-As}(u \cdot \nabla u)(t-s)||_{\mathcal{H}^1} ds [\psi]_{\mathbf{VMO}}. \end{aligned} \tag{4.22}$$

To bound $D^p e^{-As}$ we need the following auxiliary estimates. We recall that in the whole space the Stokes operator behaves as the Heat operator and thus the operators $D^p e^{-A}$ have convolution kernels

$$\begin{aligned} K_p(x) &= \sum_{|\alpha|=2r\ ,\ r=0,\ldots,s} c_r x^\alpha e^{-|x|^2/u} \quad \text{where} \quad p = 2s, \\ K_p(x) &= \sum_{|\alpha|=2r+1\ ,\ r=0,\ldots,s} c_r x^\alpha e^{-|x|^2/u} \quad \text{where} \quad p = 2s+1, \end{aligned} \tag{4.23}$$

Where the c_r are easy to compute, but their exact expression is not relevant for our computations. It follows that

$$|D^p e^{-As}| \leq C_o \frac{1}{s^{p/2+n/2}}.$$

Taking the supremum over all ψ with **VMO** norm equal to one in (4.22) yields

$$(4.24)\ ||D^p u(t) - D^p e^{-At/2} u(t/2)||_{\mathcal{H}_1} \leq C_o \int_{t/2}^{t} \frac{1}{s^{p/2+n/2}} ||(u \cdot \nabla u)(t-s)||_{\mathcal{H}^1} ds.$$

Thus we have

$$(4.25)\quad \begin{aligned} & ||D^p u(t) - D^p e^{-At/2} u(t/2)||_{\mathcal{H}^1} \\ = & \sup_{[\psi]_{\mathbf{VMO}=1}} < D^p u(t) - D^p e^{-At/2} u(t/2), \psi > \\ \leq & \, C_o \int_{t/2}^{t} \frac{1}{s^{p/2}} ||(u \cdot \nabla u)(t-s)||_{\mathcal{H}^1} ds \end{aligned}$$

From theorem (4.2) and inequality (4.25) it follows that

$$(4.26)\quad ||D^p u(t) - D^p e^{-At/2} u(t/2)||_{\mathcal{H}^1} \leq C_o \int_{t/2}^{t} \frac{1}{s^{p/2+n/2}} ||u||_2 ||\nabla u||_2 ds$$

Under the conditions on the hypothesis it follows from theorem (2.1) , since $\int_{I\!R^n} |u_o|^2 dx = 0$

$$\begin{aligned} ||u(t-s)||_2^2 &\leq C_o (t-s+1)^{-n/2-1}, \\ ||D^p u(t-s)||_2^2 &\leq C_o (t-s+1)^{-(n/2+p+1)}. \end{aligned}$$

Now replacing these last two inequalities in (4.26) yields after integrating

$$(4.27)\quad ||D^p u(t) - D^p e^{-At/2} u(t/2)||_{\mathcal{H}^1} \leq C_1 (t/2)^{-p/2-n/2}$$

Next we need a decay estimate on

$$||D^p e^{-At/2} u_o||_{\mathcal{H}^1}.$$

From (4.23) it follows that $K_p(x)$ decay exponentially it thus the kernels satisfy the conditions from Theorem 4 of [15], page 115. Hence one can conclude that

$$(4.28)\quad ||D^p e^{-A} u_o||_{\mathcal{H}^1} \leq C ||u_0||_{\mathcal{H}^1}$$

where $C > 0$ and independent from u_o. Let$u_{t,o}(x) = u_o(\sqrt(t)x)$. Similar steps as in [8] yield

$$(4.29)\quad ||u_{t,o}||_{\mathcal{H}^1} = t^{-n/2} ||u_o||_{\mathcal{H}^1},\ \ ||D^p e^{-A} u_{t,o}||_{\mathcal{H}^1} = t^{p/2-n/2} ||D^p e^{-At} u_o||_{\mathcal{H}^1}$$

Thus from (4.23), (4.28), (4.29)

$$(4.30)\quad ||D^p e^{-At/2} u_o||_{\mathcal{H}^1} \leq C_o t^{-p/2} ||u(t/2)||_{\mathcal{H}^1}$$

Since the solutions are bounded in $\mathcal{H}^1$ (see [8])

$$(4.31)\quad ||D^p e^{-At/2} u_o||_{\mathcal{H}^1} \leq C_o t^{-p/2} ||u(t/2)||_{\mathcal{H}^1}$$

Now combining the inequalities from (4.18), (4.27) and (4.31) yields the desired decay

$$(4.32)\quad ||D^p u||_{\mathcal{H}^1} = C_o t^{-p/2}$$

This completes the proof.

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