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AN IMPROVEMENT OF A BORWEIN- ERDÉLYI- KÓS RESULT*

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Abstract. This paper considers the problem of finding, given n, the smallest m such that there exists a polynomial $\phi(x)$ of degree m satisfying $|\phi(-1)| > \sum_{i=0}^{n} |\phi(i)|$. It is shown that $m \geq \lfloor\sqrt{n \ln 2}\rfloor - 1$. For polynomials non-negative on $[0, n]$ we find the best possible value of m and show that

$$2\lfloor\sqrt{n \ln 2}\rfloor - 3 \leq m \leq 2\lfloor\sqrt{n \ln 2}\rfloor + 2, \;\; n \geq 3,$$

improving an earlier result of Borwein, Erdélyi and Kós [3]. As a consequence we sharpen some bounds concerning the Prouhet-Terry-Escott problem, polynomials with restricted coefficients and the sequence reconstruction problem.

1. Introduction. A question of finding for a given n a polynomial $\phi(x)$ of the least degree satisfying

$$(*) \qquad |\phi(-1)| > \sum_{i=0}^{n} |\phi(i)|,$$

has arisen in connection with so-called Littlewood-type problems [3], and has many applications, see e.g. [2, 3, 5]. Probably the most interesting of them is: having found such a polynomial of degree m then there are only trivial solutions for the Prouhet-Terry-Escott (PTE) problem of size $n+1$ and degree m. The PTE problem of size $n+1$ and degree d asks for non-trivial solutions $u_i, w_i \in \mathbb{N} \cap [0, n+1]$, $i = 0, \ldots, q \leq n+1$ to the system:

$$u_0^h + u_1^h + \cdots + u_q^h = w_0^h + w_1^h + \cdots + w_q^h, h = 0, \ldots, d;$$
$$u_0 < u_1 < \cdots < u_q, w_0 < w_1 < \cdots < w_q.$$

A trivial solution is given by $u_i = w_i$, $i = 0, \ldots, q$.

Notice also that the famous Vinogradov mean value theorem gives an upper bound on the total number of the solutions of this system. Let $\mathcal{A}_m^n$ (resp. $\mathcal{B}_m^n$) denote the set of polynomials (resp. polynomials non-negative on $[0, n]$) of degree m satisfying (*).

It is easily checked (see e.g. [5]) that the above PTE system has a non-trivial solution iff there is a non-zero sequence $\delta_0, \ldots \delta_{n+1}$ with $\delta_i \in \{-1, 0, 1\}$, such that $\sum_{i=0}^{n+1} \delta_i g(i) = 0$, for any polynomials $g(x)$ of degree m. Thus, whenever $\mathcal{A}_m^n \neq \emptyset$ then PTE problem has only trivial solutions for $d \geq m$.

Another consequence is that a polynomial $\sum_{j=0}^{n+1} a_j x^j$ with $|a_0| = 1$, $|a_j| \leq 1$ can have at most m-fold zero at 1 [3]. Condition $\mathcal{A}_m^n \neq \emptyset$ provides also a bound for the Sequence Reconstruction Problem (see [5] and Theorem 1.5 below).

In Borwein, Erdélyi and Kós [3] an ingenious example is constructed which gives a non-negative polynomial of degree $\lfloor\frac{16}{7}\sqrt{n+1}\rfloor + 4$ satisfying $|\phi(-1)| > \sum_{i=0}^{n} \phi(i)$. It turns out that this was an extremely good guess, since, as we will show, no such a polynomial exists for $m \leq \lfloor\sqrt{n \ln 2}\rfloor - 2$. We improve on their result by considering polynomials which are non-negative on $[0, n]$ and finding such polynomials of the least

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degree which satisfy (*) - see Theorem 4.1. Using estimates of this minimum degree we find:

THEOREM 1.1.

$$m \geq 2\lfloor\sqrt{n\ln 2}\rfloor + 2 \Rightarrow \mathcal{B}_m^n \neq \emptyset$$
$$m < 2\lfloor\sqrt{n\ln 2}\rfloor - 3 \Rightarrow \mathcal{B}_m^n = \emptyset$$

Since $|\phi(-1)| > \sum_{i=0}^{n} |\phi(i)|$, $deg(\phi(x)) = m$, implies $\phi^2(-1) > \sum_{i=0}^{n} \phi^2(i)$, and hence $\mathcal{A}_{2m}^n \neq \emptyset$, we get

THEOREM 1.2. *If* $m \leq \lfloor\sqrt{n\ln 2}\rfloor - 2$ *then* $\mathcal{A}_m^n = \emptyset$.

THEOREM 1.3. *The PTE problem of size* $n+1$ *has only trivial solutions for degree* $m \geq 2\lfloor\sqrt{n\ln 2}\rfloor + 2$.

THEOREM 1.4. $\sum_{j=0}^{n+1} a_j x^j$, $|a_0| = 1$, $|a_j| \leq 1$ *has at most* $2\lfloor\sqrt{n\ln 2}\rfloor + 2$ *zeros at 1.*

THEOREM 1.5. *Any word of length n is uniquely determined by all its* $\binom{n}{m}$ *subwords of length m, provided* $m \geq 2\lfloor\sqrt{n\ln 2}\rfloor + 3$.

Concerning Theorems 1.3 and 1.4 notice that the known lower bound is $\Omega(\sqrt{\frac{n}{\ln n}})$ [1] (see also [3]).

We consider polynomials non-negative on $[0, n]$ as we can apply Lukács' Theorem (see [7], p.4) to these polynomials and then use the Hahn polynomials in order to obtain polynomials with the required minimum degrees. We state a minor variation of Lukács' Theorem which we use in this paper.

THEOREM 1.6. *(a) Let* $f(x)$ *be a polynomial of degree* m *with real coefficients which is non-negative on* $[-1, 1]$. *Then there exists a polynomial* $h(z)$, $z = e^{i\theta}$ *of degree* m *such that* $f(\cos(\theta)) = |h(z)|^2$.

Let $k = \lfloor\frac{m+1}{2}\rfloor$ *then:*

(b) $f(x) = p(x)^2 + (1-x^2)q(x)^2$ *where* $p(x)$, $q(x)$ *are of degree* $k, k-1$ *respectively.*

(c) Let $f(x)$ *be a polynomial of degree* m *with real coefficients where* $p(x)$, $q(x)$ *are of degree* k, $k-1$ *respectively.*

Proof. Part (a) is found in [7].

Part (b): If m is even then we have $f(\cos(\theta)) = |h(z)|^2 = |e^{-im\theta/2}h(z)|^2$ and on expanding out in powers of $e^{i\theta}$ the result follows on using the Chebyshev polynomials - see proof in [7]. If $m = 2k-1$ is odd then we use $f(\cos(\theta)) = |e^{-i(k-1)\theta}h(z)|^2$ instead and expand. This is the only change for the odd case from the standard proof.

Part (c): $g(x) = f(n(x+1)/2)$ satisfies conditions of (b) and so we have $g(x) = p(x)^2 + (1-x^2)q(x)^2$ for suitable polynomials p, q. Since $f(x) = g((x-1)/(n-1))$ we find on substitution that $f(x) = p((x-1)/(n-1))^2 + x(n-x)(2q(x)/n)^2$. Hence result. □

2. The Hahn Polynomials. The references for this section are [4],[6]. We consider the Hahn polynomials for α, $\beta > -1$, $n \in \mathbb{N}$:

$$H_k(x;\alpha,\beta,n) = \sum_{j=0}^{k} (-1)^j \frac{\binom{k}{j}\binom{k+\alpha+\beta+j}{j}}{\binom{j+\alpha}{j}\binom{n}{j}} \binom{x}{j}.$$

For fixed α, β they are orthogonal on $[0, \dots, n]$ with weight $\rho^{(\alpha,\beta)}(x) = \binom{x+\alpha}{x}\binom{n-x+\beta}{n-x}$, i.e.

$$\sum_{i=0}^{n} \rho^{(\alpha,\beta)}(i) H_k(i;\alpha,\beta,n) H_q(i;\alpha,\beta,n) = \delta_{k,q} d_k^{(\alpha,\beta)}$$

where

$$d_k^{(\alpha,\beta)} = \frac{n+k+\alpha+\beta+1}{2k+\alpha+\beta+1} \frac{\binom{k+\beta}{k}\binom{n+k+\alpha+\beta}{n}}{\binom{k+\alpha}{k}\binom{n}{k}}.$$

We shall work with the following two special cases of the Hahn polynomials. First, we have the discrete Chebyshev polynomials [6]:

$$T_k(x) = H_k(x;0,0,n) = \sum_{j=0}^{k} (-1)^j \frac{\binom{k}{j}\binom{k+j}{j}}{\binom{n}{j}} \binom{x}{j},$$

$$\rho^{(0,0)}(x) = 1, \; d_k^{(0,0)} = d_k = \frac{n+k+1}{2k+1} \frac{\binom{n+k}{n}}{\binom{n}{k}}.$$

The other special case is given by:

$$R_k(x) = H_k(x-1;1,1,n-2) = \sum_{j=0}^{k} (-1)^j \frac{\binom{k}{j}\binom{k+2+j}{j}}{(j+1)\binom{n-2}{j}} \binom{x-1}{j},$$

$$\rho^{(1,1)}(x) = x(n-x), \; d_k^{(1,1)} = D_k = \frac{n+k+1}{2k+3} \frac{\binom{n+k}{n-2}}{\binom{n-2}{k}}.$$

We also need to find $T_k(-1)$, $R_k(-1)$. To do this we need the following:

LEMMA 2.1.

$$\sum_{j=0}^{k} \frac{\binom{k}{j}\binom{k+t+j}{j}}{\binom{n}{j}} = \frac{\binom{n+k+t+1}{k}}{\binom{n}{k}}.$$

Proof. On expanding the binomial coefficients and multiplying both sides by $\binom{n}{k}$ we find the above is equivalent to $f(n,k,t) = \sum_{j=0}^{k} \binom{n-j}{k-j}\binom{k+t+j}{j} = \binom{n+k+t+1}{k}$. Using $\binom{n-j}{k-j} = \binom{n-1-j}{k-j} + \binom{n-1-j}{k-1-j}$ we obtain:

$$\begin{aligned} f(n,k,t) &= \sum_{j=0}^{k} \left(\binom{n-1-j}{k-j} + \binom{n-1-j}{k-1-j} \right) \binom{k+t+j}{j} \\ &= f(n-1,k,t) + f(n-1,k-1,t+1). \end{aligned}$$

The result follows by induction on $n+k+t$. □

LEMMA 2.2.

$$T_k(-1) = \frac{\binom{n+k+1}{k}}{\binom{n}{k}}, \; k = 0, \dots, n;$$

$$R_k(-1) = \frac{\binom{n+k+1}{k}}{\binom{n-2}{k}}, \; k = 0, \dots, n-2.$$

Proof. Apply Lemma 2.1 with $t = 0$, $t = 2$ respectively. □

3. Finding the Optimal Polynomial. Let $\mathcal{P}_k^n$ denote the set of all polynomials of degree k which are non-nega tive on $\{0,\dots,n\}$. We find the minimum value over all polynomials $\phi(x) \in \mathcal{P}_k^n$ of:

$$\Delta(\phi) = \sum_{i=0}^{n} \phi(i) - |\phi(-1)|.$$

3.1. Even Case. Let $\phi_{2k}(x) \in \mathcal{P}_{2k}^n$. By Lukács' Theorem we have polynomials $p_k(x)$, $q_{k-1}(x)$ of degrees k, $k-1$ respectively such that $\phi_{2k}(x) = p_k^2(x) + x(n-x)q_{k-1}^2(x)$.

LEMMA 3.1. *Let* $\phi \in \mathcal{P}_{2k}^n$,.
(a) if $\phi(-1) > 0$ *then*

$$\Delta(\phi) < 0 \Rightarrow \sum_{i=0}^{k} T_i^2(-1)/d_i > 1 \Leftrightarrow \sum_{i=0}^{k} (2i+1)\frac{(n+i+1)!(n-i)!}{(n+1)!^2} > 1;$$

(b) if $\phi(-1) < 0$ *then*

$$\Delta(\phi) < 0 \Rightarrow \sum_{i=0}^{k-1} R_i^2(-1)/D_i > 1/(n+1)$$
$$\Leftrightarrow \sum_{i=0}^{k-1} (2i+3)(i+2)(i+1)\frac{(n+i+1)!(n-i-2)!}{(n+1)!n!} > 1.$$

Proof. (a) We note that

$$\Delta(\phi) = \sum_{x=0}^{n} p_k^2(x) - p_k^2(-1) + \sum_{x=0}^{n} x(n-x)q_{k-1}^2(x) + (n+1)q_{k-1}^2(-1).$$

Since the last two terms are a sum of positive terms we have $\Delta(\phi) \geq \Delta(p_k^2)$.

Now write $p_k(x) = \sum_{i=0}^{k} a_i T_i(x)$ and let $F_k = \sum_{j=0}^{k} T_j^2(-1)/d_j$. Using the orthogonality properties of the Hahn polynomials and the Cauchy-Schwarz inequality we have:

$$\Delta(p_k^2) = \sum_{i=0}^{k} a_i^2 d_i - (\sum_{j=0}^{k} a_j T_j(-1))^2 \geq \sum_{i=0}^{k} a_i^2 d_i - (\sum_{j=0}^{k} a_j^2 d_j)F_k = (1-F_k)(\sum_{i=0}^{k} a_i^2 d_i).$$

Hence $F_k \leq 1 \Rightarrow \Delta(p_k^2) \geq 0 \Rightarrow \Delta(\phi) \geq 0$.

Substituting the values for $T_i(-1)$ we obtain:

$$F_k = \sum_{i=0}^{k} (2i+1)\frac{(n+i+1)!(n-i)!}{(n+1)!^2}$$

and (a) follows.

(b) In this case we have

$$\Delta(\phi) = \sum_{x=0}^{n} p_k^2(x) + p_k^2(-1) + \sum_{x=0}^{n} x(n-x)q_{k-1}^2(x) - (n+1)q_{k-1}^2(-1).$$

Hence we have $\Delta(\phi) \leq \Delta(x(n-x)q_{k-1}^2(x))$ and we write $q_{k-1} = \sum_{i=0}^{k-1} b_i R_i(x)$. Let $G_{k-1} = \sum_{j=0}^{k-1}(n+1)R_j^2(-1)/D_j$. Following similar reasoning to (a) we find

$$\Delta(x(n-x)q_{k-1}^2) \geq (1-G_{k-1})(\sum_{i=0}^{k-1} b_i^2 D_i).$$

Hence $G_{k-1} \leq 1 \Rightarrow \Delta(\phi) \geq 0$.

Substituting the values found for $R_i(-1)$ we obtain:

$$G_{k-1} = \sum_{i=0}^{k-1}(2i+3)(i+2)(i+1)\frac{(n+i+1)!(n-i-2)!}{(n+1)!n!}$$

and (b) follows. □

THEOREM 3.2. *Given n the least k such that* $\exists\phi \in \mathcal{P}_{2k}^n$ *and* $\Delta(\phi) < 0$ *is given by* $k = \mathrm{Min}\{\mathrm{i} : \mathrm{F_i} > 1\}$

Proof. First we show that the inequality:

$$F_k > G_{k-1}, \quad k < \sqrt{n \ln 2}$$

is true for $n \geq 3$ and hence we need only consider the case $\phi(-1) \geq 0$, $n \geq 3$. For $n = 3, 4$ it can be checked directly and so we assume $n \geq 5$.

Let $W_j = (n+j+2)(n-j-1) - (n+1)(j+2)(j+1)$. After a small amount of algebra we find:

$$F_k - G_{k-1} = 1/(n+1) + \sum_{j=0}^{k-1}(2j+3)(n+j+1)!(n-j-2)!W_j/(n+1)!^2.$$

Now for $j \leq n^2/\sqrt{n+2} - 2$. It is easy to show that $n^2/\sqrt{n+2} - 2 \geq \sqrt{n \ln 2}$, $n \geq 5$. The theorem follows by the last Lemma if we can find an example of a Case (a) polynomial of the given degree $k = \mathrm{Min}\{\mathrm{i} : \mathrm{F_i} > 1\}$ which satisfies the conditions.

We see easily that the polynomial

$$\phi(x) = p_k^2(x), \; p_k(x) = \sum_{i=0}^{k} \frac{T_i(-1)}{d_i} T_i(x)$$

satisfies $\Delta(\phi) < 0$ where $k = \mathrm{Min}\{\mathrm{i} : \mathrm{F_i} > 1\}$. □

3.2. Estimating the optimal even degree. We establish the following result

LEMMA 3.3.

$$e^{j(j+1)/(n+1)} < \frac{\binom{n+j+1}{j}}{\binom{n}{j}} < e^{j(j+1)/n}, \quad 1 \leq j \leq \sqrt{n}$$

This can be more precisely stated as:

$$(a) \; e^{j(j+1)/(n+1)} < \frac{\binom{n+j+1}{j}}{\binom{n}{j}}, \; j \geq 1$$

$$(b) \; e^{j(j+1)/n} > \frac{\binom{n+j+1}{j}}{\binom{n}{j}}, \; 1 \leq j \leq \sqrt{n}$$

Proof. By induction. We use the inequality $e^{x/(1+x/2)} < 1+x$, $x > 0$ throughout this proof without comment.

(a) Certainly true for $j = 1$. Now

$$e^{j(j+1)/(n+1)} = e^{(j-1)j/(n+1)}e^{2j/(n+1)} < \frac{\binom{n+j}{j-1}}{\binom{n}{j-1}}e^{2j/(n+1)}$$

by induction. But

$$e^{2j/(n+1)} < 1 + 2j/(n-j+1) = (n+j-1)/(n-j+1)$$

and so

$$e^{j(j+1)/(n+1)} < \frac{\binom{n+j}{j-1}}{\binom{n}{j-1}}(n+j-1)/(n-j+1) = \frac{\binom{n+j+1}{j}}{\binom{n}{j}}.$$

Hence result.

(b). Certainly (b) is true for $j = 1$. Suppose that $j \le \sqrt{n}$ and (b) is true for $j-1$. Then we have

$$e^{j(j+1)/n} = e^{(j-1)j/n}e^{2j/n} > \frac{\binom{n+j}{j-1}}{\binom{n}{j-1}}e^{2j/n}.$$

Hence if we can show that

$$e^{2j/n} > (n+j+1)/(n-j+1) = 1 + 2j/(n-j+1)$$

we obtain

$$e^{j(j+1)/n} > \frac{\binom{n+j}{j-1}}{\binom{n}{j-1}}(n+j+1)/(n-j+1) = \frac{\binom{n+j+1}{j}}{\binom{n}{j}}.$$

Now let $x = 2j/n$ in $e^{2j/n} - (n+j+1)/(n-j+1)$ to obtain

$$e^x - 1 - 2nx/(2n-nx+2) > x + x^2/2 - 2nx/(2n-nx+2) =$$

$x^2/2 - x(nx-2)/(2n-nx+2) > 0$ if $nx^2 - 2x - 4 < 0$, i.e.

$$x < \frac{1+\sqrt{4n+1}}{2} \Leftrightarrow j < \sqrt{n+1/4} + 1/2.$$

But $j \le \sqrt{n}$ by the assumption, hence result. □

3.2.1. Finding bounds for $\gamma(n)$. Using Lemma 3.2 we now estimate the first value of k such that $F_k > 1$. It is well-known that for a monotone function $f(x)$,

$$min(f(0), f(m)) \le \sum_{i=0}^{m} f(i) - \int_0^m f(z)dz \le max(f(0), f(m)).$$

Using this we get

$$F_k = \sum_{j=0}^{k}(2j+1)\frac{(n+j+1)!(n-j)!}{(n+1)!^2} > \frac{2j+1}{n+1}e^{(j+1)j/(n+1)}$$

$$> \int_0^k \frac{2x+1}{n+1}e^{(x+1)x/(n+1)}dx + 1/(n+1) = e^{(k+1)k/(n+1)} - 1 + 1/(n+1).$$

Hence $e^{(k+1)k/(n+1)} - 1 + 1/(n+1) > 1 \Rightarrow F_k > 1$. Now

$$\begin{aligned} e^{(k+1)k/(n+1)} - 1 + 1/(n+1) > 1 \Rightarrow \\ k(k+1) > (n+1)\ln(2 - 1/(n+1)) > (n+1)(\ln 2 - 1/(2n+1)). \end{aligned}$$

This implies

$$(k+1/2)^2 > n\ln 2 + \ln 2 - (n+1)/(2n+1) + 1/4 > n\ln 2 + 1/3, \;\; n \geq 3$$

i.e.

$$k > \sqrt{n\ln 2 + 1/3} - 1/2 \Rightarrow F_k > 1, \;\; n \geq 3.$$

We also have using Lemma 3.2 and assuming that $k < \sqrt{n}$:

$$F_k < \sum_{j=0}^{k} \frac{2j+1}{n+1} e^{(j+1)j/n} < \int_0^{k+1} \frac{2x+1}{n+1} e^{(x+1)x/n} dx = \frac{n}{n+1}(e^{(k+2)(k+1)/n} - 1).$$

Hence we have

$$\begin{aligned} & e^{(k+2)(k+1)/n} - 1 < (n+1)/n \\ \Leftrightarrow\ & (k+2)(k+1) < n(\ln(2+1/n)) < n(\ln(2) + 1/(2n-1)) \\ \Rightarrow\ & k < \sqrt{n\ln(2) + n/(2n-1) + 1/4} - 3/2 < \sqrt{n\ln(2)+1} - 3/2. \end{aligned}$$

Hence we have shown $k < \sqrt{n\ln(2)+1} - 3/2 \Rightarrow F_k < 1$. If we let $\gamma(n) = \mathrm{Min}\{i : F_i > 1\}$ then if we let $z_n = \lfloor \sqrt{n\ln(2)} \rfloor$ we have:

$$z_n - 2 \leq \gamma(n) \leq z_n + 1, \;\; n \geq 3$$

4. General Case.

Notation: Let $\mu(n) = \mathrm{Min}\{i : \mathcal{B}_i^n \neq \emptyset\}$. Recall that $\gamma(n) = \mathrm{Min}\{i : F_i > 1\}$. We have shown in Section 3 that $\mu(n) \leq 2\gamma(n)$ Note that if $k_* = \gamma(n)$ i.e $\exists\, \phi(x) \in \mathcal{P}_{2k_*}^n$ satisfying $\Delta(\phi) < 0$, then $\mu(n) \geq 2k_* - 1$. Clearly we have $\mu(n) \neq 2s,\; s < k_*$ and $\mu(n) \neq 2s-1, s < k_*$. This last case follows from the first as if $\phi_1(x) \in \mathcal{P}_{2s-1}^n, \Delta(\phi) < 0$ then $\phi_2(x) = ax^{2s} + \phi_1(x) \in \mathcal{P}_{2s}^n$ and $\Delta(\phi_2) < 0$ for small enough $a \in \mathbb{R}^+$, e.g. $a = -0.5\Delta(\phi_1)/\Delta(x^{2s})$. However it is possible to have $\mu(n) = 2k_* - 1$ for some n and the following theorem gives the necessary condition on n.

Notation:

$$\begin{aligned} b_{k-1} &= (k+1)(n-k)/(n(n-1)) \\ M_k &= \frac{b_{k-1}^2 D_{k-1}(1+G_{k-1})}{1+G_{k-2}} \\ m_k &= \frac{d_k(1-F_k)}{1-F_{k-1}} \\ Q_k &= m_k/M_k \end{aligned}$$

THEOREM 4.1. *Given $n \in \mathbb{N}$. Let $k_* = \gamma(n)$.*

$$\begin{aligned} Q_{k_*} < -1 &\Rightarrow \mu(n) = 2k_* - 1 \\ Q_{k_*} \geq -1 &\Rightarrow \mu(n) = 2k_* \end{aligned}$$

Proof. Since we have $\mu(n) \leq 2k_*$ from the even case, we consider odd degree polynomials. Let $\phi(x) \in \mathcal{P}^n_{2k-1}$. By Lukasc' Theorem we can write: $\phi(x) = p(x)^2 + x(n-x)q(x)^2$, where $p(x), q(x)$ are of degree k, $k-1$ respectively. This enables us to re-use the calculations for the even case, but we pay for this by having to normalize the polynomial. On using the Hahn polynomials as in the even case we have:

$$p_k(x) = T_k(x) + \sum_{i=0}^{k-1} a_i T_i(x)$$

$$q_{k-1}(x) = b_{k-1}R_{k-1}(x) + \sum_{i=0}^{k-2} b_i R_i(x)$$

where $b_{k-1} = (k+1)(n-k)/(n(n-1))$ is a fixed constant ensuring that $T_k^2(x) - b_{k-1}^2 R_{k-1}^2(x)$ has no term in x^{2k}. But we note that the coefficient of x^{2k-1} is determined by these choices and is non-zero. This gives a normalization of the polynomial $\phi(x)$.

As before we have to consider the cases $\phi(-1) > 0$, $\phi(-1) < 0$.

Case (a) $\phi(-1) > 0$.

$\Delta(\phi) = \sum_{x=0}^{n} p_k^2(x) - p_k^2(-1) + \sum_{x=0}^{n} x(n-x)q_{k-1}^2(x) + (n+1)q_{k-1}^2(-1)$. But now we cannot assume that $q_i(x) = 0, x \in \{-1, 1, ..\}$ as we have a non-zero term in $R_{k-1}(x)$. But we can minimize this expression independently to get a value independent of $p_k(x)$. On minimizing using the decomposition into Hahn polynomials we obtain that $\sum_{x=0}^{n}(n-x)q_{k-1}^2(x) + (n+1)q_{k-1}^2(-1)$ has the minimum value:

$$M_k = \frac{b_{k-1}^2 D_{k-1}(1+G_{k-1})}{1+G_{k-2}}$$

This value occurs at $b_i = \beta_{k-1}R_i(-1)/D_i$, $i = 0, \ldots, k-2$, where

$$\beta_{k-1} = -(n+1)b_{k-1}R_{k-1}(-1)/(1+G_{k-2}).$$

Hence we have $\Delta(\phi) \geq \Delta(p_k^2(x)) + M_k$ and we now minimize

$$\Delta(p_k^2(x)) = \sum_{i=0}^{k-1} a_i^2 d_i + d_k - (T_k(-1) + \sum_{j=0}^{k-1} a_j T_j(-1))^2$$

to obtain the minimum value:

$$m_k = \frac{d_k(1-F_k)}{1-F_{k-1}}$$

and $a_i = \alpha_{k-1}T_i(-1)/d_i$, $i = 0, \ldots, k-1$, where $\alpha_{k-1} = T_{k-1}/(1-F_{k-1})$. Hence we have $M_k + m_k \geq 0 \Rightarrow \Delta(\phi) \geq 0$. We note that $m_k < 0 \Leftrightarrow k = k_*$. A necessary condition for $\Delta(\phi) < 0$ is that $M_k + m_k < 0 \Leftrightarrow m_k < -M_k < 0 \Rightarrow k = k_*$.

We see directly that $\mathcal{B}^n_{2k_*-1} \neq \emptyset \Leftrightarrow Q_{k_*} < -1$ and an example of such a polynomial is given by using the coefficients:

$$a_i = \alpha_{k_*-1}T_i(-1)/d_i, \ i = 0, \ldots, k_* - 1$$
$$b_i = \beta_{k_*-1}R_i(-1)/D_i, \ i = 0, \ldots, k_* - 2$$

where $\alpha_{k_*-1} = T_{k_*-1}/(1-F_{k_*-1})$, $\beta_{k_*-1} = -(n+1)b_{k_*-1}R_{k_*-1}(-1)/(1+G_{k_*-2})$.

Case (b) $\phi(-1) < 0$.

$\Delta(\phi) = \sum_{x=0}^{n} p_k^2(x) + p_k^2(-1) + \sum_{x=0}^{n} x(n-x)q_{k-1}^2(x) - (n+1)q_{k-1}^2(-1)$. We obtain that the minimum k such that $\Delta(\phi) < 0$ is given by the least k such that $N_k + n_k < 0$ where

$$N_k = \frac{b_{k-1}^2 D_{k-1}(1-G_{k-1})}{1-G_{k-2}}, \quad n_k = \frac{d_k(1+F_k)}{1+F_{k-1}}$$

We now show that $S_k = N_k + n_k - (M_k + m_k) \geq 0$ for all $k \leq \gamma(n)$ and Case (a) gives the least k. After some algebra we quickly find that:

$$\begin{aligned} S_k &= 2\beta_{k-1}^2 D_{k-1}\frac{G_{k-2}-G_{k-1}}{1-G_{k-2}^2} + 2d_k\frac{F_k - F_{k-1}}{1-F_{k-1}^2} = -2\beta_{k-1}^2\frac{R_{k-1}(-1)^2}{1-G_{k-2}^2} + 2\frac{T_k(-1)^2}{1-F_{k-1}^2} \\ &\geq \frac{2(T_k(-1)^2 - \beta_{k-1}^2 R_{k-1}(-1)^2)}{1-G_{k-2}^2}, \quad k \leq \gamma(n). \end{aligned}$$

The last inequality follows from $0 < G_{k-2} \leq F_{k-1} < 1, \ k \leq \gamma(n)$; see proof of Theorem 3.2. Using Lemma 2.1 we see that $\beta_{k-1}R_{k-1}(-1) \leq T_k(-1) \Leftrightarrow n^2 \geq nk$ and the result follows. □

5. Calculations and Examples. Mathematica was used for all calculations. We find on calculation using the test $F_k > 1$ for the minimum degree $2k$ of even degree polynomials ϕ non-negative on $[0, n]$ and satisfying $\Delta(\phi) < 0$ that $k = \lfloor\sqrt{n \ln 2}\rfloor$ is correct in about 99% of cases from $n = 300$ to $n = 1000$. The other values calculated are all $\lfloor\sqrt{n \ln 2}\rfloor + 1$.

5.1. Odd Degree polynomials. The following values of n up to 605 have odd degree minimum degree polynomials ϕ which are positive on $[0, n]$ and satisfy $\Delta(\phi) < 0$. The values of n are given in ranges; $[a, b]$ meaning all integers between a, b including a, b.

$[5,7]$, $[12,15]$, $[22,27]$, $[35,42]$
$[143,157]$, $[174,189]$, $[207,224]$, $[243,261]$
$[282,302]$, $[324,345]$, $[368,391]$, $[416,440]$
$[467,492]$, $[520,547]$, $[576,605]$

All other values of n have even degree minimum degree polynomials.

5.2. Examples of Minimum Polynomials: $n = 23$. If we calculate the minimum degree even dimension polynomial ϕ_{even} given in Section 3 we find we get a polynomial of degree 8:

$$\frac{(75900 - 46754x + 7599x^2 - 454x^3 + 9x^4)^2}{5309162496}.$$

However the minimum odd degree polynomial is of degree 7 and is:

$$\begin{gathered}(6020175780326366208 - 4683235859203903260x + 1393432015941831656x^2 - \\ 204308880340318863x^3 + 16253918538185423x^4 - 712839540298293x^5 + \\ 16129965347897x^6 - 146153389200x^7)/12347200692845568\end{gathered}$$

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