METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 7, No. 4, pp. 727–746, December 2000
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ON THE ASYMPTOTICS FOR LATE COEFFICIENTS IN UNIFORM ASYMPTOTIC EXPANSIONS OF INTEGRALS WITH COALESCING SADDLES*

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Abstract. We will construct Borel transforms for uniform asymptotic expansions of integrals with coalescing saddles. These Borel transforms are in terms of new functions that have coalescing branch points. The Riemann sheet structures of these clusters of branch points plus a Cauchy type integral representation enable us to obtain asymptotic expansions for the late coefficients of the uniform asymptotic expansions.

Two examples involving a Pearcey-type integral are included.

1. Introduction. We will study integrals of the form

$$I_{C(\mathbf{A})}(k,\mathbf{A}) = \int_{C(\mathbf{A})} e^{kf(z,\mathbf{A})} g(z)\, dz, \tag{1.1}$$

where $C(\mathbf{A})$ is an infinite path of integration connecting valleys of the exponential, and f and g are analytic functions of z near $C(\mathbf{A})$. In addition, f and C depend on parameters

$$\mathbf{A} = \{A_1, A_2, \cdots\}, \tag{1.2}$$

whose variation causes some of the saddles $z_n(\mathbf{A})$ of f to coalesce, and smooth displacements of the path of integration. We will assume that a cluster of N saddles dominates the large k asymptotics of the integral, and that the N saddles will coalesce at a critical value $\mathbf{A}_0$. Our analysis will also include the case $N = 1$.

In the case $N \geqslant 2$ and $\mathbf{A}$ in a neighbourhood of $\mathbf{A}_0$ the asymptotics of (1.1) cannot be expressed in simple asymptotic expansions. It is well known how to obtain uniform asymptotic expansions for this integral. These expansions are for $k \to \infty$ and they are uniformly valid for $\mathbf{A}$ in a neighbourhood of $\mathbf{A}_0$. Even in the case $N = 2$, the coefficients in the uniform asymptotic expansions are complicated functions of their arguments. Hence, asymptotics for the late coefficients is useful to compute these coefficients and to obtain more information on the divergence of the uniform asymptotic expansions.

One of the main tools in obtaining asymptotics for late coefficients in recent publications has been the Borel transform. In the case of simple (non uniform) asymptotic expansions, the Borel transform is a well-known tool. It produces analytic functions with simple branch points. Via the Laplace transform of the Borel transform exponentially improved asymptotic expansions (hyperasymptotics) and (hyper-) asymptotic expansions for late coefficients can be obtained, which then can be used to compute Stokes multipliers. For more details see [7].

Originally, Borel transforms were used to obtain functions (solutions) with a prescribed simple asymptotic expansion. Since the Borel transform of a uniform asymptotic expansion produces a very complicated singularity structure, the Borel transform approach has not been used in uniform asymptotics.

* Received Nov. 3, 1999; revised July 28, 2000.

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In this paper we will use the Borel transform approach. First we introduce in §2 the uniform asymptotic expansion, and obtain via the methods given in [2] integral representations for the coefficients in these uniform asymptotic expansions. The Borel transform will be in terms of functions that have coalescing branch points. These functions are introduced in §3, and we will give a Cauchy-type integral theorem that will enable us in §5 to obtain new integral representations for the coefficients. The Borel transform itself is given in §4 in terms of an integral.

The results of the first five sections are very generally valid. To obtain asymptotic expansions for the late coefficients we need more information on $f(z,\mathbf{A})$ and $g(z)$. With this extra information we are able to determine the local Riemann sheet structures, and finally obtain asymptotic expansions for the late coefficients. These asymptotic expansions have the resurgence property, that is, the coefficients in these expansions are the same coefficients as in the uniform asymptotic expansions of the original integral $I_C(k)$, but where the contour passes through other dominating clusters of saddles.

We conclude this paper with two examples involving a Pearcey-type integral with two coalescing saddles, and one distant saddle.

The main advantage of studying the Borel transform for integrals is that we obtain a simple integral representation for the Borel transform. The integral representation is very helpful in obtaining global information on the singularities of the Borel transform. The next step is to obtain the Borel transform for uniform asymptotic expansions for solutions of differential equations. This paper can be used to determine the functions that are involved in describing the singularities of the Borel transform, and the Riemann sheet structures.

The theory of uniform asymptotics for solutions of difference equations is almost nonexistent. The Borel transform might be used to obtain new results for difference equations. The advantage of using Borel transforms is that the difference between these transforms for the solution of differential and difference equations is not very great.

2. The uniform asymptotic expansion. Without loss of generality, we will assume that for $\mathbf{A}=\mathbf{A}_0$ our cluster of N saddles will coalesce at the origin. The first step to obtain an asymptotic expansion is the mapping of $f(z,\mathbf{A})$ in the neighbourhood of the cluster of N saddle points onto a polynomial‡

$$f(z,\mathbf{A}) = f_0(\mathbf{A}) + F_N(u,\mathbf{X}), \tag{2.1}$$

where

$$\mathbf{X}=\{X_1,\cdots,X_{N-1}\} \qquad \text{and} \qquad F_N(u,\mathbf{X}) = \frac{u^{N+1}}{N+1} + \sum_{n=1}^{N-1} X_n \frac{u^n}{n}. \tag{2.2}$$

In this paper we shall frequently omit the $\mathbf{A}$ and $\mathbf{X}$ dependences of f and F_N.

The 0 in $f_0(\mathbf{A})$ refers to the coalescing point of the N saddles. We map the saddles of f onto the saddles of F_N. This guarantees that the mapping $z \to u(z,\mathbf{A})$ is locally one-to-one and gives N equations determing the N unknowns $f_0(\mathbf{A})$, $\mathbf{X}(\mathbf{A})$.

‡ This is a generalisation of the Chester, Friedman and Ursell ansatz. See [4] and [10]. The generalisation is well known in singularity theory [1].

Thus (1.1) becomes

$$I_{C(\mathbf{A})}(k,\mathbf{A}) = e^{kf_0(\mathbf{A})}\int_{C_N} e^{kF_N(u,\mathbf{X})}G_0(u)\,du, \tag{2.3}$$

where

$$G_0(u) = g(z(u))\frac{dz}{du}, \tag{2.4}$$

and C_N connects two valleys of the exponential. For simplicity we assume that the path C_N starts at $\infty e^{-\pi i/(N+1)}$ and ends at $\infty e^{\pi i/(N+1)}$. In general we can always write the image of $C(\mathbf{A})$ as a finite sum of $e^{2m\pi i/(N+1)}C_N$, m an integer.

Next, we use Bleistein's method [3], that is, we decompose $G_0(u)$ into

$$G_0(u) = \sum_{n=0}^{N-1} a_{0n}u^n + F_N'(u)H_0(u). \tag{2.5}$$

The n-sum contains the contribution of the saddles, and the final term in (2.5) vanishes at the saddle points.

On substituting into (2.3) by means of (2.5), we obtain

$$\begin{aligned} I_{C(\mathbf{A})}(k,\mathbf{A}) &= e^{kf_0(\mathbf{A})}\sum_{n=0}^{N-1} a_{0n}\int_{\infty e^{-\pi i/(N+1)}}^{\infty e^{\pi i/(N+1)}} e^{kF_N(u,\mathbf{X})}u^n\,du \\ &\quad + e^{kf_0(\mathbf{A})}\int_{\infty e^{-\pi i/(N+1)}}^{\infty e^{\pi i/(N+1)}} e^{kF_N(u,\mathbf{X})}F_N'(u)H_0(u)\,du \\ &= e^{kf_0(\mathbf{A})}\sum_{n=0}^{N-1} a_{0n}k^{-(n+1)/(N+1)}\left(\delta_{n0} + n\frac{\partial}{\partial\xi_n}\right) \\ &\quad \cdot W_N\left(\left\{X_p k^{1-p/(N+1)}\right\}_{p=1}^{N-1}\right) \\ &\quad + (-k)^{-1}e^{kf_0(\mathbf{A})}\int_{\infty e^{-\pi i/(N+1)}}^{\infty e^{\pi i/(N+1)}} e^{kF_N(u,\mathbf{X})}G_1(u)\,du, \end{aligned} \tag{2.6}$$

where we use the recursion scheme

$$H_{r-1}'(u) = G_r(u) = \sum_{n=0}^{N-1} a_{rn}u^n + F_N'(u)H_r(u), \tag{2.7}$$

and the canonical integrals

$$W_N(\xi_1,\cdots,\xi_{N-1}) = \int_{\infty e^{-\pi i/(N+1)}}^{\infty e^{\pi i/(N+1)}} \exp\left(\frac{u^{N+1}}{N+1} + \sum_{p=1}^{N-1}\xi_p\frac{u^p}{p}\right)du. \tag{2.8}$$

The two simplest cases are

$$W_1 = i\sqrt{2\pi}, \qquad W_2(\xi_1) = 2\pi i\,\mathrm{Ai}(-\xi_1). \tag{2.9}$$

We observe that the final line of (2.6) is very similar to the right hand side of (2.3).

Hence, we can iterate the procedure and obtain the uniform asymptotic expansion

$$\begin{aligned} I_{C(\mathbf{A})}(k,\mathbf{A}) \sim & e^{kf_0(\mathbf{A})} \sum_{n=0}^{N-1} k^{-(n+1)/(N+1)} \left(\delta_{n0} + n\frac{\partial}{\partial \xi_n}\right) \\ & \cdot W_N\left(\left\{X_p k^{1-p/(N+1)}\right\}_{p=1}^{N-1}\right) \sum_{r=0}^{\infty} \frac{a_{rn}}{(-k)^r}, \end{aligned} \tag{2.10}$$

as $k \to \infty$.

In this procedure the coefficients $a_{rn}(\mathbf{X})$ are produced in the recursion scheme (2.7). To obtain an explicit formula for them we follow the steps in [2], which is a generalisation of [8]. First we write

$$G_r(u) = \frac{1}{2\pi i}\oint_{\{u,u_m\}} G_0(v)R_r(u,v)\,dv, \tag{2.11}$$

where the contour of integration encircles u and the saddle points u_m of $F_N(u,\mathbf{X})$ in the positive sense, and the rational function $R_r(u,v)$ is given by

$$\begin{aligned} R_r(u,v) &= \left[\frac{-1}{F_N'(v)}\frac{\partial}{\partial v}\right]^r \frac{1}{v-u} \\ &= \frac{(-)^r r!}{2\pi i}\oint_v \frac{F_N'(w)\,dw}{\left(F_N(w)-F_N(v)\right)^{r+1}(w-u)}. \end{aligned} \tag{2.12}$$

Hence,

$$G_r(u) = \frac{(-)^r r!}{(2\pi i)^2}\oint_{\{u,u_m\}} dv \oint_v dw \frac{G_0(v)F_N'(w)}{\left(F_N(w)-F_N(v)\right)^{r+1}(w-u)}. \tag{2.13}$$

We note that

$$\frac{F_N'(w)}{w-u_m} = \sum_{n=0}^{N-1} u_m^n L_n(w,\mathbf{X}), \tag{2.14}$$

where the polynomials $L_n(w,\mathbf{X})$ are given by

$$\begin{aligned} L_n(w,\mathbf{X}) &= \frac{1}{2\pi i}\oint_{\{0,w\}} \frac{F_N'(x)\,dx}{x^{n+1}(x-w)} \\ &= w^{N-n-1} + \sum_{s=0}^{N-n-3} X_{s+n+2}w^s. \end{aligned} \tag{2.15}$$

Since

$$G_r(u_m) = \sum_{n=0}^{N-1} a_{rn}u_m^n, \tag{2.16}$$

we obtain from (2.13) and (2.14) the integral representation

$$a_{rn} = \frac{(-)^r r!}{(2\pi i)^2}\oint_{\{u_m\}} dv \oint_v dw \frac{G_0(v)L_n(w,\mathbf{X})}{\left(F_N(w)-F_N(v)\right)^{r+1}}, \tag{2.17}$$

and we use (2.1) to write this representation as

$$a_{rn}=\frac{(-)^r r!}{(2\pi i)^2}\oint_{\{z_m\}}dz\oint_{w(z)}dw\,\frac{g(z)L_n(w,\mathbf{X})}{\left(F_N(w)-f(z)+f_0\right)^{r+1}}. \tag{2.18}$$

In (2.18) the z-contour of integration encircles the cluster of N saddle points of $f(z)$, and for each point z on this contour the w-contour of integration is a small loop encircling $w(z)$, where $w(z)$ is the solution of $F_N(w)=f(z)-f_0$. Recall that this mapping is one-to-one in a neighbourhood of the cluster of saddles.

In [2] representation (2.18) was used to obtain asymptotic estimates for the coefficients a_{rn} as $r\to\infty$. To obtain similar results we introduce a Borel transform of the uniform asymptotic expansion (2.10). With this Borel transform we will be able to obtain new integral representations for the coefficients, which will be used to obtain asymptotic expansions for a_{rn} as $r\to\infty$.

3. The singularities in the Borel plane. Before we introduce the Borel transform itself we have to introduce a new class of functions. In the case of asymptotics for integrals with simple saddles, the singularities in the Borel plane are simple branch points. The situation will be more complicated in the case of coalescing saddles.

Let

$$U^N_{-r,n}(t,\mathbf{X})=-\frac{r!}{2\pi i}\int_{\infty e^{-\pi i/(N+1)}}^{\infty e^{\pi i/(N+1)}}\frac{u^n}{\left(F_N(u,\mathbf{X})-t\right)^{r+1}}\,du, \tag{3.1}$$

$r=0,1,2,\cdots$, $n=0,\cdots,N-1$. This definition is for $\operatorname{ph} t\in(-\pi,\pi)$ and t bounded away from the origin. The analytic continuation can be obtained by taking as the contour of integration a small loop around $u(t)$, where $u(t)$ is the solution of $F_N(u,\mathbf{X})=t$ such that $u(t)=(N+1)^{1/(N+1)}t^{1/(N+1)}$ in the case $\mathbf{X}=\mathbf{0}$

$$U^N_{-r,n}(t,\mathbf{X})=\frac{r!}{2\pi i}\oint_{u(t)}\frac{u^n}{\left(F_N(u,\mathbf{X})-t\right)^{r+1}}\,du. \tag{3.2}$$

Hence, the only t-singularities of $U^N_{-r,n}(t,\mathbf{X})$ are branch points at $F_N(u_m,\mathbf{X})$, where u_m are the saddle points of $F_N(u,\mathbf{X})$.

The two simplest cases are

$$U^1_{0,0}(t)=\frac{1}{\sqrt{2t}},$$

$$U^2_{0,0}(t,X_1)=\frac{\left(\sqrt{\left(\frac{3}{2}t\right)^2+X_1^3}+\frac{3}{2}t\right)^{1/3}+\left(\sqrt{\left(\frac{3}{2}t\right)^2+X_1^3}-\frac{3}{2}t\right)^{1/3}}{2\sqrt{\left(\frac{3}{2}t\right)^2+X_1^3}}, \tag{3.3}$$

$$U^2_{0,1}(t,X_1)=\frac{\left(\sqrt{\left(\frac{3}{2}t\right)^2+X_1^3}+\frac{3}{2}t\right)^{2/3}-\left(\sqrt{\left(\frac{3}{2}t\right)^2+X_1^3}-\frac{3}{2}t\right)^{2/3}}{2\sqrt{\left(\frac{3}{2}t\right)^2+X_1^3}}.$$

In the case $N=2$ the saddle points are $u=\pm iX_1^{1/2}$. The reader can check that $t=F_2\left(\pm iX_1^{1/2},X_1\right)=\pm\frac{2}{3}iX_1^{3/2}$ are the only singularities of $U^2_{0,n}(t,X_1)$.

We note that

$$\frac{\partial}{\partial t}U_{r,n}^N(t,\mathbf{X}) = U_{r-1,n}^N(t,\mathbf{X}), \tag{3.4}$$

and since we are mainly interested in the singular part of $U_{r,n}^N(t,\mathbf{X})$ we can use (3.4) to define $U_{r,n}^N(t,\mathbf{X})$ in the case $r = 1, 2, \cdots$. But we also give the integral representation

$$U_{r+1,n}^N(t,\mathbf{X}) = \frac{1}{r!2\pi i}\oint_{u(t)} \frac{F_N'(u)\int_0^u (F_N(u)-F_N(v))^r v^n\,dv}{F_N(u)-t}\,du, \tag{3.5}$$

$r = 0, 1, 2, \cdots$ and $n = 0, \cdots, N-1$. The reader can check that (3.4) holds. The endpoint 0 in the v-integral is arbitrary; for (3.4) to hold, this should be a fixed point. We choose the endpoint to be the same as the point of coalescence of the N saddles of $F_N(u,\mathbf{X})$.

In addition to the functions $U_{r,n}^N(t,\mathbf{X})$ we also introduce functions $V_{r,n}^N(t,\mathbf{X})$, which are simple linear combinations of $U_{r,n}^N(t,\mathbf{X})$. Let

$$\begin{aligned} V_{-r,n}^N(t,\mathbf{X}) &= \frac{r!}{2\pi i}\oint_{u(t)} \frac{L_n(u,\mathbf{X})}{(F_N(u,\mathbf{X})-t)^{r+1}}\,du, \\ V_{r+1,n}^N(t,\mathbf{X}) &= \frac{1}{r!2\pi i}\oint_{u(t)} \frac{F_N'(u)\int_0^u (F_N(u)-F_N(v))^r L_n(v)\,dv}{F_N(u)-t}\,du, \end{aligned} \tag{3.6}$$

$r = 0, 1, 2, \cdots$ and $n = 0, \cdots, N-1$. Again we have

$$\frac{\partial}{\partial t}V_{r,n}^N(t,\mathbf{X}) = V_{r-1,n}^N(t,\mathbf{X}). \tag{3.7}$$

The t-singularities of the functions $U_{r,n}^N(t,\mathbf{X})$ and $V_{r,n}^N(t,\mathbf{X})$ are branch points at $F_N(u_m,\mathbf{X})$. Hence, these branch points depend on $\mathbf{X}$ and they coalesce at the origin in the case $\mathbf{X} = \mathbf{0}$. In that case we have

$$\begin{aligned} U_{r,n}^N(t,\mathbf{0}) &= V_{r,N-n-1}^N(t,\mathbf{0}) \\ &= \frac{\Gamma((n+1)/(N+1))}{\Gamma(r+(n+1)/(N+1))}(N+1)^{(n-N)/(N+1)}t^{r+(n-N)/(N+1)}. \end{aligned} \tag{3.8}$$

The reader can obtain this result by taking $\mathbf{X} = \mathbf{0}$ and $r = 0$ in (3.1) and using (3.4). The right-hand side of (3.8) shows us that the functions $U_{r,n}^N(t,\mathbf{X})$ and $V_{r,n}^N(t,\mathbf{X})$ as functions of t live on $N+1$ Riemann sheets, that is, if we walk $N+1$ times around the cluster of singularities near the origin, then we arrive back at our starting point.

The main reason that we are interested in these functions is that the Laplace transform of $U_{r,n}^N(t,\mathbf{X})$ is equal to the asymptotic basis of the uniform asymptotic expansion (2.10). Thus in the case $\Re k > 0$ we have

$$\begin{aligned} &\frac{1}{2\pi i}\int_C e^{kt}U_{r,n}^N(t,\mathbf{X})\,dt \\ =&k^{-(n+1)/(N+1)}(-k)^{-r}\left(\delta_{n0} + n\frac{\partial}{\partial \xi_n}\right)W_N\left(\left\{X_p k^{1-p/(N+1)}\right\}_{p=1}^{N-1}\right), \end{aligned} \tag{3.9}$$

$n = 0, \cdots, N-1$, where C is a contour that starts at $\infty e^{-\pi i}$, encircles the t-singularities of $U^N_{r,n}(t,\mathbf{X})$, and ends at $\infty e^{\pi i}$. This follows from

$$\begin{aligned}&\frac{1}{2\pi i}\int_C e^{kt}U^N_{0,n}(t,\mathbf{X})\,dt\\ =&\frac{1}{(2\pi i)^2}\int_C\int_{\infty e^{-\pi i/(N+1)}}^{\infty e^{\pi i/(N+1)}}\frac{e^{kt}u^n}{t-F_N(u,\mathbf{X})}\,du\,dt\\ =&\frac{1}{2\pi i}\int_{\infty e^{-\pi i/(N+1)}}^{\infty e^{\pi i/(N+1)}} e^{kF_N(u,\mathbf{X})}u^n\,du\\ =&k^{-(n+1)/(N+1)}\left(\delta_{n0}+n\frac{\partial}{\partial\xi_n}\right)W_N\left(\left\{X_pk^{1-p/(N+1)}\right\}_{p=1}^{N-1}\right),\end{aligned}\tag{3.10}$$

in the case $r=0$. The other cases can be obtained by using (3.4).

The main result that we need to obtain integral representations for the coefficients in the uniform asymptotic expansions is the following theorem.

THEOREM 1. *Let C^N be a loop that encircles the t-singularities of $U^N_{r,m}(t,\mathbf{X})$ $N+1$ times. Then*

$$\frac{1}{2\pi i}\int_{C^N}U^N_{r,m}(t,\mathbf{X})V^N_{s,n}(t,\mathbf{X})\,dt=(-)^r\delta_{m,n}\delta_{r,-s},\tag{3.11}$$

where r and s are integers and $m,n\in\{0,\cdots,N-1\}$.

Proof.

Case 1: $r+s<0$. In this case we can use the fact that

$$U^N_{r,m}(t,\mathbf{X})=\mathcal{O}(t^{r+(m-N)/(N+1)}),\qquad V^N_{s,n}(t,\mathbf{X})=\mathcal{O}(t^{s-(n+1)/(N+1)}),\tag{3.12}$$

as $t\to\infty$. Hence, the integral on the left-hand side of (3.11) is zero.

Case 2: $r+s>0$. We use (3.4) and (3.7) and obtain

$$\frac{1}{2\pi i}\int_{C^N}U^N_{r,m}(t,\mathbf{X})V^N_{s,n}(t,\mathbf{X})\,dt=\frac{(-)^s}{2\pi i}\int_{C^N}U^N_{r+s,m}(t,\mathbf{X})V^N_{0,n}(t,\mathbf{X})\,dt.\tag{3.13}$$

Let $|\tau|$ be large and $u(\tau)$ be the solution of $F_N(u,\mathbf{X})=\tau^{N+1}$ such that $u(\tau)=(N+1)^{1/(N+1)}\tau$ if $\mathbf{X}=\mathbf{0}$. Integral representation (3.5) can be written as

$$\begin{aligned}U^N_{r+1,m}(\tau^{N+1},\mathbf{X})&=\frac{1}{r!2\pi i}\oint_{u(\tau)}\frac{F_N'(u)\int_0^u(F_N(u)-F_N(v))^r v^m\,dv}{F_N(u)-\tau^{N+1}}\,du\\ &=\frac{1}{r!}\int_0^{u(\tau)}(F_N(u(\tau))-F_N(v))^r v^m\,dv,\end{aligned}\tag{3.14}$$

and (3.2) can be written as

$$\begin{aligned}U^N_{0,m}(\tau^{N+1},\mathbf{X})&=\frac{1}{2\pi i}\oint_{u(\tau)}\frac{u^m\,du}{F_N(u)-\tau^{N+1}}\\ &=\frac{u(\tau)^m}{F_N'(u(\tau))}.\end{aligned}\tag{3.15}$$

Similarly,

$$V^N_{0,n}(\tau^{N+1},\mathbf{X})=\frac{L_n(u(\tau))}{F_N'(u(\tau))}.\tag{3.16}$$

Thus, with $\tilde{\mathcal{C}}$ a large circle centred at the origin, we obtain

(3.17)
$$\frac{1}{2\pi i}\int_{C^N} U^N_{r+s,m}(t,\mathbf{X})V^N_{0,n}(t,\mathbf{X})\,dt = \frac{N+1}{2\pi i}\int_{\tilde{\mathcal{C}}} U^N_{r+s,m}(\tau^{N+1},\mathbf{X})V^N_{0,n}(\tau^{N+1},\mathbf{X})\tau^N\,d\tau$$
$$= \frac{N+1}{(r+s-1)!2\pi i}\int_{\tilde{\mathcal{C}}} \frac{L_n(u(\tau))\tau^N\int_0^{u(\tau)}\left(F_N(u(\tau))-F_N(v)\right)^{r+s-1}v^m\,dv}{F_N'(u(\tau))}\,d\tau$$
$$= \frac{1}{(r+s-1)!2\pi i}\int_{\tilde{\mathcal{C}}} L_n(u)\int_0^u \left(F_N(u)-F_N(v)\right)^{r+s-1}v^m\,dv\,du,$$

where we used the substitution $\tau^{N+1} = F_N(u,\mathbf{X})$. Since the integrand in the final line of (3.17) is entire, we see that the integral has to be zero.

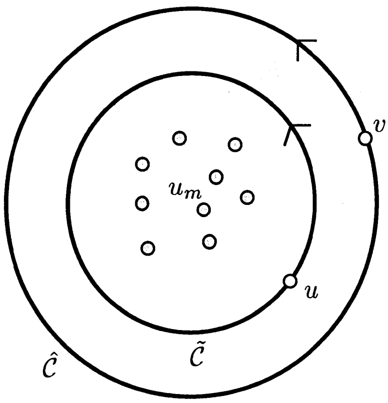

Figure 1. Contours $\tilde{\mathcal{C}}$ and $\hat{\mathcal{C}}$.

Case 3: $r+s=0$. We copy the method of case 2, use (2.15) and obtain

(3.18)
$$\frac{(-)^r}{2\pi i}\int_{C^N} U^N_{r,m}(t,\mathbf{X})V^N_{s,n}(t,\mathbf{X})\,dt = \frac{N+1}{2\pi i}\int_{\tilde{\mathcal{C}}} U^N_{0,m}(\tau^{N+1},\mathbf{X})V^N_{0,n}(\tau^{N+1},\mathbf{X})\tau^N\,d\tau$$
$$= \frac{N+1}{2\pi i}\int_{\tilde{\mathcal{C}}}\frac{u(\tau)^m L_n(u(\tau))\tau^N}{F_N'(u(\tau))^2}\,d\tau$$
$$= \frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}}\frac{u^m L_n(u)}{F_N'(u)}\,du$$
$$= \frac{1}{(2\pi i)^2}\int_{\tilde{\mathcal{C}}} du \oint_{\{0,u\}} dv\,\frac{u^m F_N'(v)}{v^{n+1}(v-u)F_N'(u)}$$
$$= \frac{1}{(2\pi i)^2}\int_{\hat{\mathcal{C}}} dv\int_{\tilde{\mathcal{C}}} du\,\frac{u^m F_N'(v)}{v^{n+1}(v-u)F_N'(u)}.$$

See Figure 1 for the contours of integration. For fixed v we can regard the u-contour of integration as a loop around $u=v$. Hence, the final line in (3.18) equals

(3.19)
$$\frac{1}{2\pi i}\oint_0 v^{m-n-1}\,dv = \delta_{m,n}. \qquad \square$$

4. The Borel transform. Usually the Borel transform is defined via a series. Here we use an integral representation. Let

$$Y(t, \mathbf{X}) = \frac{1}{2\pi i} \oint_{u(t)} \frac{G_0(u)\, du}{F_N(u, \mathbf{X}) - t}, \tag{4.1}$$

where, again, $u(t)$ is the solution of $F_N(u, \mathbf{X}) = t$ such that $u(t) = (N+1)^{1/(N+1)} t^{1/(N+1)}$ in the case $\mathbf{X} = \mathbf{0}$. To obtain the local behaviour near the origin we use (2.7) and integration by parts. After P steps we arrive at

$$Y(t, \mathbf{X}) = \sum_{r=0}^{P} \left\{ \sum_{n=0}^{N-1} a_{rn} U_{r,n}^N(t, \mathbf{X}) + \frac{t^r}{r!} H_r(0) \right\} + R_P(t, \mathbf{X}), \tag{4.2}$$

where
(4.3)

$$\begin{aligned} R_P(t, \mathbf{X}) &= \frac{1}{P! 2\pi i} \oint_{u(t)} \frac{F_N'(u) \int_0^u (F_N(u) - F_N(v))^P G_{P+1}(v)\, dv}{F_N(u) - t}\, du \\ &= \frac{(-)^{P+1}(P+1)}{(2\pi i)^2} \oint_{\{u(t), u_m\}} \oint_{\tilde{v}} \int_0^{u(t)} \frac{(t - F_N(v))^P F_N'(w) G_0(\tilde{v})}{(F_N(w) - F_N(\tilde{v}))^{P+1} (w - v)}\, dv\, dw\, d\tilde{v}. \end{aligned}$$

We used (2.13) to obtain the final integral representation.

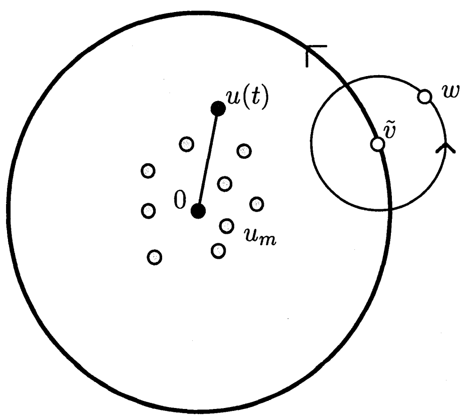

Figure 2. The three contours for (4.3).

By taking the contours as illustrated in Figure 2, we see that for t close enough to the origin $u(t)$ will be close to the origin and

$$R_P(t, \mathbf{X}) = \mathcal{O}(\rho^P), \qquad \text{as } P \to \infty, \tag{4.4}$$

where ρ is a fixed constant such that $0 < \rho < 1$. Let Ω be a neighbourhood of the origin such that if $u(t), u_1, \cdots, u_N \in \Omega$, then (4.4) holds for a fixed ρ. It is obvious that the size of Ω will depend on the radius of the w-contour of integration, and hence, on the distance of the singularities $G_0(u)$ to our cluster of N saddles. In this section, Ω will be relatively small. To obtain larger neighbourhoods we need more information on $f(z)$ and $g(z)$. In §6 we will make these extra assumptions to obtain asymptotic expansions for a_{rn} as $r \to \infty$.

With the help of (2.17) and (3.5) we can show that if $u(t), u_1, \cdots, u_N \in \Omega$, then

$$a_{rn} U_{r,n}^N(t, \mathbf{X}) = \mathcal{O}(\rho^r), \qquad \text{as } r \to \infty. \tag{4.5}$$

Let $\tilde{\Omega}$ be a neighbourhood of the origin such that if $t \in \tilde{\Omega}$ and $u_1, \cdots, u_N \in \Omega$ then $u(t) \in \Omega$. We combine the above results and obtain

$$Y(t, \mathbf{X}) = \sum_{r=0}^{\infty} \sum_{n=0}^{N-1} a_{rn} U_{r,n}^N(t, \mathbf{X}) + \tilde{Y}(t, \mathbf{X}), \tag{4.6}$$

where $\tilde{Y}(t, \mathbf{X})$ is analytic for $t \in \tilde{\Omega}$ and $u_1, \cdots, u_N \in \Omega$.

5. A new integral representation for the coefficients. All the tools to obtain integral representations for the coefficients in the uniform asymptotic expansion (2.10) are now available. Take $u_1, \cdots, u_N \in \Omega$, and let $\mathcal{C}^N$ be a loop in $\tilde{\Omega}$ that encircles the t-singularities of $U_{r,n}^N(t, \mathbf{X})$ $N+1$ times. With techniques that are similar to the ones used in the proof of Theorem 1, we obtain

$$\begin{aligned}
\frac{(-)^r}{2\pi i} \int_{\mathcal{C}^N} V_{-r,n}^N(t, \mathbf{X}) \tilde{Y}(t, \mathbf{X})\, dt &= \frac{1}{2\pi i} \int_{\mathcal{C}^N} V_{0,n}^N(t, \mathbf{X}) \frac{\partial^r}{\partial t^r} \tilde{Y}(t, \mathbf{X})\, dt \\
&= \frac{1}{2\pi i} \int_{\mathcal{C}^N} \frac{L_n\big(u(t)\big)}{F_N'\big(u(t)\big)} \frac{\partial^r}{\partial t^r} \tilde{Y}(t, \mathbf{X})\, dt \\
&= \frac{1}{2\pi i} \int_{\tilde{\mathcal{C}}} L_n(u) \frac{\partial^r}{\partial t^r} \tilde{Y}(F_N(u), \mathbf{X})\, du = 0,
\end{aligned}$$

where $\tilde{\mathcal{C}} = F_N^{-1}\left(\mathcal{C}^N\right)$ is a loop that encircles the saddle points of $F_N(u)$ once. We combine this result with Theorem 1 and (4.6) and obtain the integral representation

$$a_{rn} = \frac{(-)^r}{2\pi i} \int_{\mathcal{C}^N} Y(t, \mathbf{X}) V_{-r,n}^N(t, \mathbf{X})\, dt. \tag{5.1}$$

The restrictions $u_1, \cdots, u_N \in \Omega$ and $\mathcal{C}^N \subset \tilde{\Omega}$ can now be dropped. The only restriction that we need is that $\mathcal{C}^N$ is a loop that encircles the points $F(u_1), \cdots, F(u_N)$ $N+1$ times, and the other singularities of $Y(t, \mathbf{X})$ are in the exterior of $\mathcal{C}^N$.

6. Asymptotics for the coefficients. To obtain asymptotic expansions for the coefficients a_{rn}, as $r \to \infty$, we need more information on $f(z, \mathbf{A})$ and $g(z)$. The steepest descents paths from a saddle point z_m of $f(z, \mathbf{A})$ with phase θ, will be defined as the connected paths that start at z_m on which $\big(f(z, \mathbf{A}) - f(z_m, \mathbf{A})\big)e^{-i\theta}$ is strictly increasing. Note that if a steepest descents path hits another saddle point, then the path will split at that saddle point into two or more branches.

Let $\Delta_{z_m}(\mathbf{A})$ be the union of all ($\theta \in (-\pi, \pi]$) of these steepest descents paths starting at z_m. Again, our cluster of N saddle points is $\{z_1, \cdots, z_N\}$.

Let $\Delta_N(\mathbf{A})$ be the closure of $\Delta_{z_1}(\mathbf{A}) \cup \cdots \cup \Delta_{z_N}(\mathbf{A})$ and for $z \in \mathbb{C}$ let

$$d(z, \mathbf{A}) = \min_{1 \leqslant m \leqslant N} |f(z, \mathbf{A}) - f(z_m, \mathbf{A})|$$

be the '**distance**' to our N-cluster. In u-space this distance is $d(u,\mathbf{A}) = \min_{1\leqslant m\leqslant N} |F(u,\mathbf{X}(\mathbf{A}))-F(u_m,\mathbf{X}(\mathbf{A}))|$ and in t-space it is $d(t,\mathbf{A}) = \min_{1\leqslant m\leqslant N} |t-f(z_m,\mathbf{A})+f_0(\mathbf{A})|$.

Our main assumptions are:

(i) The phase function $f(z,\mathbf{A})$ has more than one cluster of saddle points in $\Delta_N(\mathbf{A})$, and each of these clusters collapses (reduces to one point) for $\mathbf{A}=\mathbf{A}_0$. Let R be the distance[§] from the nearest clusters in $\Delta_N(\mathbf{A})$ to our N-cluster.

(ii) There exists $\rho > 0$, fixed, such that the functions $f(z,\mathbf{A})$ and $g(z)$ have no singularities and no other saddle points in $\Delta_N(\mathbf{A})$ within distance $R+\rho$ from our N-cluster.

(iii) The functions $f(z,\mathbf{A})$ and $g(z)$ are analytic in small neighbourhoods of the nearest clusters of saddles.

Our Borel transform

$$Y(t,\mathbf{X}) = \frac{1}{2\pi i}\oint_{u(t)} \frac{G_0(u)\,du}{F_N(u,\mathbf{X})-t} = \frac{1}{2\pi i}\oint_{z(t)} \frac{g(z)\,dz}{f(z,\mathbf{A})-f_0(\mathbf{A})-t}, \tag{6.1}$$

has $N+1$ Riemann sheets with respect to our N-cluster, that is, if we walk $N+1$ times around the cluster of singularities that are due to our cluster of N saddle points, we arrive back at our starting point. Let one of the nearest clusters of M saddle points have coalescing point $z^{(M)}$. Mapping (2.1) maps $z^{(M)}$ to a unique point in u-space. Hence, the t-image of $z^{(M)}$, that is $t=f(z^{(M)},\mathbf{A})-f_0(\mathbf{A})$, is located on only one of the $N+1$ Riemann sheets. We will assume that it is located on sheet q.

As in §2 we have in the neighbourhood of this M-cluster a one-to-one mapping of the form

$$f(z,\mathbf{A}) = f_{z^{(M)}}(\mathbf{A}) + F_M(\tilde{u},\tilde{\mathbf{X}}). \tag{6.2}$$

From

$$Y(t,\mathbf{X}) = \frac{1}{2\pi i}\oint_{z(t)} \frac{g(z)\,dz}{f(z,\mathbf{A})-f_0(\mathbf{A})-t} = \frac{1}{2\pi i}\oint_{\tilde{u}(t)} \frac{\tilde{G}_0(\tilde{u})\,d\tilde{u}}{F_M(\tilde{u},\tilde{\mathbf{X}})+f_{z^{(M)}}(\mathbf{A})-f_0(\mathbf{A})-t}, \tag{6.3}$$

we see that $Y(t,\mathbf{X})$ has singularities near $t=f_{z^{(M)}}(\mathbf{A})-f_0(\mathbf{A})$. The local behaviour is

$$Y(t,\mathbf{X}) = \sum_{r=0}^{\infty}\sum_{m=0}^{M-1} \tilde{a}_{rm} U^M_{r,m}(t-f_{z^{(M)}}(\mathbf{A})+f_0(\mathbf{A}),\tilde{\mathbf{X}}) + \tilde{Y}(t-f_{z^{(M)}}(\mathbf{A})+f_0(\mathbf{A}),\tilde{\mathbf{X}}), \tag{6.4}$$

where $\tilde{Y}(t-f_{z^{(M)}}(\mathbf{A})+f_0(\mathbf{A}),\tilde{\mathbf{X}})$ is analytic near $t=f_{z^{(M)}}(\mathbf{A})-f_0(\mathbf{A})$. To determine the Riemann sheet on which these singularities are located, we have to study the steepest descents contours with respect to all phases. This will be illustrated in the examples in §7. Furthermore, the M-cluster of singularities will give us $M+1$ Riemann sheets in t-space near this cluster. Seen from this M-cluster, the N-cluster near

[§] The distance between the clusters of saddles will vary with $\mathbf{A}$. When we write that the distance between two clusters is R we mean that the distance is approximately R for $\mathbf{A}$ in a neighbourhood of $\mathbf{A}_0$, and exactly R in the collapsed case, that is, $\mathbf{A}=\mathbf{A}_0$.

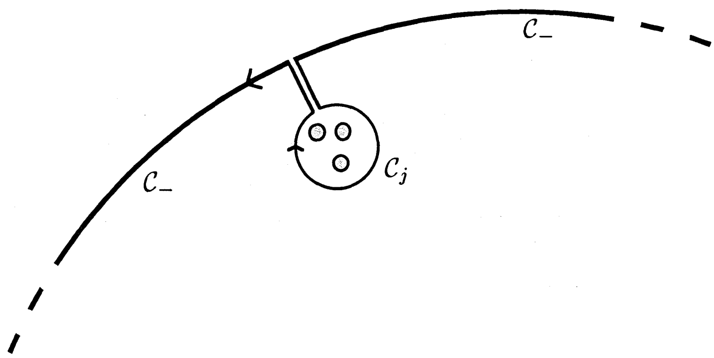

Figure 3. Contour $\mathcal{C}_-$ and $\mathcal{C}_j$.

$t = f_0(\mathbf{A})$ will be located on a specific Riemann sheet, say $\tilde{q}$. Hence, the $t - f_{z^{(M)}}(\mathbf{A}) + f_0(\mathbf{A})$ in (6.4) lives on sheet $\tilde{q}$ with respect to the M-cluster.

We assume that there are J nearest clusters of saddles, and that they have distance R to our N-cluster. Instead of taking for contour $\mathcal{C}^N$ in (5.1) a small loop encircling the cluster of singularities near the origin $N+1$ times, we take the loop depicted in Figure 3. It consists of small loops $\mathcal{C}_j$, $j = 1, \cdots, J$, around the singularities that originate from nearest clusters of M_j saddle points, plus $\mathcal{C}_-$, which is the part that has distance $R+\rho$ to the cluster of singularities near the origin. Thus for all $t \in \mathcal{C}_-$ we have $d(t, \mathbf{A}) = R + \rho$. The M_j-cluster is located on sheet q_j with respect to the N-cluster, and the N-cluster is located on sheet $\tilde{q}_j$ with respect to the M_j-cluster.

With this splitting of $\mathcal{C}^N$ we obtain

$$a_{rn} = \sum_{j=1}^{J} \frac{(-)^r}{2\pi i} \int_{\mathcal{C}_j} Y(t,\mathbf{X}) V^N_{-r,n}(t,\mathbf{X})\, dt + S_{rn}(R+\rho), \tag{6.5}$$

where

$$S_{rn}(R+\rho) = \frac{(-)^r}{2\pi i} \int_{\mathcal{C}_-} Y(t,\mathbf{X}) V^N_{-r,n}(t,\mathbf{X})\, dt. \tag{6.6}$$

First we estimate $S_{rn}(R+\rho)$. To obtain the large r asymptotics of $V^N_{-r,n}(t,\mathbf{X})$ we use the first integral in (3.6). Instead of a small loop around $u(t)$ we take steepest descents contours through one or more saddle points of $F_N(u,\mathbf{X})$. The order of each of these saddle points is between 1 and N. Laplace's method (see chapter 4.6 in [9]) shows us that for $t \in \mathcal{C}_-$

$$V^N_{-r,n}(t,\mathbf{X}) = \max_{1 \leqslant m \leqslant N} \frac{r!(r+1)^{-1/(N+1)}}{|F_N(u_m,\mathbf{X}) - t|^{r+1}} \mathcal{O}(1) = \frac{r!(r+1)^{-1/(N+1)}}{(R+\rho)^r} \mathcal{O}(1), \tag{6.7}$$

as $r \to \infty$. Since this estimate is independent of $t \in \mathcal{C}_-$ we can use it in (6.6) and obtain

$$S_{rn}(R+\rho) = \frac{r!(r+1)^{-1/(N+1)}}{(R+\rho)^r} \mathcal{O}(1), \tag{6.8}$$

as $r \to \infty$.

For the terms in the sum of (6.5) we use (6.4) and obtain

$$\text{(6.9)}\quad \frac{(-)^r}{2\pi i}\int_{\mathcal{C}_j} Y(t,\mathbf{X})V^N_{-r,n}(t,\mathbf{X})\,dt = \sum_{p=0}^{\infty}\sum_{m_j=0}^{M_j-1}\tilde{a}_{pm_j}T_{r,n,p,m_j}(t, f_{z^{(M_j)}}(\mathbf{A}) - f_0(\mathbf{A})),$$

where the coefficients originate from the M_j cluster of saddle points, and where

$$\text{(6.10)}\qquad T_{r,n,p,m}(t,\beta_j) = \frac{(-)^r}{2\pi i}\int_{\mathcal{C}_j} U^M_{p,m}(t-\beta_j,\tilde{\mathbf{X}})V^N_{-r,n}(t,\mathbf{X})\,dt.$$

In this integral representation, t is located on sheet q_j with respect to the N-cluster of singularities near the origin, and $t-\beta_j$ is located on sheet $\tilde{q}_j$ with respect to the M_j-cluster of singularities near β_j.

Instead of the finite loop $\mathcal{C}_j$ around the singularities near β_j, we take loop $\tilde{\mathcal{C}}_j$ that starts at $t = \infty e^{i\,\mathrm{ph}\,\beta_j}$, encircles the singularities near β_j once in the negative sense, and returns to its starting point. As in the derivation of (6.8) we can show that

$$\text{(6.11)}\qquad \begin{aligned}\tilde{T}_{r,n,p,m}(t,\beta_j) &\equiv \frac{(-)^r}{2\pi i}\int_{\tilde{\mathcal{C}}_j} U^M_{p,m}(t-\beta_j,\tilde{\mathbf{X}})V^N_{-r,n}(t,\mathbf{X})\,dt\\ &= T_{r,n,p,m}(t,\beta_j) + \frac{r!(r+1)^{-1/(N+1)}}{(R+\rho)^r}\mathcal{O}(1),\end{aligned}$$

as $r \to \infty$.

In the following derivation, we use integral representations (3.2) and (3.6) and we replace the small loops in these integrals by infinite loops to obtain

$$\text{(6.12)}\qquad \begin{aligned}&\tilde{T}_{r,n,p,m}(t,\beta_j)\\ &= \frac{(-)^{r-p}}{2\pi i}\int_{\tilde{\mathcal{C}}_j} U^M_{0,m}(t-\beta_j,\tilde{\mathbf{X}})V^N_{p-r,n}(t,\mathbf{X})\,dt\\ &= \frac{(r-p)!(-)^{r-p}}{(2\pi i)^3}\int_{\tilde{\mathcal{C}}_j}\int_{l_1(\tilde{q}_j)}\int_{l_2(q_j)} \frac{v^m L_n(u,\mathbf{X})\,du\,dv\,dt}{\left(F_M(v,\tilde{\mathbf{X}}) - t + \beta_j\right)\left(F_N(u,\mathbf{X}) - t\right)^{r-p+1}}\\ &= \frac{(r-p)!(-)^{r-p}}{(2\pi i)^2}\int_{l_1(\tilde{q}_j)}\int_{l_2(q_j)} \frac{v^m L_n(u,\mathbf{X})\,du\,dv}{\left(F_N(u,\mathbf{X}) - F_M(v,\tilde{\mathbf{X}}) - \beta_j\right)^{r-p+1}},\end{aligned}$$

where contour $l_1(\tilde{q})$ starts at $\infty\exp(\frac{2\tilde{q}\pi+\mathrm{ph}\,\beta_j}{M+1}i)$ and ends at $\infty\exp(\frac{(2\tilde{q}+2)\pi+\mathrm{ph}\,\beta_j}{M+1}i)$ and contour $l_2(q)$ starts at $\infty\exp(\frac{(2q-1)\pi+\mathrm{ph}\,\beta_j}{N+1}i)$ and ends at $\infty\exp(\frac{(2q+1)\pi+\mathrm{ph}\,\beta_j}{N+1}i)$. In these definitions, we prescribe that $\mathrm{ph}\,(\beta_j) \in (-\pi,\pi]$. We define

$$\text{(6.13)}\quad A^{M,N}_{m,n}\left(r,\mathbf{X},\tilde{\mathbf{X}},\beta,q,\tilde{q}\right) = \frac{r!(-)^r}{(2\pi i)^2}\int_{l_1(\tilde{q})}\int_{l_2(q)} \frac{v^m L_n(u,\mathbf{X})\,du\,dv}{\left(F_N(u,\mathbf{X}) - F_M(v,\tilde{\mathbf{X}}) - \beta\right)^{r+1}}.$$

To show that

$$\text{(6.14)}\qquad a_{rn} \sim \sum_{j=1}^{J}\sum_{p=0}^{\infty}\sum_{m_j=0}^{M_j-1}\tilde{a}_{pm_j}A^{M_j,N}_{m_j,n}\left(r-p,\mathbf{X},\tilde{\mathbf{X}}_j, f_{z^{(M_j)}}(\mathbf{A}) - f_0(\mathbf{A}), q_j, \tilde{q}_j\right),$$

as $r \to \infty$, we have to show that

$$A^{M_j,N}_{m_j,n}\left(r-p,\mathbf{X},\tilde{\mathbf{X}}_j,f_{z^{(M_j)}}(\mathbf{A})-f_0(\mathbf{A}),q_j,\tilde{q}_j\right) \gg \frac{r!(r+1)^{-1/(N+1)}}{(R+\rho)^r}, \tag{6.15}$$

as $r \to \infty$, p fixed, and that the series on the right-hand side of (6.14) has an asymptotic property as $r \to \infty$.

For $\mathbf{X}$ and $\tilde{\mathbf{X}}$ small but not fixed, the asymptotic behaviour of $A^{M,N}_{m,n}(r,\mathbf{X},\tilde{\mathbf{X}},\beta,q,\tilde{q})$, as $r \to \infty$, is very complicated, and should be expressed in terms of a uniform asymptotic approximation. But one could argue that in general the function $A^{M,N}_{m,n}(r,\mathbf{X},\tilde{\mathbf{X}},\beta,q,\tilde{q})$ is the simplest function with that kind of asymptotic behaviour.

For $\mathbf{X}$ and $\tilde{\mathbf{X}}$ small and fixed we can replace the contours of integration in (6.13) by steepest descents contours and verify that (6.15) holds. In this way we have to split the analysis of each possible unfolding of the two clusters of saddles into simple saddles. We omit the details and give just one result. In the case $\mathbf{X}=\mathbf{0}$ and $\tilde{\mathbf{X}}=\mathbf{0}$ we can compute the integrals (express them in terms of beta integrals) and obtain

(6.16)

$$\begin{aligned} &A^{M,N}_{m,n}(r,\mathbf{0},\mathbf{0},\beta,q,\tilde{q}) \\ &= -e^{\left((2\tilde{q}+1)\frac{m+1}{M+1}-2q\frac{n+1}{N+1}\right)\pi i}\beta^{\frac{m+1}{M+1}-\frac{n+1}{N+1}-r}\frac{(M+1)^{\frac{m+1}{M+1}-1}}{(N+1)^{\frac{n+1}{N+1}}}\frac{\Gamma\left(r+\frac{n+1}{N+1}-\frac{m+1}{M+1}\right)}{\Gamma\left(\frac{n+1}{N+1}\right)\Gamma\left(1-\frac{m+1}{M+1}\right)}. \end{aligned}$$

Here we have $|\beta| = |f_{z^{(M_j)}}(\mathbf{A}_0)-f_0(\mathbf{A}_0)| = R$, and hence (6.15) holds.

To show that (6.14) is an asymptotic expansion, we substitute into the left-hand side by means of a truncated version of (6.4):

(6.17)

$$\begin{aligned} &\frac{(-)^r}{2\pi i}\int_{C_j} Y(t,\mathbf{X})V^N_{-r,n}(t,\mathbf{X})\,dt \\ &=\sum_{p=0}^{P_j}\sum_{m_j=0}^{M_j-1}\tilde{a}_{pm_j}A^{M_j,N}_{m_j,n}\left(r-p,\mathbf{X},\tilde{\mathbf{X}}_j,f_{z^{(M_j)}}(\mathbf{A})-f_0(\mathbf{A}),q_j,\tilde{q}_j\right)+R_{P_j}(r,n,\mathbf{X},\tilde{\mathbf{X}}_j). \end{aligned}$$

As in (4.2) and (4.3) we have an integral representation for the tail of (6.4), and we use this integral to obtain an integral representation for $R_{P_j}(r,n,\mathbf{X},\tilde{\mathbf{X}}_j)$. Again, by taking $\mathbf{X}$ and $\tilde{\mathbf{X}}_j$ small and fixed, we replace the contours of integration by steepest descents contours, and show that $R_{P_j}(r,n,\mathbf{X},\tilde{\mathbf{X}}_j)$ is of the same order as the first neglected term. Since we have to split the analysis of each possible unfolding of the clusters of saddles, we omit the details. In [6] all these details are given for the case $N=M=2$.

Asymptotic expansion (6.14) is the main result of this section. In general, it is very unlikely that the function $A^{M,N}_{m,n}(r,\mathbf{X},\tilde{\mathbf{X}},\beta,q,\tilde{q})$ can be simplified. One way of computing these functions is to expand the integrand in (6.13) in Taylor series about $\mathbf{X}=\mathbf{0}$ and $\tilde{\mathbf{X}}=\mathbf{0}$. In this way we obtain a Taylor series for $A^{M,N}_{m,n}(r,\mathbf{X},\tilde{\mathbf{X}},\beta,q,\tilde{q})$ in which the coefficients are double integrals that can be expressed in terms of beta integrals. One of these Taylor series is given in the next section.

When $N=1$ or $M=1$, the function $A^{M,N}_{m,n}(r,\mathbf{X},\tilde{\mathbf{X}},\beta,q,\tilde{q})$ can be simplified. We will show some details in the examples in the next section.

The function $A^{M,N}_{m,n}(r,\mathbf{X},\tilde{\mathbf{X}},\beta,q,\tilde{q})$ can also be simplified in the special case $N=$

$M = 2$ and $\mathbf{X} = \tilde{\mathbf{X}}$. In [6] it is shown how to express this function in a single integral that can be seen as a generalised Airy integral.

7. An example: a Pearcey-type integral with two coalescing saddles. The two examples in this section are both for the Pearcey-type integral

$$I_C(k,a) = \int_C e^{kf(z,a)}\,dz, \tag{7.1}$$

where $f(z,a) = \frac{1}{4}z^4 + \frac{2}{3}z^3 - \frac{1}{2}a^2z^2 - 2a^2z$. This function has saddle points $z = \pm a, -2$. Thus for a small there is a cluster of two saddle points near the origin. In the first example we take

$$I_{C_1}(k,a) = \int_{\infty e^{-\pi i/4}}^{\infty e^{\pi i/4}} e^{kf(z,a)}\,dz. \tag{7.2}$$

The cluster of saddle points will dominate the large-k asymptotics. We use the mapping

$$f(z,a) = f_0(a) + \tfrac{1}{3}u^3 + X_1 u, \tag{7.3}$$

and prescribe that the saddle points $z = \pm a$ should correspond to $u = \pm i\sqrt{X_1}$. We obtain $f_0(a) = -\frac{1}{4}a^4$, $X_1 = e^{-\pi i}2^{2/3}a^2$, and when we use the method described in §2

$$\begin{aligned} I_{C_1}(k,a) \sim 2\pi i e^{-a^4k/4}\Bigg(& k^{-1/3}\mathrm{Ai}(2^{2/3}k^{2/3}a^2)\sum_{r=0}^{\infty}\frac{a_{r0}}{(-k)^r} \\ & -k^{-2/3}\mathrm{Ai}'(2^{2/3}k^{2/3}a^2)\sum_{r=0}^{\infty}\frac{a_{r1}}{(-k)^r}\Bigg), \end{aligned} \tag{7.4}$$

as $k \to \infty$. To compute the 'exact' values of a_{rn} we use (2.17). The w integral in (2.17) can be computed exactly and it will give us a rational function with poles at $v = \pm i\sqrt{X_1}$. By expanding $f(z,a)$ in a Taylor series at $z = \pm a$ and the right-hand side of (7.3) in a Taylor series at $u = \pm i\sqrt{X_1}$, we obtain a Taylor series for $G_0(u) = \frac{dz}{du}$ at $u = \pm i\sqrt{X_1}$. This Taylor series times the rational function will give us a_{rn}.

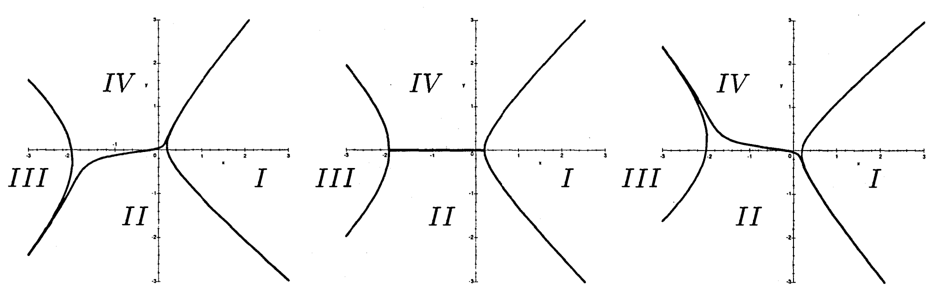

Figure 4. The steepest descents paths with phases $\frac{11}{10}\pi$ (left), π (centre) and $\frac{9}{10}\pi$ (right).

The Borel transform

(7.5)
$$Y(t,a) = \frac{1}{2\pi i} \oint_{z(t)} \frac{g(z)\,dz}{f(z,a) - f_0(a) - t}$$

has a cluster of branch points at $t = f(\pm a, a) - f_0(a) = \mp\frac{4}{3}a^3$, and a single branch point at $t = f(-2,a) - f_0(a) = -\frac{4}{3} + 2a^2 + \frac{1}{4}a^4$. We still have to determine on which Riemann sheets these branch points are located. In the following analysis we assume $a > 0$. In Figure 4 we give several steepest descents paths. These paths split the complex plane into four regions. The image $t = f(z,a) - f_0(a)$ of region I is given in Figure 5. It clearly shows that with respect to the cluster of branch points, the single branch point is not located on sheet 0.

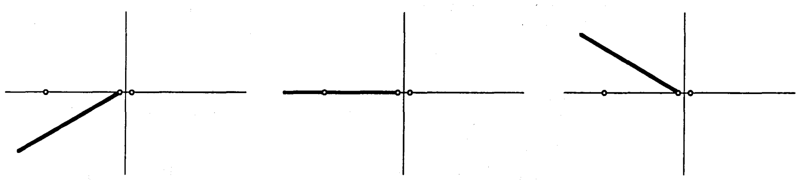

Figure 5. The image of region I.

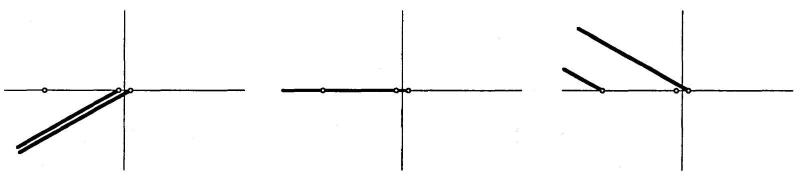

Figure 6. The image of region II.

To reach the image of region II in the t-plane, we have to walk once (clockwise) around $t = \mp\frac{4}{3}a^3$. The image of region II is given in Figure 6, and it is clear that there is a branch point on that Riemann sheet at $t = -\frac{4}{3} + 2a^2 + \frac{1}{4}a^4$. Thus $q = 1$.

On the other hand, the image of region III shows that with respect to the single branch point, the cluster of branch points are not on sheet 0. When we walk once around the single branch point in the negative direction, we reach the image of region IV, and it shows that with respect to the single branch point, the cluster of branch points is located on sheet 1. Thus $\tilde{q} = 1$.

The coefficients $\tilde{a}_{p0}$ corresponding to the saddle point $z = -2$ follow from the mapping

(7.6)
$$f(z,a) = f_{-2}(a) + \tfrac{1}{2}w^2,$$

where $f_{-2}(a) = f(-2,a) = -\frac{4}{3} + 2a^2$, and can be computed via (2.17). Thus we have the asymptotic expansion

(7.7)
$$a_{rn} \sim \sum_{p=0}^{\infty} \tilde{a}_{p0} A_{0,n}^{1,2}(r-p, X_1, 0, -\tfrac{4}{3} + 2a^2 + \tfrac{1}{4}a^4, 1, 1), \qquad n = 0, 1,$$

where the asymptotic scale can be simplified as

(7.8)
$$A^{1,2}_{0,n}(r,X_1,0,\beta,1,1) = \frac{\Gamma(r+\frac{1}{2})(-)^r}{2\pi i\sqrt{2\pi}} \int_{\infty e^{2/3\pi i}}^{\infty e^{4/3\pi i}} u^{1-n}\left(\tfrac{1}{3}u^3 + X_1 u - \beta\right)^{-r-1/2} du.$$

As in [6] this function can be seen as a generalised Airy integral. The asymptotics for $r \to \infty$ can be expressed in terms of the generalised Airy integrals of [5], but it is not obvious whether these generalised Airy integrals are simpler than the right-hand side of (7.8). We compute the right-hand side of (7.8) by expanding the integrand into a Taylor series in X_1:

(7.9)
$$A^{1,2}_{0,n}(r,X_1,0,\beta,1,1) =$$
$$\frac{(-)^{r+1}}{2^{\frac{1}{2}}\pi^{\frac{3}{2}}} \sum_{m=0}^{\infty} \frac{\Gamma\left(r+\frac{4m+2n-1}{6}\right)\Gamma\left(\frac{m-n+2}{3}\right)}{m!} \sin\left((m-n+2)\frac{2\pi}{3}\right) \frac{3^{(m-n-1)/3}(-X_1)^m}{(-\beta)^{r+\frac{2}{3}m+\frac{1}{3}n-\frac{1}{6}}}.$$

This series converges for $|X_1| < \left(\frac{3}{2}|\beta|\right)^{2/3}$. The reader can check that the first term in this expansion agrees with (6.16).

Note that (7.7) incorporates the resurgence property: the coefficients in the asymptotic expansion for a_{rn}, the coefficients originating from the cluster of saddles, are $\tilde{a}_{p0}$, the coefficients originating from the simple saddle point.

In the case $a = \frac{2}{5}$ the 'exact' value of $a_{15,0}$ is

(7.10)
$$a_{15,0} = -9.955462116 \times 10^8.$$

We take P terms on the right-hand side of (7.7), and to compute the $A^{1,2}_{0,0}(\cdots)$ functions, we use 30 terms in the sum of (7.9). The results are given in Figure 7. This figure shows that even the first term in (7.7) gives already a good approximation, and that the right-hand side of (7.7) is a divergent asymptotic series.

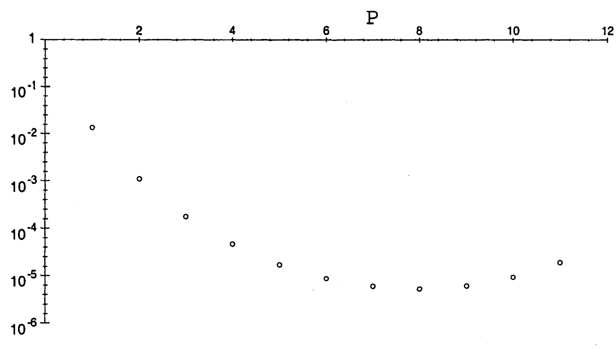

Figure 7. The relative error in asymptotic approximation (7.7) with P terms.

The second example is the integral

(7.11)
$$I_{C_2}(k,a) = \int_{\infty e^{3\pi i/4}}^{\infty e^{5\pi i/4}} e^{kf(z,a)}\, dz.$$

This time the main contribution to the large k asymptotics comes from the saddle point $z = -2$. To obtain the asymptotic expansion for this integral, we use mapping (7.6) and the method described in §2 (or the method of steepest descents) and obtain

$$I_{C_2}(k,a) \sim \frac{i}{\sqrt{2\pi k}} e^{(2a^2-4/3)k} \sum_{r=0}^{\infty} \frac{\tilde{a}_{r0}}{(-k)^r}, \tag{7.12}$$

as $k \to \infty$.

In this example there will be a distant cluster of singularities near $t = f_0(a) - f(-2,a) = \frac{4}{3} - 2a^2 - \frac{1}{4}a^4$. The Riemann sheet structure is, of course, the same as in the first example. Thus $q = \tilde{q} = 1$. Hence, now we have

$$\begin{aligned}\tilde{a}_{r0} \sim \sum_{p=0}^{\infty} \Big(& a_{p0} A_{0,0}^{2,1}(r-p, 0, X_1, \tfrac{4}{3} - 2a^2 - \tfrac{1}{4}a^4, 1, 1) \\ & + a_{p1} A_{1,0}^{2,1}(r-p, 0, X_1, \tfrac{4}{3} - 2a^2 - \tfrac{1}{4}a^4, 1, 1)\Big),\end{aligned} \tag{7.13}$$

as $r \to \infty$. We use

$$A_{m,0}^{2,1}(r, 0, X_1, \beta, 1, 1) = (-)^r A_{0,1-m}^{1,2}(r, X_1, 0, -\beta, 1, 1), \tag{7.14}$$

and (7.9) to compute the asymptotic scale in (7.13) via its Taylor series in X_1.

Note that (7.13) also incorporates the resurgence property: the coefficients in the asymptotics for $\tilde{a}_{r0}$, as $r \to \infty$, are a_{pn}.

Again, we take $a = \frac{2}{5}$ and compute the 'exact' value

$$\tilde{a}_{15,0} = 3.130922055 \times 10^9. \tag{7.15}$$

As in the first expansion we take P terms in (7.13) and we use 30 terms in the Taylor series to compute $A_{m,0}^{2,1}(\cdots)$. The relative error is given in Figure 8.

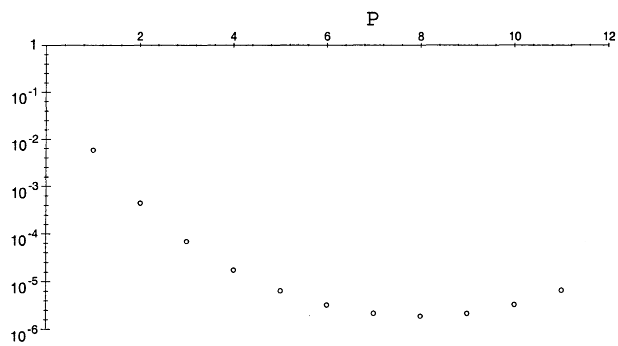

Figure 8. The relative error in asymptotic approximation (7.13) with P terms.

Acknowledgments. The author wishes to thank the referees and the editor for helpful suggestions in the presentation of this paper. He is also pleased to acknowledge the support of the Nuffield Foundation for a visit to the University of Maryland in the summer of 1997 under grant NUF–NAL 253 000 R33808.

REFERENCES

[1] V. I. ARNOLD, *Catastrophe Theory*, Third edition, Springer-Verlag, Berlin, 1992.

[2] M. V. BERRY AND C. J. HOWLS, *Unfolding the high orders of asymptotic expansions with coalescing saddles: singularity theory, crossover and duality*, Proc. Roy. Soc. London, Ser. A, 443 (1993), pp. 107–126.

[3] N. BLEISTEIN, *Uniform asymptotic expansions of integrals with stationary points near algebraic singularity*, Comm. Pure Appl. Math., 19 (1966), pp. 353–370.

[4] C. CHESTER, B. FRIEDMAN, AND F. URSELL, *An extension of the method of steepest descents*, Proc. Cambridge Philos. Soc., 53 (1957), pp. 599–611.

[5] P. G. DRAZIN AND W. H. REID, *Hydrodynamic Stability*, Cambridge University Press, Cambridge, 1981.

[6] C. J. HOWLS AND A. B. OLDE DAALHUIS, *On the resurgence properties of the uniform asymptotic expansion of Bessel functions of large order*, Proc. Roy. Soc. London, Ser. A, 455 (1999), pp. 3917–3930.

[7] A. B. OLDE DAALHUIS, *Hyperasymptotic solutions of higher order differential equations with a singularity of rank one*, Proc. Roy. Soc. London, Ser. A, 454 (1998), pp. 1–29.

[8] A. B. OLDE DAALHUIS AND N. M. TEMME, *Uniform Airy-type expansions of integrals*, SIAM J. Math. Anal., 25 (1994), pp. 304–321.

[9] F. W. J. OLVER, *Asymptotics and Special Functions*, Academic Press, New York, 1974 (Reprinted by AK Peters, Wellesley, 1997).

[10] R. WONG, *Asymptotic Approximations of Integrals*, Academic Press, New York, 1989.