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DIAGONAL ORTHOGONAL POLYNOMIAL SEQUENCES*

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Abstract. We deal with a problem linked to the generalized coherent pairs problem [3,12], when the two orthogonal sequences are identical, then called *diagonal sequences.* We exhaustively describe all the diagonal sequences (see Definition 1.5, below) associated with $\phi(x) = x-c$, with index s , $1 \leq s \leq 3$. In particular, we prove that the diagonal forms arising are classical forms, sum of a Dirac measure and a Laguerre (resp. Jacobi) form. Other solutions arise , $(x-c)w$ where w , depending on c , is a shifting of a classical form. But c must be chosen to make $(x-c)w$ regular. It is an open problem, except for some particular cases.

Introduction. In [3] Iserles et al. introduced the concept of the coherent pair, for solving problems in the theory of Sobolev inner products. This concept and the more general notion of generalized coherent pair, see [1], are special cases of a global definition given in [10]. It reads as follows.

Let $\{B_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$ be monic orthogonal polynomial sequences and ϕ a monic polynomial with $t = \deg \phi$. When there exists an integer $s \geq 0$ such that

$$(*) \qquad \phi(x)P_n(x) = \sum_{\nu=n-s}^{n+t} \lambda_{n,\nu} B_\nu^{[1]}(x), \quad \lambda_{n,n-s} \neq 0, \quad n \geq s,$$

with $B_n^{[1]}(x) = (n+1)^{-1}B'_{n+1}(x)$, $n \geq 0$, then we shall say that the pair $\left(\{P_n\}_{n\geq 0}, \{B_n\}_{n\geq 0}\right)$ is *a coherent pair associated with ϕ with index s*. Relation $(*)$ is itself a particular case of a finite–type relation between two polynomial sequences [10].

Here we deal with *diagonal sequences*, that is to say, when in $(*)$ we have $P_n = B_n$, $n \geq 0$. The case $\phi(x) = 1$ is well-known. In this occurrence the relation $(*)$ characterizes classical orthogonal sequences (Hermite, Laguerre, Bessel and Jacobi) where necessarily $0 \leq s \leq 2$, see corollary 2.3. It is the aim of the present papier to describe the case $t = 1$ completely and to determine all diagonal sequences arising.

The first section contains material of a preliminary and introductory character. The second section gives some general results about diagonal polynomial sequences. In particular, we prove that the second sequence of a coherent pair is always a diagonal sequence. Section 3 deals with the case $t = 1$. We exhaustively describe the cases which arise. We prove that a diagonal sequence either is classical or is semi-classical with the class $\sigma = 1$. Finally, section 4 gives an example of computing the coefficients $\lambda_{n,\nu}$.

1. Preliminaries and notations. Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $< u, f >$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular we denote by $(u)_n := < u, x^n >$, $n \geq 0$ the moments of u .

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Let us introduce some useful operations in $\mathcal{P}'$. For any form u, any polynomial h and any $c \in \mathbb{C}$, we let $Du = u'$, hu and $(x-c)^{-1}u$, be the forms defined by duality

$$< u', f > := - < u, f' >; \quad < hu, f > := < u, hf >, \ f \in \mathcal{P},$$
$$< (x-c)^{-1}u, f > := < u, \theta_c(f) >, \ f \in \mathcal{P},$$

where $\theta_c(f)(x) = \dfrac{f(x) - f(c)}{x - c}$.
Let $\{B_n\}_{n\geq 0}$ be a sequence of monic polynomials, $\deg(B_n) = n$, $n \geq 0$ (polynomial sequence : PS) and let $\{u_n\}_{n\geq 0}$ be its *dual sequence* $u_n \in \mathcal{P}'$ defined by $< u_n, B_m > := \delta_{n,m}$, $n, m \geq 0$. Let us recall the following result [5,6].

LEMMA 1.1. *For any $u \in \mathcal{P}'$ and any integer $m \geq 1$, the following statements are equivalent:*
i) $< u, B_{m-1} > \neq 0$, $< u, B_n > = 0$, $n \geq m$.
ii) There exist $\lambda_\mu \in \mathbb{C}$, $0 \leq \mu \leq m-1$, $\lambda_{m-1} \neq 0$ such that $u = \sum_{\mu=0}^{m-1} \lambda_\mu u_\mu$.

As a consequence, the dual sequence $\{u_n^{[1]}\}_{n\geq 0}$ of $\{B_n^{[1]}\}_{n\geq 0}$ where $B_n^{[1]}(x) = (n+1)^{-1}B'_{n+1}(x)$, $n \geq 0$ is given by

$$(u_n^{[1]})' = -(n+1)u_{n+1}, \ n \geq 0. \tag{1.1}$$

Similarly, the dual sequence $\{\tilde{u}_n\}_{n\geq 0}$ of $\{\widetilde{B}_n\}_{n\geq 0}$ with $\widetilde{B}_n(x) = a^{-n}B_n(ax+b)$, $n \geq 0$, $a \neq 0$, is given by $\tilde{u}_n = a^n(h_{a^{-1}} \circ \tau_{-b})u_n$, $n \geq 0$ where

$$< \tau_{-b}u, f > := < u, \tau_b f > = < u, f(x-b) > \quad , \quad u \in \mathcal{P}', \ f \in \mathcal{P}, \ b \in \mathbb{C},$$
$$< h_a u, f > := < u, h_a f > = < u, f(ax) > \quad , \quad u \in \mathcal{P}', \ f \in \mathcal{P}, \ a \in \mathbb{C} - \{0\}.$$

The form u is called *regular* if we can associate with it a polynomial sequence $\{B_n\}_{n\geq 0}$ such that $< u, B_m B_n > = r_n \delta_{n,m}$, $n, m \geq 0$; $r_n \neq 0$, $n \geq 0$. The sequence $\{B_n\}_{n\geq 0}$ is orthogonal with respect to u. Necessarily, $u = \lambda u_0$. In this case, we have

$$u_n = (< u_0, B_n^2 >)^{-1} B_n u_0, \ n \geq 0. \tag{1.2}$$

When u is regular, if A is a polynomial such that $Au = 0$, then $A = 0$.

A form u is called *semi-classical* when it is regular and there exist two polynomials E and F, E monic, $\deg(F) \geq 1$, such that $(Eu)' + Fu = 0$. The pair (E, F) is not unique. The previous equation is simplified (see [8]), if and only if there exists a root ξ of E such that

$$\begin{cases} E'(\xi) + F(\xi) = 0, \\ < u, \theta_\xi^2(E) + \theta_\xi(F) > = 0\,. \end{cases} \tag{1.3}$$

Then u fulfils the equation $\big(\theta_\xi(E)u\big)' + \big\{\theta_\xi^2(E) + \theta_\xi(F)\big\}u = 0$.
We call the *class* of u, the minimum value of the integer $\max\big(\deg(E) - 2, \deg(F) - 1\big)$ for all possible pairs (E, F). The pair$(\widehat{E}, \widehat{F})$ giving the class $\sigma \geq 0$ is unique. When $\sigma = 0$, the form u is *classical*, (Hermite, Laguerre, Bessel, Jacobi) and $\deg \widehat{E} \leq 2$, $\deg \widehat{F} = 1$. Any shift leaves invariant the semi-classical character. Indeed, the shifted form $\tilde{u} = (h_{a^{-1}} \circ \tau_{-b})u$ fulfils the equation $\big(\widetilde{E}\tilde{u}\big)' + \widetilde{F}\tilde{u} = 0$ where $\widetilde{E}(x) = a^{-\deg E}E(ax+b)$, $\widetilde{F}(x) = a^{1-\deg E}F(ax+b)$ [8,9].

Let ϕ be a monic polynomial with $\deg(\phi) = t \geq 0$ and let $\{B_n\}_{n\geq 0}$ be a (PS) with its dual sequence $\{u_n\}_{n\geq 0}$; for $n \geq t$ we have $< \phi u_n, B_{n-t} >=< u_n, \phi B_{n-t} >= 1$ then $\phi u_n \neq 0$, $n \geq t$, but generally, there can exist values of n, $0 \leq n < t$ such that $\phi u_n = 0$.

DEFINITION 1.2. *We say that the polynomial sequence* $\{B_n\}_{n\geq 0}$ *is compatible with* ϕ, *if we have* $\phi u_n \neq 0$, $n \geq 0$ *[10].*

REMARK. Any orthogonal sequence is compatible with any monic polynomial.

LEMME 1.3. *If the (PS)* $\{B_n\}_{n\geq 0}$ *is orthogonal, then the sequence* $\{B_n^{[1]}\}_{n\geq 0}$ *is compatible with any monic polynomial.*

Proof. Suppose that there exists an integer n, $0 \leq n < t$ such that $\phi u_n^{[1]} = 0$. After differentiating and on account of (1.1) and (1.2), we obtain

$$\phi' u_n^{[1]} = \frac{(n+1)}{< u_0, B_{n+1}^2 >} \phi B_{n+1} u_0, \ \ n \geq 0.$$

Multiplying the previous relation by ϕ, we get $\frac{(n+1)}{< u_0, B_{n+1}^2 >} \phi^2 B_{n+1} u_0 = 0$ and the regularity of u_0 implies $\frac{(n+1)}{< u_0, B_{n+1}^2 >} \phi^2 B_{n+1} = 0$, which is impossible. □

Let $\{P_n\}_{n\geq 0}$ be a (PS) with its dual sequence $\{v_n\}_{n\geq 0}$. Since $\{B_n^{[1]}\}_{n\geq 0}$ is a basis, we get

$$\phi(x) P_n(x) = \sum_{\nu=0}^{n+t} \lambda_{n,\nu} B_\nu^{[1]}(x), \ n \geq 0, \tag{1.4}$$

with $\lambda_{n,\nu} =< \phi u_\nu^{[1]}, P_n >$, $0 \leq \nu \leq n+t$, $n \geq 0$.

DEFINITION 1.4. [10] *Let* $\{B_n\}_{n\geq 0}$ *and* $\{P_n\}_{n\geq 0}$ *be monic orthogonal polynomial sequences (MOPS) and* ϕ *a monic polynomial. When there is an integer* $s \geq 0$ *such that*

$$\phi(x) P_n(x) = \sum_{\nu=n-s}^{n+t} \lambda_{n,\nu} B_\nu^{[1]}(x), \quad \lambda_{n,n-s} \neq 0, \quad n \geq s, \tag{1.5}$$

we shall say that the pair $\left(\{P_n\}_{n\geq 0}, \{B_n\}_{n\geq 0}\right)$ *is a coherent pair associated with* ϕ *with index* s. *Equivalently, the pair* (v_0, u_0) *is a coherent pair associated with* ϕ *with index* s.

The case $\phi = 1$, $s = 1$ is treated in [13]. See also [4,11,12,14]. For the case $\phi = 1$, $s = 2$, see [1].

DEFINITION 1.5. *Let* $\{B_n\}_{n\geq 0}$ *be a (MOPS) and* ϕ *a monic polynomial. When there exists an integer* $s \geq 0$ *such that the sequence* $\{B_n\}_{n\geq 0}$ *fulfils*

$$\phi(x) B_n(x) = \sum_{\nu=n-s}^{n+t} \lambda_{n,\nu} B_\nu^{[1]}(x), \quad \lambda_{n,n-s} \neq 0, \quad n \geq s, \tag{1.6}$$

we shall say that the sequences $\left(\{B_n\}_{n\geq 0},\ \{B_n\}_{n\geq 0}\right)$ *is a self coherent pair associated with* ϕ *with index* s. *In this case, the form* u_0 *is called a self coherent form associated* *with ϕ with index s. For the sake of convenience, we shall say also that $\{B_n\}_{n\geq 0}$ is a diagonal sequence (associated with ϕ with index s).*

Recall the following fundamental result

PROPOSITION 1.6. [10] *Let ϕ be as a above. For any (MOPS) $\{P_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 0}$, the following statements are equivalent*
i) The pair of polynomial sequences $\Big(\{P_n\}_{n\geq 0},\ \{B_n\}_{n\geq 0}\Big)$ is a coherent pair associated with ϕ with index s.
ii) There exist a monic polynomial sequence $\{\Omega_{n+s}\}_{n\geq 0}$, $\deg \Omega_{n+s} = n+s$, $n \geq 0$ and non-zero constants k_n, $n \geq 0$ such that

$$\phi u_n^{[1]} = k_n \Omega_{n+s} v_0, \quad n \geq 0. \tag{1.7}$$

In this case, we have

$$k_n = \frac{\lambda_{n+s,n}}{< v_0, P^2_{n+s} >}, \qquad \Omega_{n+s}(x) = \sum_{\nu=0}^{n+s} \frac{\lambda_{\nu,n}}{\lambda_{n+s,n}} \frac{< v_0, P^2_{n+s} >}{< v_0, P^2_{\nu} >} P_\nu(x), \qquad n \geq 0. \tag{1.8}$$

When $P_n = B_n$, $n \geq 0$, the relation (1.7) characterizes diagonal sequences.

Now, we can prove that the second sequence of a coherent pair is always a diagonal sequence.

PROPOSITION 1.7. *Suppose that $\Big(\{P_n\}_{n\geq 0}, \{B_n\}_{n\geq 0}\Big)$ is a coherent pair associated with ϕ and with index s, then there exist two polynomials $\mathcal{E}$ and $\mathcal{F}$ such that $\{B_n\}_{n\geq 0}$ is a diagonal sequence associated with $\Phi = \phi^2 \mathcal{F}$ with index s', where $\deg \Phi = 2(\deg \phi + s)$, $s' = s + \deg \phi + \deg(\mathcal{E})$ and*

$$\mathcal{E} = \phi\big(d_0 B_1 \Omega_{s+1} - d_1 B_2 \Omega_s\big) (\textit{for } d_0, d_1, \textit{ see below}), \qquad \mathcal{F} = \Omega_s \Omega'_{s+1} - \Omega_{s+1}\Omega'_s. \tag{1.9}$$

Moreover, the forms v_0 and u_0 are semi-classical.

Proof. Differentiating both sides of (1.7), then according to (1.1) and (1.2), we obtain

$$\phi' u_n^{[1]} - \frac{(n+1)}{< u_0, B^2_{n+1} >} \phi B_{n+1} u_0 = \ k_n (\Omega_{n+s} v_0)', \ n \geq 0. \tag{1.10}$$

Multiplying both sides of Eq. (1.10) by the polynomial ϕ and on account of (1.7), we get

$$\Big(\phi' \Omega_{n+s} - \Omega'_{n+s}\phi\Big) v_0 - d_n \phi^2 B_{n+1} u_0 = \phi \Omega_{n+s} v'_0, \ n \geq 0, \tag{1.11}$$

where

$$d_n = \frac{n+1}{< u_0, B^2_{n+1} > k_n}, \ n \geq 0.$$

Taking $n = 0$ and $n = 1$ successively into (1.11), we obtain

$$\Big(\phi' \Omega_s - \Omega'_s \phi\Big) v_0 - d_0 \phi^2 B_1 u_0 = \phi \Omega_s v'_0, \tag{1.12}$$

$$\Big(\phi' \Omega_{s+1} - \Omega'_{s+1} \phi\Big) v_0 - d_1 \phi^2 B_2 u_0 = \phi \Omega_{s+1} v'_0. \tag{1.13}$$

Eliminating v_0' from (1.12) and (1.13), we obtain

$$\phi\mathcal{E}u_0 = \phi\mathcal{F}v_0, \tag{1.14}$$

with $\mathcal{E} = \phi(d_0B_1\Omega_{s+1} - d_1B_2\Omega_s)$ and $\mathcal{F} = \Omega_s\Omega'_{s+1} - \Omega_{s+1}\Omega'_s$. Now, from (1.7), we have $\phi\mathcal{F}\phi u_n^{[1]} = k_n\phi\mathcal{F}\Omega_{n+s}v_0$ and according to (1.14) $\Phi u_n^{[1]} = k_n\Omega_{n+s}\phi\mathcal{E}u_0 = k_ne\mathcal{W}_{n+s'}u_0$, $n \geq 0$ where $\mathcal{W}_{n+s'}$ is monic and e is a normalisation factor of $\mathcal{E}$.
By cancelling u_0 between (1.12) and (1.13), we obtain

$$(\mathcal{E}v_0)' - \Big\{\big(d_0\Omega_{s+1} - d_1B_2'\Omega_s\big)\phi + 2\big(d_0B_1\Omega_{s+1} - d_1B_2\Omega_s\big)\phi'\Big\}v_0 = 0, \tag{1.15}$$

which implies that v_0 is semi-classical and u_0 as well.

REMARK. The formula (1.15) is not optimal from the point of view of the reduction of the degrees of polynomials involved on it.

2. General results about diagonal polynomial sequences. Let ϕ be a monic polynomial $\phi(x) = \prod_{\mu=1}^{m^*}(x-c_\mu)^{m_\mu}$, $\sum_{\mu=1}^{m^*} m_\mu = t$ where m^* denotes the number of distinct roots of ϕ. Let us put

(2.1)
$$\mathcal{A}(x) = \begin{cases} 1, & t=0 \\ \prod_{\mu=1}^{m^*}(x-c_\mu), & t\geq 1 \end{cases}; \quad \mathcal{B}(x) = \begin{cases} 0, & t=0 \\ \sum_{\mu=1}^{m^*}(m_\mu+1)\prod_{\nu=1,\nu\neq\mu}^{m^*}(x-c_\nu), & t\geq 1 \end{cases}.$$

LEMMA 2.1. *For all $t \geq 0$, we have the following relation:*

$$\mathcal{A}\phi' = \phi(\mathcal{B} - \mathcal{A}'). \tag{2.2}$$

For $t = 0$, it is evident. For $t \geq 1$, we have

$$\phi'(x) = \sum_{\mu=1}^{m^*} m_\mu(x-c_\mu)^{m_\mu-1}\prod_{\nu=1,\nu\neq\mu}^{m^*}(x-c_\nu)^{m_\nu}.$$

Therefore

$$\mathcal{A}(x)\phi'(x) = \phi(x)\sum_{\mu=1}^{m^*} m_\mu\prod_{\nu=1,\nu\neq\mu}^{m^*}(x-c_\nu) = \phi(x)\{\mathcal{B}(x) - \mathcal{A}'(x)\}.$$

As a particular case of the statement of Proposition 1.7, we have

PROPOSITION 2.2. *Any diagonal sequence $\{B_n\}_{n\geq 0}$ is necessarily semi-classical and its canonical form u_0 fulfils the following equations*

$$\big(\mathcal{A}\Omega_{n+s}u_0\big)' + \big(d_n\mathcal{A}\phi B_{n+1} - \mathcal{B}\Omega_{n+s}\big)u_0 = 0, \ \ n \geq 0, \tag{2.3}$$

where

$$d_n = (n+1)\frac{< u_0, B_{n+s}^2 >}{< u_0, B_{n+1}^2 > \lambda_{n+s,n}}, \ \ n \geq 0. \tag{2.4}$$

Moreover, the sequence $\{\Omega_{n+s}\}_{n\geq 0}$ *satisfies*

$$\Omega'_{n+s}\Omega_s - \Omega_{n+s}\Omega'_s = \phi\big(d_0\Omega_{n+s}B_1 - d_n\Omega_s B_{n+1}\big),\ \ n\geq 0. \tag{2.5}$$

Proof. From (1.10) where $v_0 = u_0$, we obtain

$$\phi' u_n^{[1]} - \frac{(n+1)}{< u_0, B_{n+1}^2 >}\phi B_{n+1}u_0 = \ k_n(\Omega_{n+s}u_0)',\ \ n\geq 0.$$

Hence

$$\mathcal{A}\phi' u_n^{[1]} - \frac{n+1}{< u_0, B_{n+1}^2 >}\mathcal{A}\phi B_{n+1}u_0 = k_n\big(\mathcal{A}\Omega_{n+s}u_0\big)' - k_n\mathcal{A}'\Omega_{n+s}u_0.$$

With (2.2) and (1.7), we get (2.3) – (2.4). Taking $n = 0$ in (2.3), we have

$$\big(\mathcal{A}\Omega_s u_0\big)' + \big(d_0\mathcal{A}\phi B_1 - \mathcal{B}\Omega_s\big)u_0 = 0. \tag{2.6}$$

Cancelling out u_0' between (2.3) and (2.6), we obtain (2.5), by virtue of regularity of u_0. □

COROLLARY 2.3. *When* $\{B_n\}_{n\geq 0}$ *is a diagonal sequence given by* (1.6), *then necessarily we have*

$$\frac{1}{2}t \leq s \leq t+2. \tag{2.7}$$

Moreover

(2.8)

$$\lambda_{n+s,n} = \begin{cases} (n+1)\dfrac{< u_0, B_{n+s}^2 >< u_0, B_1^2 >}{< u_0, B_{n+1}^2 >< u_0, B_s^2 >}\lambda_{s,0}, & (d_0 - d_n = 0),\ n\geq 0;\ s\leq t+1, \\ (n+1)\dfrac{< u_0, B_{n+s}^2 >< u_0, B_1^2 >}{< u_0, B_{n+1}^2 > \eta_n}\lambda_{s,0}, & (d_0 - d_n = n),\ n\geq 0;\ s = t+2. \end{cases}$$

where $\eta_n =< u_0, B_s^2 > - n\lambda_{s,0} < u_0, B_1^2 >$.

Proof. We have $\deg\big(d_0\Omega_{n+s}B_1 - d_n\Omega_s B_{n+1}\big) = n+s+1-\mu_n$ where $0\leq \mu_n \leq n+s+1$. Therefore, from (2.5), we obtain $n+2s-1 = t+n+s+1-\mu_n$, hence $\mu_n = t+2-s$, $n\geq 1$. With $0\leq \mu_1 \leq s+2$, we get (2.7). Accordingly $\deg\big(d_0\Omega_{n+s}B_1 - d_n\Omega_s B_{n+1}\big) = n+2s-t-1$, $n\geq 1$. When $s\leq t+1$, we have $n+2s-t-1\leq n+s$, therefore $d_n = d_0$, $n\geq 0$ and when $s = t+2$, we have $d_n \neq d_0$, $n\geq 0$. Then, examination of the highest degree coefficients in the members of (2.5) gives $n = d_0 - d_n$, $n\geq 0$. Hence (2.8), according to (2.4). □

REMARK. When $t = 0$, we recover a characterization of classical forms [9].

From definitions and (2.3), we see that the class of u_0 is less than or equal to $m^* + t$, since

$$m^* + t + n = \max\Big\{\deg\big(\mathcal{A}\Omega_{n+s}\big) - 2,\ \deg\big(d_n\mathcal{A}\phi B_{n+1} - \mathcal{B}\Omega_{n+s}\big) - 1\Big\},\ \ n\geq 0. \tag{2.9}$$

It is possible to obtain a more accurate estimation. For this, we may read (2.5) as

$$\Omega_{n+s}\Big\{d_0\phi B_1 + \Omega'_s\Big\} = \Omega_s\Big\{\Omega'_{n+s} + d_n\phi B_{n+1}\Big\},\ \ n\geq 0. \tag{2.10}$$

First a lemma.

LEMMA 2.4. *Let $\{B_n\}_{n\geq 0}$ be a diagonal sequence associated with ϕ and with index s and let (E, F) be a pair of polynomials, E monic and* $\deg(F) \geq 1$ *such that $(Eu_0)' + Fu_0 = 0$. Then if we associate with the pair (E, F) the integer* $p = \max\big(\deg(E) - 2, \deg(F) - 1\big)$, *we have*

$$p = \deg(F) - 1 = \deg(E) + t - s. \tag{2.11}$$

Proof. Let $F = \sum_{\nu=0}^{r} < u_\nu, F > B_\nu$ where $r = \deg F \geq 1$. But $(Eu_0)' + Fu_0 = 0$ implies $< u_0, F >= 0$ and with (1.1) – (1.2), $\Big(Eu_0 - \sum_{\nu=0}^{r-1} g_\nu u_\nu^{[1]}\Big)' = 0$, where $g_\nu = (\nu+1)^{-1} < u_{\nu+1}, F >< u_0, B_{\nu+1}^2 >$, $0 \leq \nu \leq r-1$, $g_{r-1} \neq 0$. Hence $Eu_0 = \sum_{\nu=0}^{r-1} g_\nu u_\nu^{[1]}$. Multiplying both sides by ϕ, according to (1.7) where $v_0 = u_0$ and by virtue of regularity of u_0, we get

$$\phi E = \sum_{\nu=0}^{r-1} g_\nu k_\nu \Omega_{\nu+s}.$$

We infer $\deg\phi + \deg E = r - 1 + s = \deg F - 1 + s$. Hence (2.11), taking (2.7) into account. □

PROPOSITION 2.5. *The diagonal form u_0 associated with ϕ with index s, $t \geq 1$ is of class less than or equal to $m^* + t - 1$.*

Proof. Consider euclidian division of Ω_{n+s} by Ω_s

$$\Omega_{n+s}(x) = \Omega_s(x)Q_n(x) + R_{s-1}(n)(x), \quad n \geq 0, \tag{2.12}$$

with $\deg R_{s-1}(n) \leq s - 1$ when $R_{s-1}(n) \neq 0$, $R_{s-1}(0) = 0$. Equation (2.3) becomes

$$\big(\mathcal{A}R_{s-1}(n)u_0\big)' + \Big\{\mathcal{A}\Omega_s Q_n' - \mathcal{A}\phi\big(d_0B_1Q_n - d_nB_{n+1}\big) - \mathcal{B}R_{s-1}(n)\Big\}u_0 = 0, \quad n \geq 0. \tag{2.13}$$

Two cases arise.

1) There exists $n_0 \geq 1$ such that $R_{s-1}(n_0) \neq 0$. From (2.13), we have

$$E = \mathcal{A}R_{s-1}(n_0) \quad \text{and} \quad F = \mathcal{A}\Omega_s Q_{n_0}' - \mathcal{A}\phi\big(d_0B_1Q_{n_0} - d_{n_0}B_{n_0+1}\big) - \mathcal{B}R_{s-1}(n_0).$$

Since $\deg E \leq m^* + s - 1$, we have $\deg F - 1 \leq m^* + s - 1 + t - s = m^* + t - 1$. Hence the desired result by taking (2.11) into account.

REMARK. Since $1 \leq \deg F \leq m^* + t$, we must have $t \geq 1$.

2) For any $n \geq 1$, we have $R_{s-1}(n) = 0$. Equation (2.10) becomes

$$\Omega_s Q_n' = \phi\{d_0B_1Q_n - d_nB_{n+1}\}, \quad n \geq 0. \tag{2.14}$$

For $n = 1$ in (2.14), $\Omega_s = \phi\{d_0B_1Q_1 - d_1B_2\} := \phi Z$, whence $ZQ_n' - d_0B_1Q_n = -d_nB_{n+1}$, $n \geq 0$.

It follows $< u_0, ZQ'_n - d_0 B_1 Q_n >= -d_n < u_0, B_{n+1} >= 0$ or $< (Zu_0)' + d_0 B_1 u_0, Q_n > = 0$, $n \geq 0$, which implies

$$(Zu_0)' + d_0 B_1 u_0 = 0 \quad \text{with} \quad \deg Z \leq 2.$$

In this case, the class of u_0 is zero, i.e. u_0 is a classical form. □

COROLLARY 2.6. *Let $\{B_n\}_{n\geq 0}$ be a diagonal sequence associated with ϕ with index s. When the polynomials Ω_s and $\Omega'_s + d_0\phi B_1$ are coprime, then $\{B_n\}_{n\geq 0}$ is a classical sequence.*

Proof. Following (2.10), the assumption implies existence of a polynomial sequence $\{q_n\}_{n\geq 0}$ such that $\Omega_{n+s} = \Omega_s q_n$, $n \geq 0$. Therefore $q_n = Q_n$ and $R_{s-1}(n) = 0$, $n \geq 0$. Hence the desired result from above. □

Under certain conditions, it is possible to build a diagonal sequence from a given one.

PROPOSITION 2.7. *Let $\{B_n\}_{n\geq 0}$ be a diagonal sequence associated with ϕ with index s. Let ϕ_1 be a polynomial, $\deg \phi_1 = t_1$ and consider $\phi_1 u_0$. Suppose that $\phi_1 u_0$ is regular with $\{P_n\}_{n\geq 0}$ the (MOPS) associated with it. Then $\{P_n\}_{n\geq 0}$ is a diagonal sequence associated with $\phi_1\phi$ with index $t_1 + s$.*

Proof. Let us put $v_0 = \lambda\phi_1 u_0$ with $(v_0)_0 = 1$. Following the Proposition 2.4 of [10], we have

$$B_n(x) = \sum_{\nu=n-t_1}^{n} \vartheta_{n,\nu} P_\nu(x), \quad \vartheta_{n,n-t_1} \neq 0, \quad n \geq t_1.$$

Hence

$$B_n^{[1]}(x) = \sum_{\nu=n-t_1}^{n} \tilde{\vartheta}_{n,\nu} P_\nu^{[1]}(x), \quad \tilde{\vartheta}_{n,n-t_1} \neq 0, \quad n \geq t_1,$$

where $\tilde{\vartheta}_{n,\nu} = (n+1)^{-1}(\nu+1)\vartheta_{n+1,\nu+1}$. On account of Lemma 2.1 of [10], this is equivalent to

$$v_n^{[1]} = \sum_{\nu=n}^{n+t_1} \tilde{\vartheta}_{\nu,n} u_\nu^{[1]}, \quad n \geq 0.$$

Because the sequence $\{B_n\}_{n\geq 0}$ is diagonal relatively to ϕ with index s, consequently, from the last equality and using (1.7) where $v_0 = u_0$, we obtain

$$\phi v_n^{[1]} = \sum_{\nu=n}^{n+t_1} \tilde{\vartheta}_{\nu,n} \phi u_\nu^{[1]} = \sum_{\nu=n}^{n+t_1} \tilde{\vartheta}_{\nu,n} k_\nu \Omega_{\nu+s} u_0 = r_n \Lambda_{n+t_1+s} u_0, \tag{2.15}$$

with $\Lambda_{n+t_1+s} = \sum_{\nu=n}^{n+t_1} k_\nu \tilde{\vartheta}_{\nu,n} r_n^{-1} \Omega_{\nu+s}$, $\quad r_n = k_{n+t_1} \tilde{\vartheta}_{n+t_1,n}$, $\quad n \geq 0$.
From (2.15), we infer

$$\phi_1 \phi v_n^{[1]} = r_n \Lambda_{n+t_1+s} \phi_1 u_0 = \lambda^{-1} r_n \Lambda_{n+t_1+s} v_0, \quad n \geq 0.$$

Hence the result. □

REMARK. The relation (2.15) means that $\Big(\{B_n\}_{n\geq 0}, \{P_n\}_{n\geq 0}\Big)$ is a coherent pair associated with ϕ with index $t_1 + s$.

In the sequel, we shall use the following second order recurrence relation fulfilled by the diagonal sequence $\{B_n\}_{n\geq 0}$

$$\begin{cases} B_0(x) = 1, \quad B_1(x) = x - \beta_0, \\ B_{n+2}(x) = (x - \beta_{n+1})B_{n+1}(x) - \gamma_{n+1}B_n(x), \quad n \geq 0. \end{cases} \tag{2.16}$$

It follows
(2.17)
$$B^{[1]}_{n+1}(x) = \frac{n+1}{n+2}(x-\beta_{n+1})B^{[1]}_n(x) - \frac{n}{n+2}\gamma_{n+1}B^{[1]}_{n-1}(x) + \frac{1}{n+2}B_{n+1}(x), \quad n \geq 0.$$

3. Diagonal sequences with index s, $1 \leq s \leq 3$. In this section, we only study diagonal sequences associated with $\phi(x) = x - c$ with index s where $s = 1, 2, 3$, by virtue of (2.7). From (2.3) where $n = 0$, the form u_0 satisfies the following equation:

$$(Eu_0)' + Fu_0 = 0, \tag{3.1}$$

with $E(x) = (x-c)\Omega_s(x)$ and $F(x) = d_0(x-c)^2B_1(x) - 2\Omega_s(x)$.
Following Proposition 2.5, this equation can be simplified, since the class of u_0 is less than or equal to 1. According to the expression of $E(x)$, the equation (3.1) can be simplified either by $x - c$ or by a factor of Ω_s. Thus by virtue of (1.3), the equation (3.1) is simplified by $x - c$ if and only if

$$E'(c) + F(c) = -\Omega_s(c) = 0\ , \tag{3.2}$$
$$< u_0, \theta_c^2(E) + \theta_c(F) >= - < u_0, \theta_c(\Omega_s) > +d_0\gamma_1 = 0. \tag{3.3}$$

Then u_0 satisfies

$$(E_1u_0)' + F_1u_0 = 0, \tag{3.4}$$

where

$$E_1(x) = \Omega_s(x), \quad F_1(x) = d_0(x-c)B_1(x) - \big(\theta_c(\Omega_s)\big)(x). \tag{3.5}$$

Denoting by ξ_i, $i = 1, 2, ...s$, the roots of Ω_s, for simplifying by $x - \xi_i$, it is necessary and sufficient that

$$E'(\xi_i) + F(\xi_i) = (\xi_i - c)\Big\{\big(\theta_{\xi_i}(\Omega_s)\big)(\xi_i) + d_0(\xi_i - c)B_1(\xi_i)\Big\} = 0, \tag{3.6}$$

$$\begin{aligned} &< u_0, \theta^2_{\xi_i}(E) + \theta_{\xi_i}(F) > \\ &\quad = d_0\{\gamma_1 + (\xi_i - c)^2\} - < u_0, \theta_{\xi_i}(\Omega_s) > + (\xi_i - c) < u_0, \theta^2_{\xi_i}(\Omega_s) >= 0. \end{aligned} \tag{3.7}$$

Then u_0 fulfils

$$(E_2u_0)' + F_2u_0 = 0, \tag{3.8}$$

where
(3.9)
$$\begin{aligned} E_2(x) &= (x-c)\big(\theta_{\xi_i}(\Omega_s)\big)(x), \\ F_2(x) &= d_0\{(x - 2c + \xi_i)B_1(x) + (\xi_i - c)^2\} - \big(\theta_{\xi_i}(\Omega_s)\big)(x) + (\xi_i - c)\big(\theta^2_{\xi_i}(\Omega_s)\big)(x). \end{aligned}$$

Two cases arise:
I. $c \notin \{\xi_i\}$; **II.** $c \in \{\xi_i\}$.
First, we deal with the case

I. $\Omega_s(c) \neq 0$

Necessarily, there exists i, for instance $i = 1$, such that (3.6) – (3.7) are fulfilled. With the following shape

$$(\theta_{\xi_1}(\Omega_s))(x) = \delta_{3-s,0}(x-c)^2 + a_1(x-c) + a_0, \tag{3.10}$$

where $a_0 := (\theta_{\xi_1}(\Omega_s))(c)$, $a_1 := (\theta_c\theta_{\xi_1}(\Omega_s))(c) = (\theta_{\xi_1}(\Omega_s))'(c)$; the relations (3.6) – (3.7) become

$$a_0 + (\xi_1 - c)\Big\{d_0 B_1(\xi_1) + a_1 + \delta_{3-s,0}(\xi_1 - c)\Big\} = 0, \tag{3.11}$$

$$\begin{aligned}(d_0 - \delta_{3-s,0})\gamma_1 &+ (d_0 + \delta_{3-s,0})(\xi_1 - c)^2 \\ &+ \delta_{3-s,0}(\beta_0 - c)(\xi_1 - \beta_0) + a_1(\xi_1 - \beta_0) - a_0 = 0.\end{aligned} \tag{3.12}$$

Consequently, on account of (3.10), for (3.9) we have

(3.13)
$$\begin{cases} E_2(x) = (x-c)\{\delta_{3-s,0}(x-c)^2 + a_1(x-c) + a_0\}, \\ F_2(x) = (d_0 - \delta_{3-s,0})(x-c)^2 + \Big\{\delta_{3-s,0}(\xi_1 - c) + d_0(\xi_1 - \beta_0) - a_1\Big\}(x-c) \\ \qquad + (d_0 + \delta_{3-s,0})(\xi_1 - c)^2 + \big(a_1 - (\beta_0 - c)d_0\big)(\xi_1 - c) - a_0. \end{cases}$$

We distinguish the three cases $s = 1, 2, 3$.

I$_1$. $s = 1$, $\Omega_1(x) = x - \xi_1$

Here $(\theta_{\xi_1}(\Omega_1))(x) = 1$, therefore $a_1 = 0$, $a_0 = 1$. Hence, on the basis of (3.11)–(3.13)

$$E_2(x) = x - c, \quad F_2(x) = d_0\{x^2 + (\xi_1 - 2c - \beta_0)x + \beta_0(2c - \xi_1) - \gamma_1\}. \tag{3.14}$$

It follows $E_2'(c) + F_2(c) = -1$, which means that the form u_0 is of class $\sigma = 1$.

I$_2$. $s = 2$, $\Omega_2(x) = (x - \xi_1)(x - \xi_2)$

Here $(\theta_{\xi_1}(\Omega_2))(x) = x - \xi_2$, therefore $a_1 = 1$, $a_0 = c - \xi_2$. Taking (3.11)–(3.13) into account, we have

$$E_2(x) = (x-c)(x-\xi_2), \quad F_2(x) = d_0(x-c)^2 + \{d_0(\xi_1 - \beta_0) - 1\}(x-c) + 2(\xi_2 - c). \tag{3.15}$$

It follows $E_2'(c) + F_2(c) = \xi_2 - c \neq 0$. But, we also have

$$F_2(x) = d_0(x - \xi_2)^2 + \Big\{d_0\big[2(\xi_2 - c) + \xi_1 - \beta_0\big] - 1\Big\}(x - \xi_2) + (\xi_2 - c)\big(d_0 X + 1\big),$$

where $X = \xi_1 - \beta_0 + \xi_2 - c$. We deduce

$$E_2'(\xi_2) + F_2(\xi_2) = (\xi_2 - c)\big(2 + d_0 X\big), \quad < u_0, \theta_{\xi_2}^2 E_2 + \theta_{\xi_2} F_2 >= d_0 S,$$

with $S = \xi_1 + \xi_2 - 2c$. On the other hand, the relation (3.11) becomes

$$S - (\xi_1 - c)\big(d_0 X + 2\big) = -d_0(\xi_1 - c)(\xi_2 - c). \tag{3.16}$$

In any case, the class of u_0 is $\sigma = 1$, since $S = 0$, $2 + d_0 X = 0$ is not possible from (3.16).

I$_3$. $s = 3$, $\Omega_3(x) = (x - \xi_1)(x - \xi_2)(x - \xi_3)$

Here $\big(\theta_{\xi_1}(\Omega_3)\big)(x) = (x-\xi_2)(x-\xi_3)$, therefore $a_1 = 2c-\xi_2-\xi_3$, $a_0 = (\xi_2-c)(\xi_3-c)$. With (3.11)–(3.13), we obtain

$$\begin{aligned}
E_2(x) &= (x-c)(x-\xi_2)(x-\xi_3),\\
F_2(x) &= (d_0-1)(x-c)^2 + \big\{\xi_1 - c + \xi_2 - c + \xi_3 - c + d_0(\xi_1 - \beta_0)\big\}(x-c)\\
&\quad - 2(\xi_2 - c)(\xi_3 - c).
\end{aligned}$$

It follows $E_2'(c) + F_2(c) = -(\xi_2 - c)(\xi_3 - c) \neq 0$. Moreover

$$\begin{aligned}
F_2(x) =& (d_0-1)(x-\xi_2)^2\\
&+ \big\{(d_0+1)S + (d_0-2)(\xi_2 - c) + \xi_3 - c - d_0(\beta_0 - c)\big\}(x-\xi_2)\\
&+ (\xi_2 - c)\big\{(d_0+1)S - (\xi_2 - c) - (\xi_3 - c) - d_0(\beta_0 - c)\big\},
\end{aligned}$$

on the basis of (3.11) which becomes

$$(\xi_3 - c)S + (\xi_1 - c)\big\{d_0 X - 2(\xi_3 - c) + \xi_1 - c - (d_0+1)(\xi_2 - c)\big\} = 0. \tag{3.17}$$

Consequently, we have

$$\begin{aligned}
&E_2'(\xi_2) + F_2(\xi_2) = (\xi_2 - c)\big\{(d_0+1)S - 2(\xi_3 - c) - d_0(\beta_0 - c)\big\} = (\xi_2 - c)T,\\
&\big(\theta_{\xi_2}^2 E_2\big)(x) + \big(\theta_{\xi_2} F_2\big)(x) = d_0(x - \beta_0) + (d_0+1)S,
\end{aligned}$$

where

$$T = (d_0+1)S - 2(\xi_3 - c) - d_0(\beta_0 - c) = (d_0+1)X - 2(\xi_3 - c) + \beta_0 - c. \tag{3.18}$$

Therefore $< u_0, \theta_{\xi_2}^2 E_2 + \theta_{\xi_2} F_2 >= (d_0+1)S$. Further, from the definitions and (3.18), we have

$$(\xi_3 - c)S + (\xi_1 - c)T = (\xi_1 - c)(d_0 + 2)(\xi_2 - c). \tag{3.19}$$

Let us prove either $(d_0+1)S \neq 0$ or $T \neq 0$. First, suppose $S = 0$ and $T = 0$. Then, from (3.19) and (3.18), we obtain $d_0 + 2 = 0$ and $\beta_0 = \xi_3$. Consequently, (3.12) becomes $-3\gamma_1 = 0$ which is a contradiction. Now, suppose $d_0 + 1 = 0$ and $T = 0$. Then u_0 fulfils $\big((x-c)(x-\xi_3)u_0\big)' - (x - \beta_0)u_0 = 0$ where $c \neq \xi_3$ and $\beta_0 - c = 2(\xi_3 - c)$, taking (3.18) into account. With a suitable shift, we can choose $c = 1$, $\xi_3 = -1$, therefore $\beta_0 = -3$ and u_0 satisfies $\big((x^2-1)u_0\big)' - (x+3)u_0 = 0$. It follows that u_0 is the Jacobi form with parameters $(-2, 1)$, which is not regular [2,8,9]. Hence the desired result. Similarly for the root ξ_3. In this case, the class of u_0 is also $\sigma = 1$.
Thus, when $\Omega_s(c) \neq 0$, the class of u_0 is $\sigma = 1$. We shall see a shorter proof below.

Does a form w exist such that $u_0 = \tau(x - d)w$ where $\tau \neq 0$ and d are chosen for making w a shift of a classical form? It is easy to see the following results.

For $\mathbf{I}_1$, we have $\tau = d_0(\xi_1 - c)$, $d = c$ and w fulfils

$$w' + \big\{d_0(x-c) - (\xi_1 - c)^{-1}\big\}w = 0.$$

By a shift, we obtain the Hermite form.

For $\mathbf{I}_2$, we have $\tau = -d_0(\xi_1 - c)(\xi_2 - c)^{-1}$, $d = c$ and w fulfils the following equation

$$\big((x-\xi_2)w\big)' + \big\{d_0(x-c) + (\xi_2 - c)(\xi_1 - c)^{-1}\big\}w = 0.$$

By a suitable shift, we obtain a Laguerre form.

Finally for $\mathbf{I}_3$, when $d_0+1 \neq 0$, we have $\tau = (d_0+1)(\xi_1-c)(\xi_2-c)^{-1}(\xi_3-c)^{-1}$, $d = c$ and w fulfils

$$\big((x-\xi_2)(x-\xi_3)w\big)' + (d_0+1)\big(x-c-\tau^{-1}\big)w = 0.$$

By a suitable shift, we obtain a Bessel form, when $\xi_2 = \xi_3$ and a Jacobi form, when $\xi_2 \neq \xi_3$.

When $d_0 + 1 = 0$, it is not possible to determine $\tau \neq 0$, $d \in \mathbb{C}$ for making w essentially classical.

In any case, it remains to determine the values of c for which $\tau(x-c)w$ is regular. A little about this problem is the fact that if $\{Z_n\}_{n\geq 0}$ denotes the (MOPS) with respect to w, then $(x-c)w$ is regular if and only if $Z_n(c) \neq 0$, $n \geq 1$ [2].
For instance, in the case $\mathbf{I}_1$, following Chihara's notation, if $\{\widehat{H}_n\}_{n\geq 0}$ denotes the (MOPS) associated with the Hermite form $\mathcal{H}$ fulfilling $\mathcal{H}' + 2x\mathcal{H} = 0$, we have $w = \big(h_{a^{-1}} \circ \tau_{-b}\big)\mathcal{H}$, $Z_n(x) = a^{-n}\widehat{H}_n(ax+b)$, where $2a^2 = d_0$, $2ab = -d_0 c - (\xi_1 - c)^{-1}$. Thus $ac + b = -\big(2a(\xi_1 - c)\big)^{-1}$ must not be a zero of any monic Hermite polynomial. When $d_0 < 0$, $\xi_1 \in \mathbb{R}$, then any real $c \neq \xi_1$ is suitable. But when $d_0 > 0$, $\xi_1 \in \mathbb{R}$, the problem is open for obtaining non singular real c through a constructive way. For the other cases, there are similar results.

II. $\Omega_s(c) = 0$

LEMMA 3.1. *When the equation* (3.1) *is simplified by the factor $x-c$, then it can be simplified a second time by $x-c$ and the form u_0 satisfies*

$$(E^* u_0)' + F^* u_0 = 0, \tag{3.20}$$

where

$$E^*(x) = \big(\theta_c \Omega_s\big)(x), \quad F^*(x) = d_0 B_1(x). \tag{3.21}$$

Consequently, the form u_0 is classical $(\sigma = 0)$.

Proof. Since the equation (3.1) is simplified by the factor $x-c$, the form u_0 satisfies (3.4) with (3.5). Moreover

$$E_1'(c) + F_1(c) = 0 \quad , \quad < u_0, \theta_c^2 E_1 + \theta_c F_1 > = < u_0, d_0 B_1 > = 0.$$

Therefore, we have (3.20) and (3.21). Necessarily, the form u_0 is classical. □

This leads to

THEOREM 3.2. *The diagonal form u_0 associated with $x-c$ is classical, if and only if the conditions* (3.2) – (3.3) *are satisfied.*

Proof. Sufficiency follows from the previous Lemma. When the diagonal form u_0 is classical, from (1.7), we have $(x-c)u_0^{[1]} = k_0 \Omega_s u_0$ and there is a monic polynomial Φ, $\deg \Phi \leq 2$ and $k_0' \neq 0$ such that $u_0^{[1]} = k_0' \Phi u_0$ [9]. The regularity implies $k_0'(x - c)\Phi(x) = k_0 \Omega_s(x)$, hence $\Omega_s(c) = 0$ and consequently $\Phi = \theta_c(\Omega_s)$, $k_0' = k_0 = (d_0\gamma_1)^{-1}$. Moreover

$$< u_0, \theta_c(\Omega_s) > = < u_0, \Phi > = < \Phi u_0, 1 > = k_0^{-1} < u_0^{[1]}, 1 > = d_0 \gamma_1. \qquad \square$$

$\mathbf{II}_1$. $s = 1$, $\quad \Omega_1(x) = x - c$

Inevitably, the form u_0 satisfies (3.4) with $d_0\gamma_1 = 1$. From the previous Lemma, we get

$$E^*(x) = 1, \quad F^*(x) = \gamma_1^{-1}B_1(x).$$

Through a suitable shift, we obtain the Hermite form.

II$_2$. $s = 2, \quad \Omega_2(x) = (x - \xi_1)(x - c), \quad (\xi_2 = c)$
Here$\big(\theta_c(\Omega_2)\big)(x) = x - \xi_1$.

II$_{21}$. $< u_0, x - \xi_1 >= d_0\gamma_1$

From the previous Lemma, the form u_0 satisfies (3.20) with $E^*(x) = x - \xi_1$, $F^*(x) = d_0B_1(x)$. It is the Laguerre case when $\xi_1 = 0$, $d_0 = 1$ and putting $\beta_0 = \alpha + 1$.

II$_{22}$. $< u_0, x - \xi_1 >\neq d_0\gamma_1$

Then u_0 fulfils (3.8) and $\big(\theta_{\xi_1}(\Omega_2)\big)(x) = x - c$, hence $a_1 = 1$, $a_0 = 0$ in (3.10). The conditions (3.11) – (3.12) and (3.13) respectively become

$$d_0(\xi_1 - \beta_0) + 1 = 0, \quad 0 \neq d_0\gamma_1 + \xi_1 - \beta_0 = -d_0(\xi_1 - c)^2 \tag{3.22}$$

$$E_2(x) = (x - c)^2 \quad , \quad F_2(x) = d_0(x - c)^2 - 2(x - c). \tag{3.23}$$

It follows

$$E_2'(c) + F_2(c) = 0, \quad < u_0, \theta_c^2E_2 + \theta_cF_2 >=< u_0, d_0(x - c) - 1 >= d_0(\xi_1 - c) \neq 0,$$

taking (3.22) into account.
Putting $(x - c)u_0 = \vartheta w_0$ with $(w_0)_0 = 1$, we have $\vartheta = \beta_0 - c \neq 0$. For, if $\beta_0 = c$, we should have $d_0(\xi_1 - c) + 1 = 0$, therefore $0 \neq d_0\gamma_1 = 0$ from (3.22). Thus the form w_0 satisfies

$$\big((x - c)w_0\big)' + \big\{d_0(x - c) - 2\big\}w_0 = 0. \tag{3.24}$$

Through a suitable shift, we can take $c = 0$ and $d_0 = 1$. Then $w_0 = \mathcal{L}(1)$, the Laguerre form with parameter $\alpha = 1$. On account of the definition of w_0, we have

$$u_0 = \delta + \vartheta x^{-1}w_0, \tag{3.25}$$

where $\delta = \delta_0$ defined by $< \delta_0, f >= f(0)$.
We denote by $\{R_n\}_{n\geq 0}$ the (MOPS) associated with w_0. We know that u_0 is regular if and only if

$$-\frac{R_n(0)}{R_{n-1}^{(1)}(0)} \neq \vartheta, \quad n \geq 1, \tag{3.26}$$

where $\{R_n^{(1)}\}_{n\geq 0}$ is the associated sequence of $\{R_n\}_{n\geq 0}$ defined by

$$R_n^{(1)}(x) :=< w_0, \frac{R_{n+1}(x) - R_{n+1}(\xi)}{x - \xi} >, \ n \geq 0,$$

and in this case, we have [7]

$$B_{n+1}(x) = R_{n+1}(x) + \varpi_nR_n(x), \quad n \geq 0, \tag{3.27}$$

where

$$\varpi_n = -\frac{R_{n+1}(0) + \vartheta R_n^{(1)}(0)}{R_n(0) + \vartheta R_{n-1}^{(1)}(0)}, \quad n \geq 0. \tag{3.28}$$

It can be seen that [2]

$$R_n(x)=\sum_{\nu=0}^{n}(-1)^{n-\nu}\frac{n!}{\nu!}\binom{n+1}{n-\nu}x^{\nu},\quad n\geq 0.$$

Therefore

$$R_n^{(1)}(0)=<w_0,\frac{R_{n+1}(\xi)-R_{n+1}(0)}{\xi}>=\sum_{\nu=0}^{n}(-1)^{n-\nu}\frac{(n+1)!}{(\nu+1)!}\binom{n+2}{n-\nu}(w_0)_{\nu}.$$

But $(w_0)_n=(n+1)!$. Thus

$$R_n^{(1)}(0)=(n+1)!(n+2)!\sum_{\nu=0}^{n}\frac{(-1)^{n-\nu}}{(n-\nu)!}\frac{1}{(\nu+2)!}. \tag{3.29}$$

We have

$$\sum_{\nu=0}^{n}\frac{(-1)^{n-\nu}}{(n-\nu)!}\frac{1}{(\nu+2)!}=\Xi_{n+2}-\left\{\frac{(-1)^{n+1}}{(n+1)!}+\frac{(-1)^{n+2}}{(n+2)!}\right\},\quad n\geq 0, \tag{3.30}$$

where

$$\Xi_n=\sum_{\nu=0}^{n}\frac{(-1)^{n-\nu}}{(n-\nu)!\nu!}=\frac{1}{n!}\sum_{\nu=0}^{n}(-1)^{n-\nu}\binom{n}{\nu}=\frac{(1-1)^n}{n!}=\begin{cases}0, & n\geq 1,\\ 1, & n=0.\end{cases} \tag{3.31}$$

Consequently, for (3.29) we obtain

$$R_n^{(1)}(0)=(-1)^n(n+1)!(n+1),\quad n\geq 0. \tag{3.32}$$

Then, for (3.28)

$$\varpi_n=(n+2)\frac{1-\frac{n+1}{n+2}\vartheta}{1-\frac{n}{n+1}\vartheta},\quad n\geq 0, \tag{3.33}$$

where $\vartheta\neq\dfrac{n+1}{n}$, $n\geq 1$ (cf. (3.26)).

REMARK. About (3.31), more generally, it is possible to show that $\Xi_n(\rho)=\sum_{\nu=0}^{n}(-1)^{n-\nu}a_{n-\nu}(\rho)a_{\nu}(\rho)$with $a_n(\rho)=\frac{\Gamma(\rho)}{\Gamma(n+\rho)}$, $n\geq 0$, fulfils $\Xi_{2n+1}(\rho)=0$, $\Xi_{2n+2}(\rho)=\dfrac{\rho-1}{n+\rho}\dfrac{\Gamma(\rho)}{\Gamma(2n+2+\rho)}$, $n\geq 0$.

Now, since the sequence $\{R_n\}_{n\geq 0}$ fulfils

$$\begin{aligned}&R_0(x)=1,\quad R_1(x)=x-\zeta_0\\&R_{n+2}(x)=(x-\zeta_{n+1})R_{n+1}(x)-\rho_{n+1}R_n(x),\quad n\geq 0,\end{aligned} \tag{3.34}$$

with $\zeta_n=2n+2$, $\rho_{n+1}=(n+2)(n+1)$, $n\geq 0$ and taking the following formulas from [7] into account

$$\beta_0=\zeta_0-\varpi_0=\vartheta;\ \beta_{n+1}=c+\varpi_n+\frac{\rho_{n+1}}{\varpi_n},\quad \gamma_{n+1}=-\varpi_n(c+\varpi_n-\zeta_n),\ n\geq 0, \tag{3.35}$$

(at present $c = 0$), we obtain with (3.33)

$$\left.\begin{aligned} \beta_{n+1} &= \frac{(n+1)\big(n+2-(n+1)\vartheta\big)^2 + (n+2)\big(n+1-n\vartheta\big)^2}{\big(n+1-n\vartheta\big)\big(n+2-(n+1)\vartheta\big)} \\ \gamma_{n+1} &= \frac{(n+1)^2\big\{(n-1)(1-\vartheta)+1\big\}\big\{(n+1)(1-\vartheta)+1\big\}}{(n+1-n\vartheta)^2} \end{aligned}\right\}, \quad n \geq 0.$$

Necessarily $\vartheta \neq 1$ since $\xi_1 \neq 0$ by virtue of (3.22) where $d_0 = 1$ and $c = 0$. Then, putting $(1-\vartheta)^{-1} := \alpha + 1$, we have

$$\begin{aligned} \beta_n &= 2n+1-\frac{\alpha}{(n+\alpha)(n+\alpha+1)} \\ \gamma_{n+1} &= \frac{(n+1)^2(n+\alpha)(n+\alpha+2)}{(n+\alpha+1)^2}, \end{aligned} \quad n \geq 0, \tag{3.36}$$

with $\alpha \neq -n$, $n \geq 0$. From (3.25), we have

$$< u_0, f > = \frac{f(0)}{\alpha+1} + \frac{\alpha}{\alpha+1}\int_0^{+\infty} e^{-x} f(x)dx.$$

II$_3$. $s = 3$, $\quad \Omega_3(x) = (x-\xi_1)(x-\xi_2)(x-c), \quad (\xi_3 = c)$

Here $\big(\theta_c(\Omega_3)\big)(x) = (x-\xi_1)(x-\xi_2)$.

II$_{31}$. $< u_0, (x-\xi_1)(x-\xi_2) > = d_0\gamma_1$

Following Lemma 3.1, u_0 is classical; it is the Bessel form when $\xi_1 = \xi_2 = 0$ and the Jacobi form when $\xi_1 = -1$, $\xi_2 = +1$.

II$_{32}$. $< u_0, (x-\xi_1)(x-\xi_2) > \neq d_0\gamma_1$

The form u_0 is not classical by virtue of Theorem 3.2, since the relation (3.3) is not fulfilled. Here $\big(\theta_{\xi_1}(\Omega_3)\big) = (x-c)(x-\xi_2)$, hence $a_1 = c - \xi_2$, $a_0 = 0$ in (3.10). The conditions (3.11) – (3.12) become

$$(\xi_1 - c)\big\{d_0(\xi_1-\beta_0) + \xi_1 - \xi_2\big\} = 0 \tag{3.37}$$

$$0 \neq (d_0-1)\gamma_1 - (\xi_1-\beta_0)(\xi_2-\beta_0) = -(\beta_0-c)^2 - (d_0+1)(\xi_1-c)^2. \tag{3.38}$$

Following (3.13) and taking (3.37) into account, we get

$$\begin{aligned} E_2(x) &= (x-c)^2(x-\xi_2) \\ F_2(x) &= (d_0-1)(x-c)^2 + \big\{\xi_1 - c + \xi_2 - c + d_0(\xi_1-\beta_0)\big\}(x-c). \end{aligned} \tag{3.39}$$

We infer $E_2'(c) + F_2(c) = 0$, $\quad \big(\theta_c^2 E_2\big)(x) + \big(\theta_c F_2\big)(x) = d_0(x-c) + \xi_1 - c + d_0(\xi_1-\beta_0)$, therefore

$$< u_0, \theta_c^2 E_2 + \theta_c F_2 > = (d_0+1)(\xi_1-c).$$

Necessarily, we must have $(d_0+1)(\xi_1-c) \neq 0$, otherwise u_0 would be classical. Then (3.37) reads

$$(d_0+1)(\xi_1-\beta_0) = \xi_2 - \beta_0. \tag{3.40}$$

Likewise, we have

$$E_2'(\xi_2) + F_2(\xi_2) = (d_0+2)(\xi_2-c)^2, \quad < u_0, \theta_{\xi_2}^2 E_2 + \theta_{\xi_2} F_2 > = (d_0+1)(\xi_1 - c + \xi_2 - c).$$

Necessarily, either $(d_0+2)(\xi_2-c)^2 \neq 0$ or $(d_0+1)(\xi_1-c+\xi_2-c) \neq 0$, otherwise u_0 would be classical.
Putting $(x-c)u_0 = \tilde{w}$, we get

$$\big((x-c)(x-\xi_2)\tilde{w}\big)' + \{(d_0-1)(x-c)+2(\xi_2-c)\}\tilde{w} = 0.$$

The Bessel case is not possible; for, if $c = \xi_2 = 0$, we should have $(d_0-1)x = -2(\alpha x+1)$. Consequently $c \neq \xi_2$ and we choose $c=-1$ and $\xi_2 = +1$. Then $d_0 - 1 = -(\alpha+\beta+2)$, $d_0+3 = \alpha-\beta$, therefore $\alpha = 1$ and $\beta = -(d_0+2)$. We have the Jacobi case with parameters $(1,\beta)$ and the form $\tilde{w}$ is regular if and only if $\beta \neq -n$, $n \geq 1$; we can put $\tilde{w} = \vartheta w_0$ with $(w_0)_0 = 1$, since $\beta_0 + 1 = 4(3+\beta)^{-1} \neq 0$. Thus, we obtain

$$u_0 = \delta_{-1} + \vartheta(x+1)^{-1}w_0. \tag{3.41}$$

We denote by $\{R_n\}_{n\geq 0}$ the (MOPS) associated with $w_0 = \mathcal{J}(1,\beta)$. We have [8,9]

$$R_0(x) = 1, \quad R_1(x) = x - \frac{1-\beta}{3+\beta}$$
$$R_{n+2}(x) = (x-\zeta_{n+1})R_{n+1}(x) - \rho_{n+1}R_n(x), \quad n \geq 0,$$

with

$$\begin{aligned} \zeta_{n+1} &= \frac{1-\beta^2}{(2n+\beta+3)(2n+\beta+5)}, \\ \rho_{n+1} &= 4\frac{(n+1)(n+2)(n+\beta+1)(n+\beta+2)}{(2n+\beta+2)(2n+\beta+3)^2(2n+\beta+4)}, \end{aligned} \quad n \geq 0. \tag{3.42}$$

The form u_0 is regular if and only if $-R_n(-1)\big(R^{(1)}_{n-1}(-1)\big)^{-1} \neq \vartheta$, $n \geq 1$ and $B_{n+1}(x) = R_{n+1}(x) + \varpi_n R_n(x)$, $n \geq 0$ where [7]

$$\varpi_n = -\frac{R_{n+1}(-1)+\vartheta R^{(1)}_n(-1)}{R_n(-1)+\vartheta R^{(1)}_{n-1}(-1)}, \quad n \geq 0. \tag{3.43}$$

It can be seen that [2]

$$R_n(x) = n!\frac{\Gamma(n+\beta+2)}{\Gamma(2n+\beta+2)}\sum_{\nu=0}^{n}\binom{n+1}{\nu}\binom{n+\beta}{n-\nu}(x-1)^\nu(x+1)^{n-\nu}, \quad n \geq 0. \tag{3.44}$$

Next

$$R^{(1)}_n(-1) = < w_0, \frac{R_{n+1}(\xi)-R_{n+1}(-1)}{\xi+1} >, \quad n \geq 0.$$

Since

$$\frac{R_{n+1}(\xi)-R_{n+1}(-1)}{\xi+1} = \alpha_{n+1}\Big\{\sum_{\nu=0}^{n} b_{n+1,\nu}(\xi-1)^\nu(\xi+1)^{n-\nu} + (n+2)\sum_{\nu=0}^{n}(-2)^{n-\nu}(\xi-1)^\nu\Big\},$$

we have

$$
(3.45)\qquad \begin{aligned} R_n^{(1)}(-1) = \alpha_{n+1}\Big\{ & \sum_{\nu=0}^{n} b_{n+1,\nu} < w_0, (\xi-1)^{\nu}(\xi+1)^{n-\nu} > \\ & + (n+2)\sum_{\nu=0}^{n}(-2)^{n-\nu} < w_0, (\xi-1)^{\nu} > \Big\}, \quad n \geq 0. \end{aligned}
$$

where

$$
\alpha_n = n!\frac{\Gamma(n+\beta+2)}{\Gamma(2n+\beta+2)}, \quad b_{n,\nu} = \binom{n+1}{\nu}\binom{n+\beta}{n-\nu}.
$$

We need the following lemma

LEMMA 3.3. *Denoting*

$$
(3.46)\qquad \mathcal{M}_n^{\sigma}(\alpha,\beta) := < \omega, (x+\sigma)^n >, \quad \sigma = \pm 1, \quad n \geq 0,
$$

where $\omega := \mathcal{J}(\alpha,\beta)$ *is the Jacobi form, we have*

$$
(3.47)\qquad \mathcal{M}_n^{-1}(\alpha,\beta) = (-2)^n \frac{\Gamma(\beta+1+n)}{\Gamma(\beta+1)}\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+\beta+2+n)}, \quad n \geq 0
$$

$$
(3.48)\qquad \mathcal{M}_n^{+1}(\alpha,\beta) = 2^n \frac{\Gamma(\alpha+1+n)}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+\beta+2+n)}, \quad n \geq 0.
$$

Moreover

$$
(3.49)\qquad \begin{aligned} < \omega, (x-1)^q(x+1)^p > &= \mathcal{M}_q^{-1}(\alpha,\beta)\mathcal{M}_p^{+1}(\alpha,\beta+q) \\ &= \mathcal{M}_q^{-1}(\alpha+p,\beta)\mathcal{M}_p^{+1}(\alpha,\beta), \quad p,q \geq 0. \end{aligned}
$$

Proof. The form ω fulfils the following equation [9]

$$
\big((x^2-1)\omega\big)' + \big\{-(\alpha+\beta+2)x + \alpha - \beta\big\}\omega = 0.
$$

It easily follows

$$
\mathcal{M}_{n+1}^{\sigma}(\alpha,\beta) = \sigma\frac{2n+2+\alpha+\beta+\sigma(\alpha-\beta)}{n+\alpha+\beta+2}\mathcal{M}_n^{\sigma}(\alpha,\beta), \quad n \geq 0,
$$

with $\mathcal{M}_0^{\sigma}(\alpha,\beta) = 1$. Whence (3.47) and (3.48). Next, the form $u = (x-1)^q\omega$ fulfils

$$
\big((x^2-1)u\big)' + \big\{-(\alpha+\beta+2+q)x + \alpha - \beta - q\big\}u = 0.
$$

Thus $u = \lambda\mathcal{J}(\alpha,\beta+q)$ with $\lambda = \mathcal{M}_q^{-1}(\alpha,\beta)$. Likewise $\mathcal{M}_p^{+1}(\alpha,\beta)\mathcal{J}(\alpha+p,\beta) = (x+1)^p\mathcal{J}(\alpha,\beta)$. We deduce

$$
\begin{aligned} < \omega, (x-1)^q(x+1)^p > &=< (x-1)^q\omega, (x+1)^p >= \mathcal{M}_q^{-1}(\alpha,\beta)\mathcal{M}_p^{+1}(\alpha,\beta+q) \\ &=< (x+1)^p\omega, (x-1)^q >= \mathcal{M}_p^{+1}(\alpha,\beta)\mathcal{M}_q^{-1}(\alpha+p,\beta). \quad \square \end{aligned}
$$

COROLLARY. *For* $0 \leq \nu \leq n,\ \ n \geq 0,$

(3.50) $$< w_0, (x-1)^n >= \mathcal{M}_n^{-1}(1,\beta) = (-2)^n \frac{(\beta+1)(\beta+2)}{(n+\beta+1)(n+\beta+2)}, \quad n \geq 0$$

(3.51) $$< w_0, (x-1)^\nu (x+1)^{n-\nu} >= (\beta+1)(\beta+2)(-1)^\nu 2^n \Gamma(n-\nu+2)\frac{\Gamma(\beta+1+\nu)}{\Gamma(\beta+3+n)}, \quad 0 \leq \nu \leq n,\ n \geq 0.$$

Consequently from (3.50)–(3.51), for (3.45) we obtain

$$R_n^{(1)}(-1) = \alpha_{n+1}\{\Sigma_n^1 + (n+2)\Sigma_n^2\}, \quad n \geq 0,$$

where

$$\begin{aligned}\Sigma_n^1 &= (\beta+1)(\beta+2)\frac{2^n}{\Gamma(n+\beta+3)}\sum_{\nu=0}^{n}(-1)^\nu b_{n+1,\nu}\Gamma(n-\nu+2)\Gamma(\nu+\beta+1)\\ &= (\beta+1)(\beta+2)\frac{(-2)^n(n+2)!}{n+\beta+2}\sum_{\nu=0}^{n}\frac{(-1)^\nu}{\nu!(n+2-\nu)!}\\ &= (\beta+1)(\beta+2)\frac{(-2)^n(n+1)}{n+\beta+2}, \quad n \geq 0, \quad \text{on account of (3.30), (3.31).}\end{aligned}$$

$$\begin{aligned}\Sigma_n^2 &= \sum_{\nu=0}^{n}(-2)^{n-\nu}(-2)^\nu\frac{(\beta+1)(\beta+2)}{(\nu+\beta+1)(\nu+\beta+2)}\\ &= (\beta+1)(\beta+2)(-2)^n\sum_{\nu=0}^{n}\left(\frac{1}{\nu+\beta+1}-\frac{1}{\nu+\beta+2}\right)\\ &= (\beta+2)\frac{(-2)^n(n+1)}{n+\beta+2}, \quad n \geq 0.\end{aligned}$$

Whence

(3.52) $$R_n^{(1)}(-1) = (\beta+2)(-2)^n(n+1)(n+1)!\frac{\Gamma(n+\beta+4)}{(n+\beta+2)\Gamma(2n+\beta+4)}, \quad n \geq 0.$$

Consequently, taking into account of (3.44) and (3.52), for (3.43) we have

(3.53)
$$\varpi_n = 2\frac{(n+2)(n+\beta+2)}{(2n+\beta+3)(2n+\beta+2)}\frac{1-\vartheta\chi_{n+1}}{1-\vartheta\chi_n} \text{ with } \chi_n = \frac{1}{2}(\beta+2)\frac{n(n+\beta+2)}{(n+1)(n+\beta+1)}, \ n \geq 0$$
$$\vartheta \neq (\chi_n)^{-1}, \quad n \geq 1.$$

This last condition is equivalent to the regularity condition $\vartheta \neq -R_n(-1)\left(R_{n-1}^{(1)}(-1)\right)^{-1}$, $n \geq 1$. With $\varepsilon := \frac{1}{2}(\beta+2)\vartheta$, we easily obtain

(3.54)
$$\frac{1-\vartheta\chi_{n+1}}{1-\vartheta\chi_n} = \frac{(n+1)(n+\beta+1)}{(n+2)(n+\beta+2)}\frac{(1-\varepsilon)n^2+(1-\varepsilon)(\beta+4)n+(1-\varepsilon)(\beta+3)+\beta+1}{(1-\varepsilon)n^2+(1-\varepsilon)(\beta+2)n+\beta+1}, \quad n \geq 0.$$

The case $\varepsilon = 1$ does not arise, since (3.40) implies $\xi_1 + 1 = 0$ which is in contradiction with the assumptions. Indeed, since $\vartheta = \beta_0 + 1$ from (3.41), the assumption $\varepsilon = 1$

implies $2 = -d_0(\beta_0+1)\big(\ \beta = -(d_0+2)\big)$ and writing (3.40) as $(d_0+1)(\xi_1 - c) = \xi_2 - c + d_0(\beta_0 - c)$, we obtain with $c=-1$, $\xi_2 = +1$: $(d_0+1)(\xi_1+1) = 2-2 = 0$. Putting

$$X^2 + (\beta+2)X + (1-\varepsilon)^{-1}(\beta+1) = (X+\sigma_1)(X+\sigma_2),$$

we have

$$\beta = \sigma_1+\sigma_2-2, \quad \varepsilon = \frac{\sigma_1\sigma_2+1-\sigma_1-\sigma_2}{\sigma_1\sigma_2}.$$

Consequently

$$X^2 + (\beta+4)X + \beta + 3 + (1-\varepsilon)^{-1}(\beta+1) = (X+\sigma_1+1)(X+\sigma_2+1).$$

It follows

$$\varpi_n = 2\frac{(n+1)(n+\sigma_1+\sigma_2-1)(n+\sigma_1+1)(n+\sigma_2+1)}{(2n+\sigma_1+\sigma_2)(2n+\sigma_1+\sigma_2+1)(n+\sigma_1)(n+\sigma_2)}, \quad n \geq 0. \tag{3.55}$$

A tedious but straightforward calculus based on (3.42), (3.35) where $c=-1$ and ϖ_n is given by (3.55), leads to

$$\beta_0 = \frac{2(1+\sigma_1\sigma_2) - (\sigma_1+\sigma_2)(2+\sigma_1\sigma_2)}{\sigma_1\sigma_2(\sigma_1+\sigma_2)}, \tag{3.56}$$

(3.57)

$$\begin{cases} \beta_{n+1} = \dfrac{(4-\beta^2)M_4(n) - 2\varepsilon\sigma_1\sigma_2N_2(n)}{(2n+\sigma_1+\sigma_2)(2n+\sigma_1+\sigma_2+2)(n+\sigma_1+1)(n+\sigma_2+1)(n+\sigma_1)(n+\sigma_2)}, \\ \gamma_{n+1} = 4\dfrac{(n+1)^2(n+\sigma_1+\sigma_2-1)^2(n+\sigma_1-1)(n+\sigma_1+1)(n+\sigma_2-1)(n+\sigma_2+1)}{(2n+\sigma_1+\sigma_2-1)(2n+\sigma_1+\sigma_2)^2(2n+\sigma_1+\sigma_2+1)(n+\sigma_1)^2(n+\sigma_2)^2}, \end{cases} n \geq 0,$$

where

$$\begin{cases} M_4(n) = (n+\sigma_1)(n+\sigma_2)(n+\sigma_1+1)(n+\sigma_2+1), \\ N_2(n) = 6n^2 + 6(\beta+3)n + (\beta+3)(\beta+4), \end{cases} n \geq 0.$$

We must have $\sigma_1, \sigma_2, \sigma_1+\sigma_2 \neq -n+1$, $n \geq 0$. Finally, from (3.41) we obtain

$$u_0 = (1-\varepsilon)\delta_{-1} + \varepsilon\mathcal{J}(0,\beta), \tag{3.58}$$

because $(x+1)\mathcal{J}(0,\beta) = 2(\beta+2)^{-1}\mathcal{J}(1,\beta)$, which is a special case of the general easily proved formula

$$(x+1)^p(x-1)^q\mathcal{J}(\alpha,\beta) = \mathcal{M}_p^{+1}(\alpha,\beta+q)\mathcal{M}_q^{-1}(\alpha,\beta)\mathcal{J}(\alpha+p,\beta+q),$$

where p, q are non negative integers.

REMARK. The choice $c = +1$, $\xi_2 = -1$ leads to the similar case $u_0 = \delta_1 + \vartheta(x-1)^{-1}w_0$ with $w_0 = \mathcal{J}(\alpha, 1)$.

4. Calculating the coefficients $\lambda_{n,\nu}$. An example. We are looking for the case $\mathbf{II}_{22}$ where $c=0$ and $s=2$. From (1.6), we must have

(4.1)

$$xB_n(x) = B_{n+1}^{[1]}(x) + \lambda_{n,n}B_n^{[1]}(x) + \lambda_{n,n-1}B_{n-1}^{[1]}(x) + \lambda_{n,n-2}B_{n-2}^{[1]}(x), \quad n \geq 2,$$

$$\lambda_{n,n-2} \neq 0, \quad n \geq 2.$$

$$xB_0(x) = B_1^{[1]}(x) + \lambda_{0,0},$$
$$xB_1(x) = B_2^{[1]}(x) + \lambda_{1,1}B_1^{[1]}(x) + \lambda_{1,0}. \tag{4.2}$$

PROPOSITION 4.1. *We have*

$$\lambda_{0,0} = 2 - \frac{1}{\alpha+2}, \tag{4.3}$$

$$\lambda_{1,1} = 5 - \frac{\alpha-1}{(\alpha+1)(\alpha+3)}, \quad \lambda_{1,0} = 4 - \frac{\alpha}{(\alpha+1)(\alpha+2)}. \tag{4.4}$$

$$\lambda_{n+2,n+2} = 3n + 8 + \frac{\alpha+1}{n+\alpha+4} - \frac{\alpha}{n+\alpha+2}, \quad n \geq 0, \tag{4.5}$$

$$\lambda_{n+2,n+1} = (n+2)(3n+7) + \frac{\alpha^2}{n+\alpha+2} - \frac{\alpha(\alpha+1)}{n+\alpha+3}, \quad n \geq 0, \tag{4.6}$$

$$\lambda_{n+2,n} = (n+1)(n+2)^2\frac{(n+\alpha+1)(n+\alpha+3)}{(n+\alpha+2)^2}, \quad n \geq 0. \tag{4.7}$$

Proof. First (4.3) and (4.4). From (2.17) and (2.16), we obtain $B_1^{[1]}(x) = x - \frac{1}{2}(\beta_0+\beta_1)$, therefore, from (4.2), we get $\lambda_{0,0} = \frac{1}{2}(\beta_0+\beta_1) = \dfrac{2\alpha+3}{\alpha+2}$, in accordance with (3.36). For (4.4), by virtue of (2.17), we have $B_2^{[1]}(x) = \frac{2}{3}(x-\beta_2)B_1^{[1]}(x) - \frac{1}{3}\gamma_2 + \frac{1}{3}B_2(x)$ and on using the second expression of (4.2), we get

$$\begin{aligned} xB_1(x) &= \Big\{(x-\beta_2) + \lambda_{1,1}\Big\}B_1^{[1]}(x) + \frac{1}{3}B_2(x) + \lambda_{1,0} - \frac{1}{3}\gamma_2, \\ &= x^2 + \Big\{\lambda_{1,1} - \frac{2}{3}(\beta_0+\beta_1+\beta_2)\Big\}x - \frac{1}{2}(\beta_0+\beta_1)\Big(\lambda_{1,1} - \frac{2}{3}\beta_2\Big) \\ &\quad + \frac{1}{3}\beta_0\beta_1 + \lambda_{1,0} - \frac{1}{3}(\gamma_1+\gamma_2), \end{aligned}$$

taking (2.16) into account. It follows

$$\lambda_{1,1} = \frac{2}{3}(\beta_1+\beta_2) - \frac{1}{3}\beta_0, \quad \lambda_{1,0} = \frac{1}{3}(\gamma_1+\gamma_2) - \frac{1}{6}(\beta_0^2 - 2\beta_1^2 + \beta_0\beta_1).$$

But, from (3.36), we have

$$\beta_0 = 1 - \frac{1}{\alpha+1}, \quad \beta_1 = 3 - \frac{\alpha}{(\alpha+1)(\alpha+2)}, \quad \beta_2 = 5 - \frac{\alpha}{(\alpha+2)(\alpha+3)},$$
$$\gamma_1 = \frac{\alpha(\alpha+2)}{(\alpha+1)^2}, \quad \gamma_2 = 4\frac{(\alpha+1)(\alpha+3)}{(\alpha+2)^2}.$$

Hence (4.4).

Now, let us transform $xB_{n+2}(x)$ to obtain (4.1) where n will be replaced by $n+2$. Here, we have $B_{n+1}(x) = R_{n+1}(x) + \varpi_n R_n(x)$ where $\{R_n\}_{n\geq 0}$ is the Laguerre sequence orthogonal with respect to $\mathcal{L}(1)$ and from (3.33), we have

$$\varpi_n = (n+1)\frac{n+\alpha+2}{n+\alpha+1}, \quad n \geq 0. \tag{4.8}$$

We get

$$\begin{aligned}xB_{n+2}(x) &= x\{R_{n+2}(x) + \varpi_{n+1}R_{n+1}(x)\}\\ &= R_{n+3}(x) + \zeta_{n+2}R_{n+2}(x) + \rho_{n+2}R_{n+1}(x)\\ &\quad + \varpi_{n+1}\{R_{n+2}(x) + \zeta_{n+1}R_{n+1}(x) + \rho_{n+1}R_n(x)\}\\ &= R_{n+3}(x) + \{\zeta_{n+2} + \varpi_{n+1}\}R_{n+2}(x)\\ &\quad + \{\rho_{n+2} + \zeta_{n+1}\varpi_{n+1}\}R_{n+1}(x) + \varpi_{n+1}\rho_{n+1}R_n(x), \quad n \geq 0,\end{aligned}$$

taking (3.34) into account, with

$$\zeta_n = 2(n+1), \quad \rho_{n+1} = (n+1)(n+2), \quad n \geq 0. \tag{4.9}$$

But the sequence $\{R_n\}_{n\geq 0}$ fulfils [2]

$$R_n(x) = R_n^{[1]}(x) + nR_{n-1}^{[1]}(x), \quad n \geq 0.$$

It follows

$$\begin{aligned}xB_{n+2}(x) = R_{n+3}^{[1]}(x) &+ \{n+3+\zeta_{n+2} + \varpi_{n+1}\}R_{n+2}^{[1]}(x)\\ &+ \Big\{(n+2)(\zeta_{n+2} + \varpi_{n+1}) + \rho_{n+2} + \zeta_{n+1}\varpi_{n+1}\Big\}R_{n+1}^{[1]}(x)\\ &+ \Big\{(n+1)(\rho_{n+2} + \zeta_{n+1}\varpi_{n+1}) + \varpi_{n+1}\rho_{n+1}\Big\}R_n^{[1]}(x)\\ &+ n\varpi_{n+1}\rho_{n+1}R_{n-1}^{[1]}(x), \quad n \geq 0.\end{aligned}$$

Since

$$B_n^{[1]}(x) = R_n^{[1]}(x) + \frac{n}{n+1}\varpi_n R_{n-1}^{[1]}(x), \quad n \geq 0, \tag{4.10}$$

from this with n replaced by $n+3$, we obtain

$$\begin{aligned}xB_{n+2}(x) =& B_{n+3}^{[1]}(x) + \lambda_{n+2,n+2}R_{n+2}^{[1]}(x)\\ &+ \Big\{(n+2)(\zeta_{n+2} + \varpi_{n+1}) + \rho_{n+2} + \zeta_{n+1}\varpi_{n+1}\Big\}R_{n+1}^{[1]}(x)\\ &+ \Big\{(n+1)(\rho_{n+2} + \zeta_{n+1}\varpi_{n+1}) + \varpi_{n+1}\rho_{n+1}\Big\}R_n^{[1]}(x)\\ &+ n\varpi_{n+1}\rho_{n+1}R_{n-1}^{[1]}(x), \quad n \geq 0,\end{aligned}$$

with

$$\lambda_{n+2,n+2} = n+3+\zeta_{n+2} + \varpi_{n+1} - \frac{n+3}{n+4}\varpi_{n+3}.$$

On account of (4.8), (4.9), we get (4.5).

Next, from (4.10) where n is replaced by $n+2$, we have

$$\begin{aligned}xB_{n+2}(x) =& B_{n+3}^{[1]}(x) + \lambda_{n+2,n+2}B_{n+2}^{[1]}(x) + \lambda_{n+2,n+1}R_{n+1}^{[1]}(x)\\ &+ \Big\{(n+1)(\rho_{n+2} + \zeta_{n+1}\varpi_{n+1}) + \varpi_{n+1}\rho_{n+1}\Big\}R_n^{[1]}(x)\\ &+ n\varpi_{n+1}\rho_{n+1}R_{n-1}^{[1]}(x), \quad n \geq 0,\end{aligned}$$

with

$$\lambda_{n+2,n+1} = (n+2)(\zeta_{n+2} + \varpi_{n+1}) + \rho_{n+2} + \zeta_{n+1}\varpi_{n+1} - \frac{n+2}{n+3}\varpi_{n+2}\lambda_{n+2,n+2}, \quad n \geq 0.$$

Taking (4.8), (4.9) and (4.5) into account, we have (4.6).
Further, we get

$$\begin{aligned} xB_{n+2}(x) =& B^{[1]}_{n+3}(x) + \lambda_{n+2,n+2}B^{[1]}_{n+2}(x) + \lambda_{n+2,n+1}B^{[1]}_{n+1}(x) \\ &+ \lambda_{n+2,n}R^{[1]}_{n}(x) + n\varpi_{n+1}\rho_{n+1}R^{[1]}_{n-1}(x), \end{aligned}$$

with

$$\lambda_{n+2,n} = (n+1)(\rho_{n+2} + \zeta_{n+1}\varpi_{n+1}) + \varpi_{n+1}\rho_{n+1} - \frac{n+1}{n+2}\varpi_{n+1}\lambda_{n+2,n+1}, \quad n \geq 0.$$

By virtue of (4.8), (4.9) and (4.6), we obtain (4.7). Finally (4.10) leads to

$$\begin{aligned} xB_{n+2}(x) =& B^{[1]}_{n+3}(x) + \lambda_{n+2,n+2}B^{[1]}_{n+2}(x) + \lambda_{n+2,n+1}B^{[1]}_{n+1}(x) \\ &+ \lambda_{n+2,n}B^{[1]}_{n}(x) + \left\{n\varpi_{n+1}\rho_{n+1} - \frac{n}{n+1}\varpi_n\lambda_{n+2,n}\right\}R^{[1]}_{n-1}(x). \end{aligned}$$

But

$$n\varpi_{n+1}\rho_{n+1} - \frac{n}{n+1}\varpi_n\lambda_{n+2,n} = 0, \quad n \geq 0.$$

This yields (4.1) where n is replaced by $n+2$. □

REMARK. The case $\mathbf{II}_{32}$ goes analogously but the calculations are more complicated.

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