A FAMILY OF KRALL POLYNOMIALS

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Abstract. We consider Koornwinder's generalized Jacobi polynomials $P_n^{(j,0;M_0,M_1)}(x)$ with weight function $w(x) = x^j + M_0\,\delta(x) + M_1\,\delta(1-x)$ on the interval $[0,1]$ for M_0, M_1 arbitrary masses and $j = 0, 1, 2, \ldots$. We show that the polynomials $P_n^{(j,0;M_0,M_1)}(x)$ are eigenfunctions of a differential operator of order $2j + 6$ with polynomial coefficients.

1. Introduction. Let $P_n(x)$ be orthogonal polynomials (OP) satisfying the three-term recurrence relation [1]

$$(1.1) \qquad P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x), \quad P_0 = 1, \ P_1 = x - b_0.$$

The Favard theorem [1] states that if $u_n \neq 0$, there exists a linear moment functional $\mathcal{L}$ such that

$$(1.2) \qquad \langle \mathcal{L}, P_n(x) P_m(x) \rangle = h_n\,\delta_{nm},$$

where $h_n = u_1 u_2 \ldots u_n$ are normalization constants. The moment functional $\mathcal{L}$ is defined by its moments

$$(1.3) \qquad \langle \mathcal{L}, x^n \rangle = c_n, \quad n = 0, 1, 2, \ldots$$

In what follows we will assume that OP $P_n(x)$ are non-degenerate, i.e. $u_n \neq 0$, $n = 1, 2, \ldots$. This means, in particular, that $h_n \neq 0, n = 0, 1, \ldots$.

The Krall polynomials [10], are OP satisfying the eigenvalue problem

$$(1.4) \qquad L P_n(x) = \lambda_n\,P_n(x),$$

where the N-th-order differential operator L is defined by

$$(1.5) \qquad L = \sum_{k=0}^{N} a_k(x) \partial_x^k,$$

with $a_k(x)$ polynomials such that $deg(a_k(x)) \leq k$. We use term "Krall polynomials" in a slightly more general sense than, say in [13]. Sometimes, the term "Bochner-Krall polynomials" is used instead [12].

It can be easily shown that N must be *even* and λ_n a polynomial in n of degree $\leq N$.

The OP satisfying (1.4) for $N = 2$ are the well known classical orthogonal polynomials [2]

Krall has found all possible OP satisfying 4-th order differential equations (i.e. $N = 4$). However for $N = 6$ the problem of classifying of all Krall OP is so far unsolved (and only examples are known).

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In all the explicit examples, one observes that the "non-classical" OP satisfying (1.4) differ from the classical ones (i.e. satisfying (1.4) with $N = 2$) only by the presence of one or two concentrated masses at the endpoints of orthogonality interval. That is, all known Krall polynomials connected with the Jacobi polynomials belong to special classes of Koornwinder polynomials [9] having the following weight function on the interval $[0, 1]$

$$(1.6) \qquad w(x) = x^a(1 - x)^b + M_0\, \delta(x) + M_1\, \delta(1 - x),$$

Analogously, Krall's polynomials connected with the Laguerre polynomials have weight function $w(x) = x^a \exp(-x) + M\delta(x), \quad j = 0, 1, 2, \ldots$. Koornwinder conjectured [9] that all OP associated to the weight function (1.6) should satisfy the differential equation (1.4) for some even N. However, it was shown in [21] that (when $M_0 M_1 \neq 0$) a necessary condition for this is that both a and b be simultaneously integer or half-integer numbers.

It was shawn in [6] that Koornwinder's polynomials with $a = b = j = 0, 1, 2, \ldots$ and $M_0 = M_1$ do satisfy differential equation (1.4) of order $N = 2j + 4$.

We proposed in [21] a new method to generate Krall's polynomials. This method allowed one to construct a new family of Krall's polynomials having weight function $w(x) = x^j(1 - x)^b + M\, \delta(x)$ with $b \geq -1/2$ an arbitrary parameter, $j = 0, 1, 2, \ldots$ and M an arbitrary mass.

In this paper we find another set of Krall polynomials that are a subclass of Koornwinder's polynomials. Namely, we show that the OP corresponding to the weight function (1.6) with $a = j = 0, 1, 2, \ldots$, $b = 0$ and with M_0, M_1 arbitrary, satisfy a differential equation (1.4) of order $N = 2j + 6$. The simplest member of this family corresponding to $j = 0$ and satisfying 6-th order differential equation was found by Krall [13].

In the next section we briefly describe the method proposed in [21] and additional observations on the necessary and sufficient conditions for OP to satisfy equation (1.4).

2. An algebraic approach to Krall polynomials. Without loss of generality we can put [21]

$$(2.1) \qquad a_0 = 0.$$

This means in particular that

$$(2.2) \qquad \lambda_0 = 0.$$

The polynomials $a_n(x)$ can be written in the form

$$(2.3) \qquad a_n(x) = \sum_{s=0}^{n} \alpha_{ns} x^s, \quad n = 0, 1, \ldots, N,$$

where α_{ns} are some coefficients. Clearly $deg(a_n(x)) \leq n$ and there exists at least one m $(0 \leq m \leq N)$ such that $deg(a_m(x)) = m$.

Consider the action of the operator L on the monomials x^n:

$$(2.4) \qquad L\{x^n\} = \sum_{k=0}^{n} A_{nk} x^k.$$

The coefficients A_{nk} are the matrix elements of the operator L in the monomial basis. These coefficients read

$$(2.5) \qquad A_{nk} = \sum_{j=0}^{N-n+k} \alpha_{j+n-k,j} n(n-1)\ldots(1+k-j).$$

It is useful to introduce new coefficients $A_n^{(s)} = A_{n,n-s}$. We have

$$(2.6) \qquad A_n^{(s)} = n(n-1)(n-2)\ldots(n-s+1)\pi_{N-s}(n),$$

where

$$(2.7) \qquad \pi_{N-s}(n) = \alpha_{s0} + \sum_{j=1}^{N-s} \alpha_{s+j,j}(n-s)(n-s-1)\ldots(n-s-j+1)$$

is a polynomial in n of degree not exceeding $N-s$.

The following statements are important in the theory of Krall polynomials [21].

PROPOSITION 2.1. *Assume that the differential operator L have the representation coefficents $A_n^{(s)}$ with the only restriction that $A_n^{(0)} \neq A_m^{(0)}$ if $n \neq m$. There then exists a unique system of polynomials $P_n(x), n = 0,1,2\ldots$ satisfying equation (1.4) with*

$$(2.8) \qquad \lambda_n = A_n^{(0)}.$$

Remark. In general, these polynomials $P_n(x)$ are not orthogonal.

PROPOSITION 2.2. *Assume that $A_n^{(s)}$ are arbitrary coefficients as in (2.6), with N an even positive integer and $\pi_{N-s}(n)$ arbitrary polynomials of degree not exceeding $N-s$. At least one of these polynomials has degree $N-s$. Assume also that $A_n^{(s)} = 0$, $s > N$. There then exists a unique N-th order differential operator L having $A_n^{(s)}$ as representation coefficients. (The order N of this operator coincides with the maximal degree of all coefficients $A_n^{(s)}$ considered as polynomials in n).*

The coefficients α_{ik} of the polynomials $a_i(x)$ in definition (1.5) of the differential operator L are expressed as [21]

$$(2.9) \qquad \alpha_{s+k,k} = \left.\frac{\Delta^k \pi_{N-s}(n)}{k!}\right|_{n=s},$$

where the difference operator Δ is defined as $\Delta F(n) = F(n+1) - F(n)$. Hence the differential operator L can be reconstructed from these coefficients using (1.5) and (2.3).

Let B_{nk} be the expansion coefficients of the polynomial $P_n(x)$

$$(2.10) \qquad P_n(x) = \sum_{k=0}^{n} B_{nk} x^k.$$

Substituting (2.10) and (2.4) into (1.4) we get

$$(2.11) \qquad \sum_{s=k}^{n} B_{ns} A_{sk} = \lambda_n B_{nk}, \quad k = n - N, n - N + 1, \ldots, n.$$

We thus have $N + 1$ equations (2.11) for the coefficients B_{nk} in terms of the known coefficients A_{nk}.

PROPOSITION 2.3. *The coefficients B_{nk} are rational functions of the argument* n.

The proof of this proposition can be found in [21] and will not be repeated here. The relation (2.11) can be presented in the algebraic (matrix) form

$$(2.12) \qquad \mathbf{B}\,\mathbf{A} = \boldsymbol{\Lambda}\,\mathbf{B},$$

where $\mathbf{A}$ and $\mathbf{B}$ are infinite-dimensional lower-triangular matrices with entries A_{ik} and B_{ik}, whereas $\boldsymbol{\Lambda}$ is diagonal matrix with entries $\boldsymbol{\Lambda}_{ik} = \lambda_i\,\delta_{ik}$.

For our purposes the inverse problem is important: assume that some orthogonal polynomials $P_n(x)$ are given. The coefficients B_{nk} are then known explicitly. We would like to find the matrix $\mathbf{A}$. From (2.12) we have, formally:

$$(2.13) \qquad \mathbf{A} = \mathbf{B}^{-1}\,\boldsymbol{\Lambda}\,\mathbf{B}.$$

Note that the inverse matrix $\mathbf{B}^{-1}$ does exist. Indeed, we have the expansion

$$(2.14) \qquad x^n = \sum_{k=0}^{n} \mathbf{B}_{nk}^{-1}\,P_k(x)$$

with unique coefficients $\mathbf{B}_{nk}^{-1}$. It is hence sufficient to choose the diagonal matrix $\boldsymbol{\Lambda}$, for the matrix $\mathbf{A}$ is then given by (2.13). Clearly, for arbitrary $\mathbf{B}$ and $\boldsymbol{\Lambda}$ one cannot expect that the elements A_{nk} of the matrix $\mathbf{A}$ obtained in this way, have the required properties (2.6). As we will prove possible in some examples, one can hope that the coefficients λ_n can be appropriately chosen so as to yield simple expressions for the coefficients A_{nk}. The algorithm for reconstructing $A_n^{(s)}$ works as follows [21]. It is convenient to rewrite (2.12) in the form

$$(2.15) \qquad \sum_{i=0}^{s} B_n^{(s-i)}\,A_{n-s+i}^{(i)} = \lambda_n B_n^{(s)}, \quad s = 0, 1, 2, \ldots$$

For $s = 0$ we get the relation $A_n^{(0)} = \lambda_n$, which is not a restriction but rather a definition of λ_n.

For $s = 1$ we have from (2.15)

$$(2.16) \qquad A_n^{(1)} = (\lambda_n - \lambda_{n-1})\,B_n^{(1)} = \Omega_n\,B_n^{(1)},$$

where

$$(2.17) \qquad \Omega_n = \lambda_n - \lambda_{n-1}.$$

From (2.16) we have a restriction for λ_n. Indeed, both λ_n and $A_n^{(1)}$ should be polynomials in n, while $B_n^{(1)}$ is a rational function:

$$(2.18) \qquad B_n^{(1)} = \frac{Q^{(1)}(n)}{R^{(1)}(n)},$$

where $Q^{(1)}(n)$ and $R^{(1)}(n)$ are some polynomials in n having no common zeroes.

It follows that necessarily

$$(2.19) \qquad \Omega_n = r(n) R^{(1)}(n),$$

where $r(n)$ is a polynomial in n. Assume that Ω_n is already known and reads

$$(2.20) \qquad \Omega_n = \sum_{k=0}^{M} \omega_k n^k,$$

where ω_k are some coefficients. we can then obtain ann explicit expression for λ_n

$$(2.21) \qquad \lambda_n = \sum_{k=0}^{M} \omega_k J_k(n),$$

where

$$(2.22) \qquad J_k(n) = \sum_{j=1}^{n} j^k.$$

From (2.19) and (2.21), we get

$$(2.23) \qquad deg(\lambda_n) = deg(r(n)) + deg(R^{(1)}(n)) + 1 \le N,$$

where N is order of the differential operator L.

For $A_n^{(1)}$, we have from (2.16) and (2.18)

$$(2.24) \qquad A_n^{(1)} = r(n)\, Q^{(1)}(n).$$

For $s = 2$ we get

$$(2.25) \qquad A_n^{(2)} = (\Omega_n + \Omega_{n-1})\, B_n^{(2)} - \Omega_{n-1} B_n^{(1)} B_{n-1}^{(1)}.$$

This process can be iterated: if $A_n^{(1)}, \ldots, A_n^{(s-1)}$ have already been found, $A_n^{(s)}$ is then uniquely determined from (2.15). However in order for the polynomials $P_n(x)$ to obey the eigenvalue equation (1.4), the coefficients $A_n^{(s)}$ should reads as in (2.6) or, equivalently,

$$(2.26) \qquad A_n^{(s)} = n(n-1)(n-2)\ldots(n-s+1)\pi_{N-s}(n), \quad s = 0, 1, \ldots, N,$$

where $\pi_k(n)$ is a polynomial of degree not exceeding k. Moreover, we necessarily have

$$(2.27) \qquad A_n^{(s)} \equiv 0, \quad \text{for} \quad s > N.$$

The following criterion takes place [21].

PROPOSITION 2.4. *Assume that the coefficients $A_n^{(s)}$ are constructed via the algorithm (2.24), (2.25), ... The conditions (2.26) and (2.27) prove necessary and sufficient for the polynomials $P_n(z)$ to obey the eigenvalue equation (1.4).*

There is, however, a more convenient criterion for OP to satisfy (1.4). This criterion will be obtained in the next section using technique of linear functionals.

3. Criterion in terms of moments. Let $P_n(x)$ be monic polynomials that are solutions of (1.4) for some operator L. These polynomials need not be orthogonal. Consider a space of linear functionals σ defined on the space of polynomials by their moments $\langle \sigma, x^n \rangle = c_n$.

The functionals $p(x)\,\sigma$ (where $p(x)$ is a polynomial) and $\partial_x\,\sigma$ are defined by [15]

$$(3.1) \qquad \langle p(x)\,\sigma, f(x) \rangle = \langle \sigma, p(x)\,f(x) \rangle$$

$$(3.2) \qquad \langle \partial_x\,\sigma, f(x) \rangle = -\langle \sigma, f'(x) \rangle,$$

where $f(x)$ is an arbitrary polynomial.

There exists a dual sequence of functionals [15] σ_n, $n = 0, 1, 2, \ldots$ such that

$$(3.3) \qquad \langle \sigma_n, P_m(x) \rangle = \delta_{nm}.$$

We can expand the functional $\sigma_n = \sum_{k=n}^{\infty} S_{kn}\rho_k$ in terms of the elementary functional basis ρ_n, where

$$(3.4) \qquad \langle \rho_n, x^m \rangle = \delta_{nm}.$$

From (3.3) and (2.10) we see that the moment matrix $\mathbf{S}$ is inverse of $\mathbf{B}$:

$$(3.5) \qquad S_{ik} = \mathbf{B}_{ik}^{-1}.$$

Define the conjugate operator L^* (acting on the space of linear functionals σ) by the relation

$$(3.6) \qquad \langle L^*\,\sigma, f(x) \rangle = \langle \sigma, L\,f(x) \rangle,$$

where $f(x)$ is any polynomial.

PROPOSITION 3.1. *If the polynomials $P_n(x)$ satisfy eigenvalue equation (1.4) then the dual functionals σ_n satisfy the eigenvalue equations*

$$(3.7) \qquad L^*\,\sigma_n = \lambda_n\,\sigma_n$$

Proof. We have from (3.6), (3.3) and (1.4) that

$$(3.8) \qquad \langle L^*\,\sigma_n, P_m(x) \rangle = \langle \sigma_n, L\,P_m(x) \rangle = \lambda_n\,\delta_{nm}.$$

But the relation (3.8) means that the set of functionals $L^*\,\sigma_n$ is dual with respect to the set $P_n(x)$, so we have (3.7).

Assume now that $P_n(x)$ are orthogonal polynomials. It is well known (see, e.g. [15]) that $P_n(x)$ are OP if and only if

$$(3.9) \qquad \sigma_n = h_n^{-1}\,P_n(x)\,\sigma_0,$$

where h_n are some constants. In fact, these constants coincide with the normalization constants: $h_n = \langle \sigma_0, P_n^2(x) \rangle$. In this case the functional σ_0 plays the role of orthogonality functional for the OP $P_n(x)$:

$$(3.10) \qquad \langle \sigma_0, P_n(x)P_m(x) \rangle = h_n\delta_{nm}.$$

Let

$$(3.11) \qquad c_n = \langle \sigma_0, x^n \rangle$$

be the moments corresponding to the orthogonality functional σ_0.

If the polynomials $P_n(x)$ are orthogonal, the relation (3.7) is rewritten as

$$(3.12) \qquad L^* \{P_n(x)\, \sigma_0\} = \lambda_n\, P_n(x)\, \sigma_0 = (L\, P_n(x))\, \sigma_0, \quad n = 0, 1, 2, \ldots$$

or, equivalently,

$$(3.13) \qquad L^* \{f(x)\, \sigma_0\} = L\{f(x)\}\, \sigma_0.$$

The relation (3.13) holds for arbitrary polynomial $f(x)$. This relation can be rewritten in the equivalent form

$$(3.14) \qquad \langle \sigma_0, f(x)L\{\psi(x)\}\rangle = \langle \sigma_0, \psi(x)\, L\{f(x)\}\rangle.$$

which holds for two arbitrary polynomials $f(x)$ and $\psi(x)$. Choosing $f(x) = x^n$, $\psi(x) = x^m$ and using (3.11) and (2.4) we rewrite (3.14) in the form

$$(3.15) \qquad I(n, m) = I(m, n),$$

where $I(n, m)$ is the sum

$$(3.16) \qquad I(n, m) = \sum_{i=0}^{n} A_{ni} c_{m+i}$$

The relation (3.15) should be valid for all values $m, n = 0, 1, 2, \ldots$. In matrix form relation (3.15) reads

$$(3.17) \qquad \mathbf{AC} = \mathbf{CA}^T,$$

where $\mathbf{A}^T$ is the transpose of $\mathbf{A}$, and $\mathbf{C}$ is the Hankel matrix with entries $\mathbf{C}_{ik} = c_{i+k}$.

Thus relation (3.15) is a necessary condition for the OP $P_n(x)$ to be Krall polynomials.

Conversely, let $P_n(x)$ be some OP with moments c_n. Assume that there exist coefficients A_{nk} satisfying relation (3.15). Introduce a formal operator L having A_{nk} as its representative coefficients. (This operator need not be a differential operator). Then (3.15) is rewritten as

$$(3.18) \qquad \langle \sigma_0, x^n\, L\{x^m\}\rangle = \langle \sigma_0, x^m\, L\{x^n\}\rangle.$$

Since n, m are arbitrary, (3.18) is equivalent to (3.14). Choosing $f(x) = P_n(x)$, $\psi(x) = P_m(x)$ we have

$$(3.19) \qquad \langle \sigma_0, P_n(x)\, L\, P_m(x)\rangle = \langle \sigma_0, P_m(x)\, L\, P_n(x)\rangle.$$

Obviously for any n we have

$$(3.20) \qquad L\, P_n(x) = \sum_{i=0}^{n} \xi_{ni} P_i(x)$$

with some coefficients ξ_{ni}. Substituting (3.20) into (3.19) we get

$$(3.21) \qquad \sum_{i=0}^{m} \xi_{mi}\langle \sigma_0, P_n(x)\, P_i(x)\rangle = \sum_{i=0}^{n} \xi_{ni}\langle \sigma_0, P_m(x)\, P_i(x)\rangle.$$

Assume that $n > m$. Then the lhs of (3.21) vanishes, whereas the rhs is reduced to one term for $i = m$. From the orthogonality relation (3.10) we thus have

$$\xi_{nm} \, h_m = 0, \quad m = 0, 1, \ldots, n - 1.$$

As $h_m \neq 0$, we have $\xi_{nm} = 0, \; m = 0, 1, \ldots, n - 1$ and hence

$$(3.22) \qquad\qquad\qquad L \, P_n(x) = \xi_{nn} P_n(x).$$

But condition (3.22) is equivalent to (1.4) with $\lambda_n = \xi_{nn}$.

We thus have the following result.

PROPOSITION 3.2. *Let c_n be the moments corresponding to the OP $P_n(x)$. Assume that there exist coefficients A_{nk} satisfying the relation (3.15). Then the polynomials $P_n(x)$ satisfy the eigenvalue equation (1.4), where L is an abstract operator acting on monomials x^n by (2.4). If, in addition, the coefficients A_{nk} satisfy conditions (2.6) then L is an N-th order differential operator.*

Note that this proposition is valid for any operator L having the property (2.4) (e.g. L can be q-difference operator [19]). We will call (3.15) the basic relation.

In what follows we will use the basic relation in order to verify that the coefficients A_{nk} derived inductively from the first steps of the algorithm (2.24), (2.25) ... do satisfy relation (3.15), and hence that the corresponding OP $P_n(x)$ satisfy the differential equation (1.4).

4. Koornwinder polynomials. The monic Jacobi polynomials are defined by the formula

$$(4.1) \qquad P_n^{(a,b)}(x) = \frac{(-1)^n (a+1)_n}{(a+b+n+1)_n} \, {}_2F_1 \left(\begin{matrix} -n, n+a+b+1 \\ a+1 \end{matrix} ; x \right).$$

In what follows we will assume that $a \geq -1/2$, $b \geq -1/2$. The polynomials $P_n^{(a,b)}(x)$ are orthogonal on $(0, 1)$

$$(4.2) \qquad \int_0^1 P_n^{(a,b)}(x) P_m^{(a,b)}(x) w(x) \, dx = h_n \, \delta_{nm}$$

with the normalized weight function

$$(4.3) \qquad w^{(a,b)}(x) = \frac{\Gamma(a+b+2)}{\Gamma(a+1)\Gamma(b+1)} x^a (1-x)^b$$

(we define the Jacobi polynomials on the shifted orthogonality interval with respect to the standard one [8]). The moments of the Jacobi polynomials are

$$(4.4) \qquad c_n = \int_0^1 w(x) \, x^n \, dx = \frac{(a+1)_n}{(a+b+2)_n}.$$

We need also the values

$$(4.5) \qquad P_n^{(a,b)}(0) = \frac{(-1)^n (a+1)_n}{(a+b+n+1)_n},$$

$$(4.6) \qquad Q_n^{(a,b)}(0) = (-1)^{n+1} \frac{(a+b+1) (b+1)_n \, n!}{a \, (a+b+1)_{2n}},$$

$$(4.7) \qquad P_n^{(a,b)}(1) = \frac{(b+1)_n}{(a+b+n+1)_n},$$

$$(4.8) \qquad Q_n^{(a,b)}(1) = \frac{(a+b+1) (a+1)_n \, n!}{b \, (a+b+1)_{2n}},$$

where $Q_n^{(a,b)}(z)$, the second-kind functions corresponding to the polynomials $P_n^{(a,b)}(x)$, are defined by the formula [18]

$$(4.9) \qquad Q_n^{(a,b)}(z) = \int_0^1 \frac{P_n^{(a,b)}(x)\, w(x)dx}{z-x}.$$

From (4.1) we find for the expansion coefficients $B_n^{(s)}$:

$$(4.10) \qquad B_n^{(s)} = \frac{(-n)_s\,(-a-n)_s}{s!\,(-a-b-2n)_s}.$$

Let us examine the polynomials

$$\tilde{P}_n^{(a,b)}(x) = \mathcal{G}(0,\beta_0)\mathcal{G}(1,\beta_1)P_n^{(a,b)}(x)$$

obtained from the Jacobi polynomials by the application of two Geronimus transforms $\mathcal{G}(0,\beta_0)$ and $\mathcal{G}(1,\beta_1)$ at the endpoints of orthogonality interval.

Recall [21] that the polynomials $\tilde{P}_n^{(a,b)}(x)$ have the expression

$$(4.11) \qquad \tilde{P}_n^{(a,b)}(x) = \kappa_n^{-1} \begin{vmatrix} P_n(x) & P_{n-1}(x) & P_{n-2}(x) \\ \phi_n^{(0)} & \phi_{n-1}^{(0)} & \phi_{n-2}^{(0)} \\ \phi_n^{(1)} & \phi_{n-1}^{(1)} & \phi_{n-2}^{(1)} \end{vmatrix},$$

where $\kappa_n = \phi_{n-1}^{(0)}\phi_{n-2}^{(1)} - \phi_{n-1}^{(1)}\phi_{n-2}^{(0)}$ and where $\phi_n^{(0,1)}$ are defined by

$$\phi_n^{(0)} = Q_n^{(a,b)}(0) + \beta_0 P_n^{(a,b)}(0)$$
$$\phi_n^{(1)} = Q_n^{(a,b)}(1) + \beta_1 P_n^{(a,b)}(1)$$

with β_0, β_1 arbitrary parameters.

The polynomials $\tilde{P}_n^{(a,b)}(x)$ are monic orthogonal polynomials with normalized weight function [21]

$$(4.12) \qquad \tilde{w}(x) = \mu\left(\frac{(a+b+1)(a+b)}{ab}\, w^{(a-1,b-1)}(x) - \beta_0\,\delta(x) + \beta_1\,\delta(x-1)\right),$$

where $\mu^{-1} = (a+b)(a+b+1)/(ab) + \beta_1 - \beta_0$ and $w^{(a,b)}(x)$ is given in (4.3).

Comparing (4.12) with (1.6), we arrive at the important relation

$$(4.13) \qquad \mathcal{G}(0,\beta_0)\mathcal{G}(1,\beta_1)P_n^{(a,b)}(x) = P_n^{(a-1,b-1;M_0,M_1)}(x),$$

where

$$M_0 = -\beta_0\,\frac{ab}{(a+b)(a+b+1)}$$
$$M_1 = \beta_1\,\frac{ab}{(a+b)(a+b+1)}.$$

Thus Koornwinder polynomials $P_n^{(a-1,b-1;M_0,M_1)}(x)$ can be viewed as doubly Geronimus transformed Jacobi polynomials.

The moments of the polynomials $\mathcal{G}(0,\beta_0)\mathcal{G}(1,\beta_1)P_n^{(a,b)}(x)$ are

$$(4.14) \qquad c_n = \mu\left(\frac{(a+b)(a+b+1)}{ab}\,\frac{(a)_n}{(a+b)_n} - \beta_0\,\delta_{n0} + \beta_1\right), \quad n = 0,1,2,\dots$$

As was shown in [21], the necessary condition for the polynomials $\mathcal{G}(0, \beta_0)\mathcal{G}(1, \beta_1)P_n^{(a,b)}(x)$ to satisfy the eigenvalue equation (1.4) is that both a, b are simultaneously integers or half-integers.

In what follows we restrict ouselves to the case $a = 1, 2, 3, \ldots,\ b = 1$. Assume that the coefficients $A_n^{(s)}$ corresponding to the polynomials $\mathcal{G}(0, \beta_0)\mathcal{G}(1, \beta_1)P_n^{(a,1)}(x)$ do exist. Then the expression (3.16) (where c_n have the expression (4.14)) takes the form

(4.15)

$$
I(n, m) = \mu \left((a+1)(a+2) \sum_{s=0}^{n} \frac{A_n^{(s)}}{a+n+m-s} + \beta_1 \sum_{s=0}^{n} A_n^{(s)} - \beta_0\, A_n^{(n)}\, \delta_{0m} \right).
$$

Note that the expansion coefficients $\tilde{B}_n^{(s)}$ for the polynomials $\mathcal{G}(0, \beta_0)\mathcal{G}(1, \beta_1)P_n^{(a,b)}(x)$ are easily found from (4.10) and (4.11):

(4.16) $\tilde{B}_n^{(0)} = 1,$

(4.17) $\tilde{B}_n^{(1)} = B_n^{(1)} - D_n^{(1)}$

(4.18) $\tilde{B}_n^{(s)} = B_n^{(s)} - D_n^{(1)} B_{n-1}^{(s-1)} + D_n^{(2)} B_{n-2}^{(s-2)}, \quad s = 2, 3, \ldots, n,$

where

$$
D_n^{(1)} = \frac{\phi_n^{(0)}\phi_{n-2}^{(1)} - \phi_n^{(1)}\phi_{n-2}^{(0)}}{\phi_{n-1}^{(0)}\phi_{n-2}^{(1)} - \phi_{n-1}^{(1)}\phi_{n-2}^{(0)}},
$$

$$
D_n^{(2)} = \frac{\phi_n^{(0)}\phi_{n-1}^{(1)} - \phi_n^{(1)}\phi_{n-1}^{(0)}}{\phi_{n-1}^{(0)}\phi_{n-2}^{(1)} - \phi_{n-1}^{(1)}\phi_{n-2}^{(0)}}.
$$

5. Reconstruction of the coefficients $A_n^{(s)}$. In this section we show that the coefficients $A_n^{(s)}$ satsfying relation (3.15) do exist for the polynomials $\mathcal{G}(0, \beta_0)\mathcal{G}(1, \beta_1)P_n^{(a,1)}(x)$, where $a = 1, 2, \ldots$. In order to reconstruct these coefficients we start from relation (2.24) and choose $r(n) = 1$, we then reconstruct $A_n^{(2)}$ using (2.25) etc. Repeating this process we can guess that $A_n^{(s)}$ has the expression

(5.1) $$A_n^{(s)} = W_n^{(s)} + \beta_1 X_n^{(s)} + \beta_0 Y_n^{(s)} + \beta_0\beta_1\, Z_n^{(s)}$$

where

(5.2) $W_n^{(s)} = -(-1)_s(a+2)!^2\, \dfrac{n(n+a-s)}{a}$

$X_n^{(s)} = -(-2)_s(a-1)!(a+2)! \times$

(5.3) $\dfrac{(-n)_s\,(n+1)_{1-s}(n+a-s)(n^2+(a-1)n+2s-s^2)}{2s!}$

(5.4) $Y_n^{(s)} = \dfrac{(a+2)(-1-a)_s(1-n)_s(n)_{a+1}^2}{s!(-a-n)_s}$

(5.5) $Z_n^{(s)} = (-s(s-1) + (a+1)(n-1)(n+a+1) + (a+1)(2-n)s) \times$

$\dfrac{(n)_{a+1}^2(-a-2)_s(1-n)_s}{(a+1)(a+2)s!(-a-n)_s}$

In the next section we show that the condition (3.15) is satisfied and hence that the coefficients $A_n^{(s)}$ given by (5.1) do correspond to a differential operator having the polynomials $\mathcal{G}(0, \beta_0)\mathcal{G}(1, \beta_1)P_n^{(a,1)}(x)$ as eigenfunctions.

6. Verification of the basic relation. We should verify that $I(n,m) = I(m,n)$ with $I(n,m)$ given by (4.15). Note that when $m = 0$ the basic relation reduces to

$$(6.1) \quad (a+1)(a+2)\sum_{s=0}^{n}\frac{A_n^{(s)}}{a+n-s} + \beta_1 \sum_{s=0}^{n} A_n^{(s)} - \beta_0 \, A_n^{(n)} = 0, \quad n = 0,1,2,\ldots$$

Introduce the sums

$$(6.2) \qquad\qquad S_1(n) = \sum_{s=0}^{n} A_n^{(s)}$$

and

$$(6.3) \qquad\qquad S_2(n,m) = \sum_{s=0}^{n} \frac{A_n^{(s)}}{a+n+m-s}$$

Then the basic relation reads

$$(6.4) \quad (a+1)(a+2)S_2(n,m) + \beta_1 \, S_1(n) = (a+1)(a+2)S_2(m,n) + \beta_1 \, S_1(m),$$
$$m > 0, \ n = 0,1,2,3,\ldots$$

$$(6.5) \quad (a+1)(a+2)S_2(n,0) + \beta_1 S_1(n) - \beta_0 A_n^{(n)} = 0, \ m = 0, \ n = 0,1,2,\ldots$$

The sum $S_1(n)$ can be calculated using the Chu-Vandermonde identity [2]

$$(6.6) \qquad\qquad {}_2F_1\left(\begin{matrix} -n, \alpha \\ \beta \end{matrix}; 1\right) = \frac{(\beta - \alpha)_n}{(\beta)_n}, \ n = 0,1,2,\ldots$$

The sum $S_2(n,m)$ for $m > 0$ can be calculated using the formula

$$(6.7) \quad {}_3F_2\left(\begin{matrix} -n, \alpha, \beta \\ \alpha + 1, \beta - k \end{matrix}; 1\right) \quad = \quad \begin{cases} \frac{n!(1+\alpha-\beta)_k}{(\alpha+1)_n \, (1-\beta)_k}, \ k = 0,1,\ldots,n \\ \frac{\alpha \, n!}{(1-\beta)_{n+1}}\left(\frac{(1+\alpha-\beta)_{n+1}}{(\alpha)_{n+1}} - 1\right), \ k = n+1 \end{cases}$$

Formula (6.7) can be easily derived using well known transformation formulas for the hypergeometric function ${}_3F_2(1)$ [2]: in our case, the lhs of (6.7) can be reduced to ${}_2F_1(1)$, which can be calculated from (6.6). For $m = 0$ the sum $S_2(n,0)$ is calculated using (6.6).

We thus have for $S_1(n)$

$$(6.8) \qquad\qquad S_1(n) = \beta_0 \, (a+2)! \, (n)_{a+1} - n(a+2)!^2/a$$

The sum $S_2(n,m)$ has a more complicated expression. For $m > 0$ we have

$$(6.9)$$
$$S_2(n,m) = \beta_0\beta_1 a!(n)_{a+1}\left(\frac{(m)_{a+1}(n^2 + m^2 + (3+a)mn - 3(m+n) + 1 - a)}{(n+m-2)_{a+3}} - 1\right)$$
$$+ \beta_1 \frac{(a-1)!(a+2)!n(n+a-1)(n^2 - m - 2a - 2n + 2an + 2mn + a^2 + am)}{(a+n+m)(a+m+n-1)(a+n+m-2)}$$
$$+ \beta_0 \frac{(a+2)!(n)_{a+1}(m)_{a+1}}{(n+m-1)_{a+2}} - \frac{nm(a+2)!^2}{a(n+m+a)(n+m+a-1)}$$

For $m = 0$ one has

$$(6.10) \quad S_2(n,0) = \beta_0 \beta_1 a! \left(-(n)_{a+1} + (a+1)!(\delta_{n1} - \delta_{n2}) \right)$$
$$+ \beta_1 n(a-1)!(a+2)! + \beta_0 a!(a+2)!\delta_{n1}$$

Finally we need the expression for $A_n^{(n)}$:

$$(6.11) \quad A_n^{(n)} = (a+2)! \left((a+2)!\delta_{n1} + \beta_1(a+1)!(a+2)!(\delta_{n1} - \delta_{n2}) \right).$$

Combining expressions (6.8), (6.9), (6.10) and (6.11) we see that basic relations (6.4), (6.5) are identically satisfied.

Moreover, the coefficients $A_n^{(s)}$ satisfy the properties of the proposition **2.2** with $N = 2a + 4$. We thus have

PROPOSITION 6.1. *The polynomials* $\mathcal{G}(0, \beta_0)\mathcal{G}(1, \beta_1)P_n^{(a,1)}(x)$ *with* $a = 1, 2, 3 \ldots$ *and* β_0, β_1 *arbitrary satisfy the eigenvalue equation (1.4) where* L *is a differential operator of order* $2a + 4$.

Using relation (4.13) we arrive at the

PROPOSITION 6.2. *Koornwinder's polynomials* $P_n^{(a,0;M_0,M_1)}(x)$ *with* $a = 0, 1, 2,$ *... and arbitrary masses* M_0, M_1 *inserted at the endpoints* $x = 0, 1$, *satisfy the differential equation (1.4) of order* $2a + 6$.

Note that for $a = 0$ we return to the example of Krall polynomials satisfying a 6-th order differential equation [13], [14]. For $a > 0$ our result is perhaps new.

The coefficients α_{ik} of the differential operator L can be reconstructed via (2.9). The expression for the polynomials $a_k(x)$ in (1.5) is rather complicated. However for $k = N = 2a + 4$ and $k = N - 1 = 2a + 3$ we have the simple expressions

$$(6.12) \qquad a_N(x) = \frac{\beta_0 \beta_1}{a+2} x^{a+2}(x-1)^{a+2}$$

$$(6.13) \qquad a_{N-1}(x) = \beta_0 \beta_1 x^{a+1}(x-1)^{a+1}(3(a+1)x - 2a - 1).$$

Note added in proof.

Results related to those presented here can now be found in a paper [7] that has appeared after the submission of this article.

It is shown in [7] that for positive weight functions of the type (1.6), a finite-order differential equation exists if and only if

(i) both masses M_0, M_1 are arbitrary positive and parameters a, b are nonnegative integers. In this case the order of the differential equation is $2a + 2b + 6$;

(ii) $M_1 = 0$ and M_0 is arbitrary positive. Parameter $b > -1$ is arbitrary and $a = 0, 1, \ldots$. In this case the order of the differntial equation is $2a + 4$.

(iii) $M_0 = 0$ and M_1 is arbitrary positive. Parameter $a > -1$ is arbitrary and $b = 0, 1, \ldots$. In this case the order of the differntial equation is $2b + 4$.

In all the above cases the differential equations were presented explicitly in [7].

We considered the case (ii) in [21] and in the present paper we consider a special case of (i) when $b = 0$ and $a = 0, 1, 2, \ldots$. The interest of our results lie in particular in the fact that we use a different method to construct the explicit operators. This method was proposed in [21] and was recently applied to investigate q-Krall polynomials [19].

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