METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 8, No. 1, pp. 33–96, March 2001

003

GALOISIAN OBSTRUCTIONS TO INTEGRABILITY OF HAMILTONIAN SYSTEMS*

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Abstract. An inconvenience of all the known galoisian formulations of Ziglin's non-integrability theory is the Fuchsian condition at the singular points of the variational equations. We avoid this restriction. Moreover we prove that a necessary condition for meromorphic complete integrability (in Liouville sense) is that the identity component of the Galois group of the variational equation (in the complex domain) must be abelian. We test the efficacy of these new approaches on some examples. We will give some non academic applications in two following papers.

1. Introduction. The aim of this paper is to investigate the connection between two different integrability concepts: the (complete) integrability of Hamiltonian systems and the solvability of the linear differential equations (in the sense of the differential Galois theory). This connection is given by the linearized equation along a particular solution of the Hamiltonian system.

Now we will explain the historical motivation of our work.

During the last years the search for non-integrability criteria for Hamiltonian systems based upon a study of the behaviour of the solutions in the complex domain has acquired more and more relevance.

In 1982 Ziglin ([74]) proved a non-integrability theorem using the constrains imposed on the monodromy group of the normal variational equations along some integral curve by the existence of some first integrals. This is a result about branching of solutions: the monodromy group express the ramification of the solutions of the normal variational equation in the complex domain.

We consider a *complex* analytic symplectic manifold M of dimension $2n$ and a holomorphic Hamiltonian system X_H defined over it. Let Γ be the Riemann surface corresponding to an integral curve $z = z(t)$ (which is not an equilibrium point) of the vector field X_H. Then we can write the variational equations (VE) along Γ,

$$\dot{\eta} = \frac{\partial X_H}{\partial x}(z(t))\eta.$$

Using the linear first integral $dH(z(t))$ of the VE it is possible to reduce this variational equation (i.e. to rule out one degree of freedom) and to obtain the so called normal variational equation (NVE) that, in some suitable coordinates, we can write

$$\dot{\xi} = JS(t)\xi,$$

where, as usual,

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

is the square matrix of the symplectic form. (Its dimension is $2(n-1)$.)

*Received March 1, 2000; revised April 6, 2001.
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In general if, including the Hamiltonian, there are k analytical first integrals independent over Γ and in involution, then, in a similar way, we can reduce the number of degrees of freedom of the VE by k. The resulting equation, which admits $n-k$ degrees of freedom, is also called the normal variational equation (NVE) ([5]). Then we have the following result ([74]).

THEOREM 1 (Ziglin). *Suppose that the Hamiltonian system admits $n-k$ additional analytical first integrals, independent over a neighborhood of Γ (but not necessarily on Γ itself) We assume moreover that the monodromy group of the NVE contain a non-resonant transformation g. Then any other element of the monodromy group of the NVE sends eigendirections of g into eigendirections of g.*

We recall that a linear transformation $g \in Sp(2n, \mathbf{C})$ (the monodromy group is contained in the symplectic group) is resonant if there exists integers $r_1, ..., r_n$ such that $\lambda_1^{r_1} \cdots \lambda_n^{r_n} = 1$ with $r_1 \cdots r_n \neq 0$ (where we denoted by λ_i, λ_i^{-1} the eigenvalues of g).

There are some historical antecedents of Ziglin's Theorem. Poincaré gave a non-integrability criterion, based on the monodromy matrix of the VE along a periodic (real) integral curve: if there are k first integrals of the Hamiltonian system, independent over the integral curve, then k characteristic exponents must be zero. And, if these first integrals are moreover in involution, then necessarily $2k$ characteristic exponents must be zero ([56], pg 192-198). Furthermore, in Poincaré's work we can also find the relation between the linear first integrals of the variational equation and the solutions of this differential equation ([56], pg. 168) In fact, Poincaré results are intimately related to the reduction process from the VE to the NVE.

In 1888 S. Kowalevski obtained a new case of integrability of the rigid body system with a fixed point, imposing the condition that the general solution is a meromorphic function of the (complex) time. In fact, as part of her method, she proved that (except for some particular solutions) the only cases in which the general solution is a meromorphic function of the time are the Euler, Lagrange and Kowalevski's cases ([34]). Lyapounov generalized the Kowalevski result and proved that (except for some particular solutions) the general solution is single-valued only in the above mentioned three cases. His method relies on the analysis of the variational equations along a known solution ([39, 37]).

In 1963 Arnold and Krylov analyzed sufficient conditions for the existence of a single valued (but not complex analytical!) first integral of a linear differential equation. And, under some conditions, they proved the uniform distribution of values of the monodromy group on the corresponding invariant. Their proof is based upon the properties of the closure of the monodromy group considered as contained in a linear Lie group ([2]). We remark that this is not so far from the fact that the Galois group of the linear differential equation is the Zariski adherence of the monodromy group (see below the sections devoted to the differential Galois theory).

Ziglin himself, in a second paper, applied his theorem to the rigid body and obtain that, except for the three above mentioned cases, this system is not completely integrable. He also studied the problem of the existence of an additional partial first integral and eventually included the Goryachev-Chaplygin case. Finally he applied his method to the Hénon-Heiles system and to a particular Yang-Mills field. For this last system, Ziglin proved the non-existence of a *local* meromorphic first integral independent of the Hamiltonian in any neighborhood of the hyperbolic equilibrium point ([75]). In the present paper we also obtain the local non-integrability of some

Hamiltonian systems in a neighborhood of an equilibrium point. But in our situation the equilibrium points can be degenerate (see below the section 6 devoted to applications).

In 1985 Ito applied the Ziglin Theorem to the non-integrability of a generalization of the Hénon-Heiles system ([27]). From this moment until today many papers appeared about this subject. We shall comment briefly some of them.

Yoshida published a series of papers about the application of Ziglin theorem to some homogeneous two degrees of freedom potentials with a invariant plane. For such potentials he can project the normal variational equation over the Riemann sphere and he get a *hypergeometric equation* ([71, 72]. Later Churchill and Rod interpreted geometrically Yoshida results as a reduction by discrete symmetries of the associated holomorphic connection ([12], see also [6, 14]). There are also some papers oriented towards the applications [24, 73, 26, 65]. In a forthcoming paper, using the main results of the present article, we will improve Yoshida's results [49].

The differential Galois approach of Ziglin Theory appeared for the first time, in an independent way, in [14] and [47]. The papers [51], [5], [15] and [52] followed. Two applications of the theory (developed in [14]) to non-academic examples are [35, 59]. A common limitation of these works is the restriction to fuchsian variational equations (their singularities must be regular singular). Here we overcome this difficulty. Our basic idea is very simple: we get rid of the monodromy group and we work directly with the differential Galois group. Another problem, inherent to Ziglin original approach, is the separation between two types of first integrals: the first integrals useful in order to make the reduction and the others. (Of course if we assume the involutivity and independence of all the integrals, then from a theoretical point of view this distinction is no longer relevant.) In fact if some integrals are independent over Γ, then the differential Galois theory itself allows us to clarify the process of reduction (cf. 4.3, 4.4 below).

The majority of non-integrability criteria known to date do not take any account of the involutivity hypothesis: only the independence of the first integrals is used. So, if one excepts a Poincaré's result quoted above, we exhibit more or less for the first time an obstruction to the complete integrability in Liouville sense (taking account not only of the *number* of *independent* first integrals, like in the works of Ziglin and his followers, but also of their *involutivity*).

We can express simply our guiding idea when we began to think to this problem: if the initial Hamiltonian system X_H is completely integrable (in Liouville sense classical in mechanics) then the VE must also be integrable (but in the different sense of the differential Galois Theory, that is the corresponding Picard-Vessiot extension must be a Liouvillian extension, or equivalently the identity component of the corresponding differential Galois group must be a *solvable* algebraic group). In fact we eventually obtained a more precise result: in the complete integrability case the identity component of the differential Galois group of the VE must be *abelian*. Our proof is based on an infinitesimal method: we analyze the structure of the Lie algebra of the Galois group. This approach is clearly allowed by the galoisian formulation of Ziglin's theory: the differential Galois group is an algebraic group, and therefore a Lie group. In Ziglin's initial formulation the (monodromy) group is not a Lie group and it is impossible to use an infinitesimal method. We remark that the differential Galois group contains the monodromy group and therefore the Zariski closure of this monodromy group. But, in the irregular (i.e. non fuchsian) case, the differential Galois group can be *larger* than this Zariski closure. We stress the fact that in any search for non-integrability criteria the larger is the differential Galois group the better is the situation.

We will apply the above non-integrability result to the family of two degrees of freedom Hamiltonian potentials

$$U(x_1,x_2)=\frac{1}{3}x_1^3+\frac{1}{2}(a+bx_1)x_2^2,\ a\in\ C^*,\ b\in\ C,$$
$$U(x_1,x_2)=\frac{1}{2}x_1^n+\frac{1}{2}(ax_1^{n-4}+bx_1^{n-3}+cx_1^{n-2})x_2^2,\ n\in\mathbf{N},\ n>3.$$

In two forthcoming papers [49, 50] we will apply our non-integrability result to various non academic classical situations (homogeneous potentials, N-body problems, cosmological model...), getting not only new simple proofs of known results but also many new results. The reader may check that in all these applications it is easy to conclude using a unified and systematic approach:

–1. Select a particular solution.

–2. Write the VE and afterwards the NVE.

–3. Check if the identity component of the differential Galois group of the NVE is abelian.

As we will see, the step 2 is easy (in section 4 we will give an algorithm for obtaining the NVE from the VE). Step 3 appears quite problematic in general, but in many particular cases, as it will be seen in our applications, efficient algorithms do exist. (The prototype is the Kovacic's algorithm for second order equations.) In all the applications that the authors know, the step 1 (which is shared by all the classical proofs of non integrability) is achieved by the existence of a completely integrable subsystem (typically, by the existence of an invariant plane). Of course this is in some sense unsatisfactory from a philosophical point of view: if a system is "as far as possible" from integrability, then each integral curve will be pathological and our method does not work. However in such cases the classical methods fail for the same reason.

The paper is organized as it follows.

In sections 2 and 3 we recall the basic tools that we need (meromorphic connections, symplectic connections, tensor constructions, elementary differential Galois theory, Stokes phenomena,...) Most are well known, however we need some complements that are detailed in three appendices. (These appendices are more abstract and perhaps technically more difficult than the main body of the paper.)

In section 4 we recall the construction of the VE and we describe the reduction process and the resulting NVE. This section contains an elementary but important new result that is essential in the applications: if the identity component of the differential Galois group of the VE is abelian then the identity component of the differential Galois group of the NVE is also abelian (see Proposition 6 (ii) below).

Our central results appear in section 5. The first is Theorem 6. It is a purely algebraic result whose proof relies on symplectic algebra (Poisson algebras) and musical isomorphisms. It is used as an essential tool in the proof of our main results: Theorems 7, 8, 9 (and their corollaries). These last theorems are variations (in different situations) of the central result already described: the non abelianity of the identity component of the differential Galois group of the VE (or the NVE) is a criterion for the non complete integrability (in Liouville sense).

Section 6 is devoted to some examples.

2. Linear Differential Equations and Connections.

2.1. Meromorphic Connections. Linear connections are the intrinsic version of systems of linear differential equations. Moreover with connections it is possible to work with fiber bundles which are not necessarily trivial. A good reference for this section is [66] (see also [17], [18], [29], [43]).

Let Γ be a connected Riemann surface. We denote by $\mathcal{O}_\Gamma$ its sheaf of holomorphic functions, by Ω_Γ its sheaf of holomorphic 1-forms (corresponding to the canonical bundle) and by $\mathcal{X}_\Gamma$ its sheaf of holomorphic vector fields. (We will identify vector fields with derivations on $\mathcal{O}_\Gamma$.) We have a structure of sheaf of Lie-algebras on $\mathcal{X}_\Gamma$. There exist clearly natural structures of $\mathcal{O}_\Gamma$-modules on respectively Ω_Γ and $\mathcal{X}_\Gamma$. There exists a natural map (contraction)

$$\Omega_\Gamma \otimes_{\mathcal{O}_\Gamma} \mathcal{X}_\Gamma \to \mathcal{O}_\Gamma.$$
$$\omega \otimes v \to < \omega, v >$$

Let V be a holomorphic vector bundle of rank m on Γ. We denote by $\mathcal{O}_V$ its sheaf of holomorphic sections. Then a *holomorphic connection* is by definition a map

$$\nabla : \mathcal{O}_V \to \Omega_\Gamma \otimes_{\mathcal{O}_\Gamma} \mathcal{O}_V,$$

satisfying the Leibniz rule

$$\nabla(v + w) = \nabla v + \nabla w$$
$$\nabla f v = df \otimes v + f \nabla v.$$

(Where v, w are holomorphic sections of the fiber bundle V and f is a holomorphic function.)

By definition a section v of the fiber bundle V is *horizontal* for the connection ∇ if $\nabla v = 0$.

If the connection ∇ is fixed, then to each holomorphic vector field X over Γ, we can associate the *covariant derivative* along X

$$\nabla_X : \mathcal{O}_V \to \mathcal{O}_V.$$
$$\nabla_X : v \to < \nabla v, X > .$$

It is clearly a $\mathbf{C}$-linear map. If we denote by $End_{\mathbf{C}}(\mathcal{O}_V)$ the sheaf of spaces of $\mathbf{C}$-linear endomorphisms of the sheaf of complex vector spaces $\mathcal{O}_V$, then we get a map

$$\nabla : \mathcal{X}_\Gamma \to End_{\mathbf{C}}(\mathcal{O}_V),$$
$$X \mapsto \nabla_X,$$

such that

$$\nabla_X(v + w) = \nabla_X v + \nabla_X w,$$
$$\nabla_X(fv) = X(f)v + f\nabla_X v, \quad f \in \mathcal{O}_\Gamma.$$

As, in general, in that what follows the field X will be fixed, we will write ∇ instead of ∇_X.

We are going to compute ∇ in local coordinates. Let X be a holomorphic vector field over an open subset U of the Riemann surface Γ. Restricting U if necessary, we can suppose that there exists a holomorphic local coordinate t over U such that

$$X = \frac{d}{dt}.$$

Let $e = \{e_1, ..., e_m\}$ be a holomorphic frame of U, i.e. the data of m holomorphic sections of V over U, such that $e_1(p), ..., e_m(p) \in V_p$ are linearly independent at every point $p \in U$. Then we can set

$$\nabla e_j = -\sum_{i=1}^{m} a_{ij} e_i,$$

being (a_{ij}) a square matrix of order m whose entries are holomorphic functions over U. We write $\nabla e = -Ae$.

The matrix $A = (a_{ij})$ is by definition "the" connection matrix and it determines completely the connection: if v is a holomorphic section over U, then we can write it in coordinates

$$v = \sum_{i=1}^{m} \xi_i e_i,$$

where the ξ_i's are holomorphic functions over U, and we have

$$\nabla v = \sum_{i=1}^{m} (\frac{d\xi_i}{dt} - \sum_{j=1}^{m} a_{ij}\xi_j) e_i,$$

i.e., the connection ∇ is represented in the local coordinate t and the frame e by the linear differential operator

$$\nabla := \nabla_{\frac{d}{dt}} = \frac{d}{dt} - A.$$

Hence, we can associate to the solutions $\xi \in \mathcal{O}_U^m$ of the linear differential system

$$\frac{d\xi_i}{dt} = \sum_{j=1}^{m} a_{ij}\xi_j, \quad i = 1, ..., m,$$

the horizontal sections v of the connection

$$\nabla v = 0.$$

More precisely the map

$$\xi \mapsto \sum_{i=1}^{m} \xi_i e_i$$

induces an isomorphism of m-dimensional complex vector spaces between the space of solutions and the space of horizontal sections.

In fact we are interested not only in differential equations (or systems) with *holomorphic* coefficients, but also in differential equations (or systems) with *meromorphic* coefficients, therefore we need to extend the above concept of holomorphic connection in order to deal with poles and consequently to introduce *meromorphic* connections. We shall follow the section 4 of [66] (a more detailed analysis in the context of free coherent sheaves can be found in [41])

Let $\overline{\Gamma}$ a connected Riemann surface and V a *holomorphic* vector bundle on $\overline{\Gamma}$. In practically all our applications the situation will be as follows. Let $\Gamma \subset \overline{\Gamma}$ be an open subset such that $S = \overline{\Gamma} - \Gamma$ is a discrete subset (the singular set). We will consider meromorphic sections of the bundle V, and in general we will limit ourselves to sections whose restriction to Γ is *holomorphic*. Then at any point $s \in S$ their components in coordinates with respect to a holomorphic local frame are meromorphic functions in a neighborhood U_s, which are holomorphic on $U_s - \{s\}$, with a pole at s. Using a local holomorphic coordinate t (vanishing at s) we can identify these functions with elements of the field $\mathbf{C}\{t\}[t^{-1}]$ (that is the field $\mathbf{C}\{t\}[t^{-1}]$ with the field k_s of germs at s of meromorphic functions).

We denote by $\mathcal{M}_{\overline{\Gamma}}$ the sheaf of meromorphic functions over $\overline{\Gamma}$, by $\mathcal{M}^1_{\overline{\Gamma}} = \mathcal{M}_{\overline{\Gamma}} \otimes_{\mathcal{O}_{\overline{\Gamma}}} \Omega_{\overline{\Gamma}}$ the sheaf of meromorphic 1-forms and by $\mathcal{L}_{\overline{\Gamma}} = \mathcal{M}_{\overline{\Gamma}} \otimes_{\mathcal{O}_{\overline{\Gamma}}} \mathcal{X}_{\overline{\Gamma}}$ its sheaf of meromorphic vector fields. We have a structure of a sheaf of Lie algebras on $\mathcal{L}_{\overline{\Gamma}}$. There exist clearly natural structures of sheaves of $\mathcal{M}_{\overline{\Gamma}}$-vector spaces on respectively $\mathcal{M}^1_{\overline{\Gamma}}$ and $\mathcal{L}_{\overline{\Gamma}}$. There exists a natural map (contraction)

$$\mathcal{M}^1_{\overline{\Gamma}} \otimes_{\mathcal{M}_{\overline{\Gamma}}} \mathcal{L}_{\overline{\Gamma}} \to \mathcal{M}_{\overline{\Gamma}}.$$
$$\mu \otimes v \to < \mu, v >$$

Let V be a holomorphic vector bundle of rank m on $\overline{\Gamma}$. We denote by $\mathcal{M}_V$ its sheaf of meromorphic sections. Then a *meromorphic connection* on V is by definition a map

$$\nabla : \mathcal{M}_V \to \mathcal{M}^1_{\overline{\Gamma}} \otimes_{M_{\overline{\Gamma}}} \mathcal{M}_V,$$

satisfying the Leibniz rule

$$\nabla(v + w) = \nabla v + \nabla w$$
$$\nabla f v = df \otimes v + f \nabla v,$$

where v, w are holomorphic sections the fiber bundle V and f is a meromorphic function.

If the meromorphic connection ∇ is fixed, then to each meromorphic vector field X over Γ, we can associate the covariant derivative along X

$$\nabla_X : \mathcal{M}_V \to \mathcal{M}_V.$$
$$\nabla_X : v \mapsto < \nabla v, X > .$$

It is clearly a $\mathbf{C}$-linear map. Then if we denote by $End_{\mathbf{C}}(\mathcal{M}_V)$ the sheaf of $\mathbf{C}$-linear endomorphisms of the sheaf $\mathcal{M}_V$, then we get a map

$$\nabla : \mathcal{L}_{\overline{\Gamma}} \to End_{\mathbf{C}}(\mathcal{M}_V),$$
$$X \mapsto \nabla_X,$$

such that

$$\nabla_X(v + w) = \nabla_X v + \nabla_X w,$$
$$\nabla_X(fv) = X(f)v + f\nabla_X v, \quad f \in \mathcal{M}_\Gamma.$$

Let ∇ be a meromorphic connection over $\overline{\Gamma}$. We will say that it is *holomorphic* at a point $p \in \overline{\Gamma}$ if, for every germ at p of a *holomorphic* vector field X the space of germs at p of *holomorphic* sections of the fiber bundle V is invariant under the covariant derivative ∇X. Later we will consider connections which are meromorphic on $\overline{\Gamma}$ and holomorphic on Γ. They can have poles on the singular set S.

If we want to compute in local coordinates in a neighborhood of a singular point $s \in S$, then we choose a holomorphic coordinate t at s (vanishing at s) and we write our given vector field $X = f(t)\frac{d}{dt}$, where $f \in k_s$ (in general we cannot write X as $\frac{d}{dt}$, because the vector field X may vanish or admit a pole at the point s, as we will see later: cf.section 3). Then using a holomorphic frame e of V as above, we get a differential system

$$\nabla = f(t)\frac{d}{dt} - A(t).$$

We can introduce the meromorphically equivalent differential system

$$\frac{d}{dt} - B(t),$$

where $B = f^{-1}A$ is a *meromorphic* matrix over U.

We denote the field of (global) meromorphic functions over $\overline{\Gamma}$ by $k_{\overline{\Gamma}}$. It is important to notice that every holomorphic fiber bundle over a connected Riemann surface $\overline{\Gamma}$ is *meromorphically* trivialisable over $\overline{\Gamma}$ (i.e. globally, cf. Appendix A). Therefore its space of global meromorphic sections is isomorphic to some $k_{\overline{\Gamma}}^m$. We can in particular choose a non trivial meromorphic vector field X over $\overline{\Gamma}$. It will define a *derivation* δ over the field $k_{\overline{\Gamma}}$ and we will get a *differential field* $(k_{\overline{\Gamma}}, \delta)$. If V is a holomorphic vector bundle over $\overline{\Gamma}$ and if $\mathcal{M}_V \approx k_{\overline{\Gamma}}^m$ is its $k_{\overline{\Gamma}}$-vector space of meromorphic sections, then the covariant derivative ∇_X induces a $\mathbf{C}$-linear endomorphism of the space $\mathcal{M}_V$ and therefore it can be interpreted as a $\mathbf{C}$-linear endomorphism of the space $k_{\overline{\Gamma}}^m$. We can choose as a "local coordinate" t over $\overline{\Gamma}$ a non trivial global meromorphic function over $\overline{\Gamma}$ (it will be a true local coordinate–i.e. a local biholomorphism–but perhaps over a discrete subset). We can write $X = f(t)\frac{d}{dt}$, where $f \in k_{\overline{\Gamma}}$. Then we can choose a global meromorphic frame of V over $\overline{\Gamma}$, that is a set $e = \{e_1, \ldots, e_m\}$ of meromorphic sections of V inducing a true holomorphic frame over a non trivial open subset (necessarily dense). Finally, doing as above, we can interpret our connection as a global meromorphic differential system

$$\nabla = f(t)\frac{d}{dt} - A(t).$$

or equivalently

$$\frac{d}{dt} - B(t),$$

where $B = f^{-1}A$ is a global meromorphic matrix (whose entries belongs to $k_{\overline{\Gamma}}$).

In the preceding process it is in general necessary to introduce new poles. We will keep our notations, always denoting by S the new singular set and by Γ the new regular set (i.e. the set S can be bigger than the set of poles of our connection).

We will also need meromorphic connections on *meromorphic* bundles over a connected Riemann surface $\overline{\Gamma}'$. It is easy to adapt the preceding definitions using Appendix A. We leave the details to the reader. In our applications the more general situation will be the following: ∇ will be a meromorphic connection on a meromorphic bundle over $\overline{\Gamma}'$. By restriction we will get a meromorphic connection on a holomorphic bundle over an open dense subset $\overline{\Gamma} \subset \overline{\Gamma}'$, and by a new restriction a holomorphic connection on a holomorphic bundle over an open dense subset $\Gamma \subset \overline{\Gamma}$. The sets $\overline{\Gamma} - \Gamma$ and $\overline{\Gamma}' - \overline{\Gamma}$ will be discrete (and in general *finite*) subsets and they will correspond to the introduction of respectively *equilibrium points* and *points at infinity*.

2.2. Tensor Constructions. In this section we fix a connected Riemann surface $\overline{\Gamma}$, and a non trivial meromorphic vector field X over $\overline{\Gamma}$. We interpret this field as a derivation on the field of global meromorphic functions $k_{\overline{\Gamma}} = \mathcal{M}(\overline{\Gamma})$ over $\overline{\Gamma}$. As we explained in the preceding subsection, we can consider a meromorphic vector bundle as a vector space over $k_{\overline{\Gamma}}$.

From a given meromorphic connection ∇ defined on the vector bundle V, we can obtain an infinite number of induced meromorphic connections ([17, 18, 29, 43, 66]) by natural geometric processes. The idea is to extend naturally the connection to the tensor levels, imposing that the Leibniz rule is satisfied by the tensor products ($\nabla(u \otimes v) = \nabla u \otimes v + v \otimes \nabla v$) and that the action on a direct sum is the evident one (i.e. $\nabla(U \oplus V) = \nabla U \oplus \nabla V$). So, we can construct connections: ∇^*, $\otimes^k \nabla$, $\wedge^k \nabla$, $S^k \nabla$, acting respectively on the bundles V^*, $\otimes^k V$, $\wedge^k V$, $S^k V$. By definition $\otimes^0 V$ is the field of meromorphic functions and we endow it with the connection X (interpreted as a derivation on this field). With all these constructions we can built various direct sums and we can iterate the process... So, for example, $\wedge^3(\nabla^* \oplus S^2 \nabla)$ is an induced connection. If a subbundle is invariant by a connection, this connection is by definition a subconnection. We can also introduce subconnections and quotients in our machinery.

We observe the similarity of the above definitions with the derivations in differential geometry (Lie derivative, etc...). This is not a coincidence; in section 3 we will consider a connection as a Lie derivative.

In a natural way we can generalize the above in order to consider constructions using a family of given connections. For instance, let ∇_1, ∇_2 two meromorphic connections over the vector bundles V_1, V_2 respectively, then the tensor product $\nabla_1 \otimes \nabla_2$ is defined by the Leibniz rule as above, $\nabla_1 \otimes \nabla_2 (u \otimes v) = \nabla_1 u \otimes v + v \otimes \nabla_2 v$, where $v \in V_1$, $u \in V_2$. In an analogous way we define the direct sum of connections,...Finally we get the tensor category of the meromorphic connections over $\overline{\Gamma}$. The homomorphisms of this category are defined in the following way. A homomorphism ϕ between ∇_1, ∇_2 is a homomorphism of the underlying vector spaces (over the field $k_{\overline{\Gamma}}$) $\phi : V_1 \to V_2$, such that $\phi \nabla_1 = \nabla_2 \phi$. (For more details and formal definitions which are not needed here the interested reader can consult [18]).

Now, we will compute the connection matrices in some examples.

The dual connection ∇^* is defined from the Leibnitz rule by

$$X < \alpha, v > = < \nabla^* \alpha, v > + < \alpha, \nabla v >,$$

where $v \in V$, $\alpha \in V^*$, $<,>$ being the duality. If e and e^* are dual frames in V and V^*, respectively, then we have

$$< \nabla^* e^*, e > = \frac{d}{dt} < e^*, e > + < e^*, eA > = < e^* A^t, e >,$$

being A the connection matrix of ∇ in the frame e, i.e., $\nabla e = -Ae$. Hence, we have obtained nothing else that the adjoint differential equation: the adjoint differential equation of

$$\frac{d\xi}{dt} = A\xi$$

is by definition

$$\frac{d\eta}{dt} = -A^t \eta.$$

We observe that, in order that $\alpha = \sum_{i=1}^m \eta_i e_i^*$ be a linear first integral of

$$\nabla v = 0,$$

it is necessary and sufficient that

$$\nabla^* \alpha = 0.$$

This is a well known property of the adjoint. In a similar way, it is possible to prove that the horizontal sections of $S^k \nabla^*$ are the homogeneous polynomial first integrals of the linear equation defined by the initial connection on V.

It is usual to write ∇ instead of ∇^*, $\otimes^k \nabla$, etc..., if the vector bundles on which these connections act are clear enough and we will follow this convention.

The connection $\wedge^m \nabla$ (dim$V$=m) is defined by

$$\nabla(v_1 \wedge ... \wedge v_m) = \sum_{i=1}^m v_1 \wedge ... \wedge \nabla v_i \wedge ... \wedge v_m.$$

Then $\nabla(e_1 \wedge ... \wedge e_m) = \mathrm{tr}(A)\, e_1 \wedge ... \wedge e_m$. And we have the differential equation for the determinant of a fundamental matrix of (3) ($v_1 \wedge ... \wedge v_m = \det(v_1, ..., v_m) e_1 \wedge ... \wedge e_m$).

2.3. Symplectic Connections. Symplectic manifolds. We are mainly interested in the following particular vector bundles and connections. A holomorphic symplectic vector bundle over a complex manifold Σ is a vector bundle V such that there exists a holomorphic section Ω of $\wedge^2 V^*$ over Σ such that its restrictions to the fibers of V are non degenerate antisymmetric 2-forms. (Let $m = 2n$ be the rank of V.) It is equivalent to say that the vector bundle V admits the symplectic group $G = Sp(2n; \mathbf{C})$ as a *structure group* in the sense of Appendix C (we leave the details to the reader).

By the appendix A, a symplectic vector bundle V over a connected Riemann surface is symplectically meromorphically trivial. As above we denote by $k_{\overline{\Gamma}}$ the field of meromorphic functions over $\overline{\Gamma}$. We denote by $\mathcal{E}$ the $k_{\overline{\Gamma}}$-vector space of global meromorphic sections of V. The form Ω induces a $k_{\overline{\Gamma}}$-bilinear antisymetric map

$$\Omega; \mathcal{E} \otimes \mathcal{E} \to k_{\overline{\Gamma}}$$
$$(v, w) \mapsto \Omega(v, w).$$

If v, w are holomorphic sections of V in a neighborhood of a point $p \in \overline{\Gamma}$, then $\Omega(v, w)(p) = \Omega(v(p), w(p)) \in \mathbf{C}$. Consequently the $k_{\overline{\Gamma}}$-bilinear map

$$\Omega : \mathcal{E} \otimes \mathcal{E} \to k_{\overline{\Gamma}}$$

is *non degenerate.*

Finally, for many applications, we can identify the symplectic bundle V with the symplectic vector space $\mathcal{E}$ over the field $k_{\overline{\Gamma}}$. In this situation all the purely algebraic results on symplectic vector spaces over the numerical fields $\mathbf{R}$ or $\mathbf{C}$ remain also true here ([3]). In particular, there are symplectic bases (i.e., canonical frames given by global meromorphic sections), and with respect to a symplectic base Ω is represented by the canonical form

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

Furthermore, changes of symplectic bases are given by elements of the symplectic group $Sp(n, k_{\overline{\Gamma}}) \subset GL(2n, k_{\overline{\Gamma}})$.

By definition we will say that a (holomorphic or more generally meromorphic) connection ∇ over the *symplectic* bundle V (or (∇, V, Ω) in a more formal way) is *symplectic* if Ω is a horizontal section of $\wedge^2\nabla^*$, i.e., if it satisfies $\nabla\Omega = 0$ (for a related definition see [5]). Then, it is easy to see that, after a choice of coordinate, if we compute the connection matrix A of ∇ in a symplectic frame e, then it satisfies

$$A^t J + JA = 0$$

(To show this it is sufficient to remark that $0 = \nabla\Omega = \nabla(e^* \otimes Je^{*t})$). This condition is equivalent to the existence of a meromorphic symmetric matrix S such that $A = JS$, and the matrix A belongs to the Lie Algebra of the symplectic Lie Group with coefficients in the field $k_{\overline{\Gamma}}$. Then the equation

$$\nabla v = 0$$

is the intrinsic expression of the linear Hamiltonian system

$$\dot{\xi} = JS\xi,$$

being $\xi = (\xi_1, ..., \xi_{2n})^t$ the coordinates of v in the symplectic base.

Conversely if the matrix of the connection ∇ computed in a symplectic frame is symplectic, then $\nabla\Omega = 0$ and this connection is symplectic. Therefore our definition of a symplectic connection is equivalent to the definition of a connection with structure group $G = Sp(2n; \mathbf{C})$ that we give in Appendix A.

All the above constructions remain valid if we start with a local meromorphic connection on the vector space V over the field $\mathbf{C}\{t\}[t^{-1}]$ with the suitable dictionary: $\frac{d}{dt}$ instead X,...

We recall that a *complex* analytic symplectic manifold is a complex analytic manifold M, of complex dimension $2n$, endowed with a *closed regular* (i.e. everywhere non degenerate) holomorphic 2-form Ω. This form is a holomorphic section of the bundle $\Lambda^2 T^*M$, therefore it is equivalent to say that the tangent bundle TM admits a structure of holomorphic symplectic bundle.

To a holomorphic (resp. meromorphic) vector field X over M, we can associate a holomorphic 1-form α, using the formula

$$\alpha(Y) = \Omega(Y, X).$$

We get the musical isomorphisms of holomorphic fiber bundles

$$\flat : TM \to T^*M$$
$$\flat : X \mapsto \alpha$$

and

$$\sharp : T^*M \to TM.$$

If H is a given holomorphic function over M (the Hamiltonian), then we get a holomorphic vector field (the associated Hamiltonian vector field) $X_H = \sharp(dH)$.

In fact, for some applications (cf. the introduction of the *points at infinity* below), it is necessary to allow *meromorphic* Hamiltonians functions (then the corresponding Hamiltonian vector field will be of course also meromorphic). But it can also be necessary to allow degenerated points or poles for the canonical form Ω. Then we will consider a complex connected manifold M' of complex dimension $2n$, endowed with a *closed* meromorphic 2-form Ω. We will suppose that Ω is holomorphic and regular over a non void open subset $M \subset M'$. Then we can choose M such that $M' - M$ is an analytic (non necessarily regular) hypersurface $M_\infty \subset M'$. The form Ω is a meromorphic section of the bundle $\Lambda^2 T^*M$, therefore it is equivalent to say that the tangent bundle TM admits a structure of meromorphic symplectic bundle. The manifold $(M, \Omega_{\|M})$ is clearly a symplectic manifold. We will call M_∞ the hypersurface at infinity.

Example. Let $M' = P^1(\mathbf{C}) \times P^1(\mathbf{C})$, $M = \mathbf{C}^2$, $M_\infty = \{\infty\} \times P^1(\mathbf{C}) \cup P^1(\mathbf{C}) \times \{\infty\}$. We denote $(x, y) \in \mathbf{C}^2$ and by x', y' the coordinates at infinity over respectively the first and second factor $P^1(\mathbf{C})$ of M'. Then we set $\Omega = dx \wedge dy$ over M. It extends uniquely to a meromorphic form over M' and we have $\Omega = \frac{dx' \wedge dy'}{x'^2 y'^2}$ over a neighborhood of $\{\infty\} \times \{\infty\}$.

We go back to our general situation.

To a holomorphic (resp. meromorphic) vector field X over M', we can associate a *meromorphic* 1-form α, using the formula

$$\alpha(Y) = \Omega(Y, X).$$

By restriction to M we get the usual musical map $\flat$. This map is an isomorphism over M. Writing the application $\flat$ in coordinates in a neighbourhood of a point at infinity, we see that $\flat$ admits an inverse, this inverse is holomorphic over M but can have poles over M_∞.

We get the musical isomorphisms of *meromorphic* fiber bundles

$$\flat : TM \to T^*M$$
$$\flat : X \mapsto \alpha$$

and

$$\sharp : T^*M \to TM.$$

3. Picard-Vessiot Theory. The Galois differential theory for linear differential equations is named the Picard-Vessiot Theory. We shall recall the basic definitions and results, using some different approaches to this theory ([28, 29, 32, 43, 62]). As we will see below all of them will be useful in our paper.

3.1. Classical approach. A differential field is a field with a derivative (or derivation) $\partial = '$, i.e., an additive mapping that satisfies the Leibniz rule. Examples are $\mathcal{M}(\overline{\Gamma})$ (meromorphic functions over a connected Riemann surface $\overline{\Gamma}$) with a (non trivial) meromorphic tangent vector field X as derivation, in particular $\mathbf{C}(z) = \mathcal{M}(\mathbf{P^1})$ with $\frac{d}{dz}$ or $z\frac{d}{dz}$ as derivation, $\mathbf{C}\{x\}[x^{-1}]$ (convergent Laurent series), or $\mathbf{C}[[x]][x^{-1}]$ (formal Laurent series) with $x\frac{d}{dx}$ as derivation. We observe that there are some inclusions between the above differential fields.

We can define differential subfields, differential extensions in a direct way, imposing that the inclusions must commute with the derivations. Analogously a differential automorphism of K is an automorphism that commutes with the derivation. The field of constants of K is by definition the kernel of the derivation. In all the above examples it is the complex field $\mathbf{C}$. From now on we will suppose that it is always the case.

Let

$$\xi' = A\xi, \quad A \in Mat(m, K). \tag{3.1}$$

We shall proceed to associate to the differential system (3.1) a so called *Picard-Vessiot extension* of K. A Picard-Vessiot extension L of K associated to (3.1) is an extension L of K, such that

(a) It is differentiably generated over K by the entries $u_{ij \in L}$ of a fundamental matrix U of (3.1). (A fundamental matrix solution U is a matrix whose set of columns $u_1, ..., u_m \in L^m$ is a fundamental system of solutions of the equation (3.1), that is a system linearly independent over the field of constants $\mathbf{C}$.)

(b) The fields K and $L = K < U >$ have the *same* constants.

The existence and unicity (up to K-isomorphisms) of a Picard-Vessiot extension is due to Kolchin. In the analytical case: $K = \mathcal{M}(\overline{\Gamma})$ (where $\overline{\Gamma}$ is a connected Riemann surface), the existence of a Picard-Vessiot extension follows from the (analytic) Cauchy existence theorem for linear differential equations. More precisely we get a different Picard-Vessiot extension for each non singular point $p \in \overline{\Gamma}$. A homotopy class of continuous paths between two non singular points p, p' (avoiding singular points) induces an isomorphism between the corresponding Picard-Vessiot extensions.

As in the classical Galois theory we define the Galois group $G := Gal_K(L) := Gal(L/K)$ of the differential system (3.1) as the group of all the (differential) automorphism of L which leave fixed the elements of K. This group is isomorphic to an algebraic linear group over $\mathbf{C}$, i. e., to a subgroup of $GL(m, \mathbf{C})$ defined by some polynomial equations (in m^2 variables) over $\mathbf{C}$. Also we will say that the differential extension M/K is normal if any element in $M \setminus K$ is moved by a differential automorphism of M which leaves fixed the elements of K.

It is possible to extend the Galois correspondence between groups and extensions to this theory:

Theorem 2. *Let L/K be a Picard-Vessiot extension associated to a linear differential equation. Then there is a $1-1$ correspondence between the intermediate differential fields $K \subset M \subset L$ and the algebraic subgroups $H \subset G := Gal_K(L)$, such that $H = Gal_M(L)$ and $M = K^H$ (the subfield of L fixed by H). Furthermore, the normal extensions M/K correspond to the normal subgroups $H \subset Gal_K(L)$ and $G/H = Gal_K(M)$.*

As a corollary if we consider the algebraic closure $\overline{K}$ (of K in L), then $Gal_K(\overline{K}) = G/G^0$, where $G^0 = Gal_{\overline{K}}(L)$ is the identity component (using the Zariski topology) of the Galois group G (which corresponds to the purely transcendental part of the Picard-Vessiot extension).

Another consequence of the theorem is that if $\Lambda \subset \overline{\Gamma}$ is a connected Riemann surface contained and open in the connected Riemann surface $\overline{\Gamma}$ and L is a Picard-Vessiot extension of $\mathcal{M}(\overline{\Gamma})$, then $Gal_{\mathcal{M}(\Lambda)}(L) \subset Gal_{\mathcal{M}(\overline{\Gamma})}(L)$. Similarly the local Galois group $Gal_{\mathbf{C}\{x\}[x^{-1}]}(L) := Gal_{k_s}(L)$ at a (singular) point $s \in \overline{\Gamma} - \Gamma$ is a subgroup of the global Galois group $Gal_{\mathcal{M}(\Gamma)}(L)$ (as in the subsection 2.1, we identify the germs of meromorphic functions at a singular point s with Laurent series centered at this point).

We will say that a linear differential equation is (Picard-Vessiot) *solvable* if we can obtain a corresponding Picard-Vessiot extension and, hence, the general solution, by adjunction to K of integrals, exponential of integrals or algebraic functions of elements of K (the usual terminology is that the Picard-Vessiot extension is a Liouville one). Then, it can be proved that the equation is solvable if and only if the identity component G^0 is a *solvable group.* In particular, if the identity component is abelian, then the equation is solvable.

Furthermore, the relation between the monodromy and the Galois group (in the analytic case) is as follows. The monodromy group is contained in the Galois group and if the equation is of Fuchsian class (i.e., only has regular singular singularities), then the Galois group is dense in the monodromy group (Zariski topology). In this case we will say that the Galois group is topologically generated by the monodromy group. In the general situation, the second author found a generalization of the above and, for example, he proved that the Stokes matrices associated to an irregular singularity belong to the (local) Galois Group (see subsection 3.3).

As in the examples along this paper the irreducible equations that we shall meet will be of second order and symplectic ones, we recall here the classification of the algebraic subgroups of $SL(2,\mathbf{C})$. From [33], page 7 (or [28], page 32), it is possible to prove:

PROPOSITION 1. *Any algebraic subgroup G of $SL(2,\mathbf{C})$ is conjugate to one of the following types:*

1. *Finite,* $G^0=\{\mathbf{1}\}$, *where* $\mathbf{1}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$.
2. $G=G^0=\left\{\begin{pmatrix}1&0\\\mu&1\end{pmatrix},\ \mu\in\mathbf{C}\right\}$.
3. $G=\left\{\begin{pmatrix}\lambda&0\\\mu&\lambda^{-1}\end{pmatrix},\ \lambda \text{ is a root of unity}, \mu\in\mathbf{C}\right\}$,
$$G^0=\left\{\begin{pmatrix}1&0\\\mu&1\end{pmatrix},\ \mu\in\mathbf{C}\right\}.$$
4. $G=G^0=\left\{\begin{pmatrix}\lambda&0\\0&\lambda^{-1}\end{pmatrix},\ \lambda\in\mathbf{C}^*\right\}$.
5. $G=\left\{\begin{pmatrix}\lambda&0\\0&\lambda^{-1}\end{pmatrix},\begin{pmatrix}0&-\beta^{-1}\\\beta&0\end{pmatrix}\ \lambda,\beta\in\mathbf{C}^*\right\}$,
$$G^0=\left\{\begin{pmatrix}\lambda&0\\0&\lambda^{-1}\end{pmatrix},\ \lambda\in\mathbf{C}^*\right\}.$$
6. $G=G^0=\left\{\begin{pmatrix}\lambda&0\\\mu&\lambda^{-1}\end{pmatrix},\ \lambda\in\mathbf{C}^*,\mu\in\mathbf{C}\right\}$.
7. $G=G^0=SL(2,\mathbf{C})$.

We remark that the identity component G^0 is abelian in the cases (1)–(5) and is solvable in the cases (1)–(6).

An useful criterion for unimodularity is the following. The second order equation (with coefficients p and q in a differential field K)

$$\xi''+p\xi'+q\xi=0, \tag{3.2}$$

has a Galois group contained in $SL(2,\mathbf{C})$ if, and only if, $p = nd/d'$, $n \in \mathbf{Z}$, $d \in K$. To show this we note that for all σ in the Galois group, the Wronskian $W \in K$ if, and only if, $W = \sigma(W) = det(\sigma)W$, which is equivalent to $det(\sigma) = 1$. We get the result by the Abel formula $W' + pW = 0$.

We finish this subsection with some results about abelian extensions.

We recall that a connected commutative algebraic group (over the field C) is the direct product of its unipotent radical and a maximal torus. Therefore it has the form $\mathbf{G}_a^p \times \mathbf{G}_m^q$, being $\mathbf{G}_a \approx (C, +)$, $\mathbf{G}_m \approx (C^*, \dot{})$, the additive and multiplicative unidimensional groups. We have the following results [40].(Abelian Extensions, p. 83-84.)

Proposition 2. *Let K be a differential field with an algebraically closed field of constants C (characteristic zero). Let $K \subset L$ be a Picard-Vessiot extension. We suppose that $Gal(L/K) = H_1 \times H_2$ is a direct product of algebraic subgroups H_i. Let $L_i = L^{H_i} \subset L$ the corresponding subfields. Then $L_1 \otimes_K L_2$ is an integral domain whose fraction field is isomorphic to L.*

Proposition 3. *Let K be a differential field with an algebraically closed field of constants C (characteristic zero). Let $K \subset L$ be a Picard-Vessiot extension with connected commutative differential Galois group. Then there exists finite families a_i's, b_j's, $a_i, b_j \in L$, such that $a_i' \in K$ and $b_j'/b_j \in K$.*

3.2. The Tannakian approach. We present now the Galois theory from an intrinsic perspective, using connections ([17, 29, 43]). Let (V, ∇) be, as in section 2, a meromorphic connection over a fiber bundle of rank m. Then, we consider the horizontal sections $Sol\nabla := Sol_{p_0}\nabla$ of this connection at a fixed not singular point $p_0 \in \Gamma$ (we recall that they correspond to the solutions of a corresponding linear equation). By the general existence theory of linear differential equations $Sol\nabla$ is a vector space of dimension m over $\mathbf{C}$ (if we consider the solutions in a simply connected domain containing the point p_0). Then the mapping

$$(V, \nabla) \longrightarrow Sol\nabla$$

is called a fiber functor (it is a functor between two tensor categories: from the category of meromorphic connections with poles in a set S, with $p_0 \notin S$, to the category of complex vector spaces).

Now, as in section 2, from a given connection we construct the family of tensor constructions: (V, ∇), (V^*, ∇^*),... To this family we add the subconnections of the elements of the family. (We recall that a subconnection of a construction $(C(V), C(\nabla))$ is an object $(W, C(\nabla)_{|W})$, being W a subbundle of $C(V)$, which is invariant by $C(\nabla)$.) The next step is to consider the corresponding spaces of solutions (i.e. horizontal sections), when one applies the functor Sol, for all the elements of this extended family. We have $C(Sol\nabla) = Sol(C(\nabla))$. Then the Tannakian Galois group of the initial connection (V, ∇), $Gal\nabla$, is defined as the subgroup of $GL(Sol\nabla) \approx GL(m,\ C)$, that leaves invariant the spaces corresponding to all the subconnections of all the constructions $C(V)$ (that is all the elements of the extended family). We remark that $GL(Sol\nabla)$ acts on any construction by the usual pullback on the tensors levels. The key point is that the above Tannakian Galois group is isomorphic (as an algebraic group) to the ordinary Galois group G of the corresponding linear equation. This approach to the Picard-Vessiot theory is called the Tannakian point of view ([17, 29, 43]).

It is based upon Chevalley's interpretation of algebraic groups as subgroups of the linear group stabilizing a line in some construction ([8, 64]).

Example. Let (V, ∇, Ω) a symplectic connection with $rank\ V = 2n$ and X a holomorphic vector field over $\overline{\Gamma}$. We denote by ∂ the differential of the field of meromorphic functions $\mathcal{M}(\overline{\Gamma})$ corresponding to X. We consider the construction $(\mathcal{M}(\overline{\Gamma}) \oplus \wedge^2 V^*, \partial \oplus \wedge^2 \nabla^*)$. The line subbundle generated by $1+\Omega$ is invariant, because $\nabla\Omega = 0$ and, for $f \in \mathcal{M}_\Gamma(\overline{\Gamma})$, $\nabla(f(1+\Omega)) = X(f)(1+\Omega)$ (Ω is a horizontal section of the connection $\wedge^2\nabla^*$). Hence, the corresponding construction $\mathbf{C}(1+\Omega_0)$ that we get when we apply the fiber functor Sol is invariant by the Galois group. Therefore, the Galois group is contained in the symplectic group $Sp(Sol(V)) \approx Sp(n, \mathbf{C})$.

More generally, a linear algebraic group G' being given, if a connection ∇ admits G' as structure group, then its differential Galois group $G = Gal\ \nabla$ is (isomorphic to) a subgroup of G'. This result is due to Kolchin, who introduced the notion of G-extensions [32]. In Appendix C we will give a very simple Tannakian proof of this result.

3.3. Stokes Multipliers. The objective now is to state a theorem of Ramis (the density theorem) which relates the local Picard-Vessiot theory at an irregular singular point with the Stokes multipliers at this point ([58, 43, 46, 9]). For sake of simplicity, we will only explain the main concepts, in order to understand the theorem for the particular case of a second order differential equation (or equivalently of a system of dimension two): in this case the very delicate situation of multiplicity of levels of summation (multisummability) cannot happen. The reader can find a good introduction in [46] and the complete proof is in [44, 9].

We start with the local case and we will consider that the singular point is at infinity, $x_0 = \infty$. Furthermore, we will denote by $\hat{K} := \mathbf{C}[[x^{-1}]][x]$, $K := \mathbf{C}\{x^{-1}\}[x]$, respectively the field of formal and convergent Laurent series respectively. Then our objective is to calculate the Galois group of the equation

$$\frac{d}{dx}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = A \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad A \in Mat(2, K). \tag{3.3}$$

We also assume that the Newton polygon of the above equation has only one integer slope $k \in \mathbf{N}^*$. This is called the non ramified case and the general case can be reduced to this one by a simple argument using a ramification. By the Newton polygon of (3.3) we mean the Newton polygon of any equivalent second order scalar differential equation in $z = \frac{1}{x}$, i.e., the Newton polygon of a differential polynomial $P[D] = pD^2 + qD + r$ (the equation is $P[D]\xi = 0$), being $D = \frac{d}{dz}$, $p \in \mathbf{C}[z]$, $q, r \in \mathbf{C}\{z\}$, $p(0) = q(0) = 0$. By the Fuchs theory, it is not difficult to see that the point $z_0 = \infty$ is an irregular singular point if, and only if, in the Newton polygon there is a side with a non zero slope.

We know from the classical formal theory (Huhukara-Turrittin, Wasow [67]), that the system (3.3) admits a formal fundamental matrix solution $\hat{U}$ of (3.3), such that

$$\hat{U} = \hat{H} x^L e^Q, \quad L \in M(2, \mathbf{C}),$$

where $Q = diag(q_1, q_2)$, $q_1, q_2 \in \mathbf{C}[x]$, $LQ = QL$, and $\hat{H}$ is a formal matrix (with entries in $\hat{K}$).

Then the unique positive slope k of the Newton polygon of the system (3.3) is $k = degree(q_1 - q_2)$.

It is interesting to observe that if $q_1 = q_2 = 0$, then we are in the regular singular situation: the singular point is a regular singular one and the formal matrix $\hat{H}$ is convergent; moreover if H is its sum, then $U = Hx^L e^Q$ is an actual fundamental matrix solution. But, if q_1 or $q_2 \neq 0$, then the formal matrix $\hat{H}$ is in general divergent, and the relation between formal and actual solutions is more delicate to understand.

In order to state the Ramis density theorem we need some terminology: we need to define the *exponential torus*, the *formal monodromy* and the *Stokes multipliers*.

The exponential torus of (3.3) is an algebraic subgroup of the differential Galois group. It is defined (up to an isomorphism) as the differential Galois group of the Picard-Vessiot extension

$$\hat{K} < e^{q_1}, e^{q_2} > /\hat{K}.$$

We see that this group is the Galois group of the trivial equation (that we interpret as defined over $\hat{K}$):

$$\frac{d\xi_i}{dx} = \frac{dq_i}{dx}\xi_i, \ \ i = 1, 2.$$

This exponential torus is (isomorphic to) $\mathbf{C}^*$ or $(\mathbf{C}^*)^2$ if, respectively, the rank of the $\mathbf{Z}$-module $M_Q \subset \mathbf{C}[1/x]$ generated by $\{q_1, q_2\}$ is one or two. In the first case, the action of $\mathbf{C}^*$ is defined by

$$\lambda : e^{q_i} = e^{n_i s} \mapsto \lambda^{n_i} e^{n_i s}, \quad ,\lambda \in \mathbf{C}^*, \quad \mathbf{Z}s = M_Q,$$

and, in the second case by

$$\lambda_i : e^{q_i} \mapsto \lambda_i e^{q_i}, \quad i = 1, 2.$$

(By definition this action is constant on the coefficient field $\hat{K}$.)

The formal monodromy is the transformation $\hat{M} \in GL(2, \mathbf{C})$, such that

$$\hat{U} \mapsto \hat{U}\hat{M},$$

when we make formally the circuit

$$x \mapsto e^{2\pi i}x.$$

The formal monodromy and the exponential torus are clearly formal invariants. They are in the formal differential Galois group (ie, over $\hat{K}$) and generate topologically (in the Zariski sense) this group.

A sector at infinity is characterized by d, α, where α is the "vertex" angle and d the bisecting line, and its "radius". We will denote such a sector $S_d(\alpha)$.

To the system (3.3), we associate its singular directions. By definition, they are the maximal decay (half) lines for the functions $e^{q_1 - q_2}$ or $e^{q_2 - q_1}$. Following Stokes, it would be nice to call them "Stokes lines". Unfortunately many people call Stokes lines the boundaries of the sectors $S_d(\pi/k)$, where d is a singular direction. Here we will adopt this second definition.

We know from the classical asymptotic theory (Birkhoff, Wasow [67]) that, for any open angular sector Σ at ∞ of vertex angle $\leq \pi/k$, there exists an actual analytic matrix solution $U = Hx^L e^Q$ of the system (3.3) on Σ, such that the analytic matrix H admits the formal matrix $\hat{H}$ as an asymptotic expansion:

$$H \sim \hat{H}.$$

This is a version of the "fundamental existence theorem".

We will denote

$$U \sim \hat{U}.$$

The problem is that such a U is in general not unique and that the correspondence $\hat{U} \mapsto U$ is not very good from an algebro-differential viewpoint. It is possible to overcome this difficulty, using Ramis k-summability theory ([43], [44]).

For a given $k > 0$ and a given direction d we say that the formal series $\hat{f} = \sum_{n\geq 0} a_n x^{-n} \in \mathbf{C}[[1/x]]$ is k-summable in the direction d if there exists a germ at infinity of open sector Σ bisected by d, with vertex angle greater than π/k and a holomorphic function f on Σ, satisfying inequalities:

$$x^n \mid f(x) - \sum_{p=0}^{n-1} a_p x^{-p} \mid < C_W (n!)^{1/k} A_W^n,$$

for every $x \in W$, $n \in \mathbf{N}$, on every "proper" subsector $W \subset \Sigma$ and for some constants C_W, A_W.

The function f is then unique and is called the k-sum of $\hat{f}$ in the direction d. These definitions easily extend to Laurent series. The correspondence $\hat{f} \mapsto f$ is an injective homomorphism of differential algebras. The k-sum f admits $\hat{f}$ as an asymptotic expansion on Σ. For $\hat{f} \in K$, the k-sum is of course the usual sum. Also, we can extend the above definition to a matrix of formal series $\hat{H} = (\hat{h}_{ij})$, by imposing that each of its elements must be k-summable.

The following result is the key of the definition of the Stokes multipliers.

THEOREM 3. *([58]). Let (3.3) be a differential system as above. With notations as before, the matrix $\hat{H}$ is k-summable in every non singular direction d and if H denotes the k-sum of $\hat{H}$ in the direction d, then $U = Hx^L e^Q$ is an actual solution of (3.3) on a sector $\Sigma = S_d(\pi/k) := \{t : |x| > a,\ arg x \in (d - \pi/2k, d + \pi/2k)\}$ and we have $U \sim \hat{U}$ on Σ.*

Roughly speaking, the k-summability gives a "canonical" version of the fundamental existence theorem on sectors bisected by non singular directions. The correspondence $\hat{U} \mapsto U$ is good: they satisfy the same algebraic-differential relations over K.

It is clear that by an analytic extension it is possible to continue the analytic solution U, which is the k-sum of $\hat{U}$, over sectors $S_d(\alpha)$, with $\alpha > \pi/k$. The problem is that, in such a new sector, this solution is in general no longer asymptotic to $\hat{U}$. The lines that bound the sectors where the asymptotic relation (3.3) remains valid are called the *Stokes rays.*

When we move the direction of k-summation d between two singular lines, the k-sums U glue together by analytic continuation. But when d cross a singular direction d_s, there is in general a "jump" for U. This corresponds to the *Stokes phenomenon.* More precisely, let d_s be a singular direction and let ${d_s}^+ = d_s + \varepsilon$ (resp. ${d_s}^- = \alpha - \varepsilon$) where $\varepsilon > 0$ be neighboring non singular directions of α. The k-sums U^+ (resp. U^-) of the formal fundamental solution $\hat{U}$ in the directions ${d_s}^+$ (resp. ${d_s}^-$) glue together in actual holomorphic fundamental solutions U^+ (resp. U^-). We have $U^- = U^+ Sto_{d_s}$ where $Sto_{d_s} \in Gl(2;\mathbf{C})$. By definition Sto_{d_s} is the *Stokes matrix* (or *Stokes multiplier*) associated to the singular direction d_s. The K-isomorphism of differential fields

$$K < U^+ > \rightarrow K < U^- >$$

defined by Sto_{d_s} is a Galois isomorphism and therefore Sto_{d_s} belongs to the differential Galois group.

It is possible to see that these Stokes matrices Sto_{d_s} are *unipotent*, i.e., of the form

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix},$$

or

$$\begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}.$$

In particular, they belong to $SL(2,\ C)$.

In an analogous (but more delicate) way, we may describe the exponential torus, the formal monodromy and the Stokes matrices for a local system of differential equations of arbitrary dimension m

$$\frac{d\xi}{dx} = A\xi, \quad A \in Mat(m, K) \tag{3.4}$$

(see [44, 9]).

Then, we can state the Ramis density theorem.

THEOREM 4 ([58, 44, 9]). . *The Galois group of (3.4) is topologically generated by the exponential torus, the formal monodromy and the Stokes matrices.*

We note that among these topological generators the main source of non solvability are the Stokes multipliers. For example, it is not difficult to prove that the Zariski closure of the group generated by two matrices

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix},$$

where the complex numbers $\lambda,\ \mu$ are both different from zero, is $SL(2,\ C)$ ([11]).

3.4. Coverings and Differential Galois Groups. In some applications it is useful to replace the original differential equation over a compact connected Riemann surface by a new differential equation over the Riemann sphere $\mathbf{P^1}$ (i.e., with rational coefficients) by a change of the independent variable (i.e. the original equation is a pull-back of the new one). This new equation over $\mathbf{P^1}$ is called the algebraic form of the equation. In a more general way we will therefore consider the effect of a finite ramified covering on the Galois group of a differential equation. In Appendix B the following theorem is proved

THEOREM 5. *Let X be a connected Riemann surface. Let (X', f, X) be a* finite *ramified covering of X by a connected Riemann surface X'. Let ∇ be a meromorphic connection over X. We set $\nabla' = f^*\nabla$. Then we have a natural injective homomorphism*

$$Gal\ (\nabla') \to Gal\ (\nabla)$$

of differential Galois groups which induces an isomorphism between their Lie algebras.

We observe that, in terms of differential Galois groups, this theorem means that the identity component of the differential Galois group is invariant by the covering.

The algebraic version of the above theorem is due to N. Katz ([29]). This result is also proved in [5] (proposition 4.7) in the particular case of a fuchsian connection (see also [12, 14, 16, 6]). It is the mapping version for the so-called (in the cited references) method of reduction by discrete symmetries. Therefore this method is also valid in our more general setting. It is important to notice that, if one of the connections in the proposition is symplectic, then the identity components of the Galois groups of both connections are symplectic too.

Example. The algebraic form of the Lamé Equation is ([57, 70])

$$\frac{d^2\eta}{dx^2} + \frac{f'(x)}{2f(x)}\frac{d\eta}{dx} - \frac{Ax+B}{f(x)}\eta = 0, \tag{3.5}$$

where $f(x) = 4x^3 - g_2x - g_3$, where A, B, g_2 and g_3 are parameters such that the "discriminant" $27g_3^2 - g_2^3$ is non-zero. This equation is a Fuchsian differential equation with four singular points over the Riemann sphere (i.e. a Heun's equation).

With the well known change $x = \mathcal{P}(t)$, we get the Weierstrass form of the Lamé's equation

$$\frac{d^2\eta}{dt^2} - (A\mathcal{P}(t) + B)\eta = 0, \tag{3.6}$$

being $\mathcal{P}$ the elliptic Weierstrass function with invariants g_2, g_3. Classically the equation is written with the parameter n instead of A, being $A = n(n+1)$. This new equation is defined over a torus Π (a genus one Riemann surface) with only one singular point at the origin. Let $2w_1$, $2w_3$ the real and imaginary periods of the Weierstrass function $\mathcal{P}$ and $\mathbf{g}_1$, $\mathbf{g}_2$ their corresponding monodromies in the above equation. If $\mathbf{g}_*$ represents the monodromy around the singular point, then $\mathbf{g}_* = [\mathbf{g}_1, \mathbf{g}_2]$ ([70, 57]).

By the above theorem we see that the identity component of the Galois group is preserved by the covering $\Pi \to \mathbf{P}^1$, $t \mapsto x$.

In reference [57], chapter IX, the relation between the monodromy groups of the equations (3.5) and (3.6) is studied. From a modern point of view it is studied in [16].

3.5. Examples. In this section we illustrate Picard-Vessiot Theory with several examples. In the applications we need to know if the identity component of the associated differential Galois group is abelian, and we therefore emphasize this property.

Example 1. The hypergeometric (or Riemann) equation is the more general second order linear differential equation with three regular singular singularities. We can write it in one of its reduced forms as

$$\frac{d^2\xi}{dx^2} + \frac{c-(a+b+1)}{x(x-1)}\frac{d\xi}{dx} - \frac{ab}{x(x-1)}\xi = 0, \tag{3.7}$$

where a, b, c are complex parameters. The singular points are $0, 1, \infty$, being the exponent differences $\lambda = 1 - c$, $\nu = c - a - b$ and $\mu = b - a$ respectively.

In [30] Kimura gives the necessary and sufficient conditions on λ, ν and μ in order to have solvability for (3.7) (they must satisfy the Tables of Schwartz and Hukuhara-Ohasi). Except for one case in which one of the parameters is an arbitrary complex number and the others two are half-integers, the others values in these tables are discrete. In particular, if these conditions are not satisfied, then the identity component of the Galois group is not abelian.

EXAMPLE 2 ([43]). The Bessel equation is

$$x^2\frac{d^2\xi}{dx^2}+x\frac{d\xi}{dx}+(x^2-n^2)\xi=0, \tag{3.8}$$

with n a complex parameter. This equation is a particular confluent hypergeometric equation (after a limit process two of the singular points in a variant of the hypergeometric equation will coincide). It is one of the simplest equations with non-trivial Stokes phenomena, and for this reason is quite useful for illustrating the ideas introduced in subsection 3.3.

First, we observe that the Galois group of (3.8) is "contained" in $SL(2,\mathbf{C})$, since $1/x$ is a logarithmic derivative (see subsection 3.1). Our Bessel equation is an equation with two singular points, 0, ∞, the first one being regular and the second one irregular. We are interested mainly in the point at infinity.

There are several ways to compute the matrices Q and L of section 3.3. For example, we can follow the general constructive method of the Huhukara-Turrittin theory ([67, 7]). First, we make a formal transformation

$$\begin{pmatrix}\xi\\ \xi'\end{pmatrix}=\hat{P}\begin{pmatrix}u_1\\ u_2\end{pmatrix},$$

where $P\in Mat(2,\hat{K})$ ($\hat{K}:=\mathbf{C}[[x^{-1}]][x]$), which diagonalizes formally the equation. The solution is precisely the formal solution in equation (1), and is found step by step in a recursive way ([67, 7]). We get $q_1=ix=-q_2$ and $L=-1/2I$. The exponential torus is $\mathbf{C}^*$ and the formal mononodromy $\hat{M}=-I$.

The Stokes rays are $\mathbf{R}_+$ and $\mathbf{R}_-$, and the singular lines $i\mathbf{R}_+$, $i\mathbf{R}_-$. Hence, we have two Stokes multipliers (one for each singular line),

$$St_1=\begin{pmatrix}1&\mu\\0&1\end{pmatrix},$$

$$St_2=\begin{pmatrix}1&0\\ \lambda&1\end{pmatrix}.$$

But, for this equation the global theory (coefficients in $\mathbf{C}(x)$) and the local one (coefficients in $K=\mathbf{C}\{\{x^{-1}\}\}[x]$) are essentially the same. (The actual monodromy M_0 around 0 and around ∞ are the same, therefore the differential Galois group at the origin can be interpreted as a subgroup of the differential Galois group at infinity). It is possible to compute the actual monodromy M_0 in the classical basis at the origin (which is of course different from the basis at infinity that we introduced in the above computations). We get $M_0=diag(e^{2\pi in},e^{-2\pi in})$.

It is easy to relate the actual monodromy and the formal monodromy at infinity using the Stokes multipliers:

$$M_0=St_1\hat{M}St_2.$$

Now, as the trace is an invariant, we get

$$\mathrm{tr}M_0=2\cos(2\pi n)=-\lambda\mu-2,\ \lambda\mu=-4\cos^2\pi n.$$

Hence, if the complex number n does not belong to $\mathbf{Z}+1/2$, then the Bessel equation is not solvable. In fact, this necessary condition for solvability is also sufficient. By the classical theory (see, for example, [36]) it is well known that the Bessel functions for $n \in \mathbf{Z}+1/2$ are expressed by elementary functions: the Picard-Vessiot extension is obtained by exponential of integrals of elements of $\mathbf{C}(x)$.

EXAMPLE 3. One of the forms of the general confluent hypergeometric equation is the Whittaker equation ([70])

$$\frac{d^2\xi}{dz^2} - (\frac{1}{4} - \frac{\kappa}{z} + \frac{4\mu^2-1}{4z^2})\xi = 0, \tag{3.9}$$

with parameters κ and μ. The singular points are $z = 0$ (regular) and $z = \infty$ (irregular).

As in the case of the Bessel equation we have two singular lines associated to the irregular point for (3.9). For the computation of the Galois group the following proposition is useful ([43], subsection 3.3))

PROPOSITION 4.

There is a fundamental system of solutions such that if α, β are the two complex numbers corresponding to the two singular lines, with corresponding Stokes matrices

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix},$$

then

(i) $\alpha = 0$ if and only if, either $\kappa - \mu \in \frac{1}{2} + \mathbf{N}$ or $\kappa + \mu \in \frac{1}{2} + \mathbf{N}$.

(ii) $\beta = 0$ if and only if $-\kappa - \mu \in \frac{1}{2} + \mathbf{N}$ or $-\kappa + \mu \in \frac{1}{2} + \mathbf{N}$.

Furthermore (with respect to the same fundamental system of solutions) the group generated by the formal monodromy and the exponential torus is given by the multiplicative group

$$\left\{ \begin{pmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{pmatrix} : \delta \in \mathbf{C}^* \right\}.$$

As a consequence, we get an abelianness criterion expressed in terms of the parameters $p := \kappa + \mu - \frac{1}{2}$ and $q := \kappa - \mu - \frac{1}{2}$.

COROLLARY 1. *The identity component G^0 of (3.9) is abelian if and only if (p, q) belong to $(\mathbf{N} \times -\mathbf{N}^*) \cup (-\mathbf{N}^* \times \mathbf{N})$ (i.e., p, q are integers, one of them being positive and the other one negative).*

We observe that the abelian case (G^0 abelian) for the Whittaker equation is only possible when the *two* Stokes multipliers vanish and this corresponds to the diagonal case (4) of the classification given by Proposition 1 (the Whittaker equation is a symplectic one). If *only one* of the Stokes multipliers vanishes, then we are in case (6) of this classification, and we have solvability but the identity component of the Galois group is not abelian. If the two Stokes multipliers are different from zero, then we fall in case (7) with a Galois group isomorphic to $SL(2, \mathbf{C})$, as we remarked in subsection 3.3.

If in the Bessel equation (3.8) we make the change of dependent variable $\xi = x^{-1/2}\psi$ and of independent variable $x = z/2i$, we get a Whittaker equation

$$\frac{d^2\xi}{dz^2} - (\frac{1}{4} + \frac{4n^2-1}{4z^2})\eta = 0, \tag{3.10}$$

with parameters $\kappa = 0$ and $\mu = n$. As in the above change we only introduce algebraic functions, the identity component of the Galois group of the Bessel equation is preserved.

EXAMPLE 4. One of the more simple non-trivial linear differential equations is the Airy equation

$$\frac{d^2\xi}{dz^2} - z\xi = 0. \tag{3.11}$$

It is clear that if (3.11) is considered over $\overline{\Gamma} = \mathbf{C}$ (ie, the coefficient field is the field of meromorphic functions over the finite complex plane) it is solvable, since the general solution is entire and, in particular, meromorphic over $\mathbf{C}$ and then the Galois group is trivial.

But if we consider (3.11) over the Riemann sphere, the situation is very different: the equation (3.11) has Galois group $SL(2, \mathbf{C})$ provided the field of coefficients is $\mathbf{C}(z)$, see [28, 33, 4]. More general: any equation like (3.11) but with any odd polynomial instead of the polynomial z has also $SL(2, \mathbf{C})$ as Galois group [33]. The analytical reason for that is the complicated behaviour given by the Stokes matrices at $z = \infty$.

We observe that all the examples in this section are equations over the Riemann sphere (with coefficients in $\mathbf{C}(z)$) and then it is also possible to apply the Kovacic algorithm ([33, 20]).

4. Variational Equations.

4.1. Singular curves. Let us now come back to Hamiltonian systems. Let $X := X_H$ be the holomorphic Hamiltonian system defined on an analytic complex symplectic manifold M of dimension $2n$ (the phase space) by a Hamiltonian function H. Before going to formal constructions, we shall make some comments about the essential underlying ideas. If $x = \phi(t)$ is a germ of integral curve (but not an equilibrium point) then one consider the corresponding connected complete complex phase curve $i(\Gamma)$ in the phase space. We will denote by Γ the corresponding *abstract* Riemann surface. By an abuse of terminology we will say that Γ is an integral curve (from now on by an integral curve we will in general mean this abstract curve). On the integral curve Γ we can use the complex time t (which is defined up to a additive complex constant) as a local parameter (uniformizing coordinate). However it is essential to think Γ as an abstract Riemann surface over which we can use other local parametrizations. The only distinctive fact of a temporal parametrization is that it allows us to express the Hamiltonian field in the simplest way: $X = d/dt$ (here, using a pull back, we interpret X as a holomorphic vector field on Γ). When necessary we will carefully distinguish between the abstract Riemann surface Γ and the phase curve $i(\Gamma) \subset M$ which is the image of Γ by an immersion i. This immersion induces a bijection $\Gamma \to i(\Gamma)$. (Be careful i is not in general an embedding.)

It can happen that the complex time is a *global* parametrization of Γ. This is frequent in the applications (cf. the example 2 below). More precisely we have an analytic covering $\psi : \mathbf{C} \to \Gamma$, $t \mapsto \psi(t)$. But it is important to notice that in general in such a situation it is a *infinitely* sheeted covering (i.e. the function ψ is *transcendental*).

We will see later that in our theory it is not important to distinguish the curves up to a *finite* covering (and we will use this convention repeatedly in the applications), but it will be strictly forbidden to replace a curve by one of its infinitely sheeted coverings. Therefore in the preceding situation it is important to carefully distinguish between the integral curve Γ and the complex time line $\mathbf{C}$.

In the following, the variational equation over Γ is locally a system of linear equations with holomorphic coefficients or in an abstract way a holomorphic (symplectic) connection ∇ over Γ. (We get it by pull back from the variational equation over the phase curve $i(\Gamma)$ which is classically associated to our Hamiltonian system.)

The following step is to introduce the possibility to add singular points in order to obtain a *meromorphic* (symplectic) connection on some extended Riemann surface $\overline{\Gamma}$. It seems natural to add the equilibrium points of X that belong to the closure in the phase space M of the phase curve $i(\Gamma)$ (i.e. the possible limits points of the phase curve $i(\Gamma)$ when the time is made infinite). The problem is that the resultant extended set is not, in general, an analytic smooth curve. We will limit ourselves to the following case: we will suppose that the set of equilibrium points in the closure of $i(\Gamma)$ is discrete (or if it is not the case that we add only a discrete subset) and that the extended curve $\underline{\Gamma}$ is "locally" an *analytic* complex subset of dimension one of M. We allow *singularities* on $\underline{\Gamma}$, and in general the equilibrium points we added will precisely be such singularities. As it is well-known from algebraic and analytic geometry we can desingularize this curve and obtain a "good" Riemann surface $\bar{\Gamma}$. This Riemann surface is abstract and of course it is not contained in the phase space M. The holomorphic connection ∇ over Γ which represents the VE extends on a *meromorphic* connection over $\bar{\Gamma}$. The poles of this connection are the"above" equilibrium points and correspond to branches of the curve $\underline{\Gamma}$ at the corresponding equilibrium point.

The reader unfamiliar with (algebraic or analytic) singular curves and their non singular models can find some information in [31, 54, 69] (in particular [54] Theorem 4.1.11 is valid in our case if we replace the finite set of singular points by our discrete set and the compact analytic curve by our, in general, non compact curve).

In some problems it is interesting to add *points at infinity* to Γ or to $\bar{\Gamma}$. We add now to the symplectic manifold (M, ω) an hypersurface at infinity M_∞: $M' = M \cup M_\infty$. We suppose that M' is a complex manifold and that ω admits a *meromorphic* extension over M'. Then it is natural to add to the curve $\underline{\Gamma}$ the points of M_∞ that belong to the closure in the extended phase space M' of this curve. The resultant extended set is not, in general, an analytic smooth curve. As before, we will limit ourselves to the following case: we will suppose that the set of points at infinity in the closure of $\underline{\Gamma}$ is discrete (or if it is not the case that we add only a discrete subset) and that the extended curve $\underline{\Gamma}'$ is "locally" an *analytic* complex subset of dimension one of M'. Then, as before, we can desingularize this curve and obtain a Riemann surface $\bar{\Gamma}'$. The meromorphic connection ∇ over $\bar{\Gamma}$ which represents the VE extends on a meromorphic connection over $\bar{\Gamma}$. The poles of this connection correspond to the equilibrium points or to the points at infinity.

After these preliminaries we start with the formal definitions. Let $x = \phi(t)$ (where t is a complex parameter, not necessarily the time) be a germ of regular holomorphic parametrized curve in the phase space (i.e. ϕ is given in local coordinates by a convergent Taylor expansion in a neighborhood of $t = 0$ with $\phi'(0) \neq 0$). We have locally a homomorphism between an open disk centered in $t = 0$ and its image by ϕ. We consider two such elements ϕ_1, ϕ_2 as equivalent if there exists a germ of holomorphic function ρ at the origin such that $\rho(0) = 0$, $\rho'(0) \neq 0$ and $\phi_2 = \phi_1 \circ \rho$

(change of parametrization). We denote $\mathcal{C}$ by the set of germs of curves over the phase space up to the above equivalence.

It is possible to endow the set $\mathcal{C}$ with a natural topology. If a germ ϕ belongs to $\mathcal{C}$, and is defined by a holomorphic function $\phi(t)$ for t varying in an open disk $U \subset \mathbf{C}$, U, then for any point t_0 in U, we define $\phi_{t_0} \in \mathcal{C}$ as $\phi_{t_0}(t) := \phi(t + t_0)$. The sets $U(\phi) := \{\phi_{t_0}\}$ are a basis for the open sets in $\mathcal{C}$.

Given a germ $\phi \in \mathcal{C}$, the Riemann surface Γ that it defines is, by definition, the abstract Riemann surface defined by its connected component $i(\Gamma)$. For more details see [31], chapter 7 (in this reference the analysis is made in the context of plane curves, but it remains clearly valid without changes in our situation) or the classical H. Weyl monograph [69], page 61.

So, if we have the germ of an integral curve which passes by a point x_0, $x = \phi_{x_0}(t)$ with the initial condition $\phi_{x_0}(0) = x_0$, the Riemann surface which it defines is precisely Γ. We can identify Γ with the corresponding (connected) phase curve in the phase space $i(\Gamma)$ and we get over Γ the Hamiltonian field X (d/dt in the temporal parametrization). (More precisely this Hamiltonian field is the pull-back of the Hamiltonian field over M by the immersion $i : \Gamma \to M$.) At the points of the set $\overline{\Gamma} - \Gamma$, the vector field X is by definition zero (as they correspond to the equilibrium points belonging to the closure of $i(\Gamma)$ in the phase space M).

Example. We illustrate the above considerations with an example that will become important in the applications. Let the one degree of freedom system which is defined by the following analytical Hamiltonian (over $\mathbf{C}^2$ or over an open set of $\mathbf{C}^2$)

$$H(x, y) = \frac{1}{2}y^2 + \frac{1}{2}\varphi(x).$$

If one considers the energy level zero, we obtain an analytic subset defined by the equation $P(x, y) = y^2 + \varphi(x) = 0$. We assume that this set C is connected (if this is not the case we select one of its connected components). Then C is an analytic curve. Its singular points are exactly the equilibrium points $E := \{(0, x) : \varphi(x) = \varphi'(x) = 0\}$. So, the curve $C = \underline{\Gamma}$ is equal to the disjoint union of Γ (or more precisely $i(\Gamma)$) and E. And we obtain $\overline{\Gamma}$ as the corresponding non singular model of $\underline{\Gamma}$.

In order to perform some explicit computations, we will analyze some simple particular cases:

1) Let $\varphi(x) = \frac{2}{3}x^3$. The curve C is now

$$P(x, y) = y^2 + \frac{2}{3}x^3.$$

And

$$C = \underline{\Gamma} = \Gamma \cup \{(0, 0)\}, E = \{(0, 0)\}.$$

We note that Γ admits a temporal parametrization

$$(x, y) = (-6t^{-2}, 12t^{-3}).$$

We can desingularize the point $(0, 0)$, as usual, by using Puiseux series ([31], chapter 6), indeed we obtain only one branch $(x, y) = (-6\tilde{t}^2, 12\tilde{t}^3)$. We write $\tilde{t}$ instead of t because this parameter is not the time ($\tilde{t} = 1/t$), this is so because the temporal parametrization of Γ is rational.

Now we can calculate the Hamiltonian vector field X on $\overline{\Gamma}$. As $X = yd/dx$, on Γ, $X = d/dt$ (if we use the temporal parametrization), and

$$\frac{y(\tilde{t})}{x'(\tilde{t})}\frac{d}{d\tilde{t}} = -\tilde{t}^2\frac{d}{d\tilde{t}},$$

at the singular point s in $\overline{\Gamma} - \Gamma$.

2) Let $\varphi(x) = x^2(1-x)$. The curve $\underline{\Gamma}$ contains a homoclinic orbit $i(\Gamma)$, and the origin as an equilibrium point. We can parametrize *globally* the Riemann surface Γ using the time parametrization

$$(x(t), y(t)) = (\frac{2}{1+\cosh t}, -\frac{2\sinh t}{(1+\cosh t)^2}).$$

We remark that, despite its *transcendental appearance* (cf. the above equations), the Riemann surface Γ is *algebraic*: it is only a problem related to the selected parametrization.. (Our global time parametrization is not one to one: it is an infinitely sheeted covering.) Applying Puiseux algorithm we obtain (in fact, in the algorithm, we only need the first Newton polygon) two branches and hence two points s_1, s_2 belonging to $\overline{\Gamma} - \Gamma$ above the origin in $\underline{\Gamma}$,

$$(x, y) = (\tilde{t}, \tilde{t} + h.o.t.),$$
$$(x, y) = (\tilde{t}, -\tilde{t} + h.o.t.).$$

We can express the field X as $(\tilde{t} + ...)d/d\tilde{t}$ or $(-\tilde{t} + ...)d/d\tilde{t}$, respectively.

3) We observe that in example 1) the equilibrium point is degenerated and the field X has a zero of multiplicity two at the corresponding point above it in $\overline{\Gamma}$. However, in the example 2) the equilibrium point is non degenerate and the field X has a simple zero at the two corresponding points of $\overline{\Gamma}$. This is not casual. In fact, let $\varphi(x) = x^n + O(x^{n+1})$ (with $n \leq 2$) be the expansion of φ at the origin, then, by a simple inspection of Newton polygon we get the following facts. If n is odd, we get only one point belonging to $\overline{\Gamma} - \Gamma$, $(x, y) = (\tilde{t}^2, \tilde{t}^n + h.o.t.)$, and the field X has a zero of order $n-1$ at this point. If n is even, we get two points above $(0, 0)$ in $\overline{\Gamma} - \Gamma$, $(x, y) = (\tilde{t}, \tilde{t}^{n/2})$, $(x, y) = (\tilde{t}, -\tilde{t}^{n/2})$ and the field X has a zero of order $n/2$ at each one. Of course, the above results do not depend upon the parametrization. This simple fact is fundamental as we shall see later in the applications.

4.2. Meromorphic connection associated to the variational equation. Once we have defined Γ and the derivation X, we shall define the holomorphic connection associated to the variational equation (VE) over Γ. More generally if we add some stationary points (resp. some stationary points and some points at infinity), we shall define the *meromorphic* connection associated to the variational equation (VE) over $\bar{\Gamma}$ (resp. $\bar{\Gamma}'$).

Let T_Γ be the restriction to Γ (or more precisely to $i(\Gamma)$) of the tangent bundle T_M to the phase space M. It is a symplectic holomorphic vector bundle. (More formally the fiber bundle T_Γ is the pull back of T_M by the immersion $i : \Gamma \to M$.)

The holomorphic connection which defines the variational equation along Γ comes by pull-back from the restriction to $i(\Gamma)$ of the Lie derivative with respect to the field X:

$$\nabla v := L_X Y_{|\Gamma},$$

being Y any (holomorphic) vector field extension of the section v of the bundle $T_{i(\Gamma)}$.

The fact that the connection ∇ is symplectic follows from the definition of a Hamiltonian vector field: the symplectic form is preserved by the flow.

Using locally a time parametrization t on Γ and local canonical coordinates $(x_1, ..., x_{2n})$ on M, we have the usual definition of the variational equation (VE) along the phase curve [45] (IV A.2).

We write the Hamiltonian system $\dot{x} = J\frac{\partial H}{\partial x}$. We choose a initial value τ for the time t. Let $\phi(t, \tau, \zeta)$ be the unique solution of the Hamiltonian system defined by the initial value $\zeta \in M$: $\phi(\tau, \tau, \zeta) = \zeta$. Let $Z(t, \tau, \zeta) = \frac{\partial \phi(t,\tau,\zeta)}{\partial \zeta}$ be the Jacobian of ϕ with respect to the initial condition ζ. Then $\dot{Z} = JSZ$, where $S(t, \tau, \zeta) = \frac{\partial^2 H}{\partial x^2}(\phi(t, \tau, \zeta)) = J\mathrm{Hess}\ H(\phi(t, \tau, \zeta))$ (Hess H being the Hessian of the Hamiltonian function H). The equation

$$\dot{Z} = J\mathrm{Hess}\ HZ$$

is the variational equation. It is clearly a linear Hamiltonian system. The matrix function Z satisfies $Z(\tau, \tau, \zeta) = I$, therefore it is a fundamental solution of the variational equation.

We can express our connection ∇ in a holomorphic frame $(e_1, \ldots, e_{2n})$. We get a differential system. Choosing the coordinate frame $e_i = \frac{\partial}{\partial x_i}$ associated to some coordinates $x_1, ..., x_{2n}$, we get the differential system

$$\frac{d\xi}{dt} = A(t)\xi,$$

where

$$A(t) = \frac{\partial X}{\partial x}(x_1(t), ..., x_{2n}(t)) = J\mathrm{Hess}\ H(x_1(t), ..., x_{2n}(t)),$$

where $\frac{\partial X}{\partial x}$ is the jacobian matrix of X in coordinates (this is a direct consequence from the expression of the Lie bracket $[\sum X_i \partial/\partial x_i, \partial x_j]$). Hence we obtain the differential system which defines the VE in its usual form

Now we add to $i(\Gamma)$ a discrete set of *stationary* points (where by definition the field X_H vanishes). We suppose that $\underline{\Gamma}$ is obtained by local closure at the added points of $i(\Gamma)$ in the phase space M and that $\underline{\Gamma}$ is "locally" an analytic curve in M. We denote by $\bar{\Gamma} \to \underline{\Gamma}$ a desingularization of the curve $\underline{\Gamma}$. We will consider Γ as an open subset of $\bar{\Gamma}$. By restriction of the tangent bundle T_M, we get an holomorphic bundle $T_{\underline{\Gamma}}$ over $\underline{\Gamma}$ and by pull-back an holomorphic bundle $T_{\bar{\Gamma}}$ over $\bar{\Gamma}$.

At a point stationary point a above we cannot use the temporal parametrisation on $\bar{\Gamma}$. Then we choose an uniformizing variable u at $a \in \bar{\Gamma}$ $(u(a) = 0)$, and we write the immersion i: $u \mapsto i(u)$. We get $X(i(u)) = f(u)d/du$ $(u \neq 0)$ with $f(0) = 0$ (by definition the Hamiltonian field X vanishes at $i(0)$). Using u as a local coordinate on $i(\Gamma)$, we can write $X = d/dt = f(u)d/du$. Then the pull-back by i of the VE in a punctured neighbourhood of a is

$$\frac{d\xi}{du} = \frac{1}{f(u)}\frac{\partial X}{\partial x}(i(u)).$$

It is a *holomorphic* differential system over a punctured neighbourhood of a in $\bar{\Gamma}$. It can clearly be interpreted as a *meromorphic* differential system over a neighbourhood of a.

Such local constructions over $\bar{\Gamma}$ glue together and we get a meromorphic connection over $\bar{\Gamma}$. It *defines* the VE over $\bar{\Gamma}$.

When we add some points at infinity we can perform a similar construction over $\bar{\Gamma}'$. The only difference is that the Hamiltonian field $X = X_H$ can have a pole at a point at infinity (due to the possible singularities at infinity of the symplectic form). Then the function $f(u)$ is now in general *meromorphic*.

4.3. Reduction to the normal variational equations. The problem of the reduction of a linear system of equations goes back to the D'Alembert reduction of the order of a linear differential equation when we know a particular solution.

In the Hamiltonian case, as we shall see, the mechanism which explains the reduction is the existence of invariant unidimensional horizontal sections (of (V, ∇) or (V^*, ∇^*)) in involution.

All the bundles and connections considered in this section are meromorphic. In the process of reduction we may need to add some new singular points to the reduced connection, and in this case we will consider these new singularities as singular points of the initial VE (i.e., as points of $\overline{\Gamma} - \Gamma$). With this in mind, all the bundles and connections will be defined over the same fixed connected Riemann surface $\overline{\Gamma}'$ or $\overline{\Gamma}$ (with the same "singular set"). Also, as usual, we will identify a bundle with its (sheaf of) sections.

Let V be a symplectic vector bundle of rank $2n$. Locally we can define a symplectic form Ω which defines the symplectic structure of V. If $v_1, ..., v_k$ are *global* meromorphic sections of V linearly independent over Γ ($v_1 \wedge ... \wedge v_k \in \bigwedge^k V$ is different from zero on Γ) and in involution (i.e. $\Omega(v_i, v_j) = 0$, $i, j = 1, ..., k$), then we can obtain some subbundles of V in the following way (we remark that, by definition, the sections $v_1, ..., v_k$ have their coefficients in the field of meromorphic functions over $\overline{\Gamma}$ or $\overline{\Gamma}'$, being holomorphic over Γ).

Let F be the rank k (meromorphic) subbundle of V generated by $v_1, ..., v_k$. We get $F^{\perp}$($\perp$ with respect to the symplectic structure) as a subbundle of V. We have clearly $F \subset F^{\perp}$. The normal bundle $N := F^{\perp}/F$ is a rank $2(n-k)$ symplectic bundle which admits locally the symplectic form Ω_N defined by the projection of Ω. It is easy to see that this form is well defined and non-degenerate over Γ. (See also [5, 38], meromorphic vector bundles do not appear in these references but the constructions are similar.)

Using Lemma 2 below, we can suppose that the form Ω is globally defined (and that it is meromorphic over $\overline{\Gamma}$ and holomorphic and non degenerate over Γ). We will implicitly made these hypothesis in what follows.

The following proposition (with the same notation as above) is essentially a consequence of Propositions 1.11 and 1.6 of [5] (with small changes in the notation and taking account of the fact that we work here with meromorphic connections instead of holomorphic connections).

Proposition 5. *Let (∇, V, Ω) a symplectic connection and $v_1, ..., v_k$ an involutive set of linearly independent global horizontal sections of ∇. Then by restriction we have the subconnections $(\nabla_F = 0, F)$, $(\nabla_{F^{\perp}}, F^{\perp})$ and a symplectic connection on the normal bundle (∇_N, N, ∇_N).*

Proof. It is obvious that the bundle F is invariant by ∇. The invariance of $F^{\perp}$ by ∇ follows from the formula

$$\Omega(\nabla w, v) = X(\Omega(w, v)) - \Omega(w, \nabla v) - (\nabla\Omega)(w, v),$$

and from the fact that ∇ is a symplectic connection.

We define the connection ∇_N on $N := F^{\perp}/F$ from the action of ∇ on the representatives in $F^{\perp}$ of the classes in N. Of course it is well defined ($\nabla_F = 0$) and it is a symplectic connection.

The connection ∇_N is called the *reduced* connection and the corresponding linear differential equation the (reduced) normal equation.

We remark that although the proof of the above proposition is technically similar to these of the propositions in [5], the philosophy here is different. Here the connections ∇, ∇_N have the same singularities in $\overline{\Gamma} - \Gamma$; in particular, the differential fields of coefficients of the corresponding linear differential equations are the same (the meromorphic functions over $\overline{\Gamma}$).

Our objective now is to investigate the relation between the Galois group of the initial equation $Gal\ \nabla$ and the Galois group of the reduced equation $Gal\ \nabla_N$. We will use two different methods. The first is based upon explicit classical computations. The results such obtained are sufficient for the applications. (More precisely for these applications it is sufficient to know that if the identity component of $Gal\ \nabla$ is abelian, then the identity component of $Gal\ \nabla_N$ is also *abelian.*) The second method is based upon Tannakian arguments. It will allow us to give a more precise relation between the two differential Galois groups. This last relation seems interesting by itself, even if it is not necessary for what will follows.

Let (∇, V, Ω) be a symplectic connection. Then we have the musical isomorphism defined by Ω

$$\flat : (\nabla, V, \Omega) \longrightarrow (\nabla^*, V^*, \{,\}).$$

We set $\natural := \flat^{-1}$. The symplectic form (section) is transported to the Poisson bracket,

$$\{\alpha, \beta\} = \Omega(\natural(\alpha), \natural(\beta)).$$

(In some references it is said that V^* with the Poisson bracket is a Poisson vector bundle, see for instance [38].)

Let now $\alpha \in V^*$, $v \in V$ two sections. Then $\alpha_0 := \alpha(p_0)$, $v_0 := v(p_0)$ are elements belonging to the fibres at $p_0 \in \Gamma$, that using Cauchy's existence theorem we can identify with elements belonging respectively to the vector spaces of germs of solutions at p_0: $E^* = Sol\nabla^*$, $E = Sol\nabla$.

Lemma 1. *Let (V, ∇, Ω) be a symplectic connection and let α, $v := \flat^{-1}\alpha$ global sections of respectively the bundles V^* and V. Then the following conditions are equivalent*

(i) α is a (linear) first integral of the linear equation defined by ∇.
(ii) α is a horizontal section of ∇^* (i.e., a solution of the adjoint differential equation).
(iii) v is a horizontal section of ∇.
(iv) α_0 is invariant by the Galois group $Gal\ \nabla$.
(v) v_0 is invariant by the Galois group $Gal\ \nabla$.

Proof. The equivalence between (i) and (ii) follows from

$$X(<\alpha, w>) = <\nabla\alpha, w> + <\alpha, \nabla w>,$$

being X the holomorphic vector field on the Riemann surface $\overline{\Gamma}$, such that $\nabla := \nabla_X$. The equivalence between (ii) and (iii) follows from

$$0 = X(\Omega(v, v)) = X(<\alpha, v>) = <\nabla\alpha, v> + <\alpha, \nabla v> .$$

Now if α is a horizontal section of ∇^*, $(\mathcal{M}_{\overline{\Gamma}}(1+\alpha), \delta \oplus \nabla^*)$ is a rank one subconnection of $(\mathcal{M}_{\overline{\Gamma}} \oplus V^*, \delta \oplus \nabla^*)$. Then (as in the final example of Section 4) the complex construction corresponding by the fiber functor Sol, that is the complex line $\mathbf{C}(1+\alpha_0)$ is (pointwise) invariant by the Galois group. So we get (iv). That (ii) is a necessary condition for (iv) is clear from the fact that α_0 is a local horizontal section at a point $p_0 \in \Gamma$ that, by assumption, can be extended to a global section α. From the unicity in Cauchy's theorem it is necessarily a horizontal section.

The equivalence between (iii) and (v) is obtained in a similar way: we only need to write V instead V^* (another way for finishing the proof is to prove the equivalence between (iv) and (v) using the fact that the musical isomorphism $\flat_0$ between the vector spaces (E, Ω_0) and $(E^*, \{,\}_0)$ induces a bijection between the invariants of the Galois group in E and E^*).

Let $\alpha_1, ..., \alpha_k$ (α_i being a section of V^*) be an involutive set of (global) independent (i.e., generating a rank k subbundle) first integrals of the symplectic connection (∇, V, Ω). By the above lemma we obtain an involutive set $v_1, ..., v_k$ of independent (global) horizontal sections of (∇, V, Ω). If as above, F is the rank k subbundle of horizontal sections generated by $v_1, ..., v_k$, we can construct the subbundles and connections (∇_F, F), $(\nabla_{F^\perp}, F^\perp)$ and $(\nabla_N, N = F^\perp/F, \Omega_N)$ (in general we will write simply ∇_N). (We remark that it is easy to prove that

$$F^\perp = \{w \in V :< \alpha_i, w >= 0, i = 1, ..., k\}$$

([5])).

From the (meromorphic) triviality of the symplectic vector bundles (see appendix B) and the properties of the symplectic bases (the global meromorphic sections are a symplectic vector space over the field of global meromorphic functions $K = \mathcal{M}(\overline{\Gamma})$ over $\overline{\Gamma}$) we have

Lemma 2. *There exist a global (meromorphic) symplectic canonical frame which contains the given linearly independent and involutive horizontal sections* $v_1, ..., v_k$.

Let JS be the matrix of ∇ in a canonical frame (S is a *symmetric* matrix). We define a symplectic change of variables using some new canonical frame which contains the given linearly independent and involutive horizontal sections $v_1, ..., v_k$.

We denote the matrix of the symplectic change of variables by

$$P = (D_1 \ D_2 \ C_1 \ C_2),$$

where $C_2 = (\xi_1^t, ..., \xi_k^t)$, and the $2n$-dimensional columns vectors ξ_i, $i = 1, ..., k$, are the coordinates of v_i in the original canonical frame. Then we have ([45]):

$$P^{-1} = \begin{pmatrix} -C_1^t J \\ -C_2^t J \\ D_1^t J \\ D_2^t J \end{pmatrix}$$

$$AP - \dot{P} = JSP - \dot{P} = (JSD_1 - \dot{D}_1 \quad JSD_2 - \dot{D}_2 \quad JSC_1 - \dot{C}_1 \quad 0),$$

since C_2 is a fundamental matrix solution of the original linear equation. Hence the matrix of the transformed equation is

$$P[JS] := P^{-1}(JSP - \dot{P})$$

$$= \begin{pmatrix} C_1^t(SD_1 + J\dot{D}_1) & G & C_1^t(SC_1 + J\dot{C}_1) & 0 \\ C_2^t(SD_1 + J\dot{D}_1) & C_2^t(SD_2 + J\dot{D}_2) & C_2^t(SC_1 + J\dot{C}_1) & 0 \\ -D_1^t(SD_1 + J\dot{D}_1) & F & -D_1^t(SC_1 + J\dot{C}_1) & 0 \\ M & H & E & 0 \end{pmatrix},$$

where E, F, G, M and H are some matrices. The matrix $P[JS]$ is necessarily infinitesimally symplectic, i.e., of the form JS_1 with S_1 symmetric (see, for instance, [45], page 36,37). Hence $C_2^t(SD_i + J\dot{D}_i)$ $(i = 1, 2)$ and $C_2^t(SC_1 + J\dot{C}_1)$ are zero, $E = -G^t$, $M = F^t$, $C_1^t(SD_1 + J\dot{D}_1) = (C_1^t S - \dot{C}_1^t J)D_1$ and H, $C_1^t(SC_1 + J\dot{C}_1)$, $-D_1^t(SD_1 + J\dot{D}_1)$ are symmetric.

Reordering the new canonical frame, the matrix of the connection ∇ becomes

$$P[JS] = \begin{pmatrix} C_1^t(SD_1 + J\dot{D}_1) & C_1^t(SC_1 + J\dot{C}_1) & G & 0 \\ -D_1^t(SD_1 + J\dot{D}_1) & (-D_1^t S - \dot{D}_1^t J)C_1 & F & 0 \\ 0 & 0 & 0 & 0 \\ F^t & -G^t & H & 0 \end{pmatrix}.$$

Then the transformed differential equation $\dot{\eta} = P[JS]\eta$ in the variable $\eta = (\alpha, \beta, \gamma, \delta)$ is

$$\begin{array}{ll} \dot{\alpha} & = C_1^t(SD_1 + J\dot{D}_1)\alpha + C_1^t(SC_1 + J\dot{C}_1)\beta + G\gamma, \\ \dot{\beta} & = -D_1^t(SD_1 + J\dot{D}_1)\alpha + (-D_1^t S - \dot{D}_1^t J)C_1\beta + F\gamma, \\ \dot{\gamma} & = 0, \\ \dot{\delta} & = F^t\alpha - G^t\beta + H\gamma. \end{array}$$

The matrix of the reduced equation is

$$\begin{pmatrix} C_1^t(SD_1 + J\dot{D}_1) & C_1^t(SC_1 + J\dot{C}_1) \\ -D_1^t(SD_1 + J\dot{D}_1) & (-D_1^t S - \dot{D}_1^t J)C_1 \end{pmatrix}.$$

Then we get the Picard-Vessiot extension L/K of ∇ from two successive extensions

$$K \subset L_N \subset L,$$

where L_N/K is the Picard-Vessiot extension of the (normal) reduced equation (i.e.,of ∇_N) and L/L_N is an extension composed by two successive extensions L/L_1 and L_1/L_N by integrals.

Now, using the Galois correspondence, we get

$$Gal(L_N/K) = Gal(L/K)/Gal(L/L_N).$$

It is well known that extensions by integrals are normal purely transcendental extensions. Their Galois groups are additive abelian groups isomorphic to some G_a^r ([28]). Therefore $Gal(L/L_N)$ is Zariski connected and we get the inclusion of $Gal(L/L_N)$ in the identity component $Gal\nabla_0$ ($Gal(L/L_N)$ is the group H that will appear below in the Tannakian reduction: see the next subsection).

Finally, for $n = k$ we can solve the initial system by quadratures. This is proved in [45]. Our computations on the matrix JS are a generalization of the computations in this reference (we in fact kept the same notations).

If now we have the variational equation (VE) over an integral curve Γ of a holomorphic Hamiltonian system X_H and if $f_1 = H, f_2, ..., f_k$ is an involutive set of holomorphic and independent first integrals of X_H, then (see Section 5) we get an involutive set $\alpha_1 = dH, \alpha_2, ..., \alpha_k$ of independent first integrals of the (VE) and we can apply the results of this section. The (normal) reduced equation such obtained is then called the *normal variational equation* (NVE).

PROPOSITION 6. *Let $\alpha_1, ..., \alpha_k \in V^*$ be an involutive set of independent (global) first integrals of (∇, V, Ω). Let ∇_N be the reduced connection defined by the above set. Then we have*

(i) The linear differential equation corresponding to the connection (∇, V, Ω) is solvable if and only if the reduced equation corresponding to (∇_N, N, Ω_N) is solvable.

(ii) If the identity component of $Gal\nabla$ is abelian then the identity component of $Gal\nabla_N$ is also abelian.

Proof. Using the preceding results and the differential Galois correspondence, the equivalence (i) follows from the general group theoretic facts that any subgroup and any quotient group of a solvable group is solvable and that conversely, if a normal subgroup and the corresponding quotient group are solvable then the original group is also solvable. Claim (ii) is evident.

Claim (ii) in the preceding proposition is *essential* for the applications.

EXAMPLE. Now, as an example, we shall apply the above considerations to the variational equations of a two degree of freedom Hamiltonian system along an integral curve Γ. We want to obtain the (NVE) from the (VE) when the Hamiltonian is a natural mechanical system, $H = T + U$, $T = \frac{1}{2}(y_1^2 + y_2^2)$, $\quad U = U(x_1, x_2)$ (potential).

Then $\alpha = dH$ (over $\overline{\Gamma}$) is a linear first integral (i.e., an element of V^*). We know that this is equivalent to the fact that $\dot{z} = (y_1, y_2, -U_1, -U_2) \in V$ (where we used subindexes for the derivatives of the potential) is a known solution. There are many possibilities for the choice of the symplectic change P (which is defined by a symplectic frame admitting $\dot{z}$ as one of its elements). We can suppose that y_1 and y_2 are not identically zero (if this is not the case the NVE is obtained without any computation from the VE), then we can select a very simple solution

$$P = \begin{pmatrix} 0 & 0 & 0 & y_1 \\ -\frac{y_2}{y_1} & 0 & 0 & y_2 \\ 0 & -\frac{1}{y_1} & 1 & -U_1 \\ \frac{U_2}{y_1} & 0 & -\frac{y_1}{y_2} & -U_2 \end{pmatrix}.$$

Applying the formula obtained above, the matrix of the NVE is

$$\begin{pmatrix} -U_1/y_1 & 1 + (y_1/y_2)^2 \\ y_2 U_{12}/y_1 & U_1/y_1 \end{pmatrix}.$$

We observe that, as expected, it belongs to the symplectic Lie algebra $sp(1, K) = sl(2, K)$).

4.4. Reduction from the Tannakian point of view. We will give now a new proof of Proposition 6 using Tannakian arguments. In fact we will get a slightly more precise result. This improvement has an independent interest even if it is not used later in this paper.

We recall that a group G is *metaabelian* if its derived group G' is abelian. In particular, a metaabelian group is solvable.

The relation between the initial Galois group and the reduced Galois group is given by the following result

PROPOSITION 7. *Let $\alpha_1, ..., \alpha_k \in V^*$ be an involutive set of independent(global) first integrals of (∇, V, Ω). Let ∇_N be the reduced connection defined by the above set. Then*

$$(Gal\nabla_N)_0 \approx Gal\nabla_0/H,$$

where $Gal\nabla_0$ and $(Gal\nabla_N)_0$ are respectively the identity components of the Galois groups of ∇ and ∇_N, and H a closed normal metaabelian subgroup of $Gal\nabla_0$.

Proof. With the above notations, at the level of connections we have the natural morphisms

$$(V, \nabla) \hookrightarrow (F^{\perp}, \nabla_{F^{\perp}}) \to (N = F^{\perp}/F, \nabla_N),$$

(being the first the inclusion and the second the projection) and the isomorphism

$$\flat : (\nabla, V, \Omega) \longrightarrow (\nabla^*, V^*, \{,\}).$$

By applying the fiber functor we get the corresponding morphisms and isomorphism

$$(E, \Omega) \hookrightarrow (F_0^{\perp}, \Omega_{F_0^{\perp}}) \to (N_0 = F_0^{\perp}/F_0, \Omega_{N_0})$$
$$\flat_0 : (E, \Omega) \longrightarrow (V_0^*, \{,\}),$$

where $E = Sol\nabla$, $F_0 = Sol\nabla_F$, etc... In order to simplify the notation, we will write Ω and $\{,\}$ instead of Ω_0, $\{,\}_0$.

Let ${F_0}^*, \{,\}$ be the involutive subalgebra generated by $\alpha_1, ..., \alpha_k$. Then we obtain the morphisms

$$(E^*, \{,\}) \hookrightarrow (F_0^{*\perp}, \{,\}_{F_0^{*\perp}}) \to (N_0^* := {F_0}^{*\perp}/{F_0}^*, \{,\}_{N_0^*}),$$

where orthogonality $\perp$ is now defined by the Poisson bracket.

We have natural morphisms of algebraic groups

$$Gal\ \nabla \to Gal\ \nabla_{F^{\perp}} \to Gal\ \nabla_N.$$

By composition, we get a surjective morphism $\phi : Gal\ \nabla \to Gal\ \nabla_N$.

We get also the corresponding morphisms of Lie algebras

$$Lie\nabla \to Lie\nabla_{F^{\perp}} \to Lie\nabla_N$$

and the surjective morphism

$$\pi : Lie\nabla \to Lie\nabla_N.$$

We will see later in Section 5 that the Lie algebra $Lie\nabla$ is isomorphic to a Lie subalgebra of $S^2E^*, \{,\})$ and that modulo this isomorphism, the action of $\beta \in Lie\nabla$ on E^* is given by

$$\{\beta, .\} : \delta \mapsto \{\beta, \delta\}.$$

Then it is easy to describe the natural morphisms

$$Lie\nabla \to Lie\nabla_{F^\perp} \to Lie\nabla_N$$

by restriction and projection of $\{\beta, \}$ (we observe that $\{\beta,\alpha\}=0$, for any $\beta \in Lie\nabla$, $\alpha \in F_0^*$ or in a shorter way $\{Lie\nabla, F_0^*\}=0$) and $Lie\nabla_N$ is considered also as a Lie subalgebra of $(S^2N^*, \{,\})$, modulo a musical isomorphism.

Using the morphism of Lie algebras

$$\pi : Lie\nabla \to Lie\nabla_N$$

we get an isomorphism

$$Lie\nabla_N \approx Lie\nabla / Ker\pi.$$

We set $Ker\ \pi := \mathcal{H}$. Then $\beta \in \mathcal{H}$ if and only if $\{\beta, {F_0^*}^\perp\} \subset F_0^*$. As $\{E^*, E^*\} \subset \mathbf{C}$, by the Jacobi identity

$$\{\{\beta,\delta\},\alpha\} = \{\beta,\{\delta,\alpha\}\} - \{\delta,\{\beta,\alpha\}\},$$

(with $\beta \in Lie\nabla$, $\delta \in E^*$, $\alpha \in F_0^*$), we get the inclusion $\{Lie\nabla, E^*\} \subset {F_0^*}^\perp$. Applying again the Jacobi identity in the form

$$\{\{\beta,\beta'\},\delta\} = \{\beta,\{\beta',\delta\}\} - \{\beta',\{\beta,\delta\}\},$$

with β,$\beta' \in \mathcal{H}$ and $\delta \in E^*$ (we note that this identity is simply the expression of the action of the commutator $[A,B]$ as $AB-BA$ in the usual linear representation), one obtain $\{\{\beta,\beta'\},\delta\} \in F_0^*$. So, as the algebra $\mathcal{H}$ annihilate F_0^* we get that the derived algebra $\mathcal{H}'$ is abelian, i.e., $\mathcal{H}$ is metaabelian and, in particular, solvable.

We set $H = Ker\ \phi$. The group H is an algebraic subgroup of $Gal\ \nabla$. Using the results of the preceding subsection we can interpret H as a differential Galois group and we recall that this group is Zariski connected. Its Lie algebra is $\mathcal{H}$, therefore H is metaabelian.

From the classical Picard-Vessiot theory we get

Corollary 2. *We have the following statements*
(i) The linear differential equation corresponding to the connection (∇, V, Ω) *is solvable if and only if the reduced equation corresponding to* (∇_N, N, Ω_N) *is solvable.*
(ii) If the identity component of $Gal\nabla$ *is abelian then both, the identity component of* $Gal\nabla_N$ *and the group* H*, are also abelian.*

Proof. (i) follows from the general group theoretic fact that any subgroup and any quotient group of a solvable group is solvable and conversely, if a normal subgroup and the corresponding quotient group are solvable then the original group is solvable. Claim (ii) is a direct consequence of the above proposition.

In the above corollary (i) express the meaning of the reduction: we can solve (in the Galois differential sense) the linear equation corresponding to ∇ when we know the solutions of the linear equation corresponding to ∇_N.

5. Non integrability.

5.1. Algebraic preliminaries. Let V be a *symplectic* complex space. We set $dim_{\mathbf{C}} V = 2n$. We choose a symplectic basis $\{e_1, \dots, e_n; \varepsilon_1, \dots, \varepsilon_n\}$ and we denote by $(x_1, \dots x_n; y_1, \dots, y_n)$ the coordinates of $v \in E$ in this basis. If $< v, v' >$ is the symplectic product of $v, v' \in E$, then

$$< v, v' >= \sum_{i=1}^{n} x_i y_i' - x_i' y_i.$$

We set

$$\mathbf{C}[V] = \bigoplus_{k \geq 0} S^k V^*.$$

We endow $\mathbf{C}[V]$ with the ordinary multiplication. We get the commutative $\mathbf{C}$-algebra of polynomials on V. We denote by $\mathbf{C}(V)$ the field of fractions of $\mathbf{C}[V]$. Using the Poisson product, we endow $\mathbf{C}(V)$ with a structure of non commutative $\mathbf{C}$-algebra, the Poisson structure.

Using coordinates we can compute the Poisson product of $f, g \in \mathbf{C}(V)$:

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i}.$$

The two products on $\mathbf{C}(V)$ are related by the Leibniz rule

$$\{fg, h\} = f\{g, h\} + g\{f, h\}.$$

Let $A \subset \mathbf{C}(V)$ be a complex vector subspace. If it is stable by the Poisson product, then A is a Poisson subalgebra of $\mathbf{C}(V)$. The field of fractions of A is also a Poisson subalgebra of $\mathbf{C}(V)$.

Let A be a subset of $\mathbf{C}(V)$. If, for each pair $f, g \in A$, we have $\{f, g\} = 0$, we will say that A is an *involutive* subset. Then the complex vector subspace generated by A is also involutive and is a Poisson subalgebra. Using the Leibniz rule we verify that the subalgebra (for the ordinary product) generated by A is involutive and is also a Poisson subalgebra.

A subset of an involutive subset is clearly also an involutive subset.

Let A be a subset of $\mathbf{C}(V)$. We define the orthogonal $A^\perp$ of A in $\mathbf{C}(V)$ by

$$A^\perp = \{f \in \mathbf{C}(V) | \{f, g\} = 0 \text{ for every } g \in A\}.$$

The biorthogonal of A is $A^{\perp\perp} = (A^\perp)^\perp$.

Using the Leibniz rule and the Jacobi identity we verify immediately that $A^\perp$ is a *subalgebra* and a *Poisson subalgebra* of $\mathbf{C}(V)$. Therefore $A^{\perp\perp}$ is also a subalgebra and a Poisson subalgebra of $\mathbf{C}(V)$.

Let A be a subset of $\mathbf{C}[V]$. It is involutive if and only if we have the inclusion $A \subset A^\perp$. Moreover, if A is involutive, we have the inclusions

$$A \subset A^{\perp\perp} \subset A^\perp$$

and $A^{\perp\perp}$ is an *involutive* subalgebra of $\mathbf{C}(V)$.

Let A be a subalgebra of $\mathbf{C}(V)$ (for the ordinary product). We will say that $f \in \mathbf{C}(V)$ is *algebraic* over A if there exists a non trivial polynomial $P \in A[Z]$ such that $P(f) = 0$. The algebraic closure $\bar{A}$ of A in $\mathbf{C}(V)$ is by definition the set of the

$f \in \mathbf{C}(V)$ which are algebraic over A. The algebraic closure $\bar{A}$ of A in $\mathbf{C}(V)$ is a subfield (it is the algebraic closure of the field of fractions of A).

The following result and its corollary are essential in our paper.

PROPOSITION 8. *Let $A \subset \mathbf{C}(V)$ be an involutive subalgebra (that is a subalgebra for the ordinary product which is also an involutive subset). Let $\bar{A} \subset \mathbf{C}[V]$ be the algebraic closure of A in $\mathbf{C}(V)$. Then we have inclusions*

$$A \subset \bar{A} \subset A^{\perp\perp} \subset A^{\perp}$$

and $\bar{A}$ is an involutive subalgebra of $\mathbf{C}(V)$.

Let $\beta \in \bar{A}$. Let $P \in A[Z]$ be a *minimal* polynomial for β.

We choose $\beta' \in A^{\perp}$. From $P(\beta) = 0$, we get $\{P(\beta), \beta'\} = 0$. Using $\beta \in \bar{A}$ and the Leibniz rule we see easily that the operator $\{., \beta'\}$ is a A-linear derivation on $A[\beta]$, therefore

$$\{P(\beta), \beta'\} = \frac{\partial P}{\partial Z}(\beta)\{\beta, \beta'\} = 0.$$

As the polynomial P is minimal, we have $\frac{\partial P}{\partial Z}(\beta) \neq 0$ and therefore $\{\beta, \beta'\} = 0$. This yields $\bar{A} \subset A^{\perp\perp}$. As $A^{\perp\perp}$ is involutive, the subalgebra $\bar{A}$ is also involutive.

COROLLARY 3. *Let V be a symplectic complex space. We set $\dim_{\mathbf{C}} V = 2n$. Let $A \subset \mathbf{C}(V)$ be a subalgebra (for the ordinary product) which is generated by a finite involutive subset $\alpha = (\alpha_1, \ldots, \alpha_n)$. We suppose that the n elements $\alpha_1, \ldots, \alpha_n$ are algebraically independent. Then*

(i) A is an involutive subalgebra,

(ii) $A^{\perp}$ is an involutive subalgebra,

(iii) $\bar{A} = A^{\perp} = A^{\perp\perp}$.

Claim *(i)* is evident.

Let $f \in A^{\perp}$. It is orthogonal to $\alpha \subset A$. The n elements $\alpha_1, \ldots, \alpha_n$ are algebraically independent and in involution, therefore f and $\alpha_1, \ldots, \alpha_n$ are *algebraically dependent*; we admit this claim and we will prove it later: cf. Corollary 4 below. Then $f \in \bar{A}$. We get an inclusion $A^{\perp} \subset \bar{A}$. Using the proposition, we get also the inclusions $\bar{A} \subset A^{\perp\perp} \subset A^{\perp}$. Therefore $\bar{A} = A^{\perp} = A^{\perp\perp}$. The subalgebra $A^{\perp\perp}$ is involutive. Claim *(ii)* follows.

We set as usual

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where I_n is the (n, n) identity matrix.

We have $J^t = -J = J^{-1}$.

A $(2n, 2n)$ matrix M is symplectic if and only if

$$M^t J M = J.$$

A $(2n, 2n)$ matrix M is in the Lie algebra of the Lie group of symplectic matrices if and only if

$$M^t J + J M = 0.$$

This is equivalent to $(JM)^t = JM$, that is to the fact that the matrix JM is *symmetric*.

The symplectic structure on the symplectic space V gives the musical isomorphism

$$\flat : V \to V^*.$$

If X is the column vectors of the coordinates of $v \in V$ in the chosen symplectic basis, then the column vector of the coordinates of $\flat(v)$ in the dual basis is JX.

Using the contraction $V^* \otimes V \to \mathbf{C}$ between the first and the third factor in $V^* \otimes V \otimes V$, we get an homomorphism

$$V^* \otimes V \to End(V).$$

It is well known that it is an isomorphism and in general we will identify $V^* \otimes V$ and $End(V)$ modulo this isomorphism.

We set $\psi = id_{V^*} \otimes (\frac{1}{2}\flat)$. The map $\psi : V^* \otimes V \to V^* \otimes V^*$ is an isomorphism. We can interpret ψ as an isomorphism $End(V) \to V^* \otimes V^*$. An element $u \in End(V)$ belongs to the Lie algebra $sp(V)$ if and only if $\psi(u)$ is invariant by the symmetry

$$\begin{aligned} V^* \otimes V^* &\to V^* \otimes V^* \\ v \otimes w &\mapsto w \otimes v. \end{aligned}$$

Then $\psi(u)$ defines an element of S^2V^* and ψ induces an isomorphism

$$\phi : sp(V) \to S^2V^*.$$

If we use as before the chosen symplectic basis of V and the dual basis of V^* and if we denote respectively by M and M' the matrices of $u \in sp(V)$ and the matrix of the quadratic form corresponding to $\phi(u) \in S^2E^*$, then

$$M' = \frac{1}{2}JM.$$

We define an operation of the Lie algebra $End(V)$ on V^* by $w \mapsto -u^t(w)$. Using this operation and the natural operation of $End(V)$ on V, we get an operation of $End(V)$ on $V^* \otimes V$:

$$w \otimes v \mapsto -u^t(w) \otimes v + w \otimes u(v).$$

If we identify $V^* \otimes V$ with $End(V)$, then the corresponding operator is $[u, .]$.

Lemma 3. *(i) Modulo the isomorphism $\phi : sp(V) \to S^2V^*$, the operation of $sp(V)$ on V^* defined above corresponds to the action of S^2V^* on V^* by the Poisson product.*

(ii) Modulo the isomorphisms $\phi : sp(V) \to S^2V^*$ and $\flat : V \to V^*$, the natural operation of $sp(V)$ on V corresponds to the action of S^2V^* on V^* by the Poisson product.

We use the chosen canonical basis on V and the dual basis on V^*. Let M be the matrix of $u \in sp(V)$. Then $M' = \frac{1}{2}JM$ is a symmetric matrix. It is the matrix of the quadratic form corresponding to $\phi(u)$. We denote by X (resp. X') the column vector of the coordinates of $v \in V^*$ (resp. $v' \in V^*$) in the dual basis. We set $X^t = (x; y)$, $X'^t = (x'; y')$. We set $f(x; y) = X^t M' X$ and $g(x; y) = X'^t X$. We denote $M' = \frac{1}{2}\begin{pmatrix} A & B \\ B^t & D \end{pmatrix}$. The matrices A and D are symmetric. Then $2f(x;y) = x^tAx + y^tDy + x^tBy + y^tB^tx$ and $g(x;y) = x'^tx + y'^ty = x^tx' + y^ty'$.

We have

$$\{f,g\}(x;y) = x^t(-Ay' + Bx') + y^t(-B^ty' + Dx').$$

Then $\{f,g\}(x;y) = X^tX" = X"^tX$, where $X" = M"X'$, with $M" = \begin{pmatrix} B & -A \\ D & -B^t \end{pmatrix}$.

We have $M = -2JM' = \begin{pmatrix} -B^t & -D \\ A & B \end{pmatrix}$ and $-M^t = \begin{pmatrix} B & -A \\ D & -B^t \end{pmatrix}$. We have finally $M" = -M^t$. That ends the proof of claim *(i)*.

Claim *(ii)* follows from the equality

$$-J\begin{pmatrix} B & -A \\ D & -B^t \end{pmatrix}J = \begin{pmatrix} -B^t & -D \\ A & B \end{pmatrix}$$
$$-JM"J = M.$$

LEMMA 4. *Let $\phi : sp(V) \to S^2V^*$ be the isomorphism of complex vector spaces defined by $\psi = id_{V^*} \otimes \frac{1}{2}\flat$. If $sp(V)$ is endowed with its Lie algebra structure and if S^2V^* is endowed with its Poisson algebra structure, then ϕ is an isomorphism of Lie algebras.*

We can define a Poisson action of S^2V^* on $V^* \otimes V^*$: $\{f, g \otimes h\} = \{f,g\} \otimes h + g \otimes \{f,h\}$.

Using the preceding lemma we see that the action of $sp(V)$ on $V^* \otimes V$ defined above corresponds with the Poisson action of S^2V^* on $V^* \otimes V^*$ modulo the isomorphisms $\phi : sp(V) \to S^2V^*$ and $id_{V^*} \otimes \frac{1}{2}\flat : V^* \otimes V \to V^* \otimes V^*$. (In claim *(ii)* of the preceding lemma, we can replace the isomorphism $\flat$ by the isomorphism $\frac{1}{2}\flat$.) The isomorphism $id_{V^*} \otimes \frac{1}{2}\flat : V^* \otimes V \to V^* \otimes V^*$ induces the isomorphism ϕ and the action of $u \in sp(V)$ on $sp(V)$ is $v \mapsto [u,v]$. That ends the proof of the lemma.

Let $u \in sp(V)$. Using the action of u on V^* that we have defined above we get an action on $\mathbf{C}[V]$. We denote $f \mapsto u.f$ this action.

Let $\mathcal{G} \subset sp(V)$ be a Lie subalgebra.

We recall that $f \in \mathbf{C}[V]$ is an *invariant* of $\mathcal{G}$ if $u.f = 0$ for every $u \in \mathcal{G}$.

THEOREM 6. *Let V be a symplectic complex space. We set $\dim_{\mathbf{C}} V = 2n$. Let $\mathcal{G} \subset sp(V)$ be a Lie subalgebra. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a finite involutive subset. We suppose that the n elements $\alpha_1, \ldots, \alpha_n$ are algebraically independent. We suppose that $\alpha_1, \ldots, \alpha_n$ are invariants of $\mathcal{G}$. Then the Lie algebra $\mathcal{G}$ is abelian.*

Using a preceding lemma we see that $u.f = 0$ is equivalent to $\{\phi(u), f\} = 0$. We denote by A the subalgebra generated by α. This algebra is involutive, $A^\perp$ is involutive (cf. corollary) and $\phi(\mathcal{G}) \subset A^\perp$. As $\phi(\mathcal{G})$ is a Poisson algebra isomorphic to the Lie algebra $\mathcal{G}$, the result follows.

Let $U \subset \mathbf{C}^n$ be a connected open subset. We denote by $\mathcal{O}(U)$ the $\mathbf{C}$-algebra of holomorphic functions on U. We denote by $\mathcal{M}(U)$ the field of meromorphic functions on U, that is the fraction field of $\mathcal{O}(U)$. Let $f_1, \ldots, f_m \in \mathcal{M}(U)$. We will say that they are *functionally dependent* if there exists a non trivial relation $\sum_{i=1}^m g_i df_i = 0$ with $g_1, \ldots, g_m \in \mathcal{M}(U)$ $(g_1, \ldots, g_m \neq 0)$, that is if the meromorphic differential forms $df_1, \ldots, df_m$ are linearly dependent upon the field $\mathcal{M}(U)$.

It is easy to prove the following result.

Lemma 5. *Let* $U \subset \mathbf{C}^n$ *be a connected open subset. Let* $f_1, \ldots, f_m \in \mathcal{M}(U)$. *The following conditions are equivalent:*

(i) $f_1, \ldots, f_m$ *are functionally independent;*

(ii) there exists an open connected dense subset $V \subset U$ *such that* $f_1, \ldots, f_m \in \mathcal{O}(V)$ *and* $\mathrm{rank}_{\mathbf{C}} \quad df_1(x), \ldots, df_m(x) = m$ *for every* $x \in V$;

(iii) there exists a point $x \in U$ *such that* $f_1, \ldots, f_m$ *are holomorphic at* x *and such that* $\mathrm{rank}_{\mathbf{C}} \quad (df_1(x), \ldots, df_m(x)) = m$.

Proposition 9. *In* $C(V) \approx \mathbf{C}(x_1, \ldots, x_n)$ *functional (over some open set* U *of* V*) and algebraic dependence (over* $\mathbf{C}$*) are equivalent*

This result is well-known and it is proved (for instance) in [5], Proposition 1.16, p. 12. We will give the proof for completeness.

For proving the Proposition 9, we recall the following result ([5]), Proposition 1.15, p. 11).

Proposition 10. *Let* $L \subset K$ *be a field extension of 0-characteristic fields. Then any derivation on* L *extends to a derivation on* K.

Let $f_1, \ldots, f_m \in \mathbf{C}(x_1, \ldots, x_n)$. If $P(f_1, \ldots, f_m) = 0$ for some polynomial $P \in \mathbf{C}[Z_1, \ldots, Z_m]$, then $dP(f_1, \ldots, f_m) = \sum \frac{\partial}{\partial Z_i} P(f_1, \ldots, f_m) df_i = 0$, therefore f_1, ..., f_m are functionally dependent.

Conversely, if the f_i's are algebraically independent then $\mathbf{C}(f) = \mathbf{C}(f_1, \ldots, f_m)$ is a transcendental extension of degree m of $\mathbf{C}$ and the differential operators $\partial/\partial f_i$ are well defined on $\mathbf{C}(f)$. We have $\frac{\partial}{\partial f_i} f_j = \delta_{ij}$. We have a field extension $\mathbf{C}(f) \subset \mathbf{C}(x_1, \ldots, x_n)$, therefore the differential operators $\partial/\partial f_i$ extend to derivations D_i on $\mathbf{C}(x_1, \ldots, x_n)$. We have clearly $D_i f_j = \delta_{ij}$. We define the vector fields $X_i = \sum_{j=1,\ldots,n} D_i(x_j)\partial/\partial x_j$ $(i = 1, \ldots, m)$. For every $g \in \mathbf{C}(x_1, \ldots, x_n)$ we have $(dg, X_i) = D_i g$. Let $g_1, \ldots, g_m \in \mathbf{C}(x_1, \ldots, x_n)$ such that $\sum_{i=1,\ldots,m} g_i df_i = 0$. If we contract this relation with the vector field X_k, we get $g_k = 0$. Therefore the f_i's are functionally independent.

Proposition 11. *Let* V *be a symplectic complex space. We set* $\dim_{\mathbf{C}} V = 2n$. *Let* $\mathbf{C}(V)$ *be the field of rational functions on* V. *Let* $f_1, \ldots, f_{n+1} \in \mathbf{C}(V) \approx \mathbf{C}(x_1, \ldots, x_{2n})$ *be in involution. Then the functions* $f_1, ..., f_n, f_{n+1}$ *are functionally dependent over an open domain* $U \subset V$.

This proposition is well-known in the real (differentiable) case. In the complex case the same proof works well. We shall give the proof for completeness.

We can interpret $\flat$ as an isomorphism between the holomorphic fiber bundles TV (tangent bundle) and T^*V (cotangent bundle). We denote by $\natural$ the inverse isomorphism.

If we assume that the functions $f_1, ..., f_n, f_{n+1}$ are functionally independent, then they are regular and $\mathrm{rank}(df_1, ..., df_{n+1}) = n + 1$, on a dense open domain U.

The $\mathbf{C}$-linear forms $df_1(x), \ldots, df_{n+1}(x) \in T_x V^*$ are linearly independent for every $x \in U$. Let $x_0 \in U$. We set $f_i(x_0) = c_i \in \mathbf{C}$. The subset $\Sigma = \{f_1 = c_1, \ldots, f_{n+1} = c_{n+1}\} \subset U$ is an analytic (smooth) *submanifold* of complex dimension $n - 1$. The vector fields $Y_i = \natural df_i$ $(i = 1, \ldots n + 1)$ are *tangent to* Σ $(df_i(Y_j) = \{f_i, f_j\} = 0)$ and *linearly independent* over the complex field at each point of V. (The linear map $\natural$ induces an isomorphism between $T_x V^*$ and $T_x V$.) This implies *dim* $\Sigma \geq n + 1$ and we get a contradiction.

Corollary 4. *Let V be a symplectic complex space. We set $\dim_{\mathbf{C}} V = 2n$. Let $f_1, \ldots, f_{n+1} \in \mathbf{C}(V) \approx \mathbf{C}(x_1, \ldots, x_{2n})$ in involution. Then*

(i) $f_1, \ldots, f_{n+1}$ are algebraically dependent

(ii) if, moreover $f_1, \ldots, f_n$ are algebraically independent, then f_{n+1} is algebraic over the $\mathbf{C}$-algebra generated by $f_1, \ldots, f_n$.

The functions $f_1, \ldots, f_{n+1}$ are functionally dependent (on some open subset), therefore they are algebraically dependent (over the complex field $\mathbf{C}$).

If the functions $f_1, \ldots, f_n$ are algebraically independent, then we get a relation $P(f_1, \ldots, f_{n+1}) = A_m f_{n+1}^m + \ldots + A_0 = 0$, where $P \in \mathbf{C}[F_1, \ldots, F_{n+1}]$ and $A_0, \ldots, A_m \in \mathbf{C}[F_1, \ldots, F_n] \approx \mathbf{C}[f_1, \ldots, f_n]$ (being done the last isomorphism by the algebraic independence of $f_1, \ldots, f_n$) with $m > 0$.

With this result we can end the proof of Corollary 3 and therefore the proof of Theorem 6.

5.2. Main result. Let E be a complex vector space of dimension $m \geq 1$. As above we denote by $\mathbf{C}[E]$ the $\mathbf{C}$-algebra of polynomial functions on E, and by $\mathbf{C}(E)$ the field of rational functions on E (i.e. the quotient field of $\mathbf{C}[E]$).

Let $G \subset GL(E)$ be an algebraic subgroup. We define a left action of G on $\mathbf{C}[E]$ or $\mathbf{C}(E)$ by $(g, f) = g.f = f \circ g^{-1}$ $(g \in G\ , f \in \mathbf{C}(E))$. (It corresponds clearly to the usual action of G on the constructions over E.) Let $\mathcal{G}$ be the Lie algebra of G. If $u \in \mathcal{G} \subset End(E)$, we define its action on V^* by $-{}^t u$ and its action on $E^* \otimes E^*$ by $-{}^t u \otimes 1 - 1 \otimes^t u$. Its natural action $f \mapsto u \bullet f$ on $\mathbf{C}[E]$ (isomorphic to the symmetric tensor algebra $S^*(E^*)$) or on $\mathbf{C}(E)$ follows using evident formulas.

We define by $\mathbf{C}[E]^G$ (resp $\mathbf{C}(E)^G$) the $\mathbf{C}$-algebra of G-invariant elements of $\mathbf{C}[E]$ (resp $\mathbf{C}(E)$) (i.e. those $f \in \mathbf{C}[E]$ (resp. $\mathbf{C}(E)$) such that $g.f = f$, for all $g \in G$). If $f \in \mathbf{C}(V)^G$, then $u \bullet f = 0$ for all $u \in \mathcal{G}$.

We can clearly identify $\mathbf{C}$ with a subfield of $\mathbf{C}(E)^G$. As in [14, 5], for $r \geq 1$, we will say that an algebraic group G is r-Ziglin if $\mathrm{transdeg}_{\mathbf{C}} \mathbf{C}(E)^G \geq r$. We will say that an algebraic group G is r-involutive Ziglin if there exists r algebraically independent elements $f_1, \ldots, f_r \in \mathbf{C}(E)^G$ in involution.

We can state now our main results.

In order to facilitate the main applications to non academic problems, we will give three versions of this main result and add some corollaries for local situations. (Each statement will generalize the preceding.)

Let Γ be the abstract Riemann surface defined by a non stationary connected integral curve $i(\Gamma)$ of an analytic Hamiltonian system X_H with n degrees of freedom on a symplectic complex manifold M.

Theorem 7. *If there are n meromorphic first integrals of X_H which are in involution and independent over a neighbourhood of the curve $i(\Gamma)$ (not necessarily on Γ itself), then the Galois group of the VE over Γ is n-involutive Ziglin. This Galois group is the Zariski closure of the monodromy group. Furthermore, the identity component of the Galois group of the VE over Γ is abelian.*

The following result is a consequence of a theorem of Ziglin ([74] Theorem 2, Remark 1). As an exercise, we give here a different proof. In fact Ziglin's result is stronger: he does not need involutivity hypothesis. We will come back to Ziglin's theorem later.

Corollary 5. *We suppose that there are n meromorphic first integrals of X_H which are in involution and independent over a neighbourhood of the curve $i(\Gamma)$ (not* *necessarily on* Γ *itself). Let* g, g' *be elements of the monodromy group. We suppose that they are non resonant. Then* g *must commute with* g'.

Proof. Let G' be the Zariski closure in the Galois group G of the subgroup generated by g and g'. Let H (resp. H') be the Zariski closure of the subgroup generated by g (resp. g'). Because g and g' are non resonant, the groups H and H' are maximal torus in the symplectic group, i.e., they are conjugate to the multiplicative group of dimension n

$$T := \{diag(\lambda_1, ..., \lambda_n, \lambda_1^{-1}, ..., \lambda_n^{-1}, \lambda_i \in \mathbf{C}^*, i = 1, ..., n\}.$$

(see [5], proposition 2.13)

Therefore they are Zariski-connected. It follows that $G' \subset G^0$. Applying theorem 7 we see that G' is abelian, therefore g and g' commute.

We add now to the curve $i(\Gamma)$ a discrete set of stationary points. We get a singular curve $\underline{\Gamma} \subset M$. Let $\bar{\Gamma}$ be a non singular model of $\underline{\Gamma}$.

THEOREM 8. *If there are* n *first integrals of* X_H *which are meromorphic and in involution over a neighbourhood of the curve* $\underline{\Gamma}$ *and independent in a neighbourhood of* Γ *(not necessarily on* Γ *itself), then the Galois group of the meromorphic VE over* $\bar{\Gamma}$ *is* n*-involutive Ziglin. Furthermore, the identity component of the Galois group of the VE over* $\bar{\Gamma}$ *is abelian.*

We add now to the symplectic manifold (M, ω) a hypersurface at infinity M_∞ ($M' = M \cup M_\infty$) and to the curve $i(\Gamma)$ a discrete set of stationary points and a discrete set of points at infinity. We get a singular curve $\underline{\Gamma}' \subset M'$. (We assume as before that ω admits a meromorphic extension over M'.) Let $\bar{\Gamma}'$ be a non singular model of $\underline{\Gamma}'$.

THEOREM 9. *If there are* n *first integrals of* X_H *which are meromorphic and in involution over a neighbourhood of the curve* $\underline{\Gamma}'$ *in* M' *(in particular meromorphic at infinity) and independent in a neighbourhood of* Γ *(not necessarily on* Γ *itself), then the Galois group of the meromorphic VE over* $\bar{\Gamma}'$ *is* n*-involutive Ziglin. Furthermore, the identity component of the Galois group of the VE over* $\bar{\Gamma}'$ *is abelian.*

Be careful: when we are in the third case we can consider *three different* Galois groups corresponding to the variational equations over *three* (in general different) Riemann surfaces (Γ, $\bar{\Gamma}$ and $\bar{\Gamma}'$). (Each group contains the preceding.) But our abelian criterion is less and less precise (the set of *allowed* first integrals is *smaller* at each step...). Unfortunately it is in general difficult to compute the differential Galois group. (If the Riemann surface is open it is a transcendental problem.) The best situation is when, in the second (resp. the third case), the Riemann surface $\bar{\Gamma}$ (resp. $\bar{\Gamma}'$) is compact (and therefore algebraic). Then the connection is defined over a finite extension of the rational functions field $\mathbf{C}(z)$ and there exists an algebraic algorithm to decide if the identity component of the differential Galois group is *solvable* and more precisely there exists a procedure to find a basis for the space of Liouvillian solutions ([63, 42]). So in that situation we get the existence of a *purely algebraic criterion* (unfortunately not yet effective...) for *rational non integrability.* (Notice that, if the manifold M' is algebraic, then $\bar{\Gamma}'$ is algebraic.)

It is important to notice that, if the meromorphic VE over $\bar{\Gamma}'$ is *regular singular* (i.e. of Fuchs type), then our three differential Galois groups coincide. Then, if we are in the algebraic situation that we described above, we get an obstruction not only to the existence of n *rational* first integrals in involution, but more generally to the existence of n first integrals *meromorphic* on the initial manifold M and in involution. (An arbitrary growth at infinity is allowed.)

In fact in many practical situations, the situation is the following: the Riemann surface Γ is an affine curve (i.e. $\Gamma = \Gamma'' - S$, where Γ'' is a compact Riemann surface and S a finite subset), the VE (resp. the NVE) is a holomorphic connection ∇ on a trivial holomorphic bundle over the Riemann surface Γ and can be extended as a meromorphic connection ∇'' on a trivial bundle over Γ''. If moreover this last connection ∇'' is *regular singular*, then the differential Galois groups of ∇ and ∇" coincide. Therefore we can (theoretically...) compute algebraically the differential Galois group and we can apply Theorem 7. Of course we will have in general $\Gamma'' = \bar{\Gamma}$ or $\bar{\Gamma}'$, but, for the applications it is not necessary to verify this fact! These remark will be very useful for some important (non academic...) applications. We can conclude that Theorems 8 and 9 are really interesting when we get *irregular singularities* at the singular points (stationary points or points at infinity), in particular in the local situations that we will describe now.

In the following two corollaries we will give some *local* versions of our results.

Locally on Γ or at a regular-singular point of $\bar{\Gamma}$, Ziglin's Theorem or our main result give nothing. But, using our main result, we can get some proofs of local non-integrability at a stationary point (or at a point at infinity) in some cases (cf. below: 6. Example 1).

Let X_H be an analytic Hamiltonian system with n degrees of freedom on a symplectic complex manifold M. Let a be a stationary point of X_H. Let $\underline{\Gamma}$ be a germ of (perhaps singular) analytic curve at a which is the union of $\{a\}$ and a connected non stationary germ of phase curve. Let $\bar{\Gamma}$ be a germ of smooth holomorphic curve which is a non singular model for $\underline{\Gamma}$.

Corollary 6. *If there are n germs at a of meromorphic first integrals of X_H which are in involution in a neighbourhood of a (in particular meromorphic at infinity) and independent in a neighbourhood of a (not necessarily on Γ itself), then the local Galois group of the meromorphic germ at a of VE over $\bar{\Gamma}$ is n-involutive Ziglin. Furthermore, the identity component of the local Galois group of the germ at a of VE over $\bar{\Gamma}$ is abelian.*

We add now to the symplectic manifold (M, ω) an hypersurface at infinity M_∞ ($M' = M \cup M_\infty$) and to the curve $i(\Gamma)$ a point at infinity $\infty \in M_\infty$. We get a germ at ∞ of singular curve $\underline{\Gamma}'$. (We suppose that ω admits a meromorphic extension at ∞.)

Corollary 7. *If there are n germs at ∞ of first integrals of X_H which are meromorphic and in involution in a neighbourhood of ∞ in M' and independent in a neighbourhood of ∞ (not necessarily on Γ itself), then the local Galois group of the meromorphic germ at ∞ of VE over $\bar{\Gamma}'$ is n-involutive Ziglin. Furthermore, the identity component of the local Galois group of the germ of VE over $\bar{\Gamma}'$ is abelian.*

We will now prove Theorem 7. Later we will indicate how to modify the proof in order to get theorems 8 and 9.

Let V, ∇ be the holomorphic symplectic vector bundle and the connection corresponding to the variational equation of our Hamiltonian system along the solution Γ. On the symmetric bundle S^*V^* of polynomials we can define the structure of a

Poisson "algebra" (over the sheaf $\mathcal{O}_\Gamma$ of $\mathbf{C}$-algebras of holomorphic functions on Γ) in the following way.

Let d be the differential over the fiber, i.e.

$$d : S^k V^* \longrightarrow S^{k-1} V^* \otimes V^*,$$
$$d\alpha = \sum \frac{\partial \alpha}{\partial \eta_i} \otimes \eta_i,$$

being $\eta_1, ..., \eta_{2n}$ fiber coordinates in the bundle V^* (this is a special case of the differential of Spencer).

Then we obtain the mappings,

$$d \otimes d : S^k V^* \otimes S^r V^* \longrightarrow (S^{k-1} V^* \otimes V^*) \otimes (S^{r-1} V^* \otimes V^*)$$
$$Id \otimes \natural : (S^{k-1} V^* \otimes V^* \otimes S^{r-1} V^*) \otimes V^* \longrightarrow S^{k-1} V^* \otimes V^* \otimes S^{r-1} V^* \otimes V^*$$
$$c : S^{k-1} V^* \otimes V^* \otimes S^{r-1} V^* \otimes V \longrightarrow S^{k-1} V^* \otimes S^{r-1} V^*$$
$$sym : S^{k-1} V^* \otimes S^{r-1} V^* \longrightarrow S^{k+r-2} V^*,$$

where $\natural := \flat^{-1}$, c and sim are duality by the symplectic structure (musical isomorphism), the contraction between V and V^*, and the symmetric product.

The Poisson bracket

$$\{,\} : S^k V^* \otimes S^r V^* \longrightarrow S^{k+r-2} V^*,$$

is the composition of the four above maps. In a direct way we can prove that it is the usual Poisson bracket in coordinates, if we only derivate with respect to the fiber, i.e.

$$\{\alpha, \beta\} = \sum \frac{\partial \alpha}{\partial \eta_i} \frac{\partial \beta}{\partial \xi_i} - \frac{\partial \alpha}{\partial \xi_i} \frac{\partial \beta}{\partial \eta_i},$$

in a canonical frame and canonical coordinates ξ, η. We can extend by bilinearity $\{,\}$ to all the symmetric algebra $S^* V^*$, and obtain a Poisson algebra $(S^* V^*, \{,\})$ (more precisely a $\mathcal{O}(\Gamma)$-Poisson "algebra").

From now on we fix a point $p_0 \in \Gamma$. Let $E_0 = Sol_{p_0} \nabla$ be the space of germs at p_0 of solutions (i.e. horizontal vectors of the connection ∇). We can associate to a germ of solution its initial condition at p_0. We get an isomorphism between E_0 and $E = V_{p_0} = T_{p_0} M$. The $\mathbf{C}$-algebra $\mathbf{C}[E]$ is a complex Poisson subalgebra of the complex Poisson algebra underlying the $\mathcal{O}(\Gamma)$-Poisson algebra $(S^* V^*, \{,\})$, and the natural isomorphism $E_0 \to E$ induces an isomorphism between this Poisson algebra and the natural Poisson algebra $(\mathbf{C}[E], \{,\})$ that we defined above (using the symplectic structure on $E = V_{p_0} = T_{p_0} M$). In the following we will identify these two algebras.

The Galois group G of the variational equation acts on $\mathbf{C}[E]$ and the algebra of invariants $\mathbf{C}[E]^G$ is also a Poisson subalgebra. Indeed since G is a symplectic group, G commutes with the symplectic form and, for $\sigma \in G$, $\alpha, \beta \in \mathbf{C}[E]^G$ $\sigma\{\alpha, \beta\} = \{\sigma\alpha, \sigma\beta\} = \{\alpha, \beta\}$.

We can now replace the holomorphic bundle $S^* V^*$ (whose sections are functions which are meromorphic on the basis and *polynomial* on the fiber) by the holomorphic locally trivial bundle LV^* whose sections are functions which are meromorphic on the basis and *rational* on the fibres. We extend easily the preceding constructions to this bundle. The $\mathbf{C}$-algebra $\mathbf{C}(E)$ is a complex Poisson subalgebra of the complex Poisson algebra underlying the $\mathcal{O}(\Gamma)$-Poisson algebra $(LV^*, \{,\})$, and the natural isomorphism $E_0 \to E$ induces an isomorphism between this Poisson algebra and the natural Poisson algebra $(\mathbf{C}(E), \{,\})$ that we defined above. In the following we will identify these two algebras. The Galois group G of the variational equation acts on $\mathbf{C}(E)$, it commutes with the Poisson product and the algebra of invariants $\mathbf{C}(E)^G$ is also a Poisson subalgebra.

In the following, by definition, a first integral of the variational equation (or of the corresponding connection ∇) is a meromorphic function defined on the total space of the bundle V, which is meromorphic over the basis and rational over the fibers (i.e. a meromorphic section of the bundle LV^*) and which is constant on the solutions (i. e. horizontal sections). As the symplectic fiber bundle V is meromorphically trivial (as a symplectic bundle), such a first integral can be interpreted as an element of $\mathcal{M}(\Gamma)(\eta_1, \ldots, \eta_{2n})$ $((1,0,\ldots,0),\ldots,(0,\ldots,0,1)$ corresponding to a global meromorphic symplectic frame).

Let f be a *holomorphic first integral* defined on a neighbourhood of the analytical curve $i(\Gamma)$. Then for any point $p \in \Gamma$ we define the junior part $[f]_p$ of f at p as the first non-vanishing homogeneous Taylor polynomial of f at p with respect to some coordinate system in the phase space. This process has an invariant meaning and the junior part $[f]_p$ must be considered as a homogeneous polynomial on the tangent space $T_pM = V_p$ at p (see [5] for the details). Furthermore, the degree $k \in \mathbf{N}$ of this polynomial is the same for any point $p \in \Gamma$ ([5], Proposition 1.25). In this way, when p varies in Γ, we obtain a holomorphic first integral (polynomial on the fibres) of the variational equation defined on the bundle $T_\Gamma M = V$ ... It is a holomorphic section of S^*V^*.

Let f be now a *meromorphic first integral* defined on a neighbourhood of the analytic curve $i(\Gamma)$. Then for any point $p \in \Gamma$ we can naturally extend the map $f \mapsto [f]_p$ to the fraction fields and define the junior part $[f]_p$ of the *meromorphic* first integral f at p. This junior part $[f]_p$ must be considered as a homogeneous rational function on the tangent space $T_pM = V_p$ at p. Furthermore, the degree $k \in \mathbf{Z}$ of this homogeneous rational function is the same for any point $p \in \Gamma$ ([5], Proposition 1.25). In this way, when p varies in Γ, we obtain a meromorphic first integral (rational on the fibres and holomorphic on the basis) of the variational equation defined on the bundle $T_\Gamma M = V$. It is a holomorphic section of LV^*.

Let f, g two meromorphic first integrals in involution in a neighbourhood of the analytic curve $i(\Gamma)$. If we denote respectively by f^0, g^0 the junior parts of them at p_0, then these rational functions are also in involution. Indeed $0 = \{f, g\} = \{f_k + h.o.t., g_r + h.o.t.\} = \{f_k, g_r\} + h.o.t.$, where the first term has the degree $k+r-2$. The involutivity of f^0 and g^0 follows from this and from the definition of the junior part ([5]).

Now we are going to recall a fundamental Lemma due by Ziglin. Let f be a holomorphic function defined over a neighbourhood of the origin in a finite dimensional complex vector space E. We define the junior part f^0 of f at the origin ([5]). It is an homogeneous element of the rational function field $\mathbf{C}(E)$.

Lemma 6 (Ziglin Lemma,[74]). *Let $f_1, ..., f_r$ be a set of meromorphic functions over a neighbourhood of the origin in the complex vector space E. We suppose that they are (functionally) independent over a punctured neighbourhood of the origin (they are not necessarily independent at the origin itself). Then there exists polynomials $P_i \in \mathbf{C}[X_1, ..., X_i]$ such that, if $g_i = P_i(f_1, \ldots, f_i)$, then the r rational functions $g_1^0, \ldots, g_r^0 \in \mathbf{C}(E)$ are algebraically independent.*

The following result is proved in [74], [5].

LEMMA 7. *Let V, ∇ be the holomorphic symplectic vector bundle and the connection corresponding to the variational equation over Γ. Let f^0 be a first integral of the variational equation, holomorphic over the basis and rational over the fibers. Let $p \in \Gamma$. Then the rational function f^0_p is invariant under the action of the monodromy group $\pi_1(M;p)$.*

The point p defines a representation of the differential Galois group G of the variational equation as a closed (in Zariski sense) subgroup of $GL(V_p)$. We will write $G \subset GL(V_p)$. Then the image $\rho(\pi_1(M;p))$ of the monodromy representation at p is a Zariski dense subgroup of G. We get the following result.

LEMMA 8. *Let V, ∇ be the holomorphic symplectic vector bundle and the connection corresponding to the variational equation over Γ. Let f^0 be a first integral of the variational equation, holomorphic over the basis and rational over the fibers. Let $p \in \Gamma$. Then the rational function f^0_p is invariant under the action of the differential Galois group of ∇.*

We will need generalizations of this lemma when we will have singular points (equilibrium points or points at infinity) and when we will consider variational equations over $\bar{\Gamma}$ or $\bar{\Gamma}'$. But in such cases it is in general not true that the image of the monodromy representation is dense in the Galois group and our preceding proof no longer works. Therefore we will give below a new proof of Lemma 8 which remains valid, mutatis mutandis, in *all cases*. It is very elementary (even almost trivial...) but central in the proof of our main results. In fact we will prove a slightly more general result.

LEMMA 9. *Let V, ∇ be the holomorphic symplectic vector bundle and the connection corresponding to the variational equation over Γ. Let f^0 be a first integral of the variational equation, meromorphic over the basis and rational over the fibers. Let $p \in \Gamma$. We suppose that f^0 is holomorphic over the basis in a neighborhood of p. Then the rational function f^0_p is invariant under the action of the differential Galois group of ∇.*

We shall give two different proofs of this lemma.

First Proof. As above we choose a global meromorphic symplectic frame

$$(\phi_1, \dots, \phi_{2n})$$

for the symplectic (meromorphically trivial) holomorphic bundle V over Γ. Using this frame, we identify the field of meromorphic sections of LV^* with $\mathcal{M}(\Gamma)(\eta_1, \dots, \eta_{2n})$. We fix a point $p \in \Gamma$. The frame allows us to identify the fiber V_p with the space $\mathbf{C}^{2n}$ (with its canonical symplectic structure). We can suppose that $(\phi_1, \dots, \phi_{2n})$ are holomorphic and independent at p. Then, in a neighbourhood of p, we can choose an uniformizing variable x over the basis and write the connection ∇ as a differential system $\Delta\eta = \frac{d}{dx}\eta - A(x)\eta$, where A is a holomorphic matrix.

Let $(\zeta_1(x), \dots, \zeta_{2n}(x))$ be the set of solutions satisfying the initial conditions $\zeta_1(0) = (1, 0, \dots, 0), \dots, \zeta_{2n}(0) = (0, \dots, 0, 1)$ at the point p $(x = 0)$. The system $(\zeta_1, \dots, \zeta_{2n})$ defines uniquely a Picard-Vessiot extension $\mathcal{M}(\Gamma) < \zeta_1, \dots, \zeta_{2n} >$ of the differential field $\mathcal{M}(\Gamma)$. Using this system we get a representation of the differential Galois group G of ∇ as a subgroup of $Sp(2n; \mathbf{C}) \subset GL(V_p)$.

Let $f^0 = f^0(x;\eta_1,\ldots,\eta_{2n}) \in \mathcal{M}(\Gamma)(\eta_1,\ldots,\eta_{2n})$ be a first integral of the variational equation, meromorphic over the basis and rational over the fibers. We suppose that f^0 is holomorphic over the basis in a neighborhood of p. To a fixed $\lambda = (\lambda_1,\ldots,\lambda_{2n}) \in \mathbf{C}^{2n}$ we associate the solution $\eta_\lambda = \lambda_1\zeta_1 + \ldots, \lambda_{2n}\zeta_{2n}$. In a neighborhood of p, we have $f^0(x;\eta_\lambda(x)) = f^0(0;\lambda) \in \mathbf{C}$. We can interpret $f^0(x;\eta_\lambda)$ as an element of the Picard-Vessiot extension $\mathcal{M}(\Gamma) < \zeta_1,\ldots,\zeta_{2n} >$. In this Picard-Vessiot extension this element is a constant (i.e. it belongs to $\mathbf{C}$), therefore it is invariant under the action of G. When λ varies in $\mathbf{C}^{2n}$ the function $\lambda \mapsto f^0(0;\lambda)$ is rational: it is the expression in coordinates of the function f^0_p.

Let $\sigma \in G$. Using the definition of a differential Galois group, we get $\sigma(f^0(x;\eta_\lambda)) = f^0(x;\sigma(\eta_\lambda)) = f^0(0;\lambda) = f^0(0;\mu)$, where $\mu = \sigma(\eta_\lambda)(0)$.

Then $\sigma\zeta = B\zeta$, with $B = (b_{ij}) \in Sp(2n;\mathbf{C})$. We have $\sigma(\sum_i \lambda_i\zeta_i) = \sum_{i,j} b_{ij}\lambda_i\zeta_j$. Therefore $\mu = {}^tB\lambda$ and $f^0(0;\lambda) = f^0(0;{}^tB\lambda)$. This proves the invariance of f^0_p under the action of G.

Second Proof. We sketch a second proof based upon Tannakian arguments. Let $f^0 \in LV^*$ be a first integral of the variational equation. We first suppose that f^0 is not a polynomial. Then we can write $f^0 = \frac{h}{g}$, being $h \in S^kV^*$ and $g \in S^rV^*$ relatively prime symmetric tensors. If v is a solution of the variational equation (that is $\nabla v = 0$) then the equation $X_h(f^0(v)) = 0$ is equivalent to the equation $(S^k\nabla^*h(v))g(v) - h(v)(S^r\nabla^*g(v)) = 0$. Consequently we get two equations

$$S^k\nabla^*h = ah,$$
$$S^r\nabla^*g = ag,$$

where a is an element of the coefficient field K of the variational equation.

We set $W = S^kV^* \oplus S^rV^*$ and $\nabla_W = S^k\nabla^* \oplus S^r\nabla^*$. The one-dimensional K-vector subspace $W' = K(h+g)$ of the K-vector space W is clearly stable under the action of the connection ∇_W. We denote by $\nabla_{W'}$ the restriction of ∇_W to W'. Hence we get a rank one subconnection $(W', \nabla_{W'})$ of the connection $(W, \nabla_W) = (S^kV^* \oplus S^rV^*, S^k\nabla^* \oplus S^r\nabla^*)$. This last connection is an object of the tensor category of the (generalized) constructions over ∇. Then we can choose a point $p \in \Gamma$ and introduce the corresponding fiber functor. The space of germs at p of horizontal sections of the subconnection $(W', \nabla_{W'})$ is a complex line in the complex space of horizontal sections of the connection (W, ∇_W). From the Tannakian definition of the Galois group G, this complex line is invariant by G. This complex line is generated over $\mathbf{C}$ by an element $\varphi(h_p + g_p)$ where $\varphi' + a\varphi = 0$. The invariance of the rational function $f^0_p = \frac{h_p}{g_p}$ by the Galois group G follows immediately. (The above proof it is not far from some arguments used in J.A. Weil's Thesis ([68]) for the study of Darboux's invariants.)

If f^0 is a polynomial, we set $f = f^0$, $g = 1$. The equation $X_h(f^0(v)) = 0$ is equivalent to the equation $(S^k\nabla^*f(v)) = 0$. Then we can replace in the preceding proof the connection $K(h+g), S^k\nabla^* \oplus S^r\nabla^*)$ by the connection $(K(1+f), \delta_K \oplus S^k\nabla)$. This connection is a rank one subconnection of the connection $(K \oplus S^kV^*, \delta_K \oplus S^k\nabla)$ and we can conclude as above (here the complex line $\mathbf{C}(f_p+1)$ is invariant by G and G acts trivially on it).

Let now $f_1, ..., f_n$ be a family of meromorphic first integrals of the Hamiltonian X_H in involution and independent over a neighbourhood of $i(\Gamma)$ (not necessarily on $i(\Gamma)$ itself).

If we apply Ziglin Lemma (see Lemma 6 above) at a point $p \in \Gamma$ to our functions $f_1, ..., f_n$, by all the arguments we gave above we get n homogeneous and independent (algebraically and analytically) functions $\alpha_{1_0}, ..., \alpha_{n_0}$ in the algebra $(\mathbf{C}(E)^G, \{,\})$. In other words, the abelian Lie algebra $(A, \{.\})$ of polynomials in $\alpha_{1_0}, ..., \alpha_{n_0}$ (with complex coefficients) is a Poisson subalgebra of $(\mathbf{C}(E)^G, \{,\})$ and it is invariant by the differential Galois group Γ of the variational equation, therefore it is annihilated by the Lie algebra $\mathcal{G} = Lie\, G$ of the algebraic group G. Then we finishes using Theorem 6 of section 5.1. This ends the proof of Theorem 7. The proofs of Theorems 8 and 9 are similar, with very simple modifications. The essential difference is the following. By hypothesis the first integrals $f_1, \ldots, f_n$ are meromorphic over the manifold M (resp. M'), in particular at the stationary points (resp. at the stationary points and at the points at infinity), therefore their junior parts $f_1^0, \ldots, f_n^0$ are *meromorphic* sections over $\bar{\Gamma}$ (resp. $\bar{\Gamma}'$) of the fiber bundle LV^*. (Their restrictions over Γ will of course remain holomorphic, but in general they will have poles at the singular points and at the points at infinity.)

If between our n meromorphic integrals there are some of them which are functionally independent over Γ, then using the results of section 4, we get

Corollary 8. *Let $f_1, ..., f_n$ be a family of meromorphic first integrals of the Hamiltonian X_H in involution and independent over a neighbourhood of $i(\Gamma)$ (not necessarily on $i(\Gamma)$ itself). If moreover, for a fixed integer $k \leq n$, the k first integrals $f_1, ..., f_k$ are (functionally) independent over Γ, then the Galois group of the NVE is $n-k$-involutive Ziglin. Furthermore the identity component of the Galois group of this NVE is abelian.*

There are similar statements in the situation of Theorem 8 (resp. 9): i.e. when $f_1, ..., f_n$ are meromorphic in a neighborhood of $\underline{\Gamma}$ (resp. $\underline{\Gamma}'$). We leave the details to the reader. The proof is essentially the same for the three cases. In order to perform the reduction, we complete $f_1, ..., f_n$ into a global meromorphic symplectic frame over Γ (resp. $\bar{\Gamma}$ or $\bar{\Gamma}'$) and we apply the process described above in 4.3. We get the NVE. It can have poles, in particular at the stationary points and at the points at infinity. The Galois group G' of the NVE is a quotient of the Galois group G of the VE. The identity component G^0 of G is abelian, therefore the identity component G'^0 of G' is also abelian. More precisely G^0 is an extension of G'^0 by an algebraic group isomorphic to some additive group $\mathbf{C}^p$. We remark that G'^0 can contain a non trivial torus isomorphic to some $\mathbf{C}^{*q}$ (cf. our examples below in section 6). So we observe the possibility of quite a big difference between the first integrals eligible for the normal reduction process and the others.

We remark that, as for the main Theorem, the conclusion of the corollary is the same if we restrict the NVE to a neighbourhood of some singular point s and if the Galois group is the local Galois group. In this way we can use our results in order to obtain non-existence of local first integrals in any neighbourhood of an equilibrium point or of one point at infinity.

Now we show how Ziglin's Theorem is a direct consequence of the above corollary when we assume the *complete* integrability of the system (i.e., when in the above corollary $n = 2$).

Corollary 9. *Let f be a meromorphic first integral of the two-degrees of freedom Hamiltonian system X_H. We suppose that f and H are independent over a neighbourhood of $i(\Gamma)$ (not necessarily on $i(\Gamma)$ itself). Moreover we assume that the* *monodromy group of the NVE contains a non-resonant transformation g. Then any other transformation belonging to this monodromy group sends eigendirections of g into eigendirections of g.*

Proof. First, we assume that, as in the above results of this section, the set $i(\Gamma)$ is not reduced to an equilibrium point. Then dH remains different from zero over $i(\Gamma)$ and the reduction to the NVE is made using the one form dH (or in a dual way the vector field X_H).

Then the NVE is given by a symplectic connection over a two dimensional vector space. Hence its Galois group is an algebraic group whose identity component is abelian and we can identify this group with a subgroup of $SL(2,\mathbf{C})$. In Proposition 1, we gave the classification of the algebraic subgroups of $SL(2,\mathbf{C})$. Here the only possible cases are cases 4 and 5, because for the others either the identity component of the Galois group is not abelian else all the elements of the Galois group are resonant. It is clear that in both cases 4 and 5 we have $g \in G^0$ (we recall that the group topologically generated by a non resonant element g is a torus, more precisely here this torus is maximal and we have $G^0 \approx \mathbf{C}^*$) and the remaining transformations belonging to the Galois group either preserve the two eigendirections of g else permute them.

In fact, a stronger result is true: Ziglin's general Theorem is a corollary of our results, as was remarked by Churchill [13]. That follows from the following theorem which generalizes the above corollary.

THEOREM 10. *Ziglin's Theorem (Theorem 1) is true mutatis mutandis if we substitute Galois group by monodromy group*

For a proof, see the above reference of Churchill.

Now, as the monodromy group is contained in the Galois group, we get the Theorem 1 as a corollary.

6. Examples. Let X_H be the Hamiltonian system defined by the Hamiltonian

$$H = T + U := 1/2(y_1^2 + y_2^2) + 1/2\varphi(x_1) + 1/2\alpha(x_1)x_2^2 + h.o.t.(x_2),$$

where x_i's are coordinates and y_i's canonically conjugated momenta ($i = 1,2$). We assume that this Hamiltonian is holomorphic at the origin.

The plane $\{x_2 = y_2 = 0\}$ is invariant and the Hamiltonian restricted to this plane is of the type studied in the example of subsection 4.1. We write $x := x_1$, $y := y_1$. Then we have the analytic integral curve $y^2 + \varphi(x) = 0$. We assume that $\varphi(x) = x^n + h.o.t.$, $n \geq 2$. We want to study the NVE along this integral curve in a neighbourhood of the origin (which is an equilibrium point).

The NVE is

$$\frac{d^2\xi}{dt^2} + \alpha(x(t))\xi = 0.$$

As

$$\frac{d}{dt} = (\pm\hat{t}^{n/2} + h.o.t.)\frac{d}{d\hat{t}}$$

for n even, and

$$\frac{d}{dt} = (\frac{1}{2}\hat{t}^{n-1} + h.o.t.)\frac{d}{d\hat{t}}$$

for n odd, we obtain for the corresponding NVE

$$\frac{d^2\xi}{d\hat{t}^2} + (\frac{n}{2\hat{t}} + h.o.t.)\frac{d\xi}{d\hat{t}} + \frac{\alpha(x(\hat{t}))}{\hat{t}^n}\xi = 0, \ n \text{ even},$$

$$\frac{d^2\xi}{d\hat{t}^2} + (\frac{n-1}{\hat{t}} + h.o.t.)\frac{d\xi}{d\hat{t}} + \frac{4\alpha(x(\hat{t}))}{\hat{t}^{2n-2}}\xi = 0, \ n \text{ odd}.$$

And if

$$\alpha(x) = a_k x^k + h.o.t., \ a_k \neq 0,$$

then, by the Fuchs Theorem about regular singular singularities, we get the following result that we state as a proposition for future references.

Proposition 12. *The origin (or more precisely its corresponding points in the desingularized curve) is a regular singular point of the NVE of the above Hamiltonian system along the integral curve* $y_1^2 + \varphi(x_1) = 0$ *if and only if* $n - k \leq 2$, *being* n *and* k *the multiplicity (as a zero) of* $x = 0$ *in* φ *and* α *respectively.*

We remark that the above proposition relates the degeneration of the equilibrium points to the irregularity of the corresponding singular points of the variational equation. Hence, the *degeneration* is related to the (possible) existence of *Stokes multipliers*.

Now we shall apply Theorem 5 of subsection 3.4 to our Hamiltonian system

$$H = T + U := 1/2(y_1^2 + y_2^2) + 1/2\varphi(x_1) + 1/2\alpha(x_1)x_2^2 + h.o.t.(x_2),$$

(with two degrees of freedom), with the integral irreducible analytic curve defined by $y^2 + \varphi(x) = 0$ (as above, we drop the subindexes). Furthermore we assume that φ and α are polynomials. Then $\overline{\Gamma}$ is a compact Riemann surface (see [31]) and the usual change of variables $x \leftrightarrow t$, $x = x(t)$ ($x(t)$ is the solution of the hyperelliptic differential equation $\dot{x}^2 + \varphi(x) = 0$) gives us a pullback of the NVE over the Riemann sphere. The classics call it the *algebraic form* of the equation, [70, 57]

$$\frac{d^2\xi}{dx^2} + \frac{\varphi'(x)}{2\varphi(x)}\frac{d\xi}{dx} - \frac{\alpha(x)}{2\varphi(x)}\xi = 0.$$

We will call this equation the *algebraic* NVE.

We observe that the singular points are the branching points of the covering (i.e., the roots of φ and the point at infinity). Concerning the equilibrium points of the original Hamiltonian, we see that $x = 0$ is a singular point if $n - k > 0$ and that it is irregular if $n - k > 2$ in complete accordance with the last proposition.

Furthermore, by Theorem 5 of subsection 3.4, the identity components of the Galois groups of the NVE and of the algebraic NVE are *isomorphic*. Now if we look at the standard transformation in order to put the algebraic NVE in the normal invariant form

$$\frac{d^2\xi}{dx^2} + I(x)\xi = 0,$$

(being

$$I := q - \frac{1}{4}p^2 - \frac{1}{2}\frac{dp}{dx},$$

and

$$\frac{d^2\xi}{dx^2} + p\frac{d\xi}{dx} + q\xi = 0,$$

the original equation, where we conserve the symbol ξ for the new variable), we only introduce an algebroid function $(exp(-1/2\int p = \varphi^{-1/4})$ (a function algebraic over the rational field). Hence, the identity components of the Galois groups of the algebraic NVE and of its normal invariant form are also isomorphic. As a conclusion, we can work directly with the normal form of the algebraic NVE: if the identity component of its Galois group is *not abelian*, then our Hamiltonian system is *not integrable* (we observe that the Galois group of the normal invariant form and the Galois group of the initial NVE as well, are contained in $SL(2,\mathbf{C})$).

EXAMPLE 1. We shall apply the theory to the very simple following family of Hamiltonians with two degrees of freedom with three parameters,

$$H = T + U = \frac{1}{2}(y_1^2 + y_2^2) + \frac{c}{3}x_1^3 + \frac{1}{2}(a + bx_1)x_2^2, \quad ,a \in \ C^*, \quad b \in \ C.$$

We consider two possible cases for $c \neq 0$ and $c = 0$.

a) For $c \neq 0$, by rescaling the potential we can write the Hamiltonian function as the two-parameter family

$$H = T + U = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{3}x_1^3 + \frac{1}{2}(a + bx_1)x_2^2, \quad ,a \in \ C^*, \quad b \in \ C.$$

The corresponding Hamiltonian system admits the integral curve,

$$\Gamma: \quad \dot{x}_1^2 = -\frac{2}{3}x_1^3, \quad x_1 = -6t^{-2}, \quad y_1 = 12t^{-3}, \quad x_2 = y_2 = 0.$$

The corresponding normal variational equation is

$$\ddot{\xi} + (a - 6bt^{-2})\xi = 0.$$

We observe that there are two singular points: the origin and the point at infinity (corresponding to the initial origin). The first one is regular singular and the second one is irregular by the above proposition. In fact, as we shall see, the NVE is a confluent hypergeometric equation.

Doing the change of variables $t = \frac{iz}{2\sqrt{a}}$, we get

$$\frac{d^2\xi}{dz^2} - (\frac{1}{4} + 6b\frac{1}{z^2})\eta = 0.$$

This is a family of Whittaker equations, with *only one* parameter. In fact, as we know from subsection 3.5, it can be transformed into a family of Bessel equations. Then the identity component of the Galois group of the NVE is abelian if and only if $\mu + 1/2$ is an integer. Hence, for $b \neq \frac{1}{6}(k^2 + k)$, $k \in \mathbf{Z}$, our Hamiltonian system is not integrable: it does not admit a global first integral, meromorphic over the initial phase space $\mathbf{C}^4$, beyond the Hamiltonian. (The differential Galois groups of the NVE's over $\overline{\Gamma}$ and $\overline{\Gamma}'$ are the *same*, because the point at infinity of $\overline{\Gamma}'$, which corresponds to $t = 0$, is regular singular. But, be careful, the differential Galois groups of the NVE's over $\overline{\Gamma}$ and Γ are in general *not* the same, the identity component of the second one is always abelian.)

We observe that for $a = 0$ the above Hamiltonian is the homogeneous Hénon-Heiles Hamiltonian. It is studied from the differential Galois point of view in [52]. The situation is quite different from the preceding one: the NVE is Fuchsian.

b) For $c = 0$, the Hamiltonian system admits the very elementary integral curve,

$$\Gamma: \quad \dot{x}_1^2 = cte, \quad x_1 = dt + e, \quad y_1 = e, \quad x_2 = y_2 = 0,$$

being d and e constant.

Then the NVE is given by the equation

$$\ddot{\xi} + (a + b + edt)\xi = 0,$$

which, by rescaling the time, we can write as the Airy's equation

$$\ddot{\xi} - t\xi = 0.$$

Now, we known from subsection 3.5 that the Galois group of the Airy equation *considered over* $\mathbf{C}(t)$ is $SL(2, \mathbf{C})$. Hence, by Theorem 9 - to be more precise, by the analogous corollary to Corollary 9 corresponding to this theorem - we get the non-integrability of the Hamiltonian system above by *rational* first integrals. We remark that then the system is considered as defined over the projective space $P^4_{\mathbf{C}}$.

Probably the example b) is the simpler non-trivial application of our theorems, with a NVE of Airy's type. We are grateful to M. Audin who pointed out the existence of this nice example in [4].

Example 2. For three degrees of freedom one has the following natural generalization of example 1,

$$H = T + U = \frac{1}{2}(y_1^2 + y_2^2 + y_3^2) + \frac{1}{3}x_1^3 + \frac{1}{2}(A + Bx_1)x_2^2 + \frac{1}{2}(C + Dx_1)x_3^2,$$

where $A, C \in \mathbf{C}^*$, $B, D \in \mathbf{C}$.

This Hamiltonian system admits the integral curve,

$$\Gamma: \quad \dot{x}_1^2 = -\frac{2}{3}x_1^3, \quad x_1 = -6t^{-2}, \quad y_1 = 12t^{-3}, \quad x_2 = x_3 = y_2 = y_3 = 0.$$

The corresponding NVE is composed of two uncoupled Whittaker equations. We denote by G', G'' their differential Galois groups. It is clear that G' and G'' are *quotients* of the Galois group G of the NVE. Hence, if *one* of the parameters B or D is different of $\frac{1}{6}(k^2 + k)$, $k \in Z$, then the identity component G^0 cannot be abelian and we get non-integrability. In the same way we can generalize this to an arbitrary number of degrees of freedom and to some other examples (when the NVE splits into 2×2 systems of the same kind).

Example 3. We consider now the family of two degrees of freedom Hamiltonian systems defined as above with $\varphi(x) = x^n$, $\alpha(x) = ax^{n-4} + bx^{n-3} + cx^{n-2}$, where n is an integer, with $n > 3$, and a, b, c are complex parameters, the parameter a being different from zero.

In the same situation as above, the normal invariant form of the algebraic NVE is

$$\frac{d^2\xi}{dx^2} - ((\frac{n(n-1)}{16} + c)x^{-2} + bx^{-3} + ax^{-4})\xi = 0.$$

After the change of variables $x = \frac{\hat{x}}{2\sqrt{a}}$, we get

$$\frac{d^2\xi}{dx^2} - (\frac{1}{4a}(\frac{n(n-1)}{16} + c)x^{-2} + \frac{b}{4a}x^{-3} + \frac{1}{4}x^{-4})\xi = 0$$

(in order to simplify the notation we write again x instead of $\hat{x}$). Now, if in the Whittaker equation

$$\frac{d^2\xi}{dz^2} - (\frac{1}{4} - \frac{\kappa}{z} + \frac{4\mu^2 - 1}{4z^2})\eta = 0,$$

we do the change of variables, $z = 1/x$, we obtain

$$\frac{d^2\xi}{dx^2} - (\frac{4\mu^2 - 1}{4}x^{-2} - \kappa x^{-3} + \frac{1}{4}x^{-4})\xi = 0.$$

So, the algebraic NVE is a general Whittaker equation, with

$$4\mu = \sqrt{c + \frac{n(n-4)}{16} + \frac{1}{4}},\ \kappa = -\frac{b}{2\sqrt{a}}$$

.

Now, we recall that if

$$p := \kappa + \mu - \frac{1}{2}\ ; \quad q := \kappa - \mu - \frac{1}{2},$$

then the identity component of the Galois group of the Whittaker equation is abelian if and only if (p, q) belongs to $(\mathbf{N} \times -\mathbf{N}^*) \cup (-\mathbf{N}^* \times \mathbf{N})$ (i.e. if both p and q are integers, one of them being positive and the other one negative). Hence this last condition is a necessary condition for the integrability of the initial Hamiltonian system.

We make two remarks about the above Examples 1–3. The first one is that, because of the abelianness of the monodromy group of the NVE, it is impossible to get any non–integrability result from an analysis of the monodromy group. The second one is that, as the NVE's are confluent hypergeometric equations, then the *local Galois group* at the irregular point and the *global Galois group* coincide. Then, we have proved in fact the non-integrability of these systems in *any neighbourhood* of the origin in the complex phase space. (This origin is the equilibrium point corresponding to the irregular singular point.)

For the Example 1, we have the following numerical computations made by Carles Simó.

If (in order to have real recurrence) in the Hamiltonian of Example 1 a), we do the canonical change of variables $x_2 = ix_2'$, $y_2 = -iy_2'$, we get the new Hamiltonian

$$H = \frac{1}{2}(y_1^2 - y_2^2) + \frac{1}{3}x_1^3 - \frac{1}{2}(a + bx_1)x_2^2,$$

where we dropped the primes.

Then Figure 1 shows a Poincaré section for $x_2 = 0$, $a = 1, b = 10, h = 0.01$ (h is the energy level) in the coordinates (x_1, y_1). We observe the well known transversal homoclinical chaotic behaviour in a small neighbourhood of the origin. Apparently the Stokes phenomena for the linearized equation corresponds to phenomena of splitting of separatrices. It would be interesting to get a more precise analysis of this example.

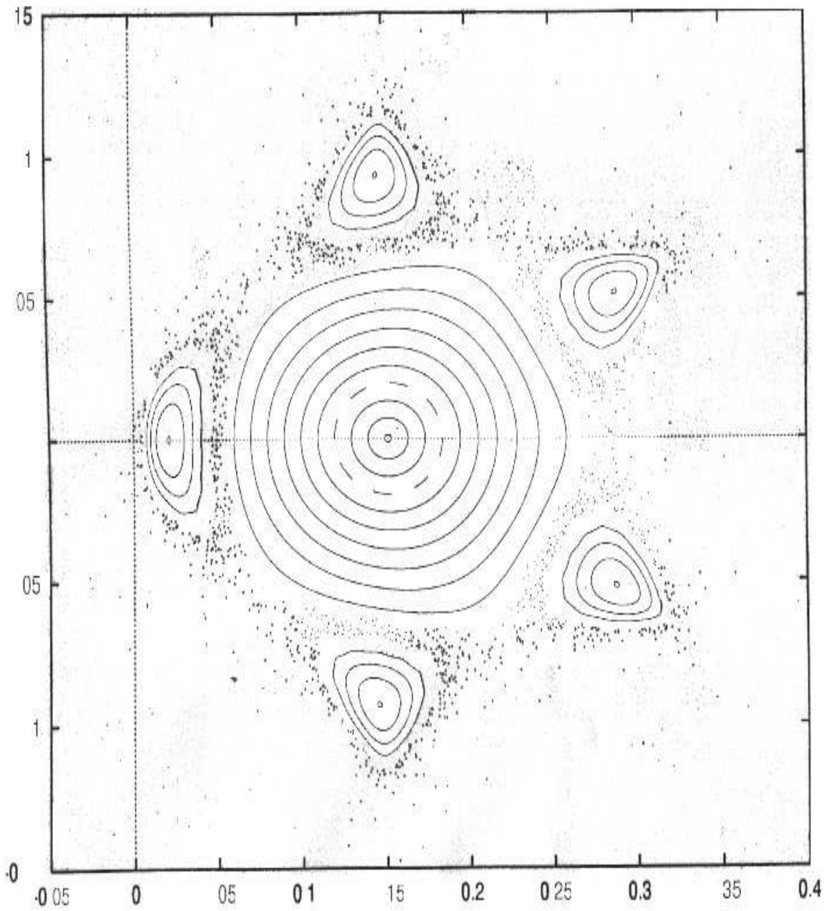

Figure 1

For some special Hamiltonians it is also possible to prove local non integrability in a Fuchsian context as it is shown by Ziglin in the following example, that we include for the sake of completeness.

EXAMPLE 4 ([74]). We recall briefly the Ziglin analysis. The starting Hamiltonian is

$$H = \frac{1}{2}(y_1^2 + y_2^2 + x_1^2 x_2^2).$$

By an elementary canonical transformation (a rotation, in order to put one of the symmetric invariant planes over $x_2 = y_2 = 0$) Ziglin obtains a Hamiltonian system with the potential (we keep the same notation for the new coordinates)

$$V(x_1, x_2) = \frac{1}{8}(x_1^2 - x_2^2)^2.$$

This is a potential of the type we studied above, with $\varphi(x_1) = 1/4x_1^4$ and $\alpha(x_1) = -1/2x_1^2$. Ziglin then considered the NVE along the family of integral curves $x_2 = y_2 = 0$, using as a parameter the energy $H > 0$ (he did *not* consider the integral curve *through* the origin). These variational equations are reduced to Lamé's type and then he apply his theorem about the monodromy group (in fact, by the subsection 5.2, he proves the non-abelianity of the identity component of the differential Galois Group). So the system under study does not have an additional holomorphic first integral in a neighbourhood of the above family of integral curves.

The key point now is that, by the (quasi-) projective structure of the Hamiltonian (the potential is a homogeneous polynomial), if the system has a holomorphic first integral, then each homogeneous polynomial in the expansion of this integral at the origin must also be a first integral. In this way Ziglin proves the *local* non-integrability of this system at the origin.

7. Final remarks. If we compare the methods proposed in this article with respect to those of other authors in previous papers, we have already observed that, in examples (1)–(3) of the above section, the *monodromy group* of the NVE is *abelian*, hence with methods based only upon the monodromy group one cannot obtain any non-integrability result. Furthermore, if we drop the involutivity assumption of the first integrals, then the Galois group is not necessarily abelian. Even more in general it is not solvable and the NVE are not Picard-Vessiot solvable too: in the reference [15] all the 2-Ziglin algebraic subgroups of $Sp(2, \mathbf{C})$ are classified. Only some of the groups in this classification are abelian. Then the other 2-Ziglin groups correspond to Hamiltonian systems which are *not completely integrable* systems in the Liouville sense.

In [49, 50] we apply some results of this paper to various non academic problems: Hamiltonian systems with homogeneous potentials, three body problems, homogeneous cosmological model,... We get very easily, in a systematic way, new proofs of classical results and many new results.

A connection of our (algebraic) criterion of non–integrability with chaotic dynamics is given in [48].

We think that it would be interesting to apply the results of this paper (and, following a suggestion of C. Simó, some conjectural generalizations to the n-th order variational equations [53]) to other classical Hamiltonian systems.

Acknowledgements. The authors are indebted to Carles Simó for the numerical computations on the example 1) of the section 6 and for many discussions and also to Jacques-Arthur Weil for many remarks. The first author was partially supported by the Spanish grant DGICYT PB94-0215, the EC grant ERBCHRXCT940460 and the Catalan grant CIRIT 1996S0GR-00105.

Appendix A. Meromorphic Bundles. Let X be a Riemann surface. We denote by $\mathcal{O}_X$ and by $\mathcal{M}_X$ the sheaves of holomorphic and meromorphic functions over X. The sheaf of holomorphic sections $\underline{V}$ of a *holomorphic vector bundle* V of rank n is a sheaf of $\mathcal{O}_X$-modules which is locally isomorphic to $\mathcal{O}_X^n$. A holomorphic vector bundle of rank n on X is also interpreted as an element of the non abelian cohomology set $H^1(X; GL(n; \mathcal{O}_X))$.

Let $G \subset GL(n; .)$ be an algebraic subgroup (defined on the field of complex numbers). We set $\mathbf{G}^{an} = G_{\mathcal{O}_X} \subset GL(n; \mathcal{O}_X)$ and $\mathbf{G}^{me} = G_{\mathcal{M}_X} \subset GL(n; \mathcal{M}_X)$. We will say that an holomorphic bundle on X admits G as structure group if it is defined by an element of $H^1(X; \mathbf{G}^{an})$.

We need *meromorphic vector bundles*. By definition the sheaf of meromorphic sections of a *meromorphic vector bundle* of rank n is a sheaf of $\mathcal{M}_X$-modules which is locally isomorphic to $\mathcal{M}_X^n$. A meromorphic vector bundle of rank n over X is also interpreted as an element of the non abelian cohomology set $H^1(X; GL(n; \mathcal{M}_X))$. If this element "belongs" to $H^1(X; \mathbf{G}^{me})$, we will say that the meromorphic vector bundle admits G as structure group. There exists an equivalent definition for a meromorphic vector bundle over a Riemann surface X, due to P. Deligne [De], 1.14 p. 52. Such a bundle is an equivalence class of *holomorphic extensions* of holomorphic bundles defined on X minus a discrete subset Σ: locally, if z is a uniformizing variable vanishing on Σ, then two extensions V_1 and V_2 of V are equivalent if the corresponding sheaves of holomorphic sections satisfy

$$z^n \underline{V}_1 \subset \underline{V}_2 \subset z^{-n} \underline{V}_1 \subset i^* \underline{V}$$

($i : X - \Sigma \to X$ being the natural inclusion).

The following result says that every meromorphic vector bundle on a Riemann surface comes from a holomorphic vector bundle.

Lemma 10. *Let X be a Riemann surface. Let $G \subset GL(n; \mathbf{C})$ be an algebraic subgroup defined on the field of complex numbers. Then the natural map*

$$H^1(X; \mathbf{G}^{an}) \to H^1(X; \mathbf{G}^{me})$$

is surjective.

The proof is easy: the set of poles of a section of $\mathbf{G}^{me}$ is discrete.

Proposition 13. *Any meromorphic vector bundle over a Riemann surface X is trivial.*

Proof. Let V^{me} be a meromorphic vector bundle over X. It comes from an holomorphic vector bundle V^{an}. If X is an open Riemann surface, then V^{an} is trivial ([21] Th. 30.4). If X is a compact connected Riemann surface, then V^{an} comes from an *algebraic* vector bundle V over the non singular projective curve X. We denote by k_X the field of rational (or meromorphic..) functions over X. The field of rational sections of the algebraic bundle V is a rank n vector space over k_X ([21] Cor. 29.17), therefore V^{me} is a trivial meromorphic bundle.

In fact we need some similar but more precise results involving vector bundles with the symplectic group as structure group. We will give them below.

If now X is a *singular* complex analytic curve, we can also define holomorphic vector bundles and meromorphic vector bundles over X along the same lines. If $\pi : \tilde{X} \to X$ is a desingularisation map (i.e. if $\tilde{X}$ is a Riemann surface and π a proper analytic map which is a finite covering ramified at worst above a discrete subset of X), then it induces an isomorphism π^* between the sheaves of meromorphic functions $\mathcal{M}_X$ and $\mathcal{M}_{\tilde{X}}$, and therefore an isomorphism π^* between the meromorphic vector bundles over X and over $\tilde{X}$.

Theorem 11 (Grauert Theorem). *Let X be a complex connected, non compact, Riemann surface. Let $\mathcal{F} = (Y, p, X)$ be a locally trivial vector (resp. principal) holomorphic fiber bundle over X with a connected complex Lie group G as structure group. Then $\mathcal{F}$ is holomorphically trivial.*

Sketch of Proof. For completeness we recall here the proof. We denote by $\mathbf{G}^{an}$ (resp. $\mathbf{G}^{c}$) he sheaf of holomorphic (resp. continuous) functions on X with values in G The open Riemann surface X is homotopically equivalent (by retraction) to a finite one-dimensional complex. On such a complex a G-fiber bundle is topologically trivial, because G is connected. Therefore the fiber bundle $\mathcal{F}$ is topologically trivial. An open Riemann surface is a Stein manifold and on such a manifold the topological classification and the analytic classification of fiber bundles with a complex Lie group as structure group coincide: the natural map

$$H^1(X; \mathbf{G}^{an}) \to H^1(X; \mathbf{G}^c)$$

is a bijection ([22, 10]). Therefore $\mathcal{F}$ is holomorphically trivial.

A complete and detailed proof is given in [61], Chapter 2.

We apply the above theorem to the symplectic group. An element of $Sp(2n, \mathbf{C})$ is a product of at most $4n - 2$ symplectic transvections ([19]) (we can also use an homomorphism between $Sp(2n, \mathbf{C})$ and the product of $SU(n)\times$ *vector space* and the connectedness of $SU(n)$). Hence

Lemma 11. *The topological group $Sp(2n, \mathbf{C})$ is connected.*

Corollary 10. *Let X be a complex connected, non compact, Riemann surface. Let $\mathcal{F} = (Y, p, X)$ be a locally trivial vector (resp. principal) holomorphic fiber bundle over X with $Sp(2n, \mathbf{C})$ as structure group. Then $\mathcal{F}$ is holomorphically trivial.*

For compact Riemann surfaces we have the following proposition.

Proposition 14. *Let X be a connected compact Riemann surface. Let $\mathcal{F} = (Y, p, X)$ be a locally trivial vector (resp. principal) holomorphic fiber bundle over X with structure group $Sp_{2n}(\mathbf{C})$. Then $\mathcal{F}$ is meromorphically trivial.*

The compact Riemann surface X is also a complex algebraic (projective) curve. We denote by $\mathbf{G}$ the sheaf of regular maps from X to the algebraic complex group G. We have a natural map

$$an : H^1(X; \mathbf{G}) \to H^1(X; \mathbf{G}^{an}).$$

The symplectic group $G = Sp_{2n}(\mathbf{C})$ satisfies condition (R) of [60] (p. 33): there exists a rational section

$$GL_{2n}(\mathbf{C})/G \to GL_{2n}(\mathbf{C})$$

(cf. [60], Example c) p. 34). Therefore we can apply Proposition 20 of[60] (p. 33): the map L is a bijection. Using an algebraic trivialization of the algebraic bundle corresponding to $\mathcal{F}$ on a convenient affine subset of the curve X, we get the result.

Let M' be a connected complex analytic manifold of complex dimension $2n$. Let Ω be a closed meromorphic form of degree two on M'. Let $M_\infty \subset M'$ be a closed analytic hypersurface (i.e. analytic subset of pure complex codimension one) of M'. We set $M = M' - M_\infty$ and we suppose that Ω is *holomorphic* and *non degenerate* over M. Then (M, Ω) is a complex symplectic manifold. We denote by $T\ M'$ (resp. $T^*\ M'$) the tangent (resp. cotangent) bundle of M'. It is a holomorphic bundle but we will use only its structure of meromorphic bundle. Then, as we noticed before, the form Ω induces a generalized musical isomorphism between the *meromorphic* bundle $T\ M'$ and the *meromorphic* bundle $T^*\ M'$: if X_1 is a meromorphic vector field on an open set $U \in M'$, then, for every meromorphic vector field X on U, $\Omega(X_1, X)$ is a meromorphic function on U, and

$$X \to \Omega(X_1, X)$$

is a k_U-linear isomorphism between the k_U-vector spaces of meromorphic sections of $T\ M'$ and $T^*\ M'$ on U. (We denoted by k_U the field of meromorphic functions on U.)

Let H be a meromorphic Hamiltonian function over the manifold M'. Let $X_H = \sharp dH$ be the corresponding Hamiltonian field. It is meromorphic over M' and its restriction to M is *holomorphic*. Let $i(\Gamma)$ be a connected non equilibrium phase curve for X_H over M. Let $\underline{\Gamma}'$ be as before a (perhaps) singular curve which is the union of $i(\Gamma)$ and of a discrete subset of equilibrium points and points at infinity. Let $\overline{\Gamma}'$ be a desingularization of $\underline{\Gamma}'$. Let $f_1, ..., f_m$ be an involutive set of first integrals ($H = f_1$) which are *meromorphic* on M'. We suppose that there are *holomorphic* and *independent* at some point of the phase curve $i(\Gamma)$. Then the system $df_1 = 0, \dots df_m = 0$ defines a *meromorphic* subbundle E of $T_{M'}$ (of rank $2n - m$). The *meromorphic* vector fields $X_1 = \sharp df_1, \dots X_m = \sharp df_m$ generate a rank m meromorphic subbundle F of E. Then $F^\perp$ is a meromorphic subbundle. As in [5] we get a structure of symplectic meromorphic bundle on the meromorphic bundle $N = (F^\perp/F)$ over Γ'. (We have only to replace holomorphic bundles by meromorphic bundles in [5].)

Finally, as in [5], we get a normal variational connection on the symplectic bundle $N = (F^\perp/F)$ over Γ'. Here the bundle and the connection are meromorphic. The bundle N is symplectically meromorphically trivializable, therefore this normal variational connection can be interpreted as a meromorphic differential equation over Γ' (the NVE).

Appendix B. Differential Galois groups and finite coverings. In this appendix we will prove that the identity component of the differential Galois group of a meromorphic connection on a Riemann surface does not change if we take inverse images by a finite ramified covering. It is an analytic version of an algebraic result of N. Katz.

PROPOSITION 15. *Let Δ be a germ of meromorphic linear system at the origin of* $\mathbf{C}$. *We denote by K the differential field of germs of meromorphic functions, by $G = Gal_K(\Delta)$ the differential Galois group of Δ and by G^0 its identity component.*

Let H be the subgroup of G topologically generated (in Zariski sense) by the exponential torus and all the Stokes multipliers of Δ.

We denote by $m \in G$ the actual monodromy of Δ, by M the Zariski closure in G of the subgroup generated by m, by M^0 the identity component of M and by H_1 the subgroup of G generated by H and M^0.

(i) The subgroups H and H_1 are Zariski closed, connected and invariant under the adjoint action of m.

(ii) The group G is topologically generated by H and m.

(iii) The group G is algebraically generated by H_1 and m, and $G^0 = H_1$.

(iv) The image of m in G/G^0 generates this finite group

Proof. The actual monodromy m and the formal monodromy $\hat{m}$ are equal up to multiplication by a product of Stokes multipliers [43, 9]. The exponential torus is (globally) invariant by the adjoint action of the formal monodromy $\hat{m}$. Then our claims follow easily from the density theorem of Ramis [43, 9], using some elementary results about linear algebraic groups [25].

LEMMA 12. *Let $\nu \in \mathbf{N}^*$. Let ∇ be a germ of meromorphic connection at the origin of the x plane* $\mathbf{C}$. *We set $x = f(t) = t^\nu$. We denote by $(X, 0)$ (resp. $(X', 0)$)*

the germ at the origin of the x (resp. t) plane. We denote by K (resp. K') the differential field of germs of meromorphic functions on X (resp. X'). We have a natural injective homomorphism of differential Galois groups

$$Gal_{K'}(f^*\nabla) \to Gal_K(\nabla)$$

which induces an isomorphism *between their Lie algebras.*

Proof. We set $G = Gal_K(\nabla)$ and $G' = Gal_{K'}(f^*\nabla)$.

The field inclusion $K \subset K'$ induces a natural map

$$\varphi : G' \to G.$$

This map is clearly injective. Let $m \in G$ be the actual monodromy of ∇.

Then the actual monodromy of $\nabla' = f^*\nabla)$ is $m' = m^\nu$.

The connections ∇ and ∇' have the same exponential torus and the same Stokes multipliers (more precisely the map φ induces isomorphisms). We use the notations of Proposition 15 for ∇ and similar notations for ∇'.

We have clearly $H = H'$, $M^0 = {M'}^0$, therefore $G^0 = H_1 = {H'}_1 = {G'}^0$.

THEOREM 12. *Let X be a connected Riemann surface. We denote by k its field of meromorphic functions. We choose a differential ∂ on k. Let $S = \{a_i\}_{i\in I} \subset X$ be a discrete subset. Let $x_0 \in X - S$. For each point $a_i \in S$, we choose a germ d_i of real half line starting from a_i and drawn on the complex line tangent to X at a_i. We denote by $\tilde{\mathcal{M}}$ the field of meromorphic functions on the universal covering $(\tilde{X}, x_0)$ of X pointed at x_0. We identify the field k with a subfield of $\tilde{\mathcal{M}}$. For $i \in I$, we denote by $\mathcal{M}_i$ the field of germs at a_i of meromorphic functions (i.e. of germs of functions meromorphic on a germ of open sector at a_i bisected by d_i). We identify the field K_i of germs at a_i of meromorphic functions with a subfield of $\mathcal{M}_i$. We extend the derivation ∂ on k to the fields $\tilde{\mathcal{M}}$, $\mathcal{M}_i$. We choose also continuous paths γ_i's joining x_0 respectively to the d_i's (that is arriving at a_i tangently to d_i).*

Let ∇ be a meromorphic connection on X with poles at most on S. We denote by ∇_i the germ at a_i of ∇. There exist a uniquely determined Picard-Vessiot extension L_0 (resp. L_i) of the differential field $(k;\partial)$ (resp. $(K_i;\partial)$) associated to ∇ (resp. ∇_i) such that $k \subset L_0 \subset \tilde{\mathcal{M}}$ (resp. $K_i \subset L_i \subset \mathcal{M}_i$). The path γ_i induces an isomorphism of differential fields Z_i between L_0 and L_i (We use Cauchy's Theorem and analytic extension along γ_i.)

We denote by G (resp. G_i) the "representation" of "the" differential Galois group $Gal_k\ \nabla$ (resp. $Gal_{K_i}\ \nabla$) associated to L_0 (resp. L_i). Using Z_i we identify the local *Galois group G_i with a subgroup of the* global *Galois group G. Let Π_1 be the usual monodromy group of ∇. Then the complex linear algebraic group G is topologically generated by the G_i's ($i \in I$) and Π_1.*

Proof. This result is a trivial extension of a classical result due to Marotte [42], Ch. II, H.30. We recall briefly the proof. We denote by H the subgroup of G generated by the G_i's ($i \in I$) and Π_1. Let $\alpha \in L_0 \subset \tilde{\mathcal{M}}_0$. It defines elements $\alpha_i \in L_i$. If α is invariant by the group H, then α_i is invariant by G_i and Π_1, therefore $\alpha_i \in K_i$: α is uniform around a_i and correspond to a germ of meromorphic function at a_i. Finally α is non ramified on S, uniform on $X - S$ and meromorphic on X (with poles at most on S). We have proved that the subfield of the Picard-Vessiot extension L_0 fixed by the subgroup $H \in G$ is K. Then from the differential Galois correspondence it follows that H is Zariski dense in G.

Theorem 13. *Let X be a connected Riemann surface. Let (X', f, X) be a* finite *ramified covering of X by a connected Riemann surface X'. Let ∇ be a meromorphic connection on X. We set $\nabla' = f^*\nabla$. Then we have a natural injective homomorphism*

$$Gal\ (\nabla') \to Gal\ (\nabla)$$

of differential Galois groups which induces an isomorphism *between their Lie algebras.*

Proof. Let k (resp. k') be the meromorphic functions field of X (resp. X'). The finite covering (X', f, X) is ramified over a finite set $\Sigma \subset X$. Let $S \subset X$ be the union of the ramification set Σ and of the set of poles of ∇. It is a discrete subset. Let $S' = f^{-1}(S) \subset X'$. It is a discrete subset. We choose a base point $x'_0 \in X' - S'$ and we set $f(x'_0) = x_0 \in X$. Then we set $G = Gal_k(\nabla)$ and $G' = Gal_{k'}(f^*\nabla)$, with conventions similar to those made above in the proof of Theorem 12.

The field inclusion $k \subset k'$ induces a natural map

$$\varphi : G' \to G.$$

This map is clearly continuous and injective and we can identify G' with a closed subgroup of G.

We have a natural injective map $\pi_1(X'-S'; x'{}_0) \to \pi_1(X-S; x_0)$. We can identify $\pi_1(X'-S'; x'_0)$ with a subgroup of $\pi_1(X-S; x_0)$. The index of this subgroup is finite.

Following our conventions, we compute G (resp. G') with the horizontal sections of ∇ meromorphic on the universal covering pointed at x_0 (resp. x'_0).

We denote by Π_1 (resp. Π'_1) the natural image of $\pi_1(X - S; x_0)$ (resp. $\pi_1(X' - S'; x'_0)$) in G (resp. G').

The global differential Galois group G (resp. G') is topologically generated by the local Galois groups G_i's and Π_1 (resp.$G'_{i'}$ and Π'_1).

Let R (rep. R') be the smallest subgroup of G (rep. G') such that it contains the identity components of all the local differential Galois groups and such that it is invariant by the adjoint action of the monodromy subgroup Π_1 (resp Π'_1). The group R (respectively R') is closed and connected.

Using Proposition 15, we see that the group G (resp. G') is topologically generated by Π and R (resp. Π' and R').

We choose continuous paths γ_i joining x_0 to each point $a_i \in S$ in $X - S$. After, for each $a'_{i'}$ above a_i, we choose a continuous path $\gamma'_{i'}$ joining x'_0 to $a'_{i'}$ in $X' - S'$. We can suppose that $\gamma'_{i'}$ is a path above γ_i followed by a path above a loop at a_i.

Applying Lemma 12 and the definition of the fundamental group as a quotient of a set of loops, we see easily that the map φ induces an isomorphism between R' and R. Therefore the natural map

$$\pi_1(X - S; x_0)/\pi_1(X' - S'; x'_0) \to G/G'$$

is Zariski dense. (The group G is topologically generated by Π and R). As the first group is finite, it follows that the map is onto and that the group G/G' is also finite. Therefore $G^0 = G'^0$.

Remark. In [29] there is an algebraic version of Theorem 13. It is possible to transpose Katz's Tannakian argument to the analytic situation. Then we get an injective homomorphism

$$G'_{K'} \to G_K \otimes_K K'$$

inducing an isomorphism of K'-Lie algebras

$$\mathcal{G}'_{K'} \to \mathcal{G}_K \otimes_K K'.$$

But this isomorphism comes by tensorization $\otimes_{\mathbf{C}} K'$ from a $\mathbf{C}$-linear natural map

$$\text{Lie } \varphi : \mathcal{G}'_{\mathbf{C}} \to \mathcal{G}_{\mathbf{C}}.$$

Therefore φ is an isomorphism of complex Lie algebras. This gives another proof of Theorem 13.

Appendix C. Connections with structure group. Let X be a Riemann surface. We denote by $\mathcal{O}_X$ (resp. $\mathcal{M}_X$) its sheaf of holomorphic (resp. meromorphic) functions.

Let $G \subset GL(n;\mathbf{C})$ be a Zariski connected complex linear algebraic group. We denote by $\mathcal{G} \subset End(n;\mathbf{C})$ its Lie algebra. We denote by $G(\mathcal{O}_X)$ (resp.$G(\mathcal{M}_X)$) the sheaf of holomorphic (resp. meromorphic) matrix functions whose values belong to $\mathcal{O}_X$ (resp. $\mathcal{M}_X$). We adopt similar notations for functions whose values belong to the Lie algebra $\mathcal{G}$.

We recall that we defined a holomorphic G-bundle over X as a holomorphic vector bundle over X admitting G as a structure group. It is defined by an element of $H^1(M; G(\mathcal{O}_X))$. We have a notion of local G-trivialization of a G-bundle. We also introduced the notion of meromorphic G-bundle (cf. Appendix A).

Let ∇ be a meromorphic connection on a G-bundle V. Using a local coordinate t and a frame corresponding to a local G-trivialization, we get a differential operator $\nabla_{\frac{d}{dt}} = \frac{d}{dt} - A$, where A is a meromorphic matrix. If the values of A belong to the Lie algebra $\mathcal{G}$, we will say that ∇ is a meromorphic connection with structure group G (or a G connection) on the G-bundle V. This definition is *independent* of the choice of a trivialization: if the values of a meromorphic invertible matrix P belong to the group G, then the values of the meromorphic matrix $P^{-1}AP - P^{-1}\frac{d}{dt}A$ belong clearly to the Lie algebra $\mathcal{G}$.

THEOREM 14. *Let ∇ be a G-meromorphic connection on a trivial G-bundle V over a connected Riemann surface X. Then its differential Galois group "is" a closed subgroup of G.*

This result is due to Kolchin, who introduced the notion of G-primitive extension [32]. We will give here a very simple Tannakian proof. Following Chevalley's Theorem [64], 5.1.3. Theorem, page 131 (cf. also [8]), the linear algebraic group $G \subset GL(n;\mathbf{C})$ is the subgroup of $GL(n;\mathbf{C})$ leaving invariant a complex line W'_0 in some construction W_0 on the complex vector space $V_0 = \mathbf{C}^n$. To the natural operation of the group G on the construction W_0 corresponds a natural operation of the Lie algebra $\mathcal{G}$ on the same construction W_0. This action clearly leaves also invariant the complex line W'_0. We denote by $G_{W_0} \subset GL(W_0)$ the natural representation of G, and by $\mathcal{G}_{W_0} \subset End(W_0)$ the corresponding Lie algebra. If we choose a basis of the complex vector space W_0 such that its first vector generates W'_0 over $\mathbf{C}$, then the Lie algebra $\mathcal{G}_{W_0}$ corresponds to the Lie algebra of the matrices whose entries which belong to the first column are zero, except perhaps the first one.

To the construction W_0 on V_0 corresponds a holomorphic vector bundle W. We obtain it from the holomorphic vector bundle V by a similar construction. To the meromorphic connection ∇ on V corresponds similarly a meromorphic connection ∇_W on W. To the complex line W'_0 corresponds a trivial sub-line bundle W' of W.

We choose a (meromorphic) uniformizing variable over X and a frame of the *trivial* G_{W_0}-bundle W such that its first element generates the sub-line bundle W'. Then the G_{W_0}-meromorphic connection ∇_W can be interpreted as a system $\frac{d}{dt} - B$, where the meromorphic matrix B takes its values into the Lie algebra $\mathcal{G}_{W_0}$. Consequently the entries of B which belong to the first column are identically zero, except perhaps the first one. Therefore the action of the meromorphic matrix B on the sheaf of meromorphic sections of the vector bundle W leaves invariant the subsheaf of meromorphic sections whose values belong to the subbundle W'. Going back to connections, we see that the meromorphic connection ∇_W leaves invariant the sub-bundle W' and consequently that it induces a *sub-connection* $(W', \nabla_{W'}) \subset (W, \nabla_W)$.

By the Tannakian definition (section 3.2) the differential Galois group H of ∇ is *defined* by the *list* of all the subspaces of all the constructions $C(V_0)$ on the vector space V_0 corresponding to the fibers (in fiber functor sense) of the underlying vector bundles of all the subconnections of the similar connections $\nabla_{C(V)}$ on the similar constructions $C(V)$. But (W'_0, W_0) *belongs to this list*, therefore H is a *closed subgroup* of the algebraic group G (which itself can be *defined* by the *only pair* (W'_0, W_0).

In our paper we need only the following result corresponding to $G = Sp(n; \mathbf{C})$. (Using Appendix A, if V is a meromorphic symplectic bundle over X, we can suppose that it is a *trivial* symplectic meromorphic bundle.)

COROLLARY 11. *Let ∇ be a symplectic meromorphic connection on a meromorphic symplectic bundle V over a connected Riemann surface X. Then its differential Galois group "is" a closed subgroup of the symplectic group.*

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