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ON THE RELATIVISTIC EULER EQUATION*

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Abstract. This work consider a more realistic equation of states. We generalize the method of DiPerna and G.-Q. Chen et al to show the existence of weak solution of the relativistic Euler equation with initial data containing the vacuum state.

1. Introduction. In this article we study the Cauchy problem to the one-dimensional relativistic Euler equation

$$\begin{aligned} &\frac{\partial}{\partial t}\frac{\rho + Pu^2/c^4}{1-u^2/c^2} + \frac{\partial}{\partial x}\frac{(\rho + P/c^2)u}{1-u^2/c^2} = 0, \\ &\frac{\partial}{\partial t}\frac{(\rho + P/c^2)u}{1-u^2/c^2} + \frac{\partial}{\partial x}\frac{P+\rho u^2}{1-u^2/c^2} = 0, \end{aligned} \tag{1.1}$$

$$\rho|_{t=0} = \rho_0(x), \qquad u|_{t=0} = u_0(x). \tag{1.2}$$

Here c is a positive constant, the speed of light, and P is a given function of ρ. The equation (1.1) governs the one dimensional motion of a perfect gas in the Minkowski space-time. When $c \to \infty$, (1.1) tends to the usual Euler equation of gas dynamics

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (P + \rho u^2)_x &= 0. \end{aligned} \tag{1.3}$$

Many mathematical investigations for this non-relativistic Euler equation were done. But the first mathematical investigation for the relativistic Euler equation (1.1) was done recently by Smoller and Temple [6]. They assume $P = \sigma^2\rho$, where σ is a positive constant $< c$. Under this assumption, they showed that if the initial data $\rho_0(x)$ and $u_0(x)$ satisfy

$$T.V. \log \rho_0 < \infty, \qquad T.V. \log \frac{c+u_0}{c-u_0} < \infty, \tag{1.4}$$

then there exists a global weak solution to the Cauchy problem (1.1) and (1.2). Here $T.V.\{f(\cdot)\}$ denotes the total variation of $f(x)$ with $x \in \mathbf{R^1}$. The result was obtained by Glimm's scheme and it is the relativistic version of Nishida's result [5] for the non-relativistic problem.

However we would like to consider a more realistic equation of states. We keep in mind the equation of state for a neutron stars, which is given by

$$P = Kc^5 f(y), \qquad \rho = Kc^3 g(y)$$

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$$f(y) = \int_0^y \frac{q^4}{\sqrt{1+q^2}} dq,$$
$$g(y) = 3\int_0^y q^2\sqrt{1+q^2} dq.$$

For this equation of state, we have $P \sim \frac{c^2}{3}\rho$ as $\rho \to \infty$ but $P \sim \frac{1}{5}K^{-2/3}\rho^{5/3}$ as $\rho \to 0$. So we assume the following properties of the function $P(\rho)$:

(A):

$$P(\rho) > 0, \qquad 0 < dP/d\rho < c^2, \qquad 0 < d^2P/d\rho^2$$

for $\rho > 0$, and

$$P = A\rho^\gamma(1 + [\rho^{\gamma-1}/c^2]_1)$$

as $\rho \to 0$. Here A and γ are positive constants and

$$\gamma = 1 + \frac{2}{2N+1},$$

N being a positive integer, and $[X]_1$ denotes a convergent power series of the form $\sum_{k\geq 1} a_k X^k$.

The result which we want to generalize to the relativistic problem is those by G.-Q. Chen et al [2]. So we assume that the initial data $\rho_0(x)$, $u_0(x)$ satisfy

$$0 \leq \rho_0(x) \leq M_0, \qquad |\frac{c}{2}\log\frac{c+u_0(x)}{c-u_0(x)}| \leq M_0. \tag{1.5}$$

A weak solution of (1.1)(1.2) is defined as follows.

We write

$$E = \frac{\rho + Pu^2/c^4}{1-u^2/c^2},$$
$$F = \frac{(\rho + P/c^2)u}{1-u^2/c^2},$$
$$G = \frac{P + \rho u^2}{1-u^2/c^2},$$
$$U = (E, F)^T, \qquad f(U) = (F, G)^T.$$

Then (1.1) can be written as

$$U_t + f(U)_x = 0. \tag{1.6}$$

Let us denote by $U_0(x)$ the initial data. Then a weak solution $U(t,x)$ is a bounded measurable function which satisfies

$$\int_0^\infty \int_{-\infty}^\infty (U\Phi_t + f(U)\Phi_x) dx dt + \int_0^\infty U_0(x)\Phi(0,x) dx = 0 \tag{1.7}$$

for any test function $\Phi \in C_0^\infty([0,+\infty)\times R)$. The existence of weak solution is the following

Main Theorem. *For any M_0 there is a positive number ϵ_0 such that if the initial data satisfy*

$$0 \le \rho_0(x) \le M_0, \qquad |\frac{c}{2}\log\frac{c+u_0(x)}{c-u_0(x)}| \le M_0$$

and if $1/c^2 \le \epsilon_0$, then a subsequence of the approximate solutions U^Δ (see section 4) converges a.e. to a limit U which is a weak solution of the relativistic Euler equation.

Corollary. *There is $\epsilon_1 > 0$ such that if*

$$0 \le \rho_0(x) \le \epsilon_1 c^{\frac{2}{\gamma-1}}, \qquad |\frac{1}{2}\log\frac{c+u_0(x)}{c-u_0(x)}| \le \epsilon_1,$$

then there is a weak solution of the relativistic Euler equation.

The paper is organized as follows. In section 2, we illustrate some basic properties of Riemann problem. In section 3, some useful entropy-entropy flux are constructed such that the Hessian matrix with respect to the flux is positive definite. The approximate solutions of the equations are constructed by Godunov scheme in section 4. In section 5, we seek an integration formula for solutions of the relativistic Euler-Poisson-Darboux equation. With this integration representations, the derivatives of entropies can be estimated in section 6. By using the results of sections 5 and 6, some important entropies can be estimated more precisely. The compactness and convergence of the approximation solutions obtained in section 4 are studied in sections 8 and 9. Finally, the proofs of some propositions are given in the appendices.

2. Riemann problems. The Riemann problem is the problem to the special initial data of the form

$$U_0(x) = \begin{cases} U_L & \text{if } x < 0, \\ U_R & \text{if } x > 0. \end{cases}$$

In order to solve this we introduce the Riemann invariants

$$w = x + y, \qquad z = x - y$$

where

$$x = \frac{c}{2}\log\frac{c+u}{c-u}, \qquad y = \int_0^\rho \frac{\sqrt{P'}}{\rho + P/c^2}d\rho.$$

Then (1.1) is diagonalized as

$$w_t + \lambda_2 w_x = 0, \qquad z_t + \lambda_1 z_x = 0,$$

where

$$\lambda_1 = \frac{u - \sqrt{P'}}{1 - \sqrt{P'}u/c^2}, \qquad \lambda_2 = \frac{u + \sqrt{P'}}{1 + \sqrt{P'}u/c^2}.$$

The possible states $U = U_R$ connected to U_L on the right by rarefaction waves are

$$R_1: \qquad w = w_L, z > z_L,$$

and

$$R_2 : \qquad w > w_L, z = z_L.$$

The Rankine Hugoniot jump condition

$$\sigma[U] = [f(U)],$$

where $[U] = U_R - U_L, [f(U)] = f(U_R) - f(U_L)$, gives the shock curve

$$\frac{(u_R - u_L)^2}{(1 - u_R^2/c^2)(1 - u_L^2/c^2)} = \frac{(\rho_R - \rho_L)(P_R - P_L)}{(\rho_L + P_L/c^2)(\rho_R + P_R/c^2)}.$$

Along this curve we have shocks

$$\begin{aligned} S_1 : &\qquad \rho_L < \rho_R, u_R < u_L, \\ S_2 : &\qquad \rho_R < \rho_L, u_R < u_L. \end{aligned}$$

The Riemann problem can be solved uniquely by using these rarefaction waves and shock waves and vaccuum state. The detailed discussion can be found in J. Chen [1].

If we look at a region of the form

$$\Sigma_B = \{(w, z) | - B \leq z \leq w \leq B\},$$

we have the following

PROPOSITION 2.1. *If the initial data U_L, U_R belong to Σ_B for some large B, then the solution of the Riemann problem is confined to Σ_B.*

Moreover if we consider the image of Σ_B in the (E, F)-space, we have

PROPOSITION 2.2. *The region Σ_B is convex in the (E, F)-plane.*

Proof. Let us consider the above hedge $F = F(E)$ which corresponds to $w = B, -B < z < B$. We have to show $d^2F/dE^2 < 0$. Along the hedge $w = B$, we have

$$u = c \tanh \frac{1}{c}(B - \int_0^\rho \frac{\sqrt{P'}}{\rho + P/c^2} d\rho),$$

from which

$$\frac{du}{d\rho} = -(1 - u^2/c^2) \frac{\sqrt{P'}}{\rho + P/c^2}.$$

By a direct calculation we have

$$\frac{dF}{dE} = \frac{u - \sqrt{P'}}{1 - \sqrt{P'}u/c^2} = \lambda_1.$$

Differentiating once more we have

$$\frac{d^2F}{dE^2} = -\frac{1 - u^2/c^2}{(1 - \sqrt{P'}u/c^2)^4} (\frac{P''}{2\sqrt{P'}} + (1 - \frac{P'}{c^2}) \frac{\sqrt{P'}}{\rho + P/c^2}) < 0.$$

The proof is complete. □

From Proposition 2.2, we have

PROPOSITION 2.3. *If $U(s), s \in [a,b]$, is confined to a region Σ_B, then the average*

$$\frac{1}{b-a}\int_a^b U(s)ds$$

belongs to Σ_B.

Let us look at the shock wave which connects the left state U_L to the right state U_R with the shock speed σ.

The right state U_R and σ are parametrized by $\rho = \rho_R$. Then we have the following fact, which will be used in Section 4.

PROPOSITION 2.4. *Along $S_1(\rho_L < \rho)$, we have $d\sigma/d\rho < 0$, and along $S_2(\rho < \rho_L)$ we have $d\sigma/d\rho > 0$.*

Proof. Without loss of generality we can assume $u_L = 0$. Then $u = u_R$ is given by

$$u = -\sqrt{\frac{[\rho][P]}{(\rho_L + P/c^2)(\rho + P_L/c^2)}},$$

where $[\rho] = \rho - \rho_L, [P] = P - P_L$. We have

$$\sigma = \frac{[F]}{[E]} = \frac{(\rho + P/c^2)u}{\rho + Pu^2/c^4 - \rho_L(1 - u^2/c^2)}.$$

By a direct but tedious computations, we have

$$\frac{d\sigma}{d\rho} = \frac{(\rho + P/c^2)(\rho_L + P_L/c^2)[\rho]X}{2(\rho + Pu^2/c^4 - \rho_L(1 - u^2/c^2))^2 u(\rho_L + P/c^2)^2(\rho + P_L/c^2)^2},$$

where

$$\begin{aligned} X = {} & (\rho + P_L/c^2)(\rho + P/c^2)P'[\rho] + (\rho + P_L/c^2)(-(\rho + P_L/c^2) + \\ & [P]/c^2)[P] - (\rho_L + P/c^2)[P]^2/c^2. \end{aligned}$$

Since $P'' > 0$ we know $[P] \leq P'[\rho]$. Thus

$$\begin{aligned} X \geq {} & (\rho + P_L/c^2)(\rho + P/c^2)[P] + (\rho + P_L/c^2)(-(\rho_L + P_L/c^2) + \\ & [P]/c^2)[P] - (\rho_L + P/c^2)[P]^2/c^2 \\ = {} & [P]((\rho + P_L/c^2)([\rho] + [P]/c^2) + ([\rho] - [P]/c^2)[P]/c^2). \end{aligned}$$

But

$$1 > \frac{[\rho] - [P]/c^2}{[\rho]} = 1 - P'(\rho_L + \theta(\rho - \rho_L))/c^2 > 0.$$

Using this, it is easy to see $X > 0$ both when $[\rho] > 0$ and when $[\rho] < 0$. Since $u < 0$, this completes the proof. □

3. Entropies. A pair of functions η and q is called an entropy- entropy flux if it satisfies the equation

$$D_U q = D_U \eta \cdot D_U f. \tag{3.1}$$

Using the Riemann invariants, we can write (3.1) as

$$\frac{\partial q}{\partial w} = \lambda_2 \frac{\partial \eta}{\partial w}, \qquad \frac{\partial q}{\partial z} = \lambda_1 \frac{\partial \eta}{\partial z}.$$

By eliminating q from the equation, we get the following second order equation:

$$\frac{\partial^2 \eta}{\partial w \partial z} + Q(J\frac{\partial \eta}{\partial w} - \frac{1}{J}\frac{\partial \eta}{\partial z}) = 0, \tag{3.2}$$

where

$$Q = \frac{1}{4\sqrt{P'}}(1 - \frac{P'}{c^2} - \frac{\rho + P/c^2}{2P'}P''),$$
$$J = \frac{1 - \sqrt{P'}u/c^2}{1 + \sqrt{P'}u/c^2}.$$

Since this equation tends to the Euler-Poisson-Darboux equation

$$\frac{\partial^2 \eta}{\partial w \partial z} + \frac{N}{w - z}(\frac{\partial \eta}{\partial w} - \frac{\partial \eta}{\partial z}) = 0 \tag{3.3}$$

as $c \to \infty$, we shall call (3.2) the relativistic Euler-Poisson-Darboux equation.

Among entropies of (3.3) when $c = \infty$ the kinetic energy

$$\eta = \frac{1}{2}\rho u^2 + \frac{P}{\gamma - 1} \tag{3.4}$$

plays an important role. Therefore we want to find an entropy of (3.2) which tends to (3.4) as $c \to \infty$. Let us look for an entropy-entropy flux of the form

$$\eta = H(\rho, u^2), \qquad q = Q(\rho, u^2)u.$$

Inserting this to the equation it is easy to find an entropy-entropy flux

$$\eta^* = -\frac{\Psi(\rho)}{(1 - u^2/c^2)^{1/2}} + c^2(\frac{\rho + Pu^2/c^4}{1 - u^2/c^2}), \tag{3.5}$$

$$q^* = (-\frac{\Psi(\rho)}{(1 - u^2/c^2)^{1/2}} + c^2\frac{\rho + P/c^2}{1 - u^2/c^2})u, \tag{3.6}$$

where

$$\Psi = exp(\int_1^\rho \frac{d\rho}{\rho + P/c^2} + K_0), \tag{3.7}$$

and K_0 is determined so that η^* tends to the kinetic energy (3.4) as $c = \infty$. We call the entropy η^* defined by (3.5) the relativistic standard entropy. The important fact is

PROPOSITION 3.1. *The Hessian $D_U^2\eta^*$ is positive definite, i.e., for any fixed B there is a positive constant k such that*

$$(\xi|D_U^2\eta^*(U)\cdot\xi) \geq k|\xi|^2,$$

for any $U \in \Sigma_B$ and $\xi = (\xi_0, \xi_1)$ with $|\xi|^2 = \xi_0^2 + \xi_1^2$. Here $(\xi|D_U^2\eta^*(U)\cdot\xi)$ is defined by $\sum_{i,j}[D_U^2\eta^*(U)]_{i,j}\xi_i\xi_j$.

Proof. The proof is due to direct but tedious calculations. We note

$$\begin{aligned}
\frac{\partial\rho}{\partial E} &= \frac{1+u^2/c^2}{1-P'u^2/c^4},\\
\frac{\partial u}{\partial E} &= -\frac{(1+P'/c^2)(1-u^2/c^2)u}{(\rho+P/c^2)(1-P'u^2/c^4)},\\
\frac{\partial\rho}{\partial F} &= -\frac{2u/c^2}{1-P'u^2/c^4},\\
\frac{\partial u}{\partial F} &= \frac{(1-u^2/c^2)(1+P'u^2/c^4)}{(\rho+P/c^2)(1-P'u^2/c^4)}.
\end{aligned}$$

Using these, we have

$$\begin{aligned}
\frac{\partial\eta^*}{\partial E} &= -\frac{\Psi}{(\rho+P/c^2)(1-u^2/c^2)^{1/2}} + c^2,\\
\frac{\partial\eta^*}{\partial F} &= \frac{\Psi u/c^2}{(\rho+P/c^2)(1-u^2/c^2)^{1/2}},\\
\frac{\partial^2\eta^*}{\partial E^2} &= \frac{\Psi/c^2}{(1-P'u^2/c^4)(1-u^2/c^2)^{1/2}(\rho+P/c^2)^2}(P'+2P'u^2/c^2+u^2),\\
\frac{\partial^2\eta^*}{\partial E\partial F} &= \frac{-\Psi/c^2}{(1-P'u^2/c^4)(1-u^2/c^2)^{1/2}(\rho+P/c^2)^2}(2P'/c^2+1+P'u^2/c^4)u,\\
\frac{\partial^2\eta^*}{\partial F^2} &= \frac{\Psi/c^2}{(1-P'u^2/c^4)(1-u^2/c^2)^{1/2}(\rho+P/c^2)^2}(1+3P'u^2/c^4).
\end{aligned}$$

Therefore we get

$$\begin{aligned}
(\xi|D_U^2\eta^*\cdot\xi) &= \eta^*_{EE}\xi_0^2 + 2\eta^*_{EF}\xi_0\xi_1 + \eta^*_{FF}\xi_1^2\\
&= \frac{\Psi/c^2}{(1-P'u^2/c^4)(1-u^2/c^2)^{1/2}(\rho+P/c^2)^2}Z,\\
Z &= (P'+2P'u^2/c^2+u^2)\xi_0^2 - 2(2P'/c^2+1+P'u^2/c^4)u\xi_0\xi_1 +\\
&\quad (1+3P'u^2/c^4)\xi_1^2\\
&\geq \frac{2P'(1-u^2/c^2)^2(1-P'u^2/c^4)}{A+C+\sqrt{(A-C)^2+4B^2}}(\xi_0^2+\xi_1^2),\\
A &= P'+2P'u^2/c^2+u^2,\\
B &= (2P'/c^2+1+P'u^2/c^4)u,\\
C &= 1+3P'u^2/c^4.
\end{aligned}$$

This completes the proof. □

4. Construction of approximate solutions. Let us construct approximate solutions using the Godunov scheme. The construction is similar if we use the Lax-Friedrichs scheme.

Suppose that the initial data $U_0(x)$ is confined to an invariant region Σ_B. Put $\Lambda_0 = \sup\{|\lambda_j(U)||j = 1, 2, U \in \Sigma_B\}$. Fixing $\Lambda_1 > \Lambda_0$, we take mesh lengths $\Delta x, \Delta t$ such that $\Delta x = \Lambda_1 \Delta t$. We denote $\Delta = \Delta x$.

Let us construct the approxomate solution $U^\Delta(t, x)$. First we put

$$U_0^\Delta(x) = U_0(x)\chi_{[-1/\Delta,1/\Delta]}.$$

We define

$$U^\Delta(+0, x) = \frac{1}{2\Delta x}\int_{2j\Delta x}^{(2j+2)\Delta x} U_0^\Delta(x)dx$$

for $2j\Delta x < x \le (2j + 2)\Delta x$. Solving the Riemann problem on each interval $[2(j-1)\Delta, 2(j+1)\Delta]$, we define $U^\Delta(t, x)$ for $0 \le t < \Delta t$. Since the Courant-Friedrichs-Levi condition is satisfied, the wave from the center $2j\Delta$ does not intersect. If $U^\Delta(t, x)$ for $0 \le t < n\Delta t$ has been defined, then we define

$$U^\Delta(n\Delta t, x) = \frac{1}{2\Delta}\int_{2j\Delta}^{(2j+2)\Delta} U^\Delta(n\Delta t - 0, x)dx$$

for $2j\Delta < x \le (2j + 2)\Delta$. Solving the Riemann problem, we define $U^\Delta(t, x)$ for $n\Delta t \le t < (n + 1)\Delta t$.

By Proposition 2.1 and 2.3, it is inductively guaranteed that U^Δ remains in Σ_B, say,

Proposition 4.1. *The approximate solution $U^\Delta(t, x)$ satisfies $U^\Delta(t, x) \in \Sigma_B$, therefore,*

$$0 \le \rho^\Delta(t, x) \le M_0, \qquad |\frac{c}{2}\log\frac{c + u^\Delta(t, x)}{c - u^\Delta(t, x)}| \le M_0.$$

Moreover we shall prove

Proposition 4.2. *For any test function Φ it holds that*

$$\int_0^\infty\int_{-\infty}^\infty (\Phi_t U^\Delta + \Phi_x f(U^\Delta))dxdt + \int_{-\infty}^\infty \Phi(0, x)U_0^\Delta(x)dx = O(\Delta^{1/2}).$$

Here and after, the constants C only depend on U_0 and M_0 but may vary from line to line. In order to prove Proposition 4.2, we prepare

Proposition 4.3. *For any shock wave from U_L to U_R with the shock speed σ and for any convex entropy η, we have*

$$\sigma[\eta] - [q] \ge 0,$$

where $[\eta] = \eta(U_R) - \eta(U_L), [q] = q(U_R) - q(U_L)$.

Proof. The right state of shocks can be parametrized by $\rho = \rho_R$. Putting

$$Q(\rho) = \sigma[\eta] - [q],$$

we shall see $dQ/d\rho \geq 0$ along $S_1 : [\rho] > 0$ and $dQ/d\rho \leq 0$ along $S_2 : [\rho] < 0$. Using the equation (3.1) and the differentiation of the Rankine-Hugoniot condition, we have

$$\begin{aligned}\frac{dQ}{d\rho} &= \frac{d\sigma}{d\rho}([\eta] - D_U\eta(U)\cdot[U]) \\ &= -\frac{d\sigma}{d\rho}\int_0^1 \theta(U-U_L|D_U^2\eta(U_L+\theta(U-U_L)\cdot(U-U_L))d\theta.\end{aligned}$$

We supposed $D_U^2\eta \geq 0$. By Proposition 2.4, we know $d\sigma/d\rho < 0$ on S_1 and $d\sigma/d\rho > 0$ on S_2. ∎

Proof of Proposition 4.2. We fix T to consider U^Δ on $0 \leq t \leq T$. First we shall show

$$\sum_{j,n}\int_{2j\Delta}^{(2j+2)\Delta} |U(n\Delta t-0,x) - U(n\Delta t+0,(2j+1)\Delta)|^2 dx \leq C. \tag{4.1}$$

Let us consider the standard entropy η^*. Then we have

$$\begin{aligned}0 &= \int \eta^*(U(T,x))dx - \int \eta^*(U(0,x))dx + L + \Sigma, \\ L &= \sum_{j,n}\int_{2j\Delta}^{(2j+2)\Delta}(\eta^*(U(n\Delta t-0,x)) - \eta^*(U(n\Delta t+0,(2j+1)\Delta)))dx, \\ \Sigma &= \int_0^T \sum_{shocks}(\sigma[\eta^*] - [q^*])dt.\end{aligned}$$

We write $U_j = U(n\Delta t+0,(2j+1)\Delta), U_1 = U(n\Delta t-0,x)$. Since

$$U_j = \frac{1}{2\Delta}\int_{2j\Delta}^{(2j+2)\Delta} U_1 dx,$$

we see

$$\begin{aligned}L &= \sum_{j,n}\int_{2j\Delta}^{(2j+2)\Delta}\int_0^1 (1-\theta)(U_1-U_j|D_U^2\eta^*(U_0+\theta(U_1-U_j))\cdot(U_1-U_j))d\theta dx \\ &\geq 0.\end{aligned}$$

On the other hand we have $\Sigma \geq 0$ from Proposition 4.3. Thus $L \leq C, \Sigma \leq C$. But from Proposition 3.1, we have $D_U^2\eta^* \geq k$. Therefore

$$C \geq L \geq \frac{k}{2}\sum_{j,n}\int_{2j\Delta}^{(2j+2)\Delta}|U_1-U_j|^2 dx.$$

Thus we get (4.1).

Now let us consider a test function Φ. Put

$$J = \int_0^\infty\int_{-\infty}^\infty(\Phi_t U^\Delta + \Phi_x f(U^\Delta))dxdt + \int_0^\infty \Phi(0,x)U_0^\Delta dx.$$

Since U^Δ is a weak solution on each time strip $n\Delta t < t < (n+1)\Delta t$, we have

$$\begin{aligned}
J &= \sum_n \int_0^\infty \Phi(n\Delta t, x)(U(n\Delta t - 0, x) - U(n\Delta t + 0, x))dx \\
&= J_1 + J_2, \\
J_1 &= \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} \Phi(n\Delta t, j\Delta)(U(n\Delta t - 0, x) - U(n\Delta t + 0, x))dx, \\
J_2 &= \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} (\Phi(t, x) - \Phi(n\Delta t, j\Delta))(U(n\Delta t - 0, x) - U(n\Delta t + 0, x))dx.
\end{aligned}$$

Since

$$U(n\Delta t + 0, x) = \frac{1}{2\Delta} \int_{2j\Delta}^{(2j+2)\Delta} U(n\Delta t - 0, x)dx$$

for $2j\Delta < x < (2j+2)\Delta$, we see $J_1 = 0$. It follows from (4.1) that

$$\begin{aligned}
|J_2| &\leq C\Delta^{1/2}||\Phi||_{C^1}(\sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} |U(n\Delta t - 0, x) - U(n\Delta t + 0, x)|^2 dx)^{1/2} \\
&\leq C'\Delta^{1/2}.
\end{aligned}$$

Here we have used $T/\Delta t = O(1/\Delta)$. □

Summing up, we have the following theorem.

THEOREM 1. *The approximate solution $U^\Delta(t, x)$ satisfies*

$$0 \leq \rho^\Delta(t, x) \leq M_0, \qquad |\frac{c}{2} \log \frac{c + u^\Delta(t, x)}{c - u^\Delta(t, x)}| \leq M_0$$

and

$$\int\int (\Phi_t U^\Delta + \Phi_x f(U^\Delta))dxdt + \int \Phi(0, x)U_0^\Delta(x) = O(\Delta^{1/2})$$

for any test function Φ.

We expect that U^Δ tends to a weak solution everywhere. For the non-relativistic gas dynamics, this was done by DiPerna [3] and G.Q.Chen et al [2]. In their proof the Darboux formula

$$\eta = \int_z^w ((w - s)(s - z))^N \phi(s)ds$$

which gives solutions of the Euler-Poisson-Darboux equation (3.3) , ϕ being arbitrary, plays an important role. Section 6 will be devoted to find such an integral formula for the relativistic Euler-Poisson-Darboux equation (3.2).

REMARK 4.1. We note that

$$\begin{aligned}
\lambda_2 - \lambda_1 &= \frac{2\sqrt{P'}(1 - u^2/c^2)}{1 - u^2P'/c^4} > 0, \\
\frac{\partial\lambda_1}{\partial z} &= \frac{1 - u^2/c^2}{2(1 - \sqrt{P'}u/c^2)^2}(1 - \frac{P'}{c^2} + \frac{(\rho + P/c^2)P''}{2P'}) > 0, \\
\frac{\partial\lambda_2}{\partial w} &= \frac{1 - u^2/c^2}{2(1 + \sqrt{P'}u/c^2)^2}(1 - \frac{P'}{c^2} + \frac{(\rho + P/c^2)P''}{2P'}) > 0
\end{aligned}$$

for $\rho > 0$ and $|u| < c$.

This says that the system is strictly hyperbolic and genuinely nonlinear on $\rho > 0$. Therefore the Glimm's theory can be applied if

$$||U_0(x) - U^*||_{L^\infty} + T.V.U_0$$

is sufficiently small, where U^* is a constant state such that $\rho^* > 0, |u^*| < c$. But the vacuum may not be covered by this application of the general theorem.

5. Generalized Darboux formula. In this section we seek an integration formula for solutions of the relativistic Euler-Poisson-Darboux equation. Let us introduce the variables

$$x = \frac{c}{2}\log\frac{c+u}{c-u}, \qquad y = \int_0^\rho \frac{\sqrt{P'}}{\rho + P/c^2}d\rho.$$

Then the relativistic Euler-Poisson-Darboux equation is

$$(EPD) \qquad \eta_{xx} - \eta_{yy} + A(x,y)\eta_y + B(x,y)\eta_x = 0,$$

where

$$A(x,y) = \frac{1}{\sqrt{P'}}(1 - \frac{P'}{c^2} - \frac{\rho + P/c^2}{2P'}P'')\frac{1 + P'u^2/c^4}{1 - P'u^2/c^4},$$
$$B(x,y) = -\frac{2u/c^2}{1 - P'u^2/c^4}(1 - \frac{P'}{c^2} - \frac{\rho + P/c^2}{2P'}P'').$$

The coefficients A and B are of the form

$$A = \frac{2N}{y} + a, \qquad a = \frac{y}{c^2}(a_0 + [x^2/c^2, y^2/c^2]_1),$$
$$B = -\frac{4N}{2N+1}\frac{x}{c^2}(1 + [x^2/c^2, y^2/c^2]_1),$$

where $[X, Y]_1$ denotes a convergent power series $\sum_{j+k\geq 1} c_{jk}X^jY^k$. In order to remove the singularity in A, we use the trick of Weinstein [7]. We introduce the sequence of variables $\eta_j, j = 0, 1, , ..., N$ by

$$\frac{\partial\eta_j}{\partial y} = y\eta_{j+1},$$

or

$$\eta_j(x,y) = I\eta_{j+1}(x,y) = \int_0^y Y\eta_{j+1}(x,Y)dY,$$

where $\eta_0 = \eta$. The sequence of formal integro-differential operators L_j is defined by

$$L_jV = V_{xx} - V_{yy} + (\frac{2(N-j)}{y} + a)V_y + BV_x +$$
$$j\tilde{a}V + \sum_{k=1}^{j} F_{j,k}I^kV_x + \sum_{k=1}^{j} H_{j,k}I^kV,$$

where

$$\tilde{a} = \frac{\partial a}{\partial y} + \frac{a}{y} = \frac{1}{c^2}[x^2/c^2, y^2/c^2]_0.$$

The coefficients $F_{j,k}$ and $H_{j,k}$ are determined inductively by

$$F_{j+1,k} = \begin{cases} F_{j,1} + \frac{1}{y}\frac{\partial B}{\partial y} & \text{if } k = 1, \\ F_{j,k} + \frac{1}{y}\frac{\partial}{\partial y}F_{j,k-1} & \text{if } k \geq 2, \end{cases}$$

$$H_{j+1,k} = \begin{cases} H_{j,1} + j\frac{1}{y}\frac{\partial \tilde{a}}{\partial y} & \text{if } k = 1, \\ H_{j,k} + \frac{1}{y}\frac{\partial}{\partial y}H_{j,k-1} & \text{if } k \geq 2. \end{cases}$$

It is easy to see that $F_{j,k}$ are of the form $\frac{x}{c^2}[x^2/c^2, y^2/c^2]_0$ and $H_{j,k}$ are of the form $\frac{1}{c^2}[x^2/c^2, y^2/c^2]_0$. By the definition we have formally

$$\frac{1}{y}\frac{\partial}{\partial y}(L_j\eta_j) = L_{j+1}\eta_{j+1}.$$

Now we consider the equation $L_N V = 0$ for $V = \eta_N$ with the initial conditions

$$V = 0, \qquad V_y = 2^{N+1}N!\phi(x), \qquad \text{at } y = 0.$$

Let $F_k = F_{N,k}$ and $H_k = H_{N,k}$, the problem is

$$(Q) \qquad V_{yy} - V_{xx} = aV_y + BV_x + N\tilde{a}V + \sum_{k=1}^{N} F_k I^k V_x + \sum_{k=1}^{N} H_k I^k V,$$
$$V = 0, \qquad V_y = 2^{N+1}N!\phi(x) \qquad \text{at } y = 0.$$

PROPOSITION 5.1. *If $\phi \in C^1(R)$, then the problem (Q) admits a unique solution V in $C^2(R \times [0, \infty))$.*

Proof. Let us denote by $H(x, y, V)$ the right hand side of the equation $L_N = 0$. Then (Q) is transformed to the integral equation

$$V(x, y) = 2^N N! \int_{x-y}^{x+y} \phi(\xi)d\xi + \frac{1}{2}\int_0^y \int_{x-y+Y}^{x+y-Y} H(X, Y, V)dXdY.$$

We can solve this integral equation by the iteration

$$V_0(x, y) = 2^N N! \int_{x-y}^{x+y} \phi(\xi)d\xi,$$
$$V^{n+1}(x, y) = 2^N N! \int_{x-y}^{x+y} \phi(\xi)d\xi + \frac{1}{2}\int_0^y \int_{x-y+Y}^{x+y-Y} H(X, Y, V^n)dXdY.$$

Fixing L arbitrarily, we consider $|x| \leq L$. Then it is easy to get the estimates

$$|V^{n+1}(x, y) - V^n(x, y)| \leq \frac{M^{n+1}y^{n+1}}{(n+1)!}.$$

Therefore V^n tends to a limit V uniformly on $|x| \leq L, 0 \leq y \leq L$. The limit is the unique solution of (Q). ∎

Now we put

$$\eta_N = V, \qquad \eta_{N-k} = I\eta_{N-k+1}.$$

Since η_{N-k} and its derivatives of order ≤ 2 all vanish on $y = 0$ for $k \geq 1$, we see $\eta = \eta_0$ gives a solution of the relativistic Euler-Poisson-Darboux equation (EPD).

Next we give an integral formula for the solution V of (Q).

PROPOSITION 5.2. *There is a C^{N+2}-function $G(x, y, \xi)$ of $|x| < \infty, y \geq 0, x-y \leq \xi \leq x+y$ such that the solution V of (Q) satisfies*

$$V(x,y) = \int_{x-y}^{x+y} G(x,y,\xi)\phi(\xi)d\xi. \tag{5.1}$$

Moreover

$$\begin{aligned} G &= 2^N N! + O(y/c^2), \\ \partial_x^{p_1}\partial_\xi^{p_2}\partial_y^{p_3} G &= O(1/c^2) \qquad \textit{for } 1 \leq p_1 + p_2 + p_3 \leq N+2. \end{aligned}$$

Proof. We consider the approximate solution $V^n(x, y)$ which appeared in the iteration of the proof of Proposition 5.1. By writing H as

$$H = (aV)_y + (BV)_x + bV + \sum (F_k I^k V)_x + \sum \tilde{H}_k I^k V,$$

where

$$\begin{aligned} b = N\tilde{a} - a_y - B_x &= \frac{1}{c^2}[x^2/c^2, y^2/c^2]_0, \\ \tilde{H}_k = H_k - (F_k)_x &= \frac{1}{c^2}[x^2/c^2, y^2/c^2]_0, \end{aligned}$$

it is easy to see inductively that there is a kernel $G^n(x, y, \xi)$ such that

$$V^n(x,y) = \int_{x-y}^{x+y} G^n(x,y,\xi)\phi(\xi)d\xi.$$

In fact $G^0 = 2^N N!$ and G^n are determined inductively by the formula

$$\begin{aligned} G^{n+1} &= 2 + \frac{1}{2}(G_I^n + G_{II}^n + G_{III}^n + \sum G_{IVk}^n + \sum G_{Vk}^n), \\ G_I &= \int_{(-x+y+\xi)/2}^{y} a(x-y+Y,Y)G(x-y+Y,Y,\xi)dY + \\ &\quad \int_{(x+y-\xi)/2}^{y} a(x+y-Y,Y)G(x+y-Y,Y,\xi)dY, \\ G_{II} &= \int_{(x+y-\xi)/2}^{y} B(x+y-Y,Y)G(x+y-Y,Y,\xi)dY - \\ &\quad \int_{(-x+y+\xi)/2}^{y} B(x-y+Y,Y)G(x-y+Y,Y,\xi)dY, \\ G_{III} &= \int\int_{D(x,y,\xi)} b(X,Y)G(X,Y,\xi)dXdY, \end{aligned}$$

where

$$\begin{aligned} D(x,y,\xi) = \{(X,Y)|X-Y \le \xi \le X+Y, x-y+Y \le X \le x+y-Y, \\ 0 \le Y \le y\}, \end{aligned}$$

$$\begin{aligned} G_{IVk} = \int_{(x-y+\xi)/2}^{y} F_k(x+y-Y,Y)J^k G(x+y-Y,Y,\xi)dY - \\ \int_{(-x+y+\xi)/2}^{y} F_k(x-y+Y,Y)J^k G(x-y+Y,Y,\xi)dY, \end{aligned}$$

where

$$JG(x,y,\xi) = \int_{|x-\xi|}^{y} YG(x,Y,\xi)dY$$

and

$$G_{Vk} = \int\int_{D(x,y,\xi)} \tilde{H}_k(X,Y)J^k G(X,Y,\xi)dXdY.$$

It is easy to see inductively that

$$|G^{n+1}(x,y,\xi) - G^n(x,y,\xi)| \le \frac{M^{n+1}y^{n+1}}{(n+1)!}.$$

therefore G^n converges to a limit G uniformly and (5.1) holds. Moreover we can differentiate G^{n+1} supposing that G^n is differentiable. In fact we have

$$\begin{aligned} G_{I,x} = &\frac{1}{2}aG((x-y+\xi)/2,(-x+y+\xi)/2,\xi) - \\ &\frac{1}{2}aG((x+y+\xi)/2,(x+y-\xi)/2,\xi) + \\ &\int_{(-x+y+\xi)/2}^{y} (aG)_x(x-y+Y,Y,\xi)dY + \\ &\int_{(x+y-\xi)/2}^{y} (aG)_x(x-Y+Y,Y,\xi)dY, \\ G_{I,\xi} = &-\frac{1}{2}aG((x-y+\xi)/2,(-x+y+\xi)/2,\xi) + \\ &\frac{1}{2}aG((x+y+\xi)/2,(x+y-\xi)/2,\xi) + \\ &\int_{(-x+y+\xi)/2}^{y} aG_\xi(x-y+Y,Y,\xi)dY + \\ &\int_{(-x+y+\xi)/2}^{y} aG_\xi(x+y-Y,Y,\xi)dY, \\ G_{I,y} = &-\frac{1}{2}aG((x-y+\xi)/2,(-x+y+\xi)/2,\xi) - \\ &\frac{1}{2}aG((x+y+\xi)/2,(x+y-\xi)/2,\xi) + \\ &2aG(x,y,\xi) - \int_{(-x+y+\xi)/2}^{y} (aG)_x(x-y+Y,Y,\xi)dY + \end{aligned}$$

$$
\begin{aligned}
&\int_{(-x+y+\xi)/2}^{y} (aG)_x(x+y-Y,Y,\xi)dY;\\
G_{II,x} = &-\frac{1}{2}BG((x+y+\xi)/2,(x+y-\xi)/2,\xi) -\\
&\frac{1}{2}BG((x-y+\xi)/2,(-x+y+\xi)/2,\xi) +\\
&\int_{(x+y-\xi)/2}^{y}(BG)_x(x+y-Y,Y,\xi)dY -\\
&\int_{(-x+y+\xi)/2}^{y}(BG)_x(x-y+Y,Y,\xi)dY,\\
G_{II,\xi} = &\frac{1}{2}BG((x+y+\xi)/2,(x+y-\xi)/2,\xi) +\\
&\frac{1}{2}BG((x-y+\xi)/2,(-x+y+\xi)/2,\xi) +\\
&\int_{(x+y-\xi)/2}^{y} BG_\xi(x+y-Y,Y,\xi)dY -\\
&\int_{(-x+y+\xi)/2}^{y} BG_\xi(x-y+Y,Y,\xi)dY,\\
G_{II,y} = &-\frac{1}{2}BG((x+y+\xi)/2,(x+y-\xi)/2,\xi) +\\
&\frac{1}{2}BG((x-y+\xi)/2,(-x+y+\xi)/2,\xi) +\\
&\int_{(x+y-\xi)/2}^{y}(BG)_x(x+y-Y,Y,\xi)dY +\\
&\int_{(-x+y+\xi)/2}^{y}(BG)_x(x-y+Y,Y,\xi)dY;\\
G_{III,x} = &\int_{(x+y-\xi)/2}^{y} bG(x+y-Y,Y,\xi)dY - \int_{(-x+y+\xi)/2}^{y} bG(x-y+Y,Y,\xi)dY,\\
G_{III,\xi} = &\int_0^{(x+y-\xi)/2} bG(\xi+Y,Y,\xi)dY + \int_0^{(-x+y+\xi)/2} bG(\xi-Y,Y,\xi)dY +\\
&\int\int_{D(x,y,\xi)} bG(X,Y,\xi)dXdY,\\
G_{III,y} = &\int_{(x+y-\xi)/2}^{y} bG(x+y-Y,Y,\xi)dY + \int_{(-x+y+\xi)/2}^{y} bG(x-y+Y,Y,\xi)dY;
\end{aligned}
$$

and the derivatives of G_{IVk} are similar to G_{II} and the derivatives of G_{IVk} are similar to G_{III}.

Then it is easy to see inductively that

$$
|G_x^{n+1} - G_x^n| + |G_\xi^{n+1} - G_\xi^n| + |G_y^{n+1} - G_y^n| \le \frac{M^n y^n}{n!}.
$$

Thus the limit G is differentiable. In a smilar manner we see

$$
\begin{aligned}
&|G_{xx}^{n+1} - G_{xx}^n| + |G_{x\xi}^{n+1} - G_{x\xi}^n| + |G_{xy}^{n+1} - G_{xy}^n| +\\
&\qquad |G_{\xi\xi}^{n+1} - G_{\xi\xi}^n| + |G_{\xi y}^{n+1} - G_{\xi y}^n| + |G_{yy}^{n+1} - G_{yy}^n|
\end{aligned}
$$

$$\leq \frac{M^{n-1}y^{n-1}}{(n-1)!}.$$

Thus G is twice continuously differentiable. In a similar manner we see that G is $N+2$-times continuously differentiable. The rough estimates stated in the propositions is obvious since the coefficients are all of $O(1/c^2)$. ▯

The solution η_{N-k} enjoys an integral representation

$$\eta_{N-k} = \int_{x-y}^{x+y} K_{N-k}(x,y,\xi)\phi(\xi)d\xi,$$

where

$$K_{N-k}(x,y,\xi) = JK_{N-k+1}(x,y,\xi) = J^k G(x,y,\xi).$$

So the solution η of the relativistic Euler-Poisson-Darboux equation is given by

$$\eta(x,y) = \int_{x-y}^{x+y} K(x,y,\xi)\phi(\xi)d\xi,$$

where

$$K(x,y,\xi) = J^N G(x,y,\xi).$$

By induction we see

$$J^k G(x,y,\xi) = \frac{2^N N!}{2^k k!}(y^2-(x-\xi)^2)^k(1+O(y/c^2)).$$

Thus we have

PROPOSITION 5.3. *There is a kernal $K(x,y,\xi)$ which is of C^{N+2}-class in $|x| < \infty, 0 \leq y, x-y \leq \xi \leq x+y$ such that*

$$\eta(x,y) = \int_{x-y}^{x+y} K(x,y,\xi)\phi(\xi)d\xi$$

gives a solution of the relativistic Euler- Poisson-Darboux equation for any smooth ϕ. Moreover

$$K(x,y,\xi) = (y^2-(x-\xi)^2)^N(1+O(y/c^2)).$$

But in order to apply this integration formula, the generalized Darboux formula, to the study of the relativistic Euler equation, more detailed estimates of the remainder are necessary.

PROPOSITION 5.4. *We have*

(1) $G_y = O(y/c^2)$.

(2) $G = 2^N N! + \frac{1}{c^2}C_0(x,c)(\xi-x) + O(y^2/c^2)$.

(3) $G_x + G_\xi = \frac{1}{c^2}C_1(x,c)(\xi-x) + O(y^2/c^2)$.

(4) $(G_x+G_\xi)_y = O(y/c^2)$.

(5) $(G_x+G_\xi)_x + (G_x+G_\xi)_\xi = \frac{1}{c^2}C_2(x,c)(\xi-x) + O(y^2/c^2)$.

(6) $G_\xi = \frac{1}{c^2}C_3(x,c) + O(y/c^2)$.

(7) $(G_x+G_\xi)_\xi = \frac{1}{c^2}C_4(x,c) + O(y/c^2)$.

Here $C_0(x),\ C_1(x),\ C_2(x), C_3(x),$ and $C_4(x)$ are functions of the form

$$[x^2/c^2]_0 + \frac{x}{c^2}[x^2/c^2]_0.$$

Proof. See Appendix A. ▯

6. Estimates of the derivatives of entropies. Let us consider the entropy η generated by ϕ of C^3-class, that is,

$$\eta(x,y) = \int_{x-y}^{x+y} K(x,y,\xi)\phi(\xi)d\xi.$$

In this section we will find estimates of the derivatives of η with respect to E, F. As auxiliary variables we introduce

$$R = y^{2N+1}, \qquad M = xy^{2N+1}. \tag{6.1}$$

We are going to prove the following

PROPOSITION 6.1. *We have*

$$\frac{\partial\eta}{\partial M} = 2^{2N+1}\int_0^1 (s-s^2)^N D\phi(x+(2s-1)y)ds + O(y^2/c^2), \tag{6.2}$$

$$\begin{aligned}\frac{\partial\eta}{\partial R} = {}& 2^{2N+1}\int_0^1 (s-s^2)^N \phi ds + \\ & 2^{2N+1}\int_0^1 (s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))D\phi ds + \\ & O(y^2/c^2),\end{aligned} \tag{6.3}$$

$$\frac{\partial^2\eta}{\partial M^2} = 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N D^2\phi ds + O(y^{-2N+1}/c^2), \tag{6.4}$$

$$\begin{aligned}\frac{\partial^2\eta}{\partial R\partial M} = {}& 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))D^2\phi ds + \\ & O(y^{-2N+1}/c^2),\end{aligned} \tag{6.5}$$

$$\begin{aligned}\frac{\partial^2\eta}{\partial R^2} = {}& 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N((-x+\frac{y}{2N+1}(2s-1))^2 + \\ & \frac{4}{(2N+1)^2}s(1-s)y^2)D^2\phi(x+(2s-1)y)ds + \\ & O(y^{-2N+1}/c^2).\end{aligned} \tag{6.6}$$

Proof. See Appendix B. ◻

Let us recall the standard entropy η^*. This is generated by

$$\phi^*(x) = A'c^2(\frac{1}{1-u^2/c^2} - \frac{1}{\sqrt{1-u^2/c^2}}),$$

where

$$A' = (2N+1)^{-2N}((2N+1)/(2N+3)A)^{\frac{2N+1}{2}}(2N-1)!!/2^{N+1}N!.$$

We note that

$$D^2\phi^*(x) = A'(1+\frac{u^2/c^2}{1-u^2/c^2})(2-\sqrt{1-u^2/c^2}) \geq A'.$$

We are going to show that the Hessian $D_U^2\eta^*$ dominates any $D_U^2\eta$.

PROPOSITION 6.2. *For each ϕ fixed in C^3 we have on each compact subset of $\{\rho \geq 0\}$*

$$|(\xi|D_U^2\eta\cdot\xi)| \leq C(\xi|D_U^2\eta^*\cdot\xi),$$

provided that c is sufficiently large.

By the assumption we have

$$\begin{aligned} R &= y^{2N+1} = K\rho(1+[\rho^{\frac{2}{2N+1}}/c^2]_1), \\ \frac{dR}{d\rho} &= K + [\rho^{\frac{2}{2N+1}}/c^2]_1, \quad \text{and} \quad \frac{d^2R}{d\rho^2} = \frac{\rho^{\frac{1-2N}{2N+1}}}{c^2}[\rho^{\frac{2}{2N+1}}/c^2]_0, \end{aligned}$$

where $K = ((2N+3)(2N+1)A)^{\frac{2N+1}{2}}$. Using these, we have

$$\begin{aligned} \frac{\partial R}{\partial E} &= \frac{dR}{d\rho}\frac{1+u^2/c^2}{1-P'u^2/c^4} \\ &= K(1+u^2/c^2) + O(y^2/c^2), \\ \frac{\partial R}{\partial F} &= -\frac{dR}{d\rho}\frac{2u/c^2}{1-P'u^2/c^4} \\ &= -K\frac{2u}{c^2} + O(y^2/c^2), \\ \frac{\partial M}{\partial E} &= -\frac{R}{\rho+P/c^2}\frac{1+P'/c^2}{1-P'u^2/c^4}u + x\frac{dR}{d\rho}\frac{1+u^2/c^2}{1-P'u^2/c^4} \\ &= K(-u+x(1+u^2/c^2)) + O(y^2/c^2), \\ \frac{\partial M}{\partial F} &= \frac{R}{\rho+P/c^2}\frac{1+P'u^2/c^4}{1-P'u^2/c^4} - \frac{dR}{d\rho}2xu/c^2\frac{1}{1-P'u^2/c^4} \\ &= K(1-2xu/c^2) + O(y^2/c^2). \end{aligned} \tag{6.7}$$

Differentiating once more, we see

$$\begin{aligned} \frac{\partial^2 R}{\partial E^2} &= -\frac{K^2}{y^{2N+1}}2u^2(1-u^2/c^2)/c^2 + O(y^{-2N+1}/c^2), \\ \frac{\partial^2 M}{\partial E^2} &= \frac{K^2}{y^{2N+1}}u(-2u^2/c^2 - 2ux(1-u^2/c^2)/c^2) + O(y^{-2N+1}/c^2), \\ \frac{\partial^2 R}{\partial E\partial F} &= \frac{K^2}{y^{2N+1}}\frac{2u}{c^2}(1-u^2/c^2) + O(y^{-2N+1}/c^2), \\ \frac{\partial^2 M}{\partial E\partial F} &= \frac{K^2}{y^{2N+1}}(2u^2/c^2 + 2xu(1-u^2/c^2)/c^2) + O(y^{-2N+1}/c^2), \\ \frac{\partial^2 R}{\partial F^2} &= -\frac{2}{c^2}\frac{K^2}{y^{2N+1}}(1-u^2/c^2) + O(y^{-2N+1}/c^2), \\ \frac{\partial^2 M}{\partial F^2} &= -\frac{K^2}{y^{2N+1}}2(u+x(1-u^2/c^2))/c^2 + O(y^{-2N+1}/c^2). \end{aligned} \tag{6.8}$$

The chain rule gives

$$\begin{aligned} \frac{\partial^2\eta}{\partial E^2} = (\frac{\partial R}{\partial E})^2\frac{\partial^2\eta}{\partial R^2} + 2\frac{\partial R}{\partial E}\frac{\partial M}{\partial E}\frac{\partial^2\eta}{\partial R\partial M} + \\ (\frac{\partial M}{\partial E})^2\frac{\partial^2\eta}{\partial M^2} + \frac{\partial^2 R}{\partial E^2}\frac{\partial\eta}{\partial R} + \frac{\partial^2 M}{\partial E^2}\frac{\partial\eta}{\partial M}, \end{aligned} \tag{6.9}$$

and so on. Inserting (6.7)nd (6.8) into (6.9), and using Proposition 6.1, we have

$$
\begin{aligned}
(\xi|D_U^2\eta\cdot\xi) = \frac{2^{2N+1}K^2}{y^{2N+1}}&\int_0^1(s-s^2)^N Z[\xi]D^2\phi ds - \frac{2K^2}{y^{2N+1}}\frac{1}{c^2}(1-u^2/c^2)\times \\
&(u\xi_0-\xi_1)^2\frac{\partial\eta}{\partial R} - \frac{2K^2}{y^{2N+1}}\frac{1}{c^2}(u+x(1-u^2/c^2))(u\xi_0-\xi_1)^2\frac{\partial\eta}{\partial M} \\
&+O(y^{-2N+1}/c^2),
\end{aligned}
$$

where

$$
\begin{aligned}
Z[\xi] &= Z_{00}\xi_0^2 + 2Z_{01}\xi_0\xi_1 + Z_{11}\xi_1^2, \\
Z_{00} &= (1+u^2/c^2)^2((-x+\frac{y}{2N+1}(2s-1))^2 + \frac{4}{(2N+1)^2}s(1-s)y^2) + \\
&\quad 2(1+u^2/c^2)(-u+x(1+u^2/c^2))(-x+\frac{y}{2N+1}(2s-1)) + \\
&\quad (-u+x(1+u^2/c^2))^2, \\
Z_{01} &= -2(1+u^2/c^2)u/c^2((-x+\frac{y}{2N+1}(2s-1))^2 + \frac{4}{(2N+1)^2}s(1-s)y^2) \\
&\quad +(1+3u^2/c^2-4x(1+u^2/c^2)u/c^2)(-x+\frac{y}{2N+1}(2s-1)) + \\
&\quad (-u+x(1+u^2/c^2))(1-2xu/c^2), \\
Z_{11} &= \frac{4u^2}{c^4}((-x+\frac{y}{2N+1}(2s-1))^2 + \frac{4}{(2N+1)^2}s(1-s)y^2) - \\
&\quad \frac{4u}{c^2}(1-2xu/c^2)(-x+\frac{y}{2N+1}(2s-1)) + (1-2xu/c^2)^2.
\end{aligned}
$$

It can be shown that

$$Z[\xi] \geq \kappa s(1-s)y^2,$$

where κ is a positive constant depending on the compact subset of $\{\rho\geq 0\}$. In fact we see

$$Z_{00}Z_{11} - Z_{01}^2 = (1-u^2/c^2)\frac{4}{(2N+1)^2}s(1-s)y^2.$$

On the other hand, we can estimate

$$
\begin{aligned}
|\frac{2K^2}{y^{2N+1}}\frac{1}{c^2}(1-u^2/c^2)\frac{\partial\eta}{\partial R}| &\leq \frac{\epsilon}{y^{2N+1}}, \\
|\frac{2K^2}{y^{2N+1}}\frac{1}{c^2}(u+x(1-u^2/c^2))\frac{\partial\eta}{\partial M}| &\leq \frac{\epsilon}{y^{2N+1}},
\end{aligned}
$$

where $\epsilon = K'/c^2$. Let us introduce the parameters

$$\zeta_0 = \xi_0, \qquad \zeta_1 = \xi_1 - u\xi_0.$$

Then we have

$$Z[\xi] = Q_{00}\zeta_0^2 + 2Q_{01}\zeta_0\zeta_1 + Q_{11}\zeta_1^2,$$

and

$$\begin{aligned} Q_{00} &= Q_{00}^{(1)}(x)(2s-1)y + Q_{00}^{(2)}(x,s)y^2, \\ Q_{01} &= Q_{01}^{(1)}(x)(2s-1)y + Q_{01}^{(2)}(x,s)y^2, \\ Q_{11} &= Z_{11} = 1 + O(1/c^2) > 0. \end{aligned}$$

Therefore if $|D^2\phi| \leq C$, we see

$$\begin{aligned} |(\xi|D_U^2\eta\cdot\xi)| \leq\; & \frac{2^{2N+1}K^2C}{y^{2N+1}}\int_0^1 (s-s^2)^N Z[\xi]ds + \\ & \frac{12\epsilon}{y^{2N+1}}\int_0^1 (s-s^2)^N\zeta_1^2 ds + O(y^{-2N+1}/c^2) \\ \leq\; & \frac{2^{2N+1}K^2C}{y^{2N+1}}\int_0^1 (s-s^2)^N (Q_{11}(1+\epsilon')\zeta_1^2 + 2Q_{01}\zeta_0\zeta_1 + Q_{00}\zeta_0^2)ds \\ & + O(y^{-2N+1}/c^2). \end{aligned}$$

But since $Q_{00}^{(0)} = Q_{01}^{(0)} = 0, \int_0^1 (s-s^2)^N(2s-1)ds = 0$, we see

$$\int_0^1 (s-s^2)^N(-2\epsilon' Q_{01}\zeta_0\zeta_1 - \epsilon' Q_{00}\zeta_0^2)ds = O(y^{-2N+1}/c^2).$$

Therefore we get

$$|(\xi|D_U^2\eta\cdot\xi)| \leq \frac{2^{2N+1}K^2C(1+\epsilon')}{y^{2N+1}}\int_0^1 (s-s^2)^N Z[\xi]ds + O(y^{-2N+1}/c^2).$$

Similarly, if $D^2\phi^* \geq \mu$, we have

$$(\xi|D_U^2\eta^*\cdot\xi) \geq \frac{2^{2N+1}K^2\mu(1-\epsilon'')}{y^{2N+1}}\int_0^1 (s-s^2)^N Z[\xi]ds + O(y^{-2N+1}/c^2).$$

Thus we get

$$|(\xi|D_U^2\eta\cdot\xi)| \leq \frac{C(1+\epsilon')}{\mu(1-\epsilon'')}(\xi|D_U^2\eta^*\cdot\xi) + O(y^{-2N+1}/c^2).$$

But we know

$$(\xi|D_U^2\eta^*\cdot\xi) \geq \kappa|\xi|^2 y^{-2N+1}.$$

Hence if c is sufficiently large we get the required estimate. $\square$

As for the first derivatives, the following conclusion is now clear.

PROPOSITION 6.3. *On each compact subset of* $\{\rho \geq 0\}$, *we have*

$$|\frac{\partial\eta}{\partial E}| + |\frac{\partial\eta}{\partial F}| \leq C.$$

7. Useful entropies. Let us consider an entropy η generated by ϕ, that is,

$$\eta(x,y) = \int_{x-y}^{x+y} K(x,y,\xi)\phi(\xi)d\xi. \tag{7.1}$$

The corresponding entropy flux q is given by integrating the differential equations

$$\frac{\partial q}{\partial w} = \lambda_2 \frac{\partial \eta}{\partial w}, \qquad \frac{\partial q}{\partial z} = \lambda_1 \frac{\partial \eta}{\partial z}.$$

We can solve these equations as

$$\begin{aligned} q &= \lambda_2 \eta - \int_z^w \frac{\partial \lambda_2}{\partial w} \eta dw \\ &= \lambda_1 \eta + \int_z^w \frac{\partial \lambda_1}{\partial z} \eta dz. \end{aligned}$$

Thus we get the formula

$$q(x,y) = \int_{x-y}^{x+y} L(x,y,\xi)\phi(\xi)d\xi, \tag{7.2}$$

where

$$\begin{aligned} L(x,y,\xi) &= \lambda_1 K(x,y,\xi) + L_1(x,y,\xi) \\ &= \lambda_2 K(x,y,\xi) + L_2(x,y,\xi), \\ L_1(x,y,\xi) &= 2\int_{(x+y-\xi)/2}^{y} \mu_1(x+y-Y,Y)K(x+y-Y,Y,\xi)dY, \\ L_2(x,y,\xi) &= -2\int_{(-x+y+\xi)/2}^{y} \mu_2(x-y+Y,Y)K(x-y+Y,Y,\xi)dY, \\ \mu_1(x,y) &= \frac{\partial \lambda_1}{\partial z} \\ &= \frac{1-u^2/c^2}{2(1-\sqrt{P'}u/c^2)}(1 - \frac{P'}{c^2} + \frac{(\rho+P/c^2)P''}{2P'}) \\ &= \frac{N+1}{2N+1} + O(1/c^2), \\ \mu_2(x,y) &= \frac{\partial \lambda_2}{\partial w} \\ &= \frac{1-u^2/c^2}{2(1+\sqrt{P'}u/c^2)}(1 - \frac{P'}{c^2} + \frac{(\rho+P/c^2)P''}{2P'}) \\ &= \frac{N+1}{2N+1} + O(1/c^2). \end{aligned}$$

In this section we will construct various kinds of useful entropies.

1) Let us put

$$\begin{aligned} \eta_k^1(x,y) &= \int_{x-y}^{x+y} K(x,y,\xi)k^{N+1}e^{k\xi}d\xi, \\ \eta_k^2(x,y) &= \int_{x-y}^{x+y} K(x,y,\xi)k^{N+1}e^{-k\xi}d\xi, \end{aligned}$$

where $k \in \mathbf{N}$.

PROPOSITION 7.1. *If $1/c^2$ is sufficiently small, we have*

$$\eta_k^1 > 0, \qquad \eta_k^2 > 0 \qquad \text{for } y > 0, \tag{7.3}$$

$$\begin{aligned} \eta_k^1 &= 2^N N! y^N (1+O(y/c^2)) e^{k(x+y)} (1+O(1/k)), \\ \eta_k^2 &= 2^N N! y^N (1+O(y/c^2)) e^{-k(x-y)} (1+O(1/k)) \end{aligned} \tag{7.4}$$

uniformly on each compact subset of $\{y > 0\}$. Moreover

$$\begin{aligned} q_k^1 &= \eta_k^1(\lambda_2 + O(1/k)), \\ q_k^2 &= \eta_k^2(\lambda_1 + O(1/k)) \end{aligned} \tag{7.5}$$

uniformly on each compact subset of $\{y \geq 0\}$ and

$$\eta_k^2 q_k^1 - \eta_k^1 q_k^2 = (2^N N!)^2 y^{2(N-1)} (\frac{2}{2N+1} + O(1/c^2)) e^{2ky} (y + O(1/k))^3. \tag{7.6}$$

Proof. Since $K = (1+O(y/c^2))(y^2 - (x-\xi)^2)^N$, we see

$$\begin{aligned} \eta_k^1 &= (1+O(y/c^2)) \int_{x-y}^{x+y} (y^2 - (x-\xi)^2)^N k^{N+1} e^{k\xi} d\xi \\ &= (1+O(y/c^2)) 2^{2N+1} y^N e^{kx} f(ky) \end{aligned}$$

where

$$\begin{aligned} f(r) &= r^{N+1} e^{-r} \int_0^1 (s(1-s))^N e^{2rs} ds \\ &= e^{-r} \int_0^r (\sigma(1 - \frac{\sigma}{r}))^N e^{2\sigma} d\sigma. \end{aligned}$$

It is easy to see

$$e^{-r} f(r) = 2^{-(N+1)} N! + O(1/r)$$

This implies (7.4). We note

$$\begin{aligned} \eta_k^1 &= (1+O(1/c^2)) 2^N N! y^{N-1} e^{k(x+y)} (y + O(1/k)) \\ \eta_k^2 &= (1+O(1/c^2)) 2^N N! y^{N-1} e^{-k(x-y)} (y + O(1/k)) \end{aligned}$$

uniformly on $\{y \geq 0\}$. Let us consider the flux. We have

$$\begin{aligned} L_2(x,y,\xi) &= -2 \int_{(-x+y+\xi)/2}^{y} \mu_2(x-y+Y,Y) K(x-y+Y,Y,\xi) dY \\ &= -2(\frac{N+1}{2N+1} + O(1/c^2)) \int_{(-x+y+\xi)/2}^{y} (Y^2 - (x-y+Y-\xi)^2)^N dY \\ &= -(\frac{1}{2N+1} + O(1/c^2))(y-x+\xi)^N (y+x-\xi)^{N+1}, \\ q_k^1 - \lambda_2 \eta_k^1 &= -(\frac{1}{2N+1} + O(1/c^2)) \times \\ & \qquad \int_{x-y}^{x+y} (y-x+\xi)^N (y+x-\xi)^{N+1} k^{N+1} e^{k\xi} d\xi. \end{aligned}$$

But

$$\begin{aligned}0 &\leq \int_{x-y}^{x+y}(y-x+\xi)^N(y+x-\xi)^{N+1}k^{N+1}e^{k\xi}d\xi\\ &= (N+1)k^N\int_{x-y}^{x+y}(y^2-(x-\xi)^2)^Ne^{k\xi}d\xi -\\ &\quad Nk^N\int_{x-y}^{x+y}(y-x+\xi)^{N-1}(y+x-\xi)^{N+1}e^{k\xi}d\xi\\ &\leq (N+1)\frac{1}{k}\int_{x-y}^{x+y}(y^2-(x-\xi)^2)^Nk^{N+1}e^{k\xi}d\xi.\end{aligned}$$

Thus

$$q_k^1-\lambda_2\eta_k^1 = O(1/k)\eta_k^1.$$

Since

$$\lambda_2-\lambda_1 = \frac{2\sqrt{P'}(1-u^2/c^2)}{1-P'u^2/c^4} = (\frac{2}{2N+1}+O(1/c^2))y,$$

we have

$$\eta_k^2q_k^1-\eta_k^1q_k^2 = \eta_k^1\eta_k^2((\frac{2}{2N+1}+O(1/c^2))y+O(1/k)).$$

This implies (7.6). ☐

2) Let ψ be a function in $C_0^\infty(-1,1)$ such that $\psi\geq 0, \psi(x)=\psi(-x), \int\psi=1$. We put

$$\begin{aligned}\phi_n^3(x) &= \psi_n(x) = n\psi(n(x-a)),\\ \phi_n^4(x) &= -D\psi_n(x),\\ \eta_n^3(x,y) &= \int_{x-y}^{x+y}K(x,y,\xi)\phi_n^3(\xi)d\xi,\\ \eta_n^4(x,y) &= \int_{x-y}^{x+y}K(x,y,\xi)\phi_n^4(\xi)d\xi.\\ \eta^3(x,y) &= K(x,y,a)\chi,\\ \eta^4(x,y) &= K_\xi(x,y,a)\chi,\\ q^3(x,y) &= L(x,y,a)\chi,\\ q^4(x,y) &= L_\xi(x,y,a)\chi,\end{aligned}$$

where

$$\chi = \begin{cases} 1 & \text{if } |x-a|<y,\\ 1/2 & \text{if } |x-a|=y,\\ 0 & \text{if } |x-a|>y.\end{cases}$$

PROPOSITION 7.2. *As $n\to\infty$, we have*

$$\eta_n^3\to\eta^3,\qquad q_n^3\to q^3,\qquad \eta_n^4\to\eta^4,\qquad q_n^4\to q^4.$$

Moreover

$$|\eta_n^3| \le M_1 y^{2N}, \qquad |q_n^3| \le M_1 y^{2N}(|x|+y), \tag{7.7}$$

$$|\eta_n^4| \le M_1 y^{2N-1}, \qquad |q_n^4| \le M_1 y^{2N-1}(|x|+y), \tag{7.8}$$

$$\eta^3 q^4 - \eta^4 q^3 = \frac{1}{2N+1}(1+O(1/c^2))(y^2-(x-a)^2)^{2N}, \tag{7.9}$$

where M_1 is a positive constant depending on ψ.

Proof. We note

$$\begin{aligned}
K_\xi &= -(\xi-x)G(x,|\xi-x|,\xi)\frac{1}{2^{N-1}(N-1)!}(y^2-(x-\xi)^2)^{N-1} + J^N G_\xi \\
&= (2N(x-\xi)+O(1/c^2)(\xi-x)^2)(y^2-(x-\xi)^2)^{N-1} + \\
&\quad O(1/c^2)(y^2-(x-\xi)^2)^N, \\
L_{1,\xi} &= 2\int_{(x+y-\xi)/2}^{y} \mu_1(x+y-Y,Y)K_\xi(x+y-Y,Y,\xi)dY.
\end{aligned}$$

The estimates (7.7), (7.8) can be seen easily. Let us consider

$$\eta^3 q^4 - \eta^4 q^3 = (KL_\xi - LK_\xi)(x,y,a).$$

Suppose $x-a \ge 0$. Then

$$\begin{aligned}
\frac{1}{2}(KL_\xi - LK_\xi) = K &\int_{(x+y-a)/2}^{y} \mu_1 K_\xi(x+y-Y,Y,a)dY - \\
K_\xi &\int_{(x+y-a)/2}^{y} \mu_1 K(x+y-Y,Y,a)dY.
\end{aligned}$$

We note

$$\frac{y}{2} \le \frac{x+y-a}{2} \le y, 0 \le x+y-Y-a.$$

Hence we have

$$\begin{aligned}
&\int_{(x+y-a)/2}^{y} \mu_1 K_\xi(x+y-Y,Y,a)dY \\
&= (\frac{N+1}{2N+1}+O(1/c^2))2N\int_{(x+y-a)/2}^{y}(x+y-Y-a)(Y^2 - \\
&\quad (x+y-Y-a)^2)^{N-1}dY + O(1/c^2)\int_{(x+y-a)/2}^{y}(Y^2-(x+y-Y-a)^2)^N dY \\
&= (\frac{N(N+1)}{2(2N+1)}+O(1/c^2))(x+y-a)^{N-1}(-x+y+a)^N\frac{1}{N(N+1)} \times \\
&\quad (y+(2N+1)(x-a)) + O(1/c^2)(-x+y+a)(y^2-(x-a)^2)^N.
\end{aligned}$$

Thus

$$\begin{aligned}
&K\int_{(x+y-a)/2}^{y} \mu_1 K_\xi dY \\
&= (\frac{1}{2(2N+1)}+O(1/c^2))(y^2-(x-a)^2)^{2N-1}(-x+y+a) \times \\
&\quad (y+(2N+1)(x-a)) \\
&+ O(1/c^2)(-x+y+a)(y^2-(x-a)^2)^{2N}.
\end{aligned}$$

Also we have

$$
\begin{aligned}
&K_\xi \int_{(x+y-a)/2}^{y} \mu_1 K dY \\
=& \Big(\frac{N}{2N+1} + O(1/c^2)\Big)(x-a)(-x+y+a)(y^2-(x-a)^2)^{2N-1} + \\
& O(1/c^2)(-x+y+a)(y^2-(x-a)^2)^{2N}.
\end{aligned}
$$

Hence

$$
\frac{1}{2}(KL_\xi - LK_\xi) = \Big(\frac{1}{2(2N+1)} + O(1/c^2)\Big)(y^2-(x-a)^2)^{2N}.
$$

Here we have used

$$
\begin{aligned}
&0 \le (x-a)(y-(x-a)) \le y^2-(x-a)^2, \\
&0 \le (y-x+a)(y+(2N+1)(x-a)) \\
&\quad \le (2N+1)(y^2-(x-a)^2)
\end{aligned}
$$

provided that $0 \le x-a \le y$. When $x-a \le 0$, we can discuss in a similar manner by using L_2. $\square$

3) Let Φ be a function in $C_0^\infty(-1,1)$ such that $\int\limits_{-\infty}^{\infty} \Phi = 0$ and the support $supp\Phi$ is $[-1+\alpha, 1+\alpha]$, where α is a small positive number. We put

$$
\begin{aligned}
\psi_n(x) &= n\Phi(n(x-a)), \\
\eta_n^5(x,y) &= \int_{x-y}^{x+y} K(x,y,\xi) D^{N+1}\psi_n(\xi) d\xi, \\
q_n^5(x,y) &= \int_{x-y}^{x+y} L(x,y,\xi) D^{N+1}\psi_n(\xi) d\xi; \\
\hat{\Phi}(x) &= \frac{d}{dx}\Big(x\int_{-1}^{x} \Phi\Big), \\
\hat{\psi}_n(x) &= n\hat{\Phi}(n(x-a)), \\
\eta_n^6(x,y) &= \int_{x-y}^{x+y} K(x,y,\xi) D^{N+1}\hat{\psi}_n(\xi) d\xi, \\
q_n^6(x,y) &= \int_{x-y}^{x+y} L(x,y,\xi) D^{N+1}\hat{\psi}_n(\xi) d\xi; \\
B_n^3 &= \eta^3 q_n^5 - \eta_n^5 q^3, \\
B_n^4 &= \eta^4 q_n^5 - \eta_n^5 q^4, \\
B_n &= \eta_n^5 q_n^6 - \eta_n^6 q_n^5.
\end{aligned}
$$

Let us divide the domain $\Sigma = \{-B \le x-y \le x+y \le B\}$ into the following 5 parts.

$$
\begin{aligned}
S_0 &= \{-\frac{1}{n} < x+y-a \le \frac{1}{n}, -\frac{1}{n} \le x-y-a < \frac{1}{n}\} \cap \Sigma, \\
S_1 &= \{\frac{1}{n} < x+y-a, x-y-a < -\frac{1}{n}\} \cap \Sigma,
\end{aligned}
$$

$$S_L = \{-\frac{1}{n} < x + y - a \le \frac{1}{n}, x - y - a < -\frac{1}{n}\} \cap \Sigma,$$
$$S_R = \{\frac{1}{n} < x + y - a, -\frac{1}{n} \le x - y - a < \frac{1}{n}\} \cap \Sigma,$$
$$S = \Sigma - (S_0 \cup S_1 \cup S_L \cup S_R).$$

PROPOSITION 7.3. *We have*

$$|B_n^3| \le M_2/n, \qquad |B_n^4| \le M_2 \quad \text{on } \Sigma, \tag{7.10}$$

and

$$|B_n| \le M_2/n \quad \text{on } S_0 \cup S_1 \cup S, \tag{7.11}$$

where M_2 is a positive constant depending on Φ. Moreover, on S_L, we have

$$B_n = ny^{2N}A_1 + y^N A_2 + A_3, \tag{7.12}$$

where

$$A_1 = (\frac{(N+1)(2^N N!)^2}{2N+1} + O(1/c^2))(\int_{-1}^{n(x+y-a)} \Phi)^2,$$
$$|A_2| \le M_2(|\int_{-1}^{n(x+y-a)} \Phi| + |\Phi(n(x+y-a))|),$$
$$|A_3| \le \frac{M_2}{n}.$$

On S_R, we have

$$B_n = ny^{2N}C_1 + y^N C_2 + C_3,$$
$$C_1 = (\frac{(N+1)(2^N N!)^2}{2N+1} + O(1/c^2))(\int_{-1}^{n(x-y-a)} \Phi)^2,$$
$$|C_2| \le M_2(|\int_{-1}^{n(x-y-a)} \Phi| + |\Phi(n(x-y-a))|),$$
$$|C_3| \le \frac{M_2}{n}.$$

Proof. See Appendix C. □

If we put

$$\hat{B}_n^3 = \eta^3\eta_n^6 - \eta_n^6 q^3,$$
$$\hat{B}_n^4 = \eta^4 q_n^6 - \eta_n^6 q^4,$$

then the same estimates hold.

8. Compactness of $\eta_t + q_x$. Let us consider an entropy η generated by ϕ through the generalized Darboux formula and its flux q. In this section we will prove

LEMMA 8.1. *Let U^Δ be the approximate solutions constructed in Section 4. Then $\eta(U^\Delta)_t + q(U^\Delta)_x$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$, Ω being a bounded open subset of $\{t \ge 0\}$.*

Proof. Let Φ be a test function and we consider

$$\begin{aligned}
J &= \int\int (\eta(U^\Delta)\Phi_t + q(U^\Delta)\Phi_x)dxdt \\
&= N + L + \Sigma, \\
N &= -\int \eta(U^\Delta(+0,x)\Phi(0,x)dx, \\
L &= \sum_n \int [\eta(U^\Delta(t,x)]_{t=n\Delta t+0}^{t=n\Delta t-0}\Phi(n\Delta t,x)dx, \\
\Sigma &= \int \sum_{shock} (\sigma[\eta] - [q])\Phi dt.
\end{aligned}$$

Since U^Δ is bounded, we see

$$|N| \leq M_3||\Phi||_C.$$

Here and after, the constants M_i for $i = 3$ to 7 depend on M_0 and U_0 but may vary from line to line. Let us look at L. We see

$$\begin{aligned}
L &= L_1 + L_2, \\
L_1 &= \sum_{j,n} \Phi(n\Delta t, (2j+1)\Delta x) \int_{2j\Delta x}^{(2j+2)\Delta x} [\eta(U^\Delta)]_{t=n\Delta t+0}^{t=n\Delta t-0}dx, \\
L_2 &= \sum_{j,n} \int_{2j\Delta x}^{(2j+2)\Delta x} (\Phi(n\Delta t,x) - \Phi(n\Delta t,(2j+1)\Delta x) \times \\
&\quad [\eta(U^\Delta)]_{t=n\Delta t+0}^{t=n\Delta t-0}dx.
\end{aligned}$$

We note

$$\begin{aligned}
[\eta(U^\Delta)]_{t=n\Delta t+0}^{t=n\Delta t-0} &= D_U\eta(U^\Delta(n\Delta t+0,x))[U^\Delta] \\
&\quad + \int_0^1 (1-\theta)([U^\Delta]|D_U^2(U^\Delta(n\Delta t+0) + \theta[U^\Delta]) \cdot [U^\Delta])d\theta.
\end{aligned}$$

and

$$\int_{2j\Delta x}^{(2j+2)\Delta x} [U^\Delta]dx = 0$$

by the scheme. Therefore

$$|L_1| \leq M_3||\Phi||_C \sum_{j,n} \int \int_0^1 (1-\theta)|F(\theta,\eta)|d\theta dx,$$

where

$$F(\theta,\eta) = ([U^\Delta]|D_U^2\eta(U^\Delta(n\Delta t+0) + \theta[U^\Delta]) \cdot [U^\Delta]).$$

By Proposition 6.2 we know $|F(\theta,\eta)| \leq M_3F(\theta,\eta^*)$. But in the proof of Proposition 4.2 we know

$$\sum_{j,n} \int \int_0^1 (1-\theta)F(\theta,\eta^*)d\theta dx \leq C.$$

Thus we know

$$|L_1| \le M_3||\Phi||_C.$$

In the proof of Proposition 4.2 we know

$$\sum_{j,n} \int_{2j\Delta x}^{(2j+2)\Delta x} ||[U^\Delta]|^2 dx \le C.$$

Therefore

$$\begin{aligned}
|L_2| &\le 2^\alpha ||\Phi||_{C^\alpha} \sum_n \int (\Delta x)^\alpha |[\eta(U^\Delta)]| dx \\
&\le 2^{\alpha-1} ||\Phi||_{C^\alpha} \sum_n \int ((\Delta x)^{\alpha+\frac{1}{2}} + (\Delta x)^{\alpha-\frac{1}{2}} |[\eta(U^\Delta)]|^2) dx \\
&\le M_3 ||\Phi||_{C^\alpha} ((\Delta x)^{\alpha-\frac{1}{2}} + (\Delta x)^{\alpha-\frac{1}{2}} \sum \int |[U^\Delta]|^2 dx \\
&\le M_4 (\Delta x)^{\alpha-\frac{1}{2}} ||\Phi||_{C^\alpha},
\end{aligned}$$

where we use the boundedness of $D_U\eta$ and $n = O(1/(\Delta x))$. Next we look at Σ. Along the shock we have

$$\begin{aligned}
&\sigma[\eta(U)] - [q(U)] \\
&\qquad = \int_{\rho_L}^{\rho_R} (-\frac{d\sigma}{d\rho} \int_0^1 \theta(U - U_L| D_U^2 \eta(U_L + \theta(U - U_L))(U - U_L)) d\theta) d\rho.
\end{aligned}$$

This implies

$$|\sigma[\eta] - [q]| \le M_5(\sigma[\eta^*] - [q^*]).$$

But we know

$$\int \sum_{shock} (\sigma[\eta^*] - [q^*]) dt \le C$$

in the proof of Proposition 4.2. Therefore

$$|\Sigma| \le M_5 ||\Phi||_C.$$

Summing up , we know the compactness. □

9. Convergence of approximate solutions. We consider the approximate solutions U^Δ constructed in Section 4. Since U^Δ is bounded, there is a sequence U^{Δ_n} and a family of Young measures $\nu_{t,x}$ such that *supp* $\nu_{t,x} \subset \Sigma = \Sigma_B$ and for any continuous function f

$$f(U^{\Delta_n}(t,x)) \to \bar{f} = < \nu_{t,x}, f >$$

in L^∞ weak-star topology. By Lemma 8.1, we can apply the compensated compactness theory, and we can assume

$$(\eta q' - \eta' q)(U^{\Delta_n}) \to < \nu, q >< \nu, q' > - < \nu, \eta' >< \nu, q >$$

in L^∞ weak-star. Here $\eta, q; \eta', q'$ are arbitrary Darboux entropy pairs. Thus we have

LEMMA 9.1. *For any pairs $(\eta, q), (\eta', q')$ of Darboux entropies-entropy flux, the identity*

$$< \nu, \eta q' - \eta' q >=< \nu, \eta >< \nu, q' > - < \nu, \eta' >< \nu, q >$$

holds a.e.-(t, x), where $\nu = \nu_{t,x}$.

Since entropies we will use are countably many, we can assune that the above identity holds outside a null set which is common to all η. We fix (t, x) at which the identity holds, and we write $\nu = \nu_{t,x}$. Of course *supp* $\nu \subset \Sigma$. Suppose that *supp* $\nu \cap \{\rho > 0\} \neq \phi$. Let Σ_0 be the smallest triangle $\{z_0 \leq z \leq w \leq w_0\}$ such that *supp* $\nu \cap \{\rho > 0\} \subset \Sigma_0$. Let us denote by P_0 the state (w_0, z_0). It will be verified that $\nu = \delta_{P_0}$, (the Dirac measure).

First we show

PROPOSITION 9.1.

$$P_0 \in supp\ \nu.$$

Proof. Suppose $P_0 \notin$ *supp* ν. Since Σ_0 is the smallest triangle containing *supp* $\nu \cap \{\rho > 0\}$, $w = w_0$ and $z = z_0$ intersect with *supp* $\nu \cap \{\rho > 0\}$. On neighborhoods of these intersection points we have

$$\eta^1 \geq \frac{1}{M_6} e^{k(w_0 - \epsilon)},$$
$$\eta^2 \geq \frac{1}{M_6} e^{-k(z_0 + \epsilon)}.$$

(See Proposition 7.1). Since ν, η^1, η^2 are nonnegative, we see

$$< \nu, \eta^1 > \geq \frac{1}{M_6} e^{k(w_0 - \epsilon)},$$
$$< \nu, \eta^2 > \geq \frac{1}{M_6} e^{-k(z_0 + \epsilon)}.$$

Since$P_0 \notin$ *supp* ν, we have

$$< \nu, \eta^2 q^1 - \eta^1 q^2 > \leq M_6 e^{k(w_0 - z_0 - \delta)}.$$

Taking $2\epsilon < \delta$, we have

$$\begin{aligned} \left|\frac{< \nu, q^1 >}{< \nu, \eta^1 >} - \frac{< \nu, q^2 >}{< \nu, \eta^2 >}\right| &= \left|\frac{< \nu, \eta^2 q^1 - \eta^1 q^2 >}{< \nu, \eta^1 >< \nu, \eta^2 >}\right| \\ &\leq M_6 e^{-k(\delta - 2\epsilon)} \\ &\to 0 \end{aligned}$$

as $k \to \infty$. Let β be a sufficiently small positive number, and we put

$$\Sigma_2 = \{z_0 \leq z \leq w < w_0 - \beta\}$$
$$\Sigma_3 = \{z_0 \leq z \leq w \leq w_0, w_0 - \beta \leq w\}.$$

Then

$$\eta^1 e^{-kw} = (1 + O(1/c^2))2^N N! y^{N-1}(y + O(1/k))$$

is bounded on Σ_0 and we have

$$< \nu|_{\Sigma_2}, \eta^1 > \leq M_6 e^{k(w_0-\beta)}.$$

Taking $\epsilon = \beta/2$, we know

$$\frac{< \nu|_{\Sigma_2}, \eta^1 >}{< \nu, \eta^1 >} \leq M_6 e^{-\beta k/2} \to 0.$$

Since $\partial\lambda_2/\partial w > 0$, we know

$$\lambda_2(w, z) \geq \lambda_2(w_0 - \beta, z_0)$$

on Σ_3. Therefore we have

$$\begin{aligned} \frac{< \nu, q^1 >}{< \nu, \eta^1 >} &= \frac{< \nu|_{\Sigma_2}, \eta^1\lambda_2 >}{< \nu, \eta^1 >} + \frac{< \nu|_{\Sigma_3}, \eta^1\lambda_2 >}{< \nu, \eta^1 >} + O(1/k) \\ &\geq o(1) + \lambda_2(w_0 - \beta, z_0) \end{aligned}$$

Similarly we see

$$\frac{< \nu, q^2 >}{< \nu, \eta^2 >} \leq o(1) + \lambda_1(w_0, z_0 + \beta).$$

Therefore we have

$$\lambda_2(w_0 - \beta, z_0) - \lambda_1(w_0, z_0 + \beta) \leq 0 + o(1).$$

Passing to the limit, we know

$$\lambda_2(w_0, z_0) \leq \lambda_1(w_0, z_0).$$

But this means $P_0 \in \{\rho = 0\}$, a contradiction. □

Let us fix a such that $z_0 < a < w_0$. We have

$$\begin{aligned} < \nu, B_n^3 > &= < \nu, \eta^3 >< \nu, q_n^5 > - < \nu, \eta_n^5 >< \nu, q^3 >, \\ < \nu, B_n^4 > &= < \nu, \eta^4 >< \nu, q_n^5 > - < \nu, \eta_n^5 >< \nu, q^4 >, \\ < \nu, \eta^3 q^4 - \eta^4 q^3 > &= < \nu, \eta^3 >< \nu, q^4 > - < \nu, \eta^4 >< \nu, q^3 >, \\ < \nu, B_n > &= < \nu, \eta_n^5 >< \nu, q_n^6 > - < \nu, \eta_n^6 >< \nu, q_n^5 > . \end{aligned}$$

From (7.9) we know

$$< \nu, \eta^3 q^4 - \eta^4 q^3 >> 0$$

and from(7.10) we know

$$< \nu, B_n^3 >\to 0$$

Using these we can prove the following propositions. Proofs can be found in Chen et al [2].

PROPOSITION 9.2. *As $n \to \infty$, $< \nu, \eta_n^5 >, < \nu, q_n^5 >, < \nu, q_n^6 >, < \nu, q_n^6 >$ are bounded.*

PROPOSITION 9.3. *As $n \to \infty$, we have $< \nu, B_n > \to 0$.*

Now, taking

$$\Phi_0(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

we put

$$\Phi(x) = \frac{1}{\beta}(\Phi_0(\frac{x+\beta}{\beta}) - \Phi_0(\frac{x-\beta}{\beta}))$$

for the generating function of η_n^5. Here $\beta = (1-\alpha)/2$. We put

$$S_+ = \{z \leq w, |w-a| \leq \frac{1-3\alpha}{n}\},$$
$$S_- = \{z \leq w, |z-a| \leq \frac{1-3\alpha}{n}\}.$$

PROPOSITION 9.4. *As $n \to \infty$, we have*

$$< \nu|_{S_+}, ny^{2N} > + < \nu|_{S_-}, ny^{2N} > \to 0.$$

Proof. Put $S'_L = S_+ \cap S_L, S'_R = S_- \cap S_R$. It is sufficient to prove that

$$< \nu|_{S'_L}, ny^{2N} > + < \nu|_{S'_R}, ny^{2N} > \to 0.$$

From (7.11) we have

$$< \nu|_{S_L}, ny^{2N} A_1 + y^N A_2 > + < \nu|_{S_R}, ny^{2N} C_1 + y^N C_2 > \to 0.$$

Note

$$A_1 = (\frac{N(2^N N!)^2}{2N+1} + O(1/c^2))(\int_{-1}^{n(x+y-a)} \Phi)^2 \geq \frac{1}{M_7} > 0$$

on S'_L. Put

$$E_n = \{0 \leq y \leq (\frac{1}{n})^\mu\},$$

where μ is a positive parameter. Then $|y^N A_2| \leq M_7(1/n)^{\mu N} = o(1)$ on $S_L \cap E_n$ and $|y^N A_2| \leq M_7 ny^{2N}(1/n)^{1-\mu N}$ on $S_L - E_n$. Choose $d_n \searrow 0$ such that

$$\int_{-1+\alpha}^{1-\alpha-d_n} \Phi = -\int_{1-\alpha-d_n}^{1-\alpha} \Phi \geq (1/n)^{\mu_0}.$$

Then

$$(\int_{-1}^{H} \Phi)^2 \geq (1/n)^{2\mu_0}$$

for $|H| \leq 1-\alpha-d_n$, and

$$|\Phi(H)| + |\int_{-1}^{H} \Phi| = o(1)$$

for $1-\alpha-d_n \leq |H| \leq 1$. Put

$$S_+^n = S_L \cap \{|w-a| \leq \frac{1-\alpha-d_n}{n}\}.$$

Then $S'_L \subset S_+^n \subset S_L$ and

$$|y^N A_2| = o(1)$$

on $S_L - S_+^n$ and

$$\begin{aligned} ny^{2N}A_1 + y^N A_2 &\geq ny^{2N}(\frac{1}{M_7}(1/n)^{2\mu_0} - M_7(1/n)^{1-\mu N}) \\ &\geq 0 \end{aligned}$$

on $S_+^n - E_n$. Here we take $0 < 2\mu_0 < 1-\mu N$. Then

$$\begin{aligned} &< \nu|_{S_L}, ny^{2N}A_1 + y^N A_2 > = < \nu|_{S_L \cap E_n}, ny^{2N}A_1 > + \\ &\quad < \nu|_{S_L - E_n}, ny^{2N}A_1 + y^N A_2 > + o(1) \\ &\geq \frac{1}{M_7} < \nu|_{S'_L \cap E_n}, ny^{2N} > + < \nu|_{S_L - S_+^n \cap E_n}, ny^{2N}A_1 > + \\ &\quad < \nu|_{S'_L - E_n}, ny^{2N}A_1 + y^N A_2 > + < \nu|_{S_+^n - S'_L - E_n}, ny^{2N}A_1 + y^N A_2 > + o(1) \\ &\geq \frac{1}{M_7} < \nu|_{S'_L \cap E_n}, ny^{2N} > + < \nu|_{S'_L - E_n}, ny^{2N}(\frac{1}{M_7} - M_7(1/n)^{1-\mu N} > + o(1) \\ &\geq \frac{1}{2M_7} < \nu|_{S'_L}, ny^{2N} > + o(1). \end{aligned}$$

Similarly we know

$$< \nu|_{S_R}, ny^{2N}C_1 + y^N C_2 > \geq \frac{1}{2M_7} < \nu|_{S'_R}, ny^{2N} > + o(1).$$

Thus we see

$$< \nu|_{S'_L}, ny^{2N} > + < \nu|_{S'_R}, ny^{2N} > \to 0.$$

□

PROPOSITION 9.5. *We have*

$$\nu|_{\{\rho>0\}} = \delta_{P_0}.$$

Proof. Proposition 9.4 says that the projections $P_w\tilde{\nu}, P_z\tilde{\nu}$ of the measure $\tilde{\nu} = y^{2N}\nu$ admits the Lebesgue lower derivatives which vanish at any a. Therefore we can claim that

$$supp\ \nu \cap \{\rho > 0\} = \{P_0\}.$$

Since ν is a probability measure, we have

$$\nu|_{\{\rho>0\}} = C\delta_{P_0}.$$

But

$$C(\eta^3 q^4 - \eta^4 q^3) = C^2(\eta^3 q^4 - \eta^4 q^3)$$

at P_0. Hence $C = 1$. □

Summing up the results, we obtain the results of the main theorem.

Proof of corollary. Let α be an arbitrary positive number. We consider the change of variables

$$\begin{aligned} &\rho = \alpha^{\frac{2}{\gamma-1}}\bar{\rho}, \quad P = \alpha^{\frac{2\gamma}{\gamma-1}}\bar{P}, \\ &u = \alpha\bar{u}, \qquad c = \alpha\bar{c}, \qquad x = \alpha\bar{x}. \end{aligned} \tag{9.1}$$

Then the problem for $\bar{\rho},\ \bar{u},\ \bar{P},\ \bar{t} = t$ and $\bar{x}$ is the same to the problem for $\rho,\ u,\ p,\ t$ and x. Therefore applying the main theorem, if

$$\begin{aligned} &0 \le \bar{\rho}_0(\bar{x}) \le M_0, \qquad |\frac{\bar{c}}{2}\log\frac{\bar{c}+\bar{u}_0(\bar{x})}{\bar{c}-\bar{u}_0(\bar{x})}| \le M_0, \\ &\frac{1}{\bar{c}^2} \le \epsilon_0(M_0), \end{aligned} \tag{9.2}$$

then we have the weak solution. The conditions (9.2) are equivalent to

$$\begin{aligned} &0 \le \rho_0(x) \le \alpha^{\frac{2}{\gamma-1}}M_0, \qquad |\frac{c}{2}\log\frac{c+u_0(x)}{c-u_0(x)}| \le \alpha M_0, \\ &\frac{1}{c^2} \le \epsilon_0(M_0)/\alpha^2. \end{aligned}$$

Fix M_0 and take $\alpha = \sqrt{\epsilon_0(M_0)}c$, then the corollary holds with

$$\epsilon_1 = min(\epsilon_0(M_0)^{\frac{1}{\gamma-1}}M_0,\ \epsilon_0(M_0)^{\frac{1}{2}}M_0).$$

The proof is complete. □

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Appendix A. Proof of Proposition 5.4.

(1) Since $a = O(y/c^2)$, it is clear that $G_{I,y} = O(y/c^2)$. Next we see

$$G_{II,y} = -B((x+y+\xi)/2, (x+y-\xi)/2) + B((x-y+\xi)/2, (-x+y+\xi)/2)) + O(y/c^2).$$

On the other hand we can write

$$B = \frac{1}{c^2}B_0(x) + O(y^2/c^2)$$

and

$$\frac{x+y+\xi}{2} = x + \frac{y+Z}{2}, \qquad \frac{x-y+\xi}{2} = x + \frac{-y+Z}{2}, \qquad Z = \xi - x.$$

Therefore we see $G_{II,y} = O(y/c^2)$. It is clear that $G_{III,y} = O(y/c^2)$ and $G_{IVk,y}$, $G_{Vk,y} = O(y^2/c^2)$. $\square$

(2) It is clear that $G_I = O(y^2/c^2)$ since $a = O(y/c^2)$. Next we see

$$\begin{aligned} G_{II} = 2^N N! &\int_{(x+y-\xi)/2}^{y} B(x+y-Y,Y)dY - \\ &2^N N! \int_{(-x+y+\xi)/2}^{y} B(x-y+Y,Y)dY + O(y^2/c^2), \end{aligned}$$

since $G = 2^N N! + O(y/c^2)$. If we write

$$B = \frac{1}{c^2}B_0(x) + O(y^2/c^2), \qquad Z = \xi - x,$$

then we see

$$\begin{aligned} &\int_{(x+y-\xi)/2}^{y} B(x+y-Y,Y)dY - \int_{(-x+y+\xi)/2}^{y} B(x-y+Y,Y)dY \\ &= \frac{1}{c^2}\Big(\int_{x}^{x+\frac{y+Z}{2}} B_0(s)ds - \int_{x+\frac{-y+Z}{2}}^{x} B_0(s)ds\Big) + O(y^2/c^2) \\ &= \frac{1}{c^2}B_0(x)Z + O(y^2/c^2). \end{aligned}$$

Note $|Z| \le y$. It is clear that $G_{III}, G_{IVk}, G_{Vk} = O(y^2/c^2)$. $\square$

(3) First we see

$$\begin{aligned} G_{I,x} + G_{I,\xi} = &\int_{(-x+y+\xi)/2}^{y} ((aG)_x + aG_\xi)(x-y+Y,Y,\xi)dY + \\ &\int_{(x+y-\xi)/2}^{y} ((aG)_x + aG_\xi)(x+y-Y,Y,\xi)dY \\ &= O(y^2/c^2), \end{aligned}$$

since $a, a_x = O(y/c^2)$. Next we see

$$\begin{aligned} G_{II,x} + G_{II,\xi} = &\int_{(x+y-\xi)/2}^{y} ((BG)_x) + BG_\xi)(x+y-Y,Y,\xi)dY - \\ &\int_{(-x+y+\xi)/2}^{y} ((BG)_x + BG_\xi)(x-y+Y,Y,\xi)dY \\ &= O(y/c^2). \end{aligned}$$

It is clear that $G_{III,x}, G_{III,\xi}, G_{Vk,x}, G_{Vk,\xi} = O(y/c^2)$. $G_{IVk,x} + G_{IVk,\xi}$ is estimated in a similar manner as $G_{II,x} + G_{II,\xi}$. Therefore $G_x + G_\xi = O(y/c^2)$. Moreover, we look at

$$\begin{aligned} G_{II,x} + G_{II,\xi} = &\int_{(x+y-\xi)/2}^{y} ((BG)_x + BG_\xi)(x+y-Y,Y,\xi)dY - \\ &\int_{(-x+y+\xi)/2}^{y} ((BG)_x + BG_\xi)(x-y+Y,Y,\xi)dY \\ = &\, 2^N N! \int_{(x+y-\xi)/2}^{y} B_x(x+y-Y,Y)dY - \\ &\, 2^N N! \int_{(-x+y+\xi)/2}^{y} B_x(x-y+Y,Y)dY + O(y^2/c^2), \end{aligned}$$

since $G = 2 + O(y/c^2)$ and $G_x + G_\xi = O(y/c^2)$. Bearing in mind that $B_y = O(y/c^2)$, we see

$$\begin{aligned} &\int_{(x+y-\xi)/2}^{y} B_x(x+y-Y,Y)dY - \int_{(-x+y+\xi)/2}^{y} B_x(x-y+Y,Y)dY \\ &= -\int_{(x+y-\xi)/2}^{y} (-B_x + B_y)(x+y-Y,Y)dY - \\ &\quad \int_{(-x+y+\xi)/2}^{y} (B_x + B_y)(x-y+Y,Y)dY + O(y^2/c^2) \\ &= -2B(x,y) + B((x+y+\xi)/2, (x+y-\xi)/2) + \\ &\quad B((x-y+\xi)/2, (-x+y+\xi)/2) + O(y^2/c^2) \\ &= \frac{1}{c^2}(-2B_0(x) + B_0(x + \frac{y+Z}{2}) + B_0(x + \frac{-y+Z}{2})) + O(y^2/c^2) \\ &= \frac{1}{c^2} B_0'(x) Z + O(y^2/c^2). \end{aligned}$$

Next we look at

$$\begin{aligned} &G_{III,x} + G_{III,\xi} \\ &\quad = \int_{(x+y-\xi)/2}^{y} bG(x+y-Y,Y,\xi)dY - \int_{(-x+y+\xi)/2}^{y} bG(x-y+Y,Y,\xi)dY \\ &\quad + \int_0^{(x+y-\xi)/2} bG(\xi+Y,Y,\xi)dY - \int_0^{(-x+y+\xi)/2} bG(\xi-Y,Y,\xi)dY + \\ &\quad \int\int_{D(x,y,\xi)} bG(X,Y,\xi)dXdY. \end{aligned}$$

Putting

$$b(x,y) = \frac{1}{c^2} b_0(x) + O(y^2/c^2),$$

we see

$$\begin{aligned} G_{III,x} + G_{III,\xi} = 2^N N! (&\int_x^{x+\frac{y+Z}{2}} b_0(s)ds - \int_{x+\frac{-y+Z}{2}}^{x} b_0(s)ds + \\ &\int_{x+Z}^{x+\frac{y+Z}{2}} b_0(s)ds - \int_{x+\frac{-y+Z}{2}}^{x+Z} b_0(s)ds) + O(y^2/c^2) \end{aligned}$$

$$
\begin{aligned}
&= \frac{2^N N!}{c^2} b_0(x)(\frac{y+Z}{2} - \frac{y-Z}{2} + \frac{y-Z}{2} - \frac{y+Z}{2}) + O(y^2/c^2) \\
&= O(y^2/c^2).
\end{aligned}
$$

$G_{IVk,x} + G_{IVk,\xi}$ can be estimated in a similer manner as $G_{II,x} + G_{II,\xi}$. Finally $G_{Vk,x}, G_{Vk,\xi} = O(y^3/c^2)$ since $J^k G = O(y^2/c^2)$ for $k \geq 1$. $\square$

(4) First we see

$$
\begin{aligned}
(G_{I,x} + G_{I,\xi})_y = & \, 2((aG)_x + aG_\xi)(x,y,\xi) - \\
& \frac{1}{2}((aG)_x + aG_\xi)((x-y+\xi)/2, (-x+y+\xi)/2, \xi) - \\
& \frac{1}{2}((aG)_x + aG_\xi)((x+y+\xi)/2, (x+y-\xi)/2, \xi) - \\
& \int_{(-x+y+\xi)/2}^{y} ((aG)_x + aG_\xi)_x(x-y+Y, Y, \xi)dY + \\
& \int_{((x+y-\xi)/2}^{y} ((aG)_x + aG_\xi)_x(x+y-Y, Y, \xi)dY \\
= & \, O(y/c^2),
\end{aligned}
$$

since $a, a_x = O(y/c^2)$. Next we see

$$
\begin{aligned}
(G_{II,x} + G_{II,\xi})_y = & -\frac{1}{2}((BG)_x + BG_\xi)((x+y+\xi)/2, (x+y-\xi)/2, \xi) + \\
& \frac{1}{2}((BG)_x + BG_\xi)((x-y+\xi)/2, (-x+y+\xi)/2, \xi) + \\
& \int_{(x+y-\xi)/2}^{y} ((BG)_x + BG_\xi)_x(x+y-Y, Y, \xi)dY + \\
& \int_{(-x+y+\xi)/2}^{y} ((BG)_x + BG_\xi)_x(x-y+Y, Y, \xi)dY \\
= & \, 2^{N-1}N!B_x((x-y+\xi)/2, (-x+y+\xi)/2) - \\
& 2^{N-1}N!B_x((x+y+\xi)/2, (x+y-\xi)/2) + O(y/c^2),
\end{aligned}
$$

since $G = 2^N N! + O(y/c^2)$ and $G_x + G_\xi = O(y/c^2)$. But

$$
B_x = \frac{1}{c^2} B_0'(x) + O(y^2/c^2)
$$

and

$$
\begin{aligned}
& B_x((x-y+\xi)/2, (-x+y+\xi)/2) - B_x((x+y+\xi)/2, (x+y-\xi)/2) \\
= & \frac{1}{c^2} B_0''(x)(-y) + O(y^2/c^2) \\
= & O(y/c^2).
\end{aligned}
$$

It is clear that

$$
\begin{aligned}
(G_{III,x} + G_{III,\xi})_y = & \int_{(x+y-\xi)/2}^{y} ((bG)_x + bG_\xi)(x+y-Y, Y, \xi)dY + \\
& \int_{(-x+y+\xi)/2}^{y} ((bG)_x + bG_\xi)(x-y+Y, Y, \xi)dY \\
= & \, O(y/c^2).
\end{aligned}
$$

Similarly we can estimate $(G_{IVk,x}+G_{IVk,\xi})_y, (G_{Vk,x}+G_{Vk,\xi})_y$ bearing in mind that $(JG)_x+(JG)_\xi = J(G_x+G_\xi)$. $\square$

(5) First we see

$$\begin{aligned}(G_{I,x}+G_{I,\xi})_x &+ (G_{I,x}+G_{I,\xi})_\xi \\ &= \int_{(-x+y+\xi)/2}^{y} ((aG)_{xx}+2(aG_\xi)_x+aG_{\xi\xi})(x-y+Y,Y,\xi)dY \\ &\quad + \int_{(x+y-\xi)/2}^{y} ((aG)_{xx}+2(aG_\xi)_x+aG_{\xi\xi})(x+y-Y,Y,\xi)dY \\ &= O(y^2/c^2),\end{aligned}$$

since $a, a_x, a_{xx} = O(y/c^2)$. Next

$$\begin{aligned}&(G_{II,x}+G_{II,\xi})_x+(G_{II,x}+G_{II,\xi})_\xi \\ &= \int_{(x+y-\xi)/2}^{y} (((BG)_x+BG_\xi)_x+((BG)_x+BG_\xi)_\xi)(x+y-Y,Y,\xi)dY + \\ &\int_{(-x+y+\xi)/2}^{y} (((BG)_x+BG_\xi)_x+((BG)_x+BG_\xi)_\xi)(x+y-Y,Y,\xi)dY \\ &= O(y/c^2).\end{aligned}$$

It is easy to see

$$(G_{III,x}+G_{III,\xi})_x+(G_{III,x}+G_{III,\xi})_\xi = O(y/c^2).$$

The estimates of G_{IVk} and G_{Vk} can be seen similarly. Therefore, $(G_x+G_\xi)_x+(G_x+G_\xi)_\xi = O(y/c^2)$, Moreover, by $G_x+G_\xi = O(y/c^2)$, we see

$$\begin{aligned}(G_{II,x}+G_{II,\xi})_x &+ (G_{II,x}+G_{II,\xi})_\xi \\ &= \int_{(x+y-\xi)/2}^{y} (B_{xx}G+2B_x(G_x+G_\xi)+ \\ &\quad B((G_x+G_\xi)_x+(G_x+G_\xi)_\xi))(x+y-Y,Y,\xi)dY - \\ &\quad \int_{(-x+y+\xi)/2}^{y} (B_{xx}G+2B_x(G_x+G_\xi)+ \\ &\quad B((G_x+G_\xi)_x+(G_x+G_\xi)_\xi)(x-y+Y,Y,\xi)dY \\ &= 2^N N! \int_{(x+y-\xi)/2}^{y} B_{xx}(x+y-Y,Y)dY - \\ &\quad 2^N N! \int_{(-x+y+\xi)/2}^{y} B_{xx}(x-y+Y,Y)dY + O(y^2/c^2).\end{aligned}$$

The same discussion to that of the proof of (3) can be applied by replacing B by B_x. Let us look at $(G_{III,x}+G_{III,\xi})_x+(G_{III,x}+G_{III,\xi})_\xi$. Note that

$$\begin{aligned}(bG)_x+bG_\xi &= b_xG+b(G_x+G_\xi) \\ &= 2^N N! b_x + O(y/c^2), \\ bG &= 2^N N! b + O(y/c^2).\end{aligned}$$

Applying the discussion of the (3) by replacing b by b_x, we see

$$
\begin{aligned}
&(G_{III,x}+G_{III,\xi})_x+(G_{III,x}+G_{III,\xi})_\xi\\
&\quad=2^N N!\Big(\int_{x+Z}^{x+\frac{y+Z}{2}} b_0(s)ds-\int_{x+\frac{-y+Z}{2}}^{x+Z} b_0(s)ds\Big)+O(y^2/c^2)\\
&\quad=-2^N N! b_0(x)Z+O(y^2/c^2).
\end{aligned}
$$

The estimates of G_{IVk}, G_{Vk} are parallel. □

(6) It is sufficient to note that

$$
\begin{aligned}
G_{II,\xi}&=2^{N-1}N!(B((x+y+\xi)/2,(x+y-\xi)/2)+\\
&\quad B((x-y+\xi)/2,(-x+y+\xi)/2))+O(y/c^2)\\
&=\frac{2^{N-1}N!}{c^2}(B_0(x+\frac{y+Z}{2})+B_0(x+\frac{-y+Z}{2}))+O(y/c^2)\\
&=\frac{2^N N!}{c^2}B_0(x)+O(y/c^2). \qquad \square
\end{aligned}
$$

(7) We see

$$
(G_{I,x}+G_{I,\xi})_\xi=O(y/c^2)
$$

by $a, a_x=O(y/c^2)$. Next we see

$$
\begin{aligned}
&(G_{II,x}+G_{II,\xi})_\xi\\
&\quad=2^{N-1}N!(B_x((x+y+\xi)/2,(x+y-\xi)/2)+\\
&\qquad B_x((x-y+\xi)/2,(-x+y+\xi)/2))+O(y/c^2)\\
&\quad=\frac{2^N N!}{c^2}B_0'(x)+O(y/c^2).
\end{aligned}
$$

And we see

$$
\begin{aligned}
&(G_{III,x}+G_{III,\xi})_\xi\\
&\quad=2^N N! b((x-y+\xi)/2,(-x+y+\xi)/2)+O(y/c^2)\\
&\quad=\frac{2^N N!}{c^2}b_0(x)+O(y/c^2).
\end{aligned}
$$

Other terms can be estimated similarly. □

Appendix B. Proof of Proposition 6.1. We write

$$
\eta=2R^{\frac{1}{2N+1}}\int_0^1 K(\frac{M}{R},R^{\frac{1}{2N+1}},\frac{M}{R}+(2s-1)R^{\frac{1}{2N+1}})\phi(\frac{M}{R}+(2s-1)R^{\frac{1}{2N+1}})ds.
$$

Differentiating η with respect to M, we have

$$
\begin{aligned}
&\frac{\partial\eta}{\partial M}=(1)+(2),\\
&(1)=2R^{\frac{-2N}{2N+1}}\int_0^1 (K_x+K_\xi)(x,y,x+(2s-1)y)\phi(x+(2s-1)y)ds,\\
&(2)=2R^{\frac{-2N}{2N+1}}\int_0^1 K(x,y,x+(2s-1)y)D\phi(x+(2s-1)y)ds.
\end{aligned}
$$

Since $K(x,y,\xi)=J^NG(x,y,\xi)$, i.e.

$$K(x,y,\xi)=\int_{|x-\xi|}^{y}Y_N\int_{|x-\xi|}^{Y_N}Y_{N-1}\cdots\int_{|x-\xi|}^{Y_2}Y_1G(x,Y_1,\xi)dY_1\cdots Y_N,$$

by (3) of Proposition 5.4 we see

$$\begin{aligned}
&(K_x+K_\xi)(x,y,x+(2s-1)y)\\
&=\int_{|2s-1|y}^{y}Y_N\int_{|2s-1|y}^{Y_N}Y_{N-1}\cdots\int_{|2s-1|y}^{Y_2}Y_1\times\\
&\qquad(G_x+G_\xi)(x,Y_1,x+(2s-1)y)dY_1\cdots dY_N\\
&=\frac{C_1(x,c)}{2^NN!c^2}y^{2N+1}(2s-1)(1-(2s-1)^2)^N+O(y^{2N+2}/c^2)\\
&=-\frac{2^NC_1(x,c)}{(N+1)!c^2}y^{2N+1}\frac{d}{ds}(s-s^2)^{N+1}+O(y^{2N+2}/c^2).
\end{aligned}$$

Therefore by integration by parts we get

$$\begin{aligned}
(1)&=R^{\frac{-2N}{2N+1}}y^{2N+2}\frac{2^{N+1}C_1(x,c)}{(N+1)!c^2}\int_0^1(s-s^2)^{N+1}D\phi ds+O(y^2/c^2)\\
&=O(y^2/c^2).
\end{aligned}$$

By (2) of Proposition 5.4 we see

$$\begin{aligned}
&K(x,y,\xi)=\int_{|x-\xi|}^{y}Y_N\int_{|x-\xi|}^{Y_N}Y_{N-1}\cdots\int_{|x-\xi|}^{Y_2}Y_1G(x,Y_1,\xi)dY_1\cdots Y_N,\\
&=2^{2N}(s-s^2)^Ny^{2N}+\frac{2^NC_0(x,c)}{N!c^2}(2s-1)(s-s^2)^Ny^{2N+1}+O(y^{2N+2}/c^2).
\end{aligned}$$

Therefore by integration by parts we get

$$\begin{aligned}
(2)=\ &2^{2N+1}R^{\frac{-2N}{2N+1}}y^{2N}\int_0^1(s(1-s))^ND\phi(x+(2s-1)y)ds+\\
&R^{\frac{-2N}{2N+1}}O(y^{2N+2}/c^2).
\end{aligned}$$

Thus we have (6.2). Next we show (6.3). We have

$$\begin{aligned}
&\frac{\partial\eta}{\partial R}=(3)+(4)+(5),\\
&(3)=\frac{2}{2N+1}R^{\frac{-2N}{2N+1}}\int_0^1K(x,y,x+(2s-1)y)\phi(x+(2s-1)y)ds,\\
&(4)=2R^{\frac{-2N}{2N+1}}\int_0^1(-x(K_x+K_\xi)+\frac{1}{2N+1}y(K_y+(2s-1)K_\xi))\times\\
&\qquad\phi(x+(2s-1)y)ds,\\
&(5)=2R^{\frac{-2N}{2N+1}}\int_0^1K(x,y,x+(2s-1)y)(-x+\frac{y(2s-1)}{2N+1})\times\\
&\qquad D\phi(x+(2s-1)y)ds.
\end{aligned}$$

By (2) of Proposition 5.4 we get

$$(3) = \frac{2^{2N+1}}{2N+1}\int_0^1 (s-s^2)^N\phi(x+(2s-1)y)ds + O(y^2/c^2).$$

As for (4) we use (3) of Proposition 5.4 and

$$\begin{aligned}
&K_y + (2s-1)K_\xi \\
&\quad = yJ^{N-1}G - (2s-1)(\xi-x)G(x,|\xi-x|,\xi)J^{N-1}1 + (2s-1)J^N G_\xi \\
&\quad = 2^{2N+1}N(s-s^2)^N y^{2N-1} + \frac{2^{N-1}C_0(x,c)}{(N-1)!c^2}(2s-1)(s-s^2)^N y^{2N} + \\
&\qquad \frac{2^N C_3(x,c)}{N!c^2}(2s-1)(s-s^2)^N y^{2N} + O(y^{2N+1}/c^2)
\end{aligned}$$

(See (6) of Proposition 5.4). Then by integration by parts we have

$$(4) = \frac{2^{2N+2}N}{2N+1}\int_0^1 (s-s^2)^N\phi(x+(2s-1)y)ds + O(y^2/c^2).$$

As (2) we get

$$(5) = 2^{2N+1}\int_0^1 (s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))D\phi(x+(2s-1)y)ds + O(y^2/c^2).$$

Thus we get (6.3).

Next we show (6.4). We have

$$\begin{aligned}
&\frac{\partial^2\eta}{\partial M^2} = (6)+(7)+(8), \\
&\quad (6) = 2R^{\frac{-4N-1}{2N+1}}\int_0^1((K_x+K_\xi)_x + (K_x+K_\xi)_\xi)(x,y,x+(2s-1)y)\times \\
&\qquad\qquad \phi(x+(2s-1)y)ds, \\
&\quad (7) = 4R^{\frac{-4N-1}{2N+1}}\int_0^1 (K_x+K_\xi)(x,y,x+(2s-1)y)D\phi(x+(2s-1)y)ds, \\
&\quad (8) = 2R^{\frac{-4N-1}{2N+1}}\int_0^1 K(x,y,x+(2s-1)y)D^2\phi(x+(2s-1)y)ds.
\end{aligned}$$

By (5) of Proposition 5.4 we have

$$\begin{aligned}
&((K_x+K_\xi)_x + (K_x+K_\xi)_\xi)(x,y,x+(2s-1)y) \\
&\qquad\qquad = \frac{2^N C_2(x,c)}{N!c^2}(s-s^2)^N(2s-1)y^{2N+1} + O(y^{2N+2}/c^2).
\end{aligned}$$

Thus by integration by parts we get

$$(6) = O(y^{-2N+1}/c^2).$$

By the same discussion as (1) we see $(7) = O(y^{-2N+1}/c^2)$. By the same discussion as (2) we see

$$(8) = 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N D^2\phi(x+(2s-1)y)ds + O(y^{-2N+1}/c^2).$$

Thus we get (6.4).

Next we show (6.5). We see

$$
\begin{aligned}
&\frac{\partial^2\eta}{\partial M\partial R} = (9)+(10)+(11)+(12)+(13)+(14),\\
&(9) = -\frac{4N}{2N+1}R^{\frac{-4N-1}{2N+1}}\int_0^1 (K_x+K_\xi)(x,y,x+(2s-1)y)\phi(x+(2s-1)y)ds,\\
&(10) = 2R^{\frac{-4N-1}{2N+1}}\int_0^1(-x(K_x+K_\xi)_x+(K_x+K_\xi)_\xi)+\\
&\qquad \frac{y}{2N+1}((K_x+K_\xi)_y+(2s-1)(K_x+K_\xi)_\xi))\phi(x+(2s-1)y)ds,\\
&(11) = 2R^{\frac{4N}{2N+1}}\int_0^1(K_x+K_\xi)(-x+\frac{y}{2N+1}(2s-1))D\phi ds,\\
&(12) = -\frac{4N}{2N+1}R^{\frac{-4N-1}{2N+1}}\int_0^1 KD\phi ds,\\
&(13) = 2R^{\frac{-4N-1}{2N+1}}\int_0^1(-x(K_x+K_\xi)+\frac{y}{2N+1}(K_y+(2s-1)K_\xi))D\phi ds\\
&(14) = 2R^{\frac{-4N-1}{2N+1}}\int_0^1 K(-x+\frac{y}{2N+1}(2s-1))D^2\phi ds.
\end{aligned}
$$

We already know that $(9)=O(y^{-2N+1}/c^2)$. (Recall (1).) Next we look at (10). The first term is $O(y^{-2N+1}/c^2)$. (Recall (6)). By (3) and (7) of Proposition 5.4 we see

$$
\begin{aligned}
&(K_x+K_\xi)_y+(2s-1)(K_x+K_\xi)_\xi\\
&\quad=\frac{2^{N-1}C_1(x,c)}{(N-1)!c^2}y^{2N}(2s-1)(s-s^2)^N+\frac{2^NC_4(x,c)}{N!c^2}y^{2N}(2s-1)(s-s^2)^N-\\
&\qquad\frac{2^{N-1}C_1(x,c)}{(N-1)!c^2}y^{2N}(2s-1)^3(s-s^2)^{N-1}+O(y^{2N+1}/c^2).
\end{aligned}
$$

By integration by parts we see $(10)=O(y^{-2N+1}/c^2)$. We already know $(11)=O(y^{-2N+1}/c^2)$. Clearly

$$
(12) = -\frac{2^{2N+2}N}{2N+1}y^{-2N-1}\int_0^1(s-s^2)^ND\phi ds+O(y^{-2N+1}/c^2).
$$

We see

$$
(13) = O(y^{-2N+1}/c^2)+\frac{2}{2N+1}R^{\frac{-4N}{2N+1}}\int_0^1(K_y+(2s-1)K_\xi)D\phi ds.
$$

As (4) we have

$$
(13) = \frac{2^{2N+2}N}{2N+1}y^{-2N-1}\int_0^1(s-s^2)^ND\phi ds+O(y^{-2N+1}/c^2).
$$

Finally we see

$$
(14) = 2^{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(-x+(2s-1)\frac{y}{2N+1})D^2\phi ds+O(y^{-2N+1}/c^2).
$$

(Recall (5)). Summing up we get (6.5).

Next we show (6.6).

$$
\begin{aligned}
&\frac{\partial^2\eta}{\partial R^2} = \frac{\partial}{\partial R}(3) + \frac{\partial}{\partial R}(4) + \frac{\partial}{\partial R}(5),\\
&\frac{\partial}{\partial R}(3) = (15) + (16) + (17),\\
&\quad (15) = -\frac{4N}{(2N+1)^2} R^{\frac{-4N-1}{2N+1}} \int_0^1 K\phi ds,\\
&\quad (16) = \frac{2}{2N+1} R^{\frac{-4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N+1}(K_y + (2s-1)K_\xi))\phi ds,\\
&\quad (17) = \frac{2}{2N+1} R^{\frac{-4N-1}{2N+1}} \int_0^1 K(-x + \frac{y}{2N+1}(2s-1))D\phi ds,\\
&\frac{\partial}{\partial R}(4) = (18) + (19) + (20),\\
&\quad (18) = -\frac{4N}{2N+1} R^{\frac{-4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N+1} \times\\
&\qquad\qquad (K_y + (2s-1)K_\xi))\phi ds,\\
&\quad (19) = 2R^{\frac{-4N-1}{2N+1}} \int_0^1 K''\phi ds,
\end{aligned}
$$

where

$$
\begin{aligned}
K'' = {} & x(K_x + K_\xi) + x^2((K_x + K_\xi)_x + (K_x + K_\xi)_\xi) +\\
& \frac{y}{2N+1}((K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi) -\\
& \frac{xy}{2N+1}((K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi) +\\
& \frac{y^2}{2N+1}((K_y + (2s-1)K_\xi)_y + (2s-1)(K_y + (2s-1)K_\xi)_\xi) +\\
& \frac{y}{(2N+1)^2}(K_y + (2s-1)K_\xi),\\
(20) = {} & 2R^{\frac{-4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N+1}(K_y + (2s-1)K_\xi)) \times\\
& (-x + \frac{y}{2N+1}(2s-1))D\phi ds,\\
\frac{\partial}{\partial R}(5) = {} & (21) + (22) + (23) + (24),\\
(21) = {} & -\frac{4N}{2N+1} R^{\frac{-4N-1}{2N+1}} \int_0^1 K(-x + \frac{y}{2N+1}(2s-1))D\phi ds,\\
(22) = {} & 2R^{\frac{-4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N+1}(K_y + (2s-1)K_\xi)) \times\\
& (-x + \frac{y}{2N+1}(2s-1))D\phi ds,\\
(23) = {} & 2R^{\frac{-4N-1}{2N+1}} \int_0^1 K(x + \frac{y}{(2N+1)^2}(2s-1))D\phi ds,
\end{aligned}
$$

$$(24) = 2R^{\frac{-4N-1}{2N+1}} \int_0^1 K(-x + \frac{y}{2N+1}(2s-1))^2 D^2\phi ds.$$

First we see

$$(15) = -\frac{-2^{2N+2}N}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2),$$
$$(16) = \frac{2^{2N+2}N}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2),$$
$$(17) = \frac{2^{2N+1}}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2).$$

Thus we have

$$\frac{\partial}{\partial R}(3) = \frac{2^{2N+1}}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2).$$

Since (18) is similar to (16), we have

$$(18) = -\frac{2^{2N+3}N^2}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2).$$

Next let us look at (19). We already know

$$2R^{\frac{-4N-1}{2N+1}} \int_0^1 x(K_x + K_\xi)\phi ds = O(y^{-2N+1}/c^2),$$
$$2R^{\frac{-4N-1}{2N+1}} \int_0^1 x^2((K_x + K_\xi)_x + (K_x + K_\xi)_\xi)\phi ds = O(y^{-2N+1}/c^2).$$

Recalling (10), we see

$$\frac{2}{2N+1} R^{\frac{-4N-1}{2N+1}} y \int_0^1 ((K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi)\phi ds = O(y^{-2N+1}/c^2),$$
$$\frac{2}{2N+1} R^{\frac{-4N-1}{2N+1}} xy \int_0^1 ((K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi)\phi ds = O(y^{-2N+1}/c^2).$$

When $N = 1$, we have

$$\begin{aligned}&(K_y + (2s-1)K_\xi)_y + (2s-1)(K_y + (2s-1)K_\xi)_\xi \\ &= 8(s-s^2) + \frac{C_3}{c^2}(2s-1)y - \frac{C_0}{c^2}(2s-1)^3 y - \frac{2C_3}{c^2}(2s-1)^3 y + O(y^2/c^2).\end{aligned}$$

When $N \geq 2$, there are bounded functions $F_j(x,c)$ such that

$$\begin{aligned}&(K_y + (2s-1)K_\xi)_y + (2s-1)(K_y + (2s-1)K_\xi)_\xi \\ &= 2^{2N+1}N(2N-1)(s-s^2)^N y^{2N-2} + \frac{F_1(x,c)}{c^2}(2s-1)(s-s^2)^{N-1}y^{2N-1} + \\ &\quad \frac{F_2(x,c)}{c^2}(2s-1)(s-s^2)^{N-2}y^{2N-1} + \frac{F_3(x,c)}{c^2}(2s-1)^3(s-s^2)^{N-2}y^{2N-1} + \\ &\quad \frac{F_4(x,c)}{c^2}(2s-1)^3(s-s^2)^{N-1}y^{2N-1} + \frac{F_5(x,c)}{c^2}(2s-1)^5(s-s^2)^{N-2}y^{2N-1} + \\ &\quad O(y^{2N}/c^2).\end{aligned}$$

Thus we see

$$\frac{2R^{\frac{-4N-1}{2N+1}}}{(2N+1)^2}y^2\int_0^1((K_y+(2s-1)K_\xi)_y+(2s-1)(K_y+(2s-1)K_\xi)_\xi)\phi ds$$
$$=\frac{2^{2N+3}N^2}{(2N+1)^2}y^{-2N-1}\int_0^1(s-s^2)^N\phi ds-\frac{2^{2N+2}N}{(2N+1)^2}y^{-2N-1}\int_0^1(s-s^2)^N\phi ds+$$
$$O(y^{-2N+1}/c^2).$$

We have

$$\frac{2}{(2N+1)^2}R^{\frac{-4N-1}{2N+1}}y\int_0^1(K_y+(2s-1)K_\xi)\phi ds$$
$$=\frac{2^{2N+2}N}{(2N+1)^2}y^{-2N-1}\int_0^1(s-s^2)^N\phi ds+O(y^{-2N+1}/c^2).$$

Therefore

$$(19)=\frac{2^{2N+3}N^2}{(2N+1)^2}y^{-2N-1}\int_0^1(s-s^2)^N\phi ds+O(y^{-2N+1}/c^2).$$

We see

$$(20)=\frac{2^{2N+2}N}{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))D\phi ds+O(y^{-2N+1}/c^2).$$

Therefore

$$\frac{\partial}{\partial R}(4)=\frac{2^{2N+2}N}{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))D\phi ds+O(y^{-2N+1}/c^2).$$

Next we see

$$(21)=-\frac{2^{2N+2}N}{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(-x+\frac{y(2s-1)}{2N+1})D\phi ds+O(y^{-2N+1}/c^2),$$
$$(22)=\frac{2^{2N+2}N}{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(-x+\frac{y(2s-1)}{2N+1})D\phi ds+O(y^{-2N+1}/c^2),$$
$$(23)=2^{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(x+\frac{y(2s-1)}{(2N+1)^2})D\phi ds+O(y^{-2N+1}/c^2),$$
$$(24)=2^{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(-x+\frac{y(2s-1)}{2N+1})^2D^2\phi ds+O(y^{-2N+1}/c^2).$$

Therefore we get

$$\frac{\partial}{\partial R}(5)=2^{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(x+\frac{y}{(2N+1)^2}(2s-1))D\phi ds+$$
$$2^{2N+1}y^{-2N-1}\int_0^1(s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))^2D^2\phi ds+$$
$$O(y^{-2N+1}/c^2).$$

Summing up, we have

$$\begin{aligned}\frac{\partial^2\eta}{\partial R^2} &= 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))D\phi ds+\\ &\quad 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N(x+\frac{y}{(2N+1)^2}(2s-1))D\phi ds+\\ &\quad 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))^2D^2\phi ds\\ &= \frac{2^{2N+2}(N+1)}{(2N+1)^2}y^{-2N-1}\int_0^1 (s-s^2)^N(2s-1)yD\phi ds+\\ &\quad 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))^2D^2\phi ds\\ &= \frac{2^{2N+3}}{(2N+1)^2}y^{-2N-1}\int_0^1 (s-s^2)^{N+1}y^2D^2\phi ds+\\ &\quad 2^{2N+1}y^{-2N-1}\int_0^1 (s-s^2)^N(-x+\frac{y}{2N+1}(2s-1))^2D^2\phi ds.\end{aligned}$$

Thus we get (6.6). $\square$

Appendix C. Proof of Proposition 7.3. For the simplicity, we write $\eta_n = \eta_n^5, q_n = q_n^5, \hat{\eta}_n = \eta_n^6, \hat{q}_n = q_n^6$.
It is easy to see inductively that, for $G_j = J^jG = K_{N-j}$, we have

$$\partial_\xi^p G_j = J\partial_\xi^p G_{j-1}$$

for $j \geq p+1$ and

$$\partial_\xi^p G_p = (-1)^p(\xi-x)^pG(x,|\xi-x|,\xi) + J\partial_\xi^p G_{p-1}.$$

Therefore

$$\partial_\xi^p K = \partial_\xi^p G_N(x,y,\xi) = 0$$

for $p \leq N-1$ and $y = |x-\xi|$. Thus by integration by parts we have

$$\begin{aligned}\eta_n &= (-1)^N\partial_\xi^N K(x,y,x+y)\psi_n(x+y) -\\ &\quad (-1)^N\partial_\xi^N K(x,y,x-y)\psi_n(x-y) + F_n^1(x,y),\\ F_n^1(x,y) &= (-1)^{N+1}\int_{x-y}^{x+y}\partial_\xi^{N+1}K(x,y,\xi)\psi_n(\xi)d\xi.\end{aligned}$$

We see

$$\partial_\xi^p L_2(x,y,\xi) = -2\int_{(-x+y+\xi)/2}^{y}\mu_2\partial_\xi^p K(x-y+Y,Y,\xi)dY$$

for $p \leq N-1$. Therefore

$$\partial_\xi^p L_2(x,y,x+y) = \partial_\xi^p L_2(x,y,x-y) = 0$$

for $p \leq N-1$. Moreover we see

$$\partial_\xi^N L_2(x,y,x+y) = 0.$$

Therefore by integration by parts we have

$$\begin{aligned}\sigma_n(x,y) &= q_n(x,y) - \lambda_2\eta_n(x,y)\\ &= -(-1)^N\partial_\xi^N L_2(x,y,x-y)\psi_n(x-y) + F_n^2(x,y),\\ F_n^2(x,y) &= (-1)^{N+1}\int_{x-y}^{x+y}\partial_\xi^{N+1}L_2(x,y,\xi)\psi_n(\xi)d\xi.\end{aligned}$$

Similarly

$$\begin{aligned}\bar{\sigma}_n(x,y) &= q_n(x,y) - \lambda_1\eta_n(x,y)\\ &= (-1)^N\partial_\xi^N L_1(x,y,x+y)\psi_n(x+y) + \bar{F}_n^2(x,y),\\ \bar{F}_n^2(x,y) &= (-1)^{N+1}\int_{x-y}^{x+y}\partial_\xi^{N+1}L_1(x,y,\xi)\psi_n(\xi)d\xi.\end{aligned}$$

We note

$$\partial_\xi^N K(x,y,\xi) = (-1)^N(\xi-x)^N G(x,|x-\xi|,\xi) + J\partial_\xi^N G_{N-1}.$$

It is easy to see inductively that

$$\begin{aligned}\partial_\xi^{p+1}G_p(x,y,\xi) = (-1)^p\frac{p(p+1)}{2}&(\xi-x)^{p-1}G(x,|x-\xi|,\xi) +\\ &(\xi-x)^p H_p(x,\xi) + J\partial_\xi^{p+1}G_{p-1},\end{aligned}$$

where $H_p = O(1/c^2)$. Therefore

$$\begin{aligned}\partial_\xi^{N+1}K(x,y,\xi) = (-1)^N\frac{N(N+1)}{2}&(\xi-x)^{N-1}G(x,|\xi-x|,\xi) +\\ &(\xi-x)^N H_N(x,\xi) + J\partial_\xi^{N+1}G_{N-1}.\end{aligned}$$

i) Suppose $(x,y)\in S$. Then it is clear that η^3,η^4,q^3, q^4,η_n,q_n, $\hat{\eta}_n,\hat{q}_n$, B_n^3,B_n^4,B_n all vanish.
ii) Suppose $(x,y)\in S_0$. Then we see

$$\begin{aligned}\eta^3 &= K(x,y,a)\\ &= O((y^2-(x-a)^2)^N)\\ &= O(n^{-2N}),\\ \eta^4 &= K_\xi(x,y,a)\\ &= O(|x-a|(y^2-(x-a)^2)^{N-1}) + O((y^2-(x-a)^2)^N)\\ &= O(n^{-2N+1}),\\ \sigma^3 &= L_2(x,y,a)\\ &= -2\int_{(-x+y+a)/2}^{y}\mu_2 K(x-y+Y,Y,a)dY\\ &= O(n^{-2N-1}),\\ \sigma^4 &= L_{2,\xi}(x,y,a)\\ &= -2\int_{(-x+y+a)/2}^{y}\mu_2 K_\xi(x-y+Y,Y,a)dY\\ &= O(n^{-2N}).\end{aligned}$$

Since $y = O(1/n)$ and $\psi_n = O(n)$, we see

$$(-1)^N \partial_\xi^N K(x, y, x + y)\psi_n(x + y) - (-1)^N \partial_\xi^N K(x, y, x - y)\psi_n(x - y) = O(n^{-N+1}).$$

Since $F_n^1 = O(1)$, we have $\eta_n = O(1)$. We see

$$\partial_\xi^N L_2(x, y, x - y) = -2\int_0^y \mu_2 \partial_\xi^N K(x - y + Y, Y, x - y)dY = O(n^{-N-1}).$$

Therefore

$$-(-1)^N \partial_\xi^N L_2(x, y, x - y)\psi_n(x - y) = O(n^{-N}).$$

Since

$$\begin{aligned}\partial_\xi^{N+1} L_2(x, y, \xi) = \mu_2 \partial_\xi^N K((x - y + \xi)/2, (-x + y + \xi)/2, \xi) - \\ 2\int_{(-x+y+\xi)/2}^y \partial_\xi^{N+1} K(x - y + Y, Y, \xi)dY \\ = O((-x + y + \xi)^N) + O(x + y - \xi),\end{aligned}$$

we see

$$\begin{aligned}F_n^2(x, y) &= (-1)^{N+1}\int_{x-y}^{x+y} \partial_\xi^{N+1} L_2(x, y, \xi)\psi_n(\xi)d\xi \\ &= O(n^{-1}).\end{aligned}$$

Hence $\sigma_n = O(n^{-1})$. Therefore

$$\begin{aligned}B_n^3 &= \eta^3\sigma_n - \eta_n\sigma^3 = O(n^{-2N-1}), \\ B_n^4 &= \eta^4\sigma_n - \eta_n\sigma^4 = O(n^{-2N}), \\ B_n &= \eta_n\hat{\sigma}_n - \hat{\eta}_n\sigma_n = O(n^{-1}).\end{aligned}$$

iii) Suppose $(x, y) \in S_1$, where $x + y > a + \frac{1}{n}$ and $x - y < a - \frac{1}{n}$. Then $\psi_n(x + y) = \psi_n(x - y) = \hat{\psi}_n(x + y) = \hat{\psi}_n(x - y) = 0$. So, $\eta_n = F_n^1, \sigma_n = F_n^2$, and so on. But

$$\begin{aligned}F_n^1(x, y) &= (-1)^{N+1}\int_{x-y}^{x+y} \partial_\xi^{N+1} K(x, y, \xi)\psi_n(\xi)d\xi \\ &= (-1)^{N+1}\int_{-1}^1 (\partial_\xi^{N+1} K(x, y, a + \frac{s}{n}) - \partial_\xi^{N+1} K(x, y, a))\Phi(s)ds \\ &= O(1/n)\end{aligned}$$

since $\int \Phi = 0$ and $\partial_\xi^{N+1} K$ is Lipschitz continuous. Same estimates hold for $F_n^2, \hat{F}_n^1, \hat{F}_n^2$. Thus

$$\begin{aligned}B_n^3 &= \eta^3 F_n^2 - F_n^1\sigma^3 = O(1/n), \\ B_n^4 &= \eta^4 F_n^2 - F_n^1\sigma^4 = O(1/n), \\ B_n &= F_n^1\hat{F}_n^2 - \hat{F}_n^1 F_n^2 = O(1/n^2).\end{aligned}$$

iv) Suppose $(x, y) \in S_L$, where $|x + y - a| \le 1/n$. It is easy to see $\eta^3 = O(n^{-N}), \eta^4 = O(n^{-N+1}), \sigma^3 = O(n^{-N-1}), \sigma^4 = O(n^{-N})$. Since $n(x - y - a) < -1$, we have $\psi_n(x - y) = 0$. Thus $\eta_n = O(n), \sigma_n = F_n^2 = O(1)$. Therefore

$$\begin{aligned}B_n^3 &= \eta^3\sigma_n - \eta_n\sigma^3 = O(n^{-N}), \\ B_n^4 &= \eta^4\sigma_n - \eta_n\sigma^4 = O(n^{1-N}).\end{aligned}$$

Let us estimate $B_n = \eta_n\hat{\sigma}_n - \hat{\eta}_n\sigma_N$. Since

$$\partial_\xi^{N+1}K = (-1)^N\frac{N(N+1)}{2}(\xi - x)^{N-1}G(x, |x-\xi|, \xi) + \\ (\xi - x)^N H_N(x,\xi) + J\partial_\xi^{N+1}G_{N-1},$$

we have

$$\begin{aligned} F_n^1 &= (-1)^{N+1}\int_{x-y}^{x+y}\partial_\xi^{N+1}K(x,y,\xi)\psi_n(\xi)d\xi \\ &= (-1)^{N+1}((-1)^N\frac{N(N+1)}{2}2^N N!(a-x)^{N-1} + F'(x,a))\int_{-1}^{n(x+y-a)}\Phi \\ &\quad +O(1/n) \\ &= -\frac{N(N+1)}{2}2^N N! y^{N-1}(1+F''(x,y,a))\int_{-1}^{n(x+y-a)}\Phi + O(1/n), \end{aligned}$$

where $F' = O(1/c^2)|x-a|^N, F'' = O(1/c^2)$. On the other hand

$$\partial_\xi^N K(x,y,x+y) = (-1)^N y^N G(x,y,x+y).$$

Hence

$$\begin{aligned} \eta_n = &\, ny^N G(x,y,x+y)\Phi(n(x+y-a)) - \\ &\frac{N(N+1)}{2}2^N N! y^{N-1}(1+F''(x,y,a))\int_{-1}^{n(x+y-a)}\Phi + O(1/n). \end{aligned}$$

Since

$$\begin{aligned} \partial_\xi^{N+1}L_2(x,y,\xi) &= \mu_2\partial_\xi^N K((x-y+\xi)/2, (-x+y+\xi)/2, \xi) - \\ &\quad 2\int_{(-x+y+\xi)/2}^{y}\mu_2\partial_\xi^{N+1}K(x-y+Y, Y, \xi)dY \\ &= (\frac{N+1}{2N+1} + O(1/c^2))(-1)^N(\frac{-x+y+\xi}{2})^N \times \\ &\quad G((x+y+\xi)/2, (-x+y+\xi)/2, \xi) + O(x+y-\xi), \end{aligned}$$

we see

$$\begin{aligned} \sigma_n = F_n^2 &= (-1)^{N+1}\int_{x-y}^{x+y}\partial_\xi^{N+1}L_2(x,y,\xi)\psi_n(\xi)d\xi \\ &= -\frac{N+1}{2N+1}2^N N! y^N(1+L'(x,y,a))\int_{-1}^{n(x+y-a)}\Phi + O(1/n), \end{aligned}$$

where $L' = O(1/c^2)$. Here we have used

$$(\frac{-x+y+a}{2})^N = (y - \frac{x+y-a}{2})^N = y^N + O(1/n).$$

Similar estimates hold for $\hat{\eta}_n, \hat{\sigma}_n$. Thus

$$B_n = ny^{2N}A_1 + y^N A_2 + A_3,$$

where

$$\begin{aligned} A_1 = & -G\frac{N+1}{2N+1}2^N N!(1+L')\Phi(\beta)\int_{-1}^{\beta}\hat{\Phi} + \\ & G\frac{N+1}{2N+1}2^N N!(1+L')\hat{\Phi}(\beta)\int_{-1}^{\beta}\Phi \\ = & \frac{N+1}{2N+1}2^N N! G(1+L')(\int_{-1}^{\beta}\Phi)^2, \\ \beta = & \, n(x+y-a). \end{aligned}$$

The estimates on S_R can be obtained in a similar manner considering $\bar{\sigma}^3, \bar{\sigma}^4, \bar{\sigma}_n$. □

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