EXPONENTIALLY ILL-CONDITIONED CONVECTION-DIFFUSION EQUATIONS IN MULTI-DIMENSIONAL DOMAINS

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Abstract. The phenomenon of dynamic metastability is analyzed for a class of singularly perturbed linear convection-diffusion equation in both a one-dimensional and a two-dimensional domain. The extreme sensitivity, known as a supersensitivity, of the time-dependent and steady-state solutions to exponentially small changes in the coefficient of the differential operator is also examined. Finally, some numerical experiments are performed in the one-dimensional case that indicate that it is possible to use standard finite difference methods to compute an exponentially ill-conditioned steady-state solution provided that quadruple precision arithmetic is used in the numerical computations.

1. Introduction. Certain linear time-dependent convection-diffusion equations with small diffusivity ε exhibit the phenomenon of dynamic metastability whereby the time-dependent solution approaches a steady-state limit only over an asymptotically exponentially long time interval. In the limit $\varepsilon \to 0$, this metastable behavior and the corresponding steady-state limit can have an $O(1)$ sensitivity (i. e. a supersensitivity) to asymptotically exponentially small changes in the coefficient of the differential operator. This metastability, and the resulting supersensitivity, is analyzed for a certain class of linear singularly perturbed convection-diffusion equation in both a one and a two-dimensional spatial domain. In §2 we consider the one-dimensional case and in §4 we treat the corresponding two-dimensional problem. Some of the results for the one-dimensional problem have been given previously in [21].

For an exponentially ill-conditioned problem, an exponentially small change in the coefficient of the differential operator can have an $O(1)$ effect on the solution. Therefore, it is natural to expect that we might require that the truncation error associated with a numerical scheme for an exponentially ill-conditioned problem must be smaller than the magnitude of the exponentially small eigenvalue in order to ensure that the numerical solution resembles its continuous counterpart. With this heuristic reasoning, we might believe that high-order or spectral-type numerical methods are preferred for solving exponentially ill-conditioned problems. Such an approach, involving a high-order integral equation scheme, was implemented in [16] to compute boundary value resonance solutions, and a Galerkin spectral method was employed in [4] to compute metastable behavior for the viscous Cahn-Hilliard equation.

On the other hand, many conventional schemes have also been used to compute the solution to exponentially ill-conditioned problems. For instance, Elliott and French [9] studied metastable dynamics for the Cahn-Hilliard equation by applying the Galerkin finite element method, and Carr and Pego [7] computed the metastable evolution of a transition layer for the Allen-Cahn equation using the method of lines subroutine LSODI of the NAG library [20] with 801 grid points. In addition, a standard fourth-order finite difference scheme was employed to compute metastable dynamics for the viscous Cahn-Hilliard equation in [23]. However, the implementation in [23] was non-standard in that quadruple precision arithmetic was used in their numerical computations.

The main question that we consider is whether it is possible to accurately compute the solution to an exponentially ill-conditioned problem using a standard finite difference scheme employing quadruple rather than double precision arithmetic, or whether

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we must use a high order spectral method that ensures that the truncation error is smaller than the order of the exponentially small eigenvalue. In §3 we investigate this issue for a simple boundary layer resonance problem.

The study of metastability here complements previous studies of metastability for nonlinear reaction-diffusion problems undertaken in [2], [6], [7], [10], [14], [22], [23], [28], [29], and [30].

2. Ill-Conditioned Convection-Diffusion: One Dimension. In this section analyze an exponentially ill-conditioned linear convection-diffusion equation exhibiting metastable behavior in one dimension. The sensitivity of the steady-state solution to exponentially small changes in the coefficient of the differential operator is also examined. A related problem of this type was considered in [21].

We consider the following convection-diffusion equation for $u = u(x, t)$:

$$(2.1a) \qquad u_t = \varepsilon u_{xx} - \Psi'(x) u_x + \varepsilon^\nu g(x) e^{-a/\varepsilon} u, \quad -1 < x < 1, \quad t > 0,$$

$$(2.1b) \qquad u(-1, t) = u_l, \quad u(1, t) = u_r; \qquad u(x, 0) = u_0(x).$$

Here $a > 0$, ν, u_l and u_r are constants, $\varepsilon \to 0^+$ and $g(x)$, $\Psi(x)$ and $u_0(x)$ are smooth. We assume that the potential $\Psi(x)$ has a global minimum on $[-1, 1]$ at $x = 0$ with $\Psi(0) = 0$, $\Psi'(0) = 0$ and $\Psi''(0) > 0$. Thus, the spatial operator in (2.1a) has a simple turning point at $x = 0$. We also assume that $\Psi'(x) \neq 0$ for $x \neq 0$ and thus $\Psi'(1) > 0$ and $\Psi'(-1) < 0$. Prototypical is $\Psi(x) = x^2/2$.

When $g(x) = 0$, the equilibrium problem corresponding to (2.1) and its associated eigenvalue problem arises in determining the exit time distribution for a Brownian particle confined by the potential well Ψ. Such problems have been well-studied in [17], [18], and [19] (see the references therein).

When $g(x) = 0$, the eigenvalue problem associated with (2.1) is

$$(2.2a) \qquad L_\varepsilon \phi \equiv \varepsilon \phi_{xx} - \Psi'(x) \phi_x = -\lambda \phi, \quad -1 < x < 1; \qquad \phi(\pm 1) = 0,$$

$$(2.2b) \qquad (\phi, \phi)_w \equiv \int_{-1}^{1} \phi^2 w \, dx = 1, \qquad w \equiv e^{-\Psi/\varepsilon}.$$

The eigenvalues λ_j for $j \geq 0$ are real with $\lambda_j > 0$ and the orthogonality relations $(\phi_j, \phi_k)_w = \delta_{jk}$ for $j, k = 0, 1, ..$, hold. It is well-known (cf. [8], [17], [16]) that λ_0 is exponentially small as $\varepsilon \to 0$ and has the asymptotic estimate

$$(2.3) \qquad \lambda_0 \sim \left(\frac{\Psi''(0)}{2\pi\varepsilon} \right)^{1/2} \left[\Psi'(1) b_r(\varepsilon) e^{-\Psi(1)/\varepsilon} - \Psi'(-1) b_l(\varepsilon) e^{-\Psi(-1)/\varepsilon} \right].$$

Here $b_r(\varepsilon)$ and $b_l(\varepsilon)$ have asymptotic expansions in powers of ε with leading terms $b_r(0) = b_l(0) = 1$. To leading order, the corresponding eigenfunction ϕ_0 has the boundary layer form

$$(2.4) \qquad \phi_0 \sim M_0 \left(1 - e^{-\Psi'(1)(1+x)/\varepsilon} - e^{\Psi'(-1)(1-x)/\varepsilon} \right),$$

where M_0 is a normalization constant. Notice that, as $\varepsilon \to 0$, ϕ_0 is a constant away from the boundary layer regions.

We first consider the equilibrium problem for (2.1). For $\varepsilon \to 0$, a leading order boundary layer analysis for the equilibrium solution $U(x; \varepsilon)$ shows that $U(x; \varepsilon) \sim \tilde{u}^\varepsilon [x; A_{0e}]$, where

$$(2.5) \qquad \tilde{u}^\varepsilon [x; A_{0e}] \equiv A_{0e} + (u_r - A_{0e}) e^{-\Psi'(1)(1-x)/\varepsilon} + (u_l - A_{0e}) e^{\Psi'(-1)(1+x)/\varepsilon},$$

for some undetermined constant A_{0e}. Singular perturbation problems of this type, where a conventional application of the method of matched asymptotic expansions fails to select certain constants uniquely, were first identified in [1] and later studied extensively in [8], [11], [16], [26], and [32] (and the references therein). The relationship between this apparent indeterminacy and the ill-conditioning of the underlying operator is emphasized in [8] and [16]. More specifically, since (2.2a) has an exponentially small eigenvalue, and $L_\varepsilon \tilde{u}^\varepsilon$ is exponentially small away from the boundary layer regions near $x = \pm 1$ for any choice of A_{0e}, it follows that the correct value of A_{0e} can only be determined by incorporating the effect of exponentially small terms into the asymptotic analysis. Methods for calculating A_{0e} are given in [11], [16], [19], and [26]. Since the equilibrium problem is exponentially ill-conditioned, it is natural to expect that it will be extremely sensitive to the exponentially small term in (2.1a). This aspect has been studied in [16], [26] and [32].

For the time-dependent problem for (2.1), we note that since $\lambda_0 > 0$, the equilibrium solution is stable when a is sufficiently large. However, since λ_0 is exponentially small, the evolution of an arbitrary initial condition $u_0(x)$ to the equilibrium solution is exponentially slow. This metastable motion can be studied by using the projection method as developed in [21].

We now outline this method and some of the results obtained from it. Following [21], we seek a solution to (2.1) in the form

$$(2.6) \qquad u(x,t) = \tilde{u}^\varepsilon\left[x; A_0(t)\right] + v(x,t) \,,$$

where $\tilde{u}^\varepsilon$ is defined in (2.5). Substituting (2.6) into (2.2), we obtain that $v(x,t)$ satisfies

$$(2.7a) \qquad v_t = L_\varepsilon v - \tilde{u}_t^\varepsilon + L_\varepsilon \tilde{u}^\varepsilon + \varepsilon^\nu g(x) e^{-a/\varepsilon}(\tilde{u}^\varepsilon + v) \,, \quad -1 < x < 1, \quad t > 0 \,,$$

$$(2.7b) \qquad v(-1,t) = u_l - \tilde{u}^\varepsilon\left[-1; A_0(t)\right] \,, \qquad v(1,t) = u_r - \tilde{u}^\varepsilon\left[1; A_0(t)\right] \,,$$

together with the initial condition $v(x,0) = u_0(x) - \tilde{u}^\varepsilon\left[x; A_0(0)\right]$. We then expand $v(x,t)$ in terms of the eigenfunctions ϕ_j of (2.2a)-(2.2b) as

$$(2.8) \qquad v(x,t) = \sum_{j=0}^\infty c_j(t) \phi_j(x) \,.$$

Using the orthogonality property of the eigenfunctions, we find that $c_j(t)$ satisfies the differential equation

$$(2.9) \quad c_j' + \lambda_j c_j = (\phi_j, L_\varepsilon \tilde{u}^\varepsilon)_w - \varepsilon w v \phi_j' \Big|_{-1}^1 - (\phi_j, \tilde{u}_t^\varepsilon)_w + \varepsilon^\nu e^{-a/\varepsilon} (g\phi_j, \tilde{u}^\varepsilon + v)_w \,,$$

and has the initial value

$$(2.10) \qquad c_j(0) = \int_{-1}^1 \left(u_0(x) - \tilde{u}^\varepsilon[x; A_0(0)]\right) \phi_j w \, dx \,.$$

Since $\lambda_0 > 0$ and is exponentially small, it is necessary that $c_0(t) \equiv 0$ in order to ensure that $v \ll \tilde{u}^\varepsilon$ over exponentially long time intervals . Therefore, the right-hand sides of (2.9) and (2.10) must vanish when $j = 0$. Then, upon using $v \ll \tilde{u}^\varepsilon$ to simplify the last term on the right-hand side of (2.9), we obtain

$$(2.11) \qquad (\phi_0, \tilde{u}_t^\varepsilon)_w \sim (\phi_0, L_\varepsilon \tilde{u}^\varepsilon)_w - \varepsilon w v \phi_0' \Big|_{-1}^1 + \varepsilon^\nu e^{-a/\varepsilon} (g\phi_0, \tilde{u}^\varepsilon)_w \,,$$

together with

$$(2.12) \qquad \int_{-1}^{1} \tilde{u}^{\varepsilon}[x; A_0(0)]\phi_0 w \, dx = \int_{-1}^{1} u_0(x)\phi_0 w \, dx.$$

Equation (2.11) is a differential equation for $A_0(t)$ and (2.12) determines the initial value $A_0(0)$.

To obtain an explicit differential equation for A_0 we use the form for ϕ_0 given in (2.4) to evaluate the various terms in (2.11)-(2.12) asymptotically for $\varepsilon \to 0$ (see [21] for details). To leading order as $\varepsilon \to 0$, the analysis in [21] shows that $A_0(t)$ satisfies the limiting differential equation

(2.13a)
$$A_0' \sim -\left(\lambda_0 - \varepsilon^{\nu} g(0)e^{-a/\varepsilon}\right) A_0 + \beta \left[u_r \Psi'(1)e^{-\Psi(1)/\varepsilon} - u_l \Psi'(-1)e^{-\Psi(-1)/\varepsilon}\right],$$

with

$$(2.13b) \qquad A_0(0) \sim u_0(0), \qquad \text{where} \qquad \beta \equiv \left[\Psi''(0)/(2\pi\varepsilon)\right]^{1/2}.$$

In (2.13a) λ_0 is given in (2.3). The metastable dynamics for $u(x,t)$ is then given by $u(x,t) \sim \tilde{u}^{\varepsilon}[x; A_0(t)]$, where $\tilde{u}^{\varepsilon}$ is defined in (2.5). If a is sufficiently large, it follows that $A_0(t) \to A_{0e}$ as $t \to \infty$, where A_{0e} is obtained by setting $A_0' = 0$ in (2.13a). Higher order corrections to the asymptotic differential equation for $A_0(t)$ are given in [21].

2.1. An Example. We let $\Psi(x) = x^2/2$, $g(x) = g_0 e^{-x^2/2}$, and $\nu = -1/2$. Then, from (2.3) we calculate that

$$(2.14) \qquad \lambda_0 \sim \left(\frac{2}{\pi\varepsilon}\right)^{1/2} e^{-\varepsilon^{-1}/2}.$$

From (2.5), (2.6), and (2.13a), we obtain that for $t \gg 1$,

$$(2.15) \qquad u(x,t) \sim A_0(t) + [u_r - A_0(t)] e^{-(1-x)/\varepsilon} + [u_l - A_0(t)] e^{-(1+x)/\varepsilon},$$

where $A_0(t)$ satisfies the asymptotic differential equation

$$(2.16) \quad A_0' \sim -\left(\frac{2}{\pi\varepsilon}\right)^{1/2} e^{-\varepsilon^{-1}/2} \left[\left(1 - \left(\frac{\pi}{2}\right)^{1/2} g_0 e^{-(a-1/2)/\varepsilon}\right) A_0 - \frac{(u_l + u_r)}{2}\right].$$

Now, from (2.16) we observe that when

$$(2.17) \qquad \left(\frac{\pi}{2}\right)^{1/2} g_0 e^{-(a-1/2)/\varepsilon} < 1,$$

then $A_0(t) \to A_{0e}$ as $t \to \infty$, where

$$(2.18) \qquad A_{0e} = \frac{(u_l + u_r)}{2} \left[1 - \left(\frac{\pi}{2}\right)^{1/2} g_0 e^{-(a-1/2)/\varepsilon}\right]^{-1}.$$

Thus, if $a > 1/2$, so that the order of the exponentially small perturbing term $\varepsilon^{\nu} g(x)e^{-a/\varepsilon}$ is smaller than that of the exponentially small eigenvalue, then $A_0(t) \to$

A_{0e} as $t \to \infty$ for any $A_0(0)$. However, if $0 < a < 1/2$, then $A_0(t) \to A_{0e}$ as $t \to \infty$ only when $g_0 \le 0$. In the borderline case where $a = 1/2$, then

$$(2.19) \qquad \lim_{t \to \infty} A_0(t) = A_{0e}, \qquad \text{when} \quad g_0 < (2/\pi)^{1/2},$$

and $A_0(t) \to \infty$ as $t \to \infty$ when $g_0 > (2/\pi)^{1/2}$. This example clearly shows the effect of supersensitivity, whereby the exponentially small perturbing term can have an $O(1)$ effect on both the time-dependent and the steady-state solutions.

3. Some Numerical Experiments in One Dimension. In this section we numerically study the convergence properties of a few numerical schemes to compute solutions to an exponentially ill-conditioned boundary value problem. Our goal is to give some guidelines in designing numerical schemes to treat ill-conditioned problems. The main question we would like to address is whether it is possible to accurately compute the solution to an exponentially ill-conditioned problem using a standard finite difference method employing quadruple, or extended, precision arithmetic, instead of using a higher order spectral method.

For simplicity, we investigate this numerical issue for the following singularly perturbed boundary value resonance problem

$$(3.1a) \qquad L_\varepsilon u \equiv \varepsilon u_{xx} - x^{2m+1} p(x) u_x = 0, \quad -1 < x < 1,$$
$$(3.1b) \qquad u(-1) = u_l, \quad u(1) = u_r.$$

Here $\varepsilon > 0$ is a small parameter, $m \ge 0$ is an integer and $p(x) > 0$ on $[-1, 1]$. One main reason to consider (3.1) is that it is linear and its solution can be found analytically. Thus, we can easily compute the errors associated with the finite difference schemes.

We now introduce the three finite difference schemes used to compute solutions to (3.1). The mesh is denoted by $I^h = \{x_i : -1 = x_N < x_{-N+1} < \ldots < x_{N-1} < x_N = 1\}$ with mesh widths $h_i = x_i - x_{i-1}$ and $h = \max_i h_i$. Two meshes will be used in the computations.

- An equidistant mesh $I_e^h = \{x_i = ih : i = -N, \ldots, N, Nh = 1\}$.
- A non-equidistant mesh $I_n^h = \{x_i : i = -N, \ldots, N\}$, where with $h = 1/N$,

$$(3.2) \qquad x_i = \begin{cases} \lambda(ih + 1) - 1, & i = -N, \ldots, 0, \\ 1 - \lambda(1 - ih), & i = 1, \ldots, N. \end{cases}$$

Here the mesh generating function $\lambda(t)$ is defined by

$$(3.3) \qquad \lambda(t) = \begin{cases} \psi(t) := -a\varepsilon \ln(1 - t/q), & t \in [0, \alpha], \\ \pi(t) := \psi(\alpha) + \psi'(\alpha)(t - \alpha), & t \in (\alpha, 1], \end{cases}$$

where $q \in (0, 1)$ and $a > 0$ are constants, and $\alpha \in (0, q)$ is determined uniquely by $\pi(1) = 1$. In the computations below we chose $a = 2$, $q = 0.5$, for which $\alpha = 0.4618$. This mesh is also called the Bakhvalov mesh [3], and it allows for more mesh points near the endpoints to resolve boundary layers. Using the usual notation for divided differences,

$$D_+ u_i = (u_{i+1} - u_i)/h_{i+1}, \qquad D_- u_i = (u_i - u_{i-1})/h_i,$$
$$D_0 u_i = (u_{i+1} - u_{i-1})/(h_i + h_{i+1}), \quad D_+ D_- u_i = 2(D_+ u_i - D_- u_i)/(h_i + h_{i+1}),$$

we define the following three finite difference schemes:

- The upwind scheme on the non-equidistant mesh I_n^h,

(3.4)
$$L_1^h u_i = \varepsilon D_+ D_- u_i + P_i D' u_i = 0 \, , \quad -N < i < N \, , \quad u_{-N} = u_l \, , \quad u_N = u_r \, ,$$

where $P_i = -x_i^{2m+1} p(x_i) \, ,$ and

$$D' u_i = \left\{ \begin{array}{ll} D_+ u_i \, , & \text{if } P_i \geq 0 \, , \\ D_- u_i \, , & \text{if } P_i < 0 \, ; \end{array} \right.$$

- The coupled scheme on the non-equidistant mesh I_n^h,

(3.5a) $L_2^h u_i = \left\{ \begin{array}{ll} L_c^h u_i = 0 \, , & \text{if } \rho_i \leq 1 \, , \\ L_a^h u_i = 0 \, , & \text{if } \rho_i > 1 \, , \end{array} \right. \quad -N < i < N$

(3.5b) $u_{-N} = u_l \, , \quad u_N = u_r \, ,$

where

$$\rho_i = \left\{ \begin{array}{ll} h_i P_i / 2\varepsilon \, , & \text{if } P_i \geq 0 \, , \\ -h_{i+1} P_i / 2\varepsilon \, , & \text{if } P_i < 0 \, , \end{array} \right.$$

(3.5c) $L_c^h u_i = \varepsilon D_+ D_- u_i + P_i D_0 u_i \, ,$ (central scheme [24]),

(3.5d)
$$L_a^h u_i = \varepsilon D_+ D_- u_i + P_{i \pm \frac{1}{2}} D_\pm u_i \, , \qquad \text{(Gushchin-Shchennikow scheme [12])} \, .$$

Here we use the plus sign (minus sign) in $D_\pm$ if P_i is positive (negative).
- The Il'in scheme on the equidistant mesh I_e^h,

(3.6)
$$L_3^h u_i = \varepsilon \sigma_i D_+ D_- u_i + P_i D_0 u_i = 0 \, , \quad -N < i < N \, , \quad u_{-N} = u_l \, , \quad u_N = u_r \, ,$$

where the fitting factor σ_i is $\sigma_i \equiv \frac{P_i h}{2\varepsilon} \coth \frac{P_i h}{2\varepsilon}$.

The background and motivation behind these finite difference schemes and meshes can be found in [3], [13], [15], [24], [25], [31], and the references therein.

Numerical results for the coupled scheme computed using *double precision* arithmetic when $p(x) = 1$, $m = 0$, $u_l = -3$ and $u_r = 1$ are shown in Table 3.1. Since the exact solution $u(x)$ to (3.1) is known we can easily compute the error for different values of h and ε. The results in this table indicate that the coupled scheme is uniformly second order convergent in the parameter ε only when the number N of mesh points is not too large. On the contrary, for large N the convergence property of the scheme deteriorates, contradicting classical convergence theory that states that the numerical error of a convergent scheme will tend to zero as the mesh size $h \to 0$. Our explanation of this paradox is that the three schemes do converge uniformly in ε, but that the degeneration for large N is caused by the round-off errors of a computer due to the exponential ill-conditioning of the finite difference operators. To verify this explanation, we show the numerically computed minimum and maximum eigenvalues and the condition numbers of the coefficient matrixes of the coupled scheme in Table 3.2 for $\varepsilon = 0.02$ and for different mesh size h. Comparing the results in Table 3.2 with Table 3.1, we notice that the numerical convergence order γ begins to deviate from the analytical value of 2 only when the condition number reaches $O(10^{15})$, which is approximately the reciprocal of the (double) machine precision. Thus, we believe that

N	$\varepsilon = 0.03$		$\varepsilon = 0.02$		$\varepsilon = 0.01$	
	E^h_∞	γ	E^h_∞	γ	E^h_∞	γ
12	.2545e-1	2.41	.2184e-1	1.89	.1900e-1	2.04
24	.4789e-2	2.00	.5872e-2	2.15	.4615e-2	-7.77
48	.1199e-2	1.99	.1319e-2	1.87	1.01	-
96	.3017e-3	1.98	.3598e-3	0.22	-	-
192	.7659e-4	1.99	.3088e-3	-0.25	-	-
384	.1925e-4	1.79	.1466e-2	-1.10	-	-
768	.5558e-5	-0.76	.3136e-2	-0.97	-	-
1536	.9406e-5	-	.6162e-2	-	-	-

TABLE 3.1

Numerical results of the coupled scheme for (3.1) with $p(x) = 1$, $m = 0$, $u_l = -3$, and $u_r = 1$, using double precision arithmetic. Here, $E^h_\infty \equiv \max_i |u(x_i) - u_i|$ is the maximum error of the numerical solution on a mesh with $h = 1/N$, γ is the numerical convergence order defined by $\gamma = (\log E^h_\infty - \log E^{h/2}_\infty)/\log 2$. Note that from (2.3), the principal eigenvalue of the operator in (3.1) satisfies $\lambda_0 \sim 2.662 \times 10^{-7}$ for $\varepsilon = 0.03$, $\lambda_0 \sim 7.835 \times 10^{-11}$ for $\varepsilon = 0.02$, and $\lambda_0 \sim 1.539 \times 10^{-21}$ for $\varepsilon = 0.01$.

N	$\lambda_{\min}$	$\lambda_{\max}$	$Cond$
12	3.633e-7	778.0	4.866e9
24	3.116e-10	4.414e3	3.199e13
48	8.186e-11	2.141e4	6.314e14
96	4.450e-11	9.595e4	4.483e15
192	7.193e-11	4.113e5	1.159e16
384	1.068e-11	1.717e6	5.003e16

TABLE 3.2

Numerical results (using Matlab) of the minimum eigenvalue $\lambda_{\min}$, the maximum eigenvalue $\lambda_{\max}$, and the condition number Cond of the coefficient matrix of the coupled scheme (3.5) when $\varepsilon = 0.02$. The other parameters are the same as in Table 3.1.

this degeneration of the convergence can be avoided if we can calculate the discrete scheme more accurately. Based on this explanation, it is anticipated that numerical computations with quadruple precision arithmetic will yield better results than those with double precision. This is verified in Table 3.3, where first order convergence for the upwind scheme and second order convergence for the coupled scheme and Il'in scheme are obtained using quadruple precision arithmetic even for much larger N than those in Table 3.1.

In summary, these numerical experiments indicate that for an exponentially ill-conditioned problem, a truncation error τ^h may not lead to very large errors in the numerical solution u^h and a scheme may still be uniformly convergent with respect to ε. A rigorous analytical proof of the uniform convergence of the three schemes described above for (3.1) is given in [27]. However, a numerical method will usually inherit the ill-conditioning associated with the underlying continuous problem. This causes the peculiar phenomenon observed in the computations above that a small grid size N may give better numerical results than a larger value of N. To minimize the effects of round-off errors which may pollute the accuracy of a scheme, higher (such as quadruple) precision arithmetic is preferred in solving the discrete schemes. This heuristic explanation perhaps suggests why the quadruple precision computations of metastability in the viscous Cahn-Hilliard done in [23], while employing a standard

N	Upwind scheme (3.4)		Coupled scheme (3.5)		Il'in scheme (3.6)	
	E_∞^h	γ	E_∞^h	γ	E_∞^h	γ
48	.8204e-1	0.98	.1225e-2	1.89	.1815e-2	1.45
96	.4153e-1	0.99	.3310e-3	2.00	.6603e-3	1.97
192	.2084e-1	1.00	.8265e-4	2.00	.1675e-3	1.99
384	.1044e-1	1.00	.2064e-4	2.00	.4221e-4	1.97
768	.5222e-2	1.00	.5154e-5	2.00	.1075e-4	1.90
1536	.2611e-2	-	.1286e-5	-	.2875e-5	-

TABLE 3.3

Numerical results of the three schemes using quadruple precision arithmetic. Here, $\varepsilon = 0.01$, and the other parameters are the same as in Table 3.1.

finite difference scheme, was successful.

4. Ill-Conditioned Convection-Diffusion: Two Dimensions. We now generalize (2.1) to the case of two spatial dimensions. To this end we consider the following convection-diffusion equation in a bounded two-dimensional domain D with a smooth boundary ∂D:

$$(4.1a) \qquad u_t = \varepsilon \triangle u - \nabla \Psi \cdot \nabla u + \varepsilon^\nu g(\mathbf{x}) e^{-a/\varepsilon} u, \quad \mathbf{x} \in D, \quad t > 0,$$

$$(4.1b) \qquad u = u_b(\mathbf{x}), \quad \mathbf{x} \in \partial D; \qquad u(\mathbf{x}, 0) = u_0(\mathbf{x}).$$

Here $u = u(\mathbf{x}, t)$ with $\mathbf{x} = (x_1, x_2)$, ν and $a > 0$ are constants, $\varepsilon \to 0^+$, and $g(\mathbf{x})$, $u_b(\mathbf{x})$ and $u_0(\mathbf{x})$ are smooth. Let $\bar{D} = D \cup \partial D$. We assume that the potential $\Psi(\mathbf{x})$ has a unique global minimum on $\bar{D}$ at some interior point $\mathbf{x}_0 \in D$. At this minimum point we assume that

$$(4.2) \qquad \Psi(\mathbf{x}_0) = 0, \qquad \nabla \Psi(\mathbf{x}_0) = 0, \qquad H[\Psi(\mathbf{x}_0)] > 0,$$

where $H(\Psi) \equiv \Psi_{x_1 x_1} \Psi_{x_2 x_2} - \Psi_{x_1 x_2}^2$ is the Hessian. We also assume that $\nabla \Psi(\mathbf{x}) \neq 0$ for $\mathbf{x} \neq \mathbf{x}_0$ and that $\nabla \Psi \cdot \hat{n} > 0$ on ∂D, where $\hat{n}$ is the unit outward normal to ∂D.

In place of (2.2), the eigenvalue problem associated with (4.1) when $g(\mathbf{x}) \equiv 0$ is

$$(4.3a) \qquad L_\varepsilon \phi \equiv \varepsilon \triangle \phi - \nabla \Psi \cdot \nabla \phi = -\lambda \phi, \quad \mathbf{x} \in D; \qquad \phi = 0, \quad \mathbf{x} \in \partial D,$$

$$(4.3b) \qquad (\phi, \phi)_w \equiv \int_D \phi^2 w \, d\mathbf{x} = 1, \qquad w \equiv e^{-\Psi/\varepsilon}.$$

The eigenvalues λ_j for $j \geq 0$ are real with $\lambda_j > 0$ and $(\phi_j, \phi_k)_w = \delta_{jk}$ for $j, k = 0, 1, \ldots$

4.1. Calculation of the Principal Eigenvalue. We now asymptotically calculate the principal eigenvalue λ_0 as $\varepsilon \to 0$ for three different cases. The corresponding eigenfunction has the boundary layer form

$$(4.4) \qquad \phi_0 \sim M_0 \left(1 - e^{\gamma \eta / \varepsilon}\right), \qquad \text{where} \qquad \gamma = \gamma(s) \equiv \nabla \Psi \cdot \hat{n} \Big|_{\partial D} > 0,$$

and M_0 is a normalization constant. Here s denotes arclength along ∂D, $-\eta$ is the distance from $\mathbf{x} \in D$ to ∂D, and $\hat{n}$ is the unit outward normal to ∂D. Since $\gamma > 0$, then $\phi_0 \to M_0$ as $\eta/\varepsilon \to -\infty$. To calculate λ_0, we integrate by parts to obtain the following key identity valid for any u and v:

$$(4.5) \qquad (L_\varepsilon u, v)_w - (u, L_\varepsilon v)_w = \int_{\partial D} \varepsilon v e^{-\Psi/\varepsilon} \partial_n u \, ds - \int_{\partial D} \varepsilon u e^{-\Psi/\varepsilon} \partial_n v \, ds.$$

Here ∂_n denotes the outward normal derivative. Substituting $u = \phi_0$ and $v = 1$ into (4.5), we obtain

$$(4.6) \qquad \lambda_0 \left(\phi_0, 1\right) = -\int_{\partial D} \varepsilon e^{-\Psi/\varepsilon} \partial_n \phi_0 \, ds \,.$$

Next, since the dominant contribution to the left-hand side of (4.6) arises from the region near $\mathbf{x} = \mathbf{x}_0$ where Ψ has a non-degenerate minimum, we obtain for $\varepsilon \to 0$ that

$$(4.7) \qquad \left(\phi_0, 1\right)_w \sim 2\pi\varepsilon M_0 \left(H\left[\Psi(\mathbf{x}_0)\right]\right)^{-1/2} \,.$$

Here H is the Hessian of Ψ at $\mathbf{x}_0$ defined following (4.2) above. Substituting (4.4) and (4.7) into (4.6), we obtain the following estimate for λ_0 as $\varepsilon \to 0$:

$$(4.8) \qquad \lambda_0 \sim \frac{\left(H\left[\Psi(\mathbf{x}_0)\right]\right)^{1/2}}{2\pi\varepsilon} \int_{\partial D} e^{-\Psi/\varepsilon} \nabla\Psi \cdot \hat{n} \, ds \,.$$

Equation (4.8) shows that λ_0 is determined by the precise behavior of Ψ on ∂D.

Let $(x_1, x_2) = (x_1(s), x_2(s))$ be a parametric representation of ∂D where s denotes arclength on ∂D and $\tilde{\Psi}(s) = \Psi(x_1(s), x_2(s))$. In calculating λ_0 asymptotically, we distinguish three cases:

Case 1: Suppose that the minimum value of $\tilde{\Psi}$ on ∂D is taken at N distinct points $\mathbf{y}_j \in \partial D$, with arclength coordinate s_j, for $j = 1, .., N$. Assume that these minima are non-degenerate. Then, the principal eigenvalue λ_0 of (4.3) is exponentially small as $\varepsilon \to 0$ and has the asymptotic estimate

$$(4.9a) \qquad \lambda_0 \sim \left(\frac{H\left[\Psi(\mathbf{x}_0)\right]}{2\pi\varepsilon}\right)^{1/2} e^{-\Psi^*/\varepsilon} \sum_{j=1}^{N} |\nabla\Psi(\mathbf{y}_j)| \left(\tilde{\Psi}''(s_j)\right)^{-1/2} \,.$$

Here $\Psi^* \equiv \Psi(\mathbf{y}_j)$ for $j = 1, .., N$, and

(4.9b)
$$\tilde{\Psi}''(s_j) \equiv |\nabla\Psi|^{-2} \left[\Psi_{x_1 x_1} \Psi_{x_2}^2 - 2\Psi_{x_1 x_2} \Psi_{x_1} \Psi_{x_2} + \Psi_{x_2 x_2} \Psi_{x_1}^2 + \kappa_j |\nabla\Psi|^3\right]\bigg|_{\mathbf{x}=\mathbf{y}_j},$$

where $\kappa_j < 0$ is the curvature of ∂D at $\mathbf{y}_j$.

Case 2: Suppose that the minimum value $\Psi^\star$ of $\tilde{\Psi}$ on ∂D is achieved at the points $s_1, s_2, \ldots, s_N$ and that $\tilde{\Psi}(s) - \tilde{\Psi}(s_j) = d_j^{-2p}(s - s_j)^{2p}(1 + o(1))$ as $s \to s_j$ where $d_j > 0$ $(j = 1, 2, \ldots, N)$ and p is a positive integer. Then, in this case where we allow for the minima to be degenerate, we get for $\varepsilon \to 0$ that

$$(4.10) \qquad \lambda_0 \sim \frac{\Gamma(\frac{1}{2p})}{2\pi p} \varepsilon^{\frac{1}{2p}-1} \left(H[\Psi(\mathbf{x}_0)]\right)^{\frac{1}{2}} e^{-\Psi^\star/\varepsilon} \sum_{j=1}^{N} d_j |\nabla\Psi(\mathbf{x}_j)| \,.$$

Here $\mathbf{x}_j = (x_1(s_j), x_2(s_j))$ for $j = 1, 2, \ldots, N$.

Case 3: Suppose that the maximum value $\Psi^\star$ of Ψ on ∂D is achieved on a set $S \subset \partial D$, with a nonempty interior S^0 and that the measure of $S - S^0$ in ∂D is zero. Then, for $\varepsilon \to 0$, we have the asymptotic estimate

$$(4.11) \qquad \lambda_0 \sim \frac{1}{2\pi\varepsilon} \left(H[\Psi(\mathbf{x}_0)]\right)^{\frac{1}{2}} e^{-\Psi^\star/\varepsilon} \int_{S} |\nabla\Psi| ds \,.$$

4.2. Derivation of the Metastability Result. For $\varepsilon \to 0$, a leading order boundary layer approximation for the equilibrium solution $U(\mathbf{x}; \varepsilon)$ to (4.1) is given by

$$(4.12) \qquad U(\mathbf{x}; \varepsilon) \sim \tilde{u}^\varepsilon[\mathbf{x}; A_{0e}] \equiv A_{0e} + [u_b(s) - A_{0e}] e^{\gamma \eta / \varepsilon},$$

in an $O(\varepsilon)$ neighborhood near ∂D. Here we have written the boundary data u_b in terms of an arclength coordinate s. The constant A_{0e} in (4.12) is undetermined. This indeterminacy is caused by the exponential ill-conditioning of the elliptic operator in (4.1).

For the special case $g(\mathbf{x}) = 0$, methods to calculate A_{0e} were given in [11] and [18]. In this case, since the principal eigenvalue λ_0 is positive, the equilibrium solution to (4.1) is stable. Starting from any initial value $u_0(\mathbf{x})$, a quasi-equilibrium solution to (4.1) of the spatial form (4.12) will be obtained on an $O(1)$ time scale. Once this quasi-equilibrium solution has been formed, it will approach its stable equilibrium solution over an exponentially long time scale.

To study this metastable motion analytically for an arbitrary $g(\mathbf{x})$ we set $u(\mathbf{x}, t) = \tilde{u}^\varepsilon[\mathbf{x}, A_0(t)] + v(\mathbf{x}, t)$ in (4.1) and obtain that $v(\mathbf{x}, t)$ satisfies

$$(4.13a) \qquad v_t = L_\varepsilon v - \tilde{u}_t^\varepsilon + L_\varepsilon \tilde{u}^\varepsilon + \varepsilon^\nu g(\mathbf{x}) e^{-a/\varepsilon}(\tilde{u}^\varepsilon + v), \quad \mathbf{x} \in D, \quad t > 0,$$

$$(4.13b) \qquad v = u_b - \tilde{u}^\varepsilon, \quad \mathbf{x} \in \partial D; \quad v(\mathbf{x}, 0) = u_0(\mathbf{x}) - \tilde{u}^\varepsilon[\mathbf{x}; A_0(0)], \quad \mathbf{x} \in D.$$

We then expand v in terms of the eigenfunctions ϕ_j of (4.3) as

$$(4.14) \qquad v(\mathbf{x}, t) = \sum_{j=0}^\infty c_j(t) \phi_j(\mathbf{x}).$$

Using the othogonality properties of the eigenfunctions, we obtain that $c_j(t)$ satisfies

$$(4.15a)$$
$$c_j' + \lambda_j c_j = (\phi_j, L_\varepsilon \tilde{u}^\varepsilon)_w - \int_{\partial D} \varepsilon w v \, \partial_n \phi_j \, ds - (\phi_j, \tilde{u}_t^\varepsilon)_w + \varepsilon^\nu e^{-a/\varepsilon}(g\phi_j, \tilde{u}^\varepsilon + v)_w,$$

with the initial value

$$(4.15b) \qquad c_j(0) = \int_D (u_0(\mathbf{x}) - \tilde{u}^\varepsilon[\mathbf{x}; A_0(0)]) \phi_j w \, d\mathbf{x}.$$

Here ∂_n denotes the outward normal derivative to ∂D. Since $\lambda_0 > 0$ and is exponentially small, it is necessary that $c_0(t) \equiv 0$ in order to ensure that $v \ll \tilde{u}^\varepsilon$ over exponentially long time intervals. Thus we require that the right-hand sides of (4.15a) and (4.15b) vanish when $j = 0$. This leads to the following differential equation for $A_0(t)$

$$(4.16a) \qquad (\phi_0, \tilde{u}_t^\varepsilon)_w \sim (\phi_0, L_\varepsilon \tilde{u}^\varepsilon)_w - \int_{\partial D} \varepsilon w v \, \partial_n \phi_0 \, ds + \varepsilon^\nu e^{-a/\varepsilon}(g\phi_0, \tilde{u}^\varepsilon)_w,$$

together with the initial condition

$$(4.16b) \qquad \int_D \tilde{u}^\varepsilon[x; A_0(0)] \phi_0 w \, dx = \int_D u_0(x) \phi_0 w \, dx.$$

To obtain an explicit ordinary differential equation for $A_0(t)$, we use the form (4.12) to asymptotically evaluate the various terms in (4.16) for $\varepsilon \to 0$. We first use the identity (4.5) to calculate

$$(4.17) \qquad (\phi_0, L_\varepsilon \tilde{u}^\varepsilon)_w = -\lambda_0 (\tilde{u}^\varepsilon, \phi_0)_w - \int_{\partial D} \varepsilon \tilde{u}^\varepsilon w \, \partial_n \phi_0 \, ds .$$

Substituting (4.17) into (4.16a), we obtain

$$(4.18) \qquad (\phi_0, \tilde{u}_t^\varepsilon)_w \sim -\lambda_0 (\tilde{u}^\varepsilon, \phi_0)_w + \varepsilon^\nu e^{-a/\varepsilon} (\tilde{u}^\varepsilon, g\phi_0)_w - \int_{\partial D} \varepsilon e^{-\Psi/\varepsilon} u_b \, \partial_n \phi_0 \, ds .$$

Next, the inner products over D are readily evaluated using Laplace's method since the dominant contribution to each of them arises from the region near $\mathbf{x} = \mathbf{x}_0$. For $\varepsilon \to 0$, we calculate that

$$(4.19a) \qquad (\phi_0, \tilde{u}_t^\varepsilon)_w \sim 2\pi\varepsilon A_0' \left(H\left[\Psi(\mathbf{x}_0) \right] \right)^{-1/2} ,$$

$$(4.19b) \qquad (\tilde{u}^\varepsilon, \phi_0)_w \sim 2\pi\varepsilon A_0 \left(H\left[\Psi(\mathbf{x}_0) \right] \right)^{-1/2} ,$$

$$(4.19c) \qquad (\tilde{u}^\varepsilon, g\phi_0)_w \sim 2\pi\varepsilon A_0 g(\mathbf{x}_0) \left(H\left[\Psi(\mathbf{x}_0) \right] \right)^{-1/2} .$$

Here $A_0' \equiv dA_0/dt$. Substituting (4.19) into (4.18), and calculating the boundary integral in (4.18) asymptotically (as in the calculation of λ_0), we obtain that $A_0(t)$ satisfies the following limiting differential equation for $\varepsilon \to 0$:

$$(4.20a) \qquad A_0' \sim -(\lambda_0 - \varepsilon^\nu g(\mathbf{x}_0) e^{-a/\varepsilon}) A_0$$

$$+ \beta_1 e^{-\Psi^\star/\varepsilon} \sum_{j=1}^{N} u_b(s_j) | \nabla \Psi(\mathbf{x}_j)| \left(\tilde{\Psi}''(s_j) \right)^{-1/2} , \quad \text{in Case 1,}$$

$$(4.20b)$$

$$A_0' \sim -(\lambda_0 - \varepsilon^\nu g(\mathbf{x}_0) e^{-a/\varepsilon}) A_0 + \beta_2 e^{-\Psi^\star/\varepsilon} \sum_{j=1}^{N} u_b(s_j) | \nabla \Psi(\mathbf{x}_j)| d_j , \quad \text{in Case 2,}$$

$$(4.20c) \qquad A_0' \sim -(\lambda_0 - \varepsilon^\nu g(\mathbf{x}_0) e^{-a/\varepsilon}) A_0 + \beta_3 e^{-\Psi^\star/\varepsilon} \int_S u_b(s) | \nabla \Psi| \, ds , \quad \text{in Case 3,}$$

with initial value $A_0(0) \sim u_0(\mathbf{x}_0)$. The three cases are precisely those described in §4.1. Here, $\mathbf{x}_0$ is the global minimum of Ψ in $D \cup \partial D$, $\tilde{\Psi}''(s_j)$ is defined in (4.9b), and λ_0 is given in (4.9), (4.10) and (4.11) in Cases 1, 2, and 3, respectively. In addition, we have labeled

$$(4.21)$$

$$\beta_1 \equiv \left(\frac{H[\Psi(\mathbf{x}_0)]}{2\pi\varepsilon} \right)^{1/2} , \quad \beta_2 \equiv \frac{\Gamma(\frac{1}{2p})}{2\pi p} \varepsilon^{\frac{1}{2p}-1} \left(H[\Psi(\mathbf{x}_0)] \right)^{1/2} , \quad \beta_3 \equiv \frac{(H[\Psi(\mathbf{x}_0)])^{1/2}}{2\pi\varepsilon} ,$$

where H is the Hessian. Moreover, $\mathbf{x}_j \in \partial D$ for $j = 1, 2, \ldots, N$ are those points where Ψ is minimized on ∂D with minimum value $\Psi^\star \equiv \Psi(\mathbf{x}_j)$ for $j = 1, 2, \ldots, N$, and d_j defined by $\tilde{\Psi}(s) - \tilde{\Psi}(s_j) = d_j^{-2p}(1 + o(1))$ as $s \to s_j$ is a coefficient of the Taylor expansion of $\tilde{\Psi}(s) = \Psi(x_1(s), x_2(s))$ at a degenerate minimum $\mathbf{x}_j$.

In summary, for $\varepsilon \to 0$ and $t \gg 1$, the metastable dynamics for (4.1) is given by $u(\mathbf{x}, t) \sim \tilde{u}^\varepsilon[\mathbf{x}; A_0(t)]$ where $\tilde{u}^\varepsilon(\mathbf{x}; A_0)$ is defined by (4.12) and $A_0(t)$ satisfies (4.20).

If a is sufficiently large so that $\varepsilon^\nu g(x_0)e^{-\frac{a}{\varepsilon}} < \lambda_0$, then we obtain from (4.20) that $A_0(t) \to A_{0e}$ as $t \to \infty$, where

(4.22a)

$$A_{0e} \sim \sum_{j=1}^{N} u_b(s_j) \frac{|\nabla \Psi(\mathbf{x}_j)|}{\left[\tilde{\Psi}''(s_j)\right]^{1/2}} \left(\sum_{j=1}^{N} \frac{|\nabla \Psi(\mathbf{x}_j)|}{\left[\tilde{\Psi}''(s_j)\right]^{1/2}} - \frac{\varepsilon^\nu g(\mathbf{x}_0)}{\beta_1} e^{-(a-\Psi^\star)/\varepsilon} \right)^{-1} \quad \text{in Case 1},$$

(4.22b)

$$A_{0e} \sim \sum_{j=1}^{N} u_b(s_j) d_j |\nabla \Psi(\mathbf{x}_j)| \left(\sum_{j=1}^{N} d_j |\nabla \Psi(\mathbf{x}_j)| - \frac{\varepsilon^\nu g(\mathbf{x}_0)}{\beta_2} e^{-(a-\Psi^\star)/\varepsilon} \right)^{-1} \quad \text{in Case 2},$$

(4.22c) $\quad A_{0e} \sim \int_S u_b(s) |\nabla \Psi| \, ds \left(\int_S |\nabla \Psi| ds - \frac{\varepsilon^\nu g(\mathbf{x}_0)}{\beta_3} e^{-(a-\Psi^\star)/\varepsilon} \right)^{-1} \quad \text{in Case 3}.$

From (4.22) it is clear that A_{0e} is very sensitive with respect to changes in the value of a when $a \approx \Psi^\star$.

When $g(\mathbf{x}) \equiv 0$ these results for A_{0e} were obtaned in [18] using the formal variational method of [11] (see also [11] for specific examples). The results for the time-dependent and steady-state problems given in (4.20) and (4.22) when $g(\mathbf{x}) \neq 0$ are new.

4.3. Two Examples. We now consider two examples of the theory of §4.1 and §4.2.

EXAMPLE 4.1: Let $\mathbf{x} = (x, y)$ and let D be the ellipse $x^2/4 + y^2 = 1$. Let $\Psi(\mathbf{x}) = x^2 + y^2$. Then, the minimum of Ψ over $\bar{D}$ occurs at $\mathbf{x}_0 = \mathbf{0}$. At this point, we calculate $H[\Psi(\mathbf{x}_0)] = 4$ so that from (4.21) we get $\beta_1 = [2/(\pi\varepsilon)]^{1/2}$.

For this example, there are two points on ∂D that are closest to $\mathbf{x}_0 = \mathbf{0}$. They are located at $\mathbf{y}_1 = (0, 1)$ and $\mathbf{y}_2 = (0, -1)$. At these points, the curvatures κ_1 and κ_2 are $\kappa_1 = \kappa_2 = -1/4$. Then, from (4.9b), we calculate $\tilde{\Psi}''(s_j) = 3/2$ for $j = 1, 2$. Since $\Psi^\star = 1$, the estimate (4.9a) for λ_0 when $\varepsilon \ll 1$ reduces to

(4.23) $$\lambda_0 \sim 8 (3\pi\varepsilon)^{-1/2} e^{-1/\varepsilon}.$$

For $t \gg 1$, we obtain from (4.12) and (4.20) that

(4.24) $\quad u(\mathbf{x}, t) \sim A_0(t) + [u_b(s) - A_0(t)] e^{\gamma\eta/\varepsilon}, \qquad \gamma = \gamma(s) \equiv \nabla\Psi \cdot \hat{n} \Big|_{\partial D} > 0,$

where $A_0(t)$ satisfies the limiting differential equation

(4.25) $\quad A_0' \sim -8 (3\pi\varepsilon)^{-1/2} e^{-1/\varepsilon} \left[1 - \left(\frac{\sqrt{3\pi}}{8} \right) \varepsilon^{\nu+1/2} g(0) e^{-(a-1)/\varepsilon} \right] (A_0 - A_{0e}).$

From (4.22a), the steady-state solution is

(4.26) $\quad A_{0e} \sim \frac{1}{2} [u_b(\mathbf{y}_1) + u_b(\mathbf{y}_2)] \left[1 - \left(\frac{\sqrt{3\pi}}{8} \right) \varepsilon^{\nu+1/2} g(0) e^{-(a-1)/\varepsilon} \right]^{-1}.$

Now, from (4.25), we observe that $A_0(t) \to A_{0e}$ as $t \to \infty$ if and only if

(4.27) $$\left(\frac{\sqrt{3\pi}}{8} \right) \varepsilon^{\nu+1/2} g(0) e^{-(a-1)/\varepsilon} < 1.$$

Thus, if $\nu = -1/2$ and $a > 1$, then for $\varepsilon \to 0$ we have that $A_0(t) \to A_{0e}$ as $t \to \infty$ for any initial condition $A_0(0)$. However, if $\nu = -1/2$ and $0 < a < 1$, then $A_0(t) \to A_{0e}$ as $t \to \infty$ only when $g(\mathbf{0}) \leq 0$. In the borderline case where $a = 1$ and $\nu = -1/2$, then

$$(4.28) \qquad \lim_{t \to \infty} A_0(t) = A_{0e}, \qquad \text{when} \quad g(\mathbf{0}) < 8 \, (3\pi)^{-1/2} \, ,$$

and $A_0(t) \to \infty$ as $t \to \infty$ when $g(\mathbf{0}) > 8 \, (3\pi)^{-1/2}$. Notice that when $a > 1$, then

$$(4.29) \qquad A_{0e} \sim \frac{1}{2} \left[u_b(\mathbf{y}_1) + u_b(\mathbf{y}_2) \right] \, ,$$

is the average of the boundary data at the two closest points. This example clearly shows that $A_0(t)$ and A_{0e} depend supersensitively on a when $a \approx 1$ and $\nu = -1/2$.

ε	A_{0e} (asympotics)	A_{0e} (numerics)
0.15	0.47179	0.37496
0.14	0.47179	0.37727
0.13	0.47179	0.38061
0.12	0.47179	0.38436
0.11	0.47179	0.38879
0.10	0.47179	0.39431
0.09	0.47179	0.39943
0.08	0.47179	0.40421
0.07	0.47179	0.41240
0.06	0.47179	0.41274
0.05	0.47179	0.43011

TABLE 4.1

Numerical results (using PLTMG) of the constant A_{0e} for Example 4.2 for various ε when $a = 1/2$. The other parameter values are as given in Example 4.2.

EXAMPLE 4.2: Let $\mathbf{x} = (x, y)$ and D be the square $-1 < x < 1$, $-1 < y < 1$. We let $\Psi = (x^2 + y^2)/2$, $g(\mathbf{x}) \equiv -1$, and $\nu = -1/2$. Then, there are four points on ∂D that are closest to the minimum point $\mathbf{x}_0 = 0$ of Ψ. These points are labeled by $\mathbf{y}_1 = (1, 0)$, $\mathbf{y}_2 = (-1, 0)$, $\mathbf{y}_3 = (0, 1)$, and $\mathbf{y}_4 = (0, -1)$. We calculate $|\nabla \Psi|$ and $\widetilde{\Psi}''(s_j) = 1$ at each of these boundary points. Also, $\Psi^* = 1/2$ and $H[\Psi(\mathbf{0})] = 1$, so that from (4.21), $\beta_1 = (2\pi\varepsilon)^{-1/2}$. Thus, from (4.9a), (4.20), and (4.22a), we obtain that

$$(4.30a) \qquad \lambda_0 \sim 4 \, (2\pi\varepsilon)^{-1/2} \, e^{-\varepsilon^{-1}/2} \, ,$$

$$(4.30b) \qquad A_{0e} \sim \frac{\sum_{j=1}^{4} u_b(\mathbf{y}_j)}{4} \left[1 + \frac{\sqrt{2\pi}}{4} e^{-(a-1/2)/\varepsilon} \right] \, ,$$

$$(4.30c) \qquad A_0' \sim -4 \, (2\pi\varepsilon)^{-1/2} \, e^{-\varepsilon^{-1}/2} \left[1 + \frac{\sqrt{2\pi}}{4} e^{-(a-1/2)/\varepsilon} \right] (A_0 - A_{0e}) \, .$$

Finally, we took the boundary data $u_b = \sin^2 (2x + y)$.

For this example, we used the finite element package PLTMG [5] to obtain full numerical results for the steady-state value A_{0e}. When $a = 1/2$, the numerical results

for A_{0e} computed from PLTMG are compared in Table 4 for various values of ε with the corresponding asymptotic result (4.30b). We observe that the asymptotic and numerical results become closer only when ε is rather small. We expect that when $\varepsilon < 0.05$ the asymptotic and numerical results for A_{0e} would compare more favorably than for the larger values of ε shown in Table 4. However, we were unable to verify this prediction conclusively since, due to the exponentially small eigenvalue λ_0, PLTMG failed to converge when $\varepsilon < 0.05$. Finally, in Table 5, where we chose $\varepsilon = 0.07$, the asymptotic and numerical results for A_{0e} are compared for different values of ν and a. We would expect a more favorable agreement between the asymptotic and numerical results if we were able to compute solutions with PLTMG at a smaller value of ε. However, Table 5 does indeed show the supersensitivity of A_{0e} to ν and a when ε is small.

(ν, a)	A_{0e} (asympotics)	A_{0e} (numerics)
(-0.5,.75)	0.754160	0.687259
(-0.5,1.0)	0.767067	0.701080
(-0.5,1.25)	0.767437	0.701101
(-0.5,2.0)	0.767448	0.701112
(-0.5,4.0)	0.767448	0.701138
(-.25,0.5)	0.580374	0.512519
(0.0, 0.5)	0.658303	0.589825
(0.5, 0.5)	0.735197	0.667597
(1.0, 0.5)	0.758643	0.691938
(2.0, 0.5)	0.766825	0.701273
(4.0, 0.5)	0.767445	0.701261

TABLE 4.2

Numerical results (using PLTMG) of the constant A_{0e} for Example 4.2 for various a and ν when $\varepsilon = 0.7$. The other parameter values are as given in Example 4.2.

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