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VORTEX DYNAMICS FOR THE GINZBURG-LANDAU EQUATION WITH NEUMANN CONDITION

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Abstract. We study the Ginzburg-Landau equation, $u_t = \Delta u + (1/\epsilon^2)(1-|u|^2)u, u = (u_1, u_2)$, in a planar contractible domain with Neumann boundary condition, where $\epsilon > 0$ is a small parameter. We construct a finite-dimensional manifold of a family of approximate solutions and consider the dynamics near the manifold. If a solution lies in a sufficiently small neighborhood of the manifold, we can derive the dynamics projected on the manifold. Then the equation describing this dynamics provides a motion law of zeros to the approximate solutions and it approximates the dynamics of zeros (called vortices) of the original solution.

1. Introduction. We devote to the Ginzburg-Landau equation in a bounded domain Ω of $\mathbb{R}^2$ with the Neumann boundary condition:

$$u_t = \Delta u + \frac{1}{\epsilon^2}(1-|u|^2)u = 0, \quad (x,t) \in \Omega \times (0,\infty), \tag{1.1}$$

$$\frac{\partial u}{\partial \nu} = 0, \qquad (x,t) \in \partial\Omega \times (0,\infty), \tag{1.2}$$

where $u = (u_1(x,t), u_2(x,t))^T$, ϵ is a small positive parameter and $\partial/\partial\nu$ denotes the outer normal derivative on the smooth boundary $\partial\Omega$. This equation is a gradient equation of the following energy functional

$$E_\epsilon(u) := \frac{1}{2}\int_\Omega \left\{ |\nabla u|^2 + \frac{1}{2\epsilon^2}(1-|u|^2)^2 \right\} dx \tag{1.3}$$

in $H^1(\Omega;\mathbb{R}^2)$. Moreover the Euler equation of $E_\epsilon(u)$ is a system of elliptic equations with the Neumann boundary condition

$$\Delta u + \frac{1}{\epsilon^2}(1-|u|^2)u = 0, \quad x \in \Omega, \tag{1.4}$$

$$\frac{\partial u}{\partial \nu} = 0, \qquad x \in \partial\Omega, \tag{1.5}$$

whose solution gives an equilibrium state of (1.1). Throughout the paper we assume that the domain Ω is contractible. We also allow the complex expression $u = u_1(x,t) + iu_2(x,t)$ for a solution $u = (u_1(x,t), u_2(x,t))^T$ to (1.1).

Since a solution $u(x,t)$ of (1.1) has two components, a vector field on Ω can be defined by the solution for each time $t > 0$ and the zero set of the solution at time t is consist of discrete points in generic. Note that the degree of each zero y can be defined by $\deg(y, \partial B_\rho(y))$, where $B_\rho(y)$ is a disk with a small radius ρ. Zeros of the equation (1.1) (or (1.4)) are called vortices which play an important role for characterizing some dynamical behavior of the solution (refer to the introduction of

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[16]). Therefore the dynamics of vortices has been extensively studied by physicists and mathematicians, for instance see [6], [12], [19], [20], [21], [22], [23], [26].

Among other things, as for the equation (1.1) with the Dirichlet boundary data

$$u = g(x), \quad x \in \partial\Omega, \quad |g| = 1, \quad \deg(g; \partial\Omega) = d, \quad d \in \mathbb{Z} \setminus \{0\} \tag{1.6}$$

(hereafter we assume $d > 0$ for simplicity of notation), elaborate results were proved by Lin ([19, 20]) and Jerrard-Soner ([12]). They derived, taking a limit of a subsequence $\epsilon_n \to 0$ after time scaling $s = t/\log(1/\epsilon)$, a limit equation describing the motion law of the vortices. If we let $(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_d)$ be a configuration of the vortices, the limit equation is written as

$$\frac{d}{ds}(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_d) = -\frac{1}{\pi}\text{grad } W(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d), \tag{1.7}$$

where $W(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d)$ is the renormalized energy given by [2]. (Note that Rubinstein-Sternberg [26] first showed rigorously that the scaling $s = t/\log(1/\epsilon)$ works in the study of vortex dynamics.)

On the other hand for the Neumann problem, (1.1)-(1.2), Lin [21] derived the limit equation as done for the Dirichlet case by defining the renormalized energy appropriately. Although the renormalized energy is not explicitly given there (except for the special case, $\Omega = \{|x| < 1\}$ and d=1), it suggests important dynamical properties of vortices, for example, the annihilation of two vortices with opposite signs of degree.

To investigate the dynamics of the limit equation extensively, we need more information on the limit equation. A recent study in [16] provides an explicit form of the limit equation for the Neumann problem. In fact, after deriving the limit equation by applying the argument of [12] to the Neumann case, we succeeded to rewrite it with a Green function and Robin function of it. We present it here. Let the configuration of m-vortices be denoted by

$$(y^{(1)}(s), \cdots, y^{(m)}(s)) \in \hat{\Omega}^m := \Omega \times \cdots \times \Omega.$$

Then the limit equation can be written as follows:

$$\frac{d}{ds}y^{(j)} = \nabla S(y^{(j)}) + 2\sum_{k \neq j}^{m} d_j d_k \nabla_x G(y^{(j)}, y^{(k)}) \qquad (j = 1, \ldots, m), \tag{1.8}$$

where time s is scaled as in (1.7), d_j ($= 1$ or -1) denotes degree of the vortex $y^{(j)}$, $G(x, y)$ is the Green function of Δ with Dirichlet condition and $S(x)$ is the Robin function of it. With the aid of this form some dynamical properties (existence or nonexistence of equilibria, collision of vortices etc.) of the limit equation are also shown in [16].

In this paper we study the dynamics of solutions to (1.1)-(1.2) for sufficiently small but positive ϵ and show that the motion law of the vortices can be approximated by the limit equation under an appropriate condition. In fact given positive $\rho_0 > 0$, we construct a family of approximate solutions

$$\begin{aligned} &u = u_\epsilon(x; \boldsymbol{y}), \quad \boldsymbol{y} = (y^{(1)}, \cdots, y^{(m)}) \in Y, \\ &Y := \{\boldsymbol{y} \in \hat{\Omega}^m : |y^{(j)} - y^{(k)}| \geq 2\rho_0, \ j \neq k, \quad \text{dist}(y^{(j)}, \partial\Omega) \geq \tfrac{3}{2}\rho_0\}. \end{aligned} \tag{1.9}$$

Then we obtain a $2m$-dimensional submanifold

$$\mathcal{M}_\epsilon := \{u = u(\cdot; \boldsymbol{y}) : \boldsymbol{y} \in Y\},$$

in the space of continuous functions $C^0(\overline{\Omega};\mathbb{R}^2)$. We will prove that for a positive function $\delta(\epsilon)=o(\epsilon)$, there is a C^1 function $\boldsymbol{y}_\epsilon(\cdot)$ from the neighborhood of $\mathcal{M}_\epsilon$

$$\mathcal{U}(\mathcal{M}_\epsilon)=\{u\in C^0(\overline{\Omega};\mathbb{R}^2):\|u-u_\epsilon(\cdot;\boldsymbol{y})\|_{C^0(\overline{\Omega})}<\delta(\epsilon),\ \boldsymbol{y}\in Y\}$$

into Y such that for a solution $u(x,t)\in\mathcal{U}(\mathcal{M}_\epsilon)$, $\boldsymbol{y}(t)=\boldsymbol{y}_\epsilon(u(\cdot,t))$ satisfies

$$\log(1/\epsilon)\frac{d}{dt}y^{(j)}=-\frac{1}{\pi}\frac{\partial}{\partial y^{(j)}}V(\boldsymbol{y})+o(1), \tag{1.10}$$

$$V(\boldsymbol{y}):=-\pi\sum_{k=1}^{m}S(y^{(k)})-\pi\sum_{k=1}^{m}\sum_{j\neq k}G(y^{(j)},y^{(k)}). \tag{1.11}$$

Hence by scaling $s=t/\log(1/\epsilon)$, the leading term of this equation agrees with (1.8). To obtain the above result, we need to verify

$$E_\epsilon(u_\epsilon(\cdot,\boldsymbol{y}))=-\pi\sum_{k=1}^{m}S(y^{(k)})-\pi\sum_{k=1}^{m}\sum_{j\neq k}G(y^{(j)},y^{(k)})+o(1), \tag{1.12}$$

$$\frac{\partial}{\partial y^{(j)}}E_\epsilon(u_\epsilon(\cdot,\boldsymbol{y}))=-\frac{\partial}{\partial y^{(j)}}V(\boldsymbol{y})+o(1). \tag{1.13}$$

In fact once we obtained these relations, we can derive the above motion law near the manifold by applying the idea found in [7].

We note that one can find many papers on the dynamics of interfaces and spikes for a class of reaction-diffusion equations including Allen-Cahn equation and Gierer-Meinhardt equation (for instance see [1], [4], [9], [8], [10], and the references therein). It is also interesting that a similar form of the energy (1.12) is addressed in [27], where it is shown that a critical point of the energy gives a locus of spikes to a spike solution. In many cases of the reaction-diffusion equations it can be proved that normal direction of manifolds for approximate solutions possesses the attractivity; thus the dynamics near the manifold is controlled by that on the manifold. Such attractivity could be proved by solving eigenvalue problem of the linearized operator at the approximate solutions or the comparison method for the scalar equation.

On the other hand for the Ginzburg-Landau equation it is not so easy to handle the eigenvalue problem of the linearized operator for vortex solutions. Actually we only have a few results for it (see [15], [23], [24]) and we can't apply them to the present case. Therefore we need to study further to prove that the solution remain in a neighborhood of the manifold for a long time. Although the study developed in this article is not complete for this reason, we believe that it would work in the future study.

This paper is organized as follows. In §2 given prescribed distinct points $y^{(j)}, 1\le j\le m$, we consider a harmonic map from $\Omega\setminus\{y^{(1)},\cdots,y^{(m)}\}$ into $S^1\subset\mathbb{C}$ with Neumann condition. This map has singularity at each $y^{(j)}$. We compute the energy of the map in a domain with holes around the points $y^{(j)}, 1\le j\le m$. On the other hand in §3 we compute the energy of the symmetric vortex solution in a small disk centered at the zero of the solution. In §4 and §5 we investigate an approximate solution parametrized by $\{y^{(1)},\cdots,y^{(m)}\}$, which denote the locus of zeros, and obtain (1.12) and (1.13). By combination of those results we derive (1.10)-(1.11) describing the dynamics near the manifold of the approximate solution in §6.

2. Energy of the harmonic map with singularity. Given $p \in \Omega$, let $\varphi(x;p)$ be a solution of

$$\text{(2.1)} \qquad \begin{cases} \Delta_x \varphi = 0, & x \in \Omega, \\ \dfrac{\partial \varphi}{\partial \nu_x} = -\langle \nu_x, \nabla_x \mathrm{Arg}(x-p) \rangle, & x \in \partial\Omega, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of vectors in $\mathbb{R}^2$. Although $\mathrm{Arg}(x-p)$ is multi-valued function with a singularity at $x = p$, each representative satisfies

$$\text{(2.2)} \qquad \nabla_x \mathrm{Arg}(x-p) = \frac{(x-p)^\perp}{|x-p|^2}, \qquad (x-p)^\perp := (-(x_2-p_2), (x_1-p_1)).$$

In fact $\nabla_x \mathrm{Arg}(x-p)$ is well-defined as a single-valued vector function (see [17]). By the condition

$$\text{(2.3)} \qquad \int_{\partial\Omega} \langle \nu_x, \nabla_x \mathrm{Arg}(x-p) \rangle dS = 0,$$

(2.1) has a solution which is unique up to additive constants (see [16]).

We denote a disk with radius ρ centered at $x = \boldsymbol{a}$ by

$$B_\rho(\boldsymbol{a}) := \{|x - \boldsymbol{a}| < \rho\},$$

and denote the degree of a function $u = (u_1(x), u_2(x))^T$ around $x = \boldsymbol{a}$ by $\deg(u(\cdot); \partial B_\rho(\boldsymbol{a}))$. Let

$$\text{(2.4)} \quad \hat{\Omega}^m := \{(y^{(1)}, \ldots, y^{(m)}) \in R^{2m} : y^{(j)} \in \Omega \ \ (1 \le j \le m), \ y^{(j)} \ne y^{(k)} \ \ (j \ne k)\}.$$

Given $\boldsymbol{y} = (y^{(1)}, \ldots, y^{(m)}) \in \hat{\Omega}^m$, define

$$\text{(2.5)} \qquad \Theta(x; \boldsymbol{y}) := \sum_{j=1}^m d_j (\mathrm{Arg}(x - y^{(j)}) + \varphi(x; y^{(j)})),$$

and

$$\text{(2.6)} \qquad u_h(x; \boldsymbol{y}) = \exp(i\Theta(x; \boldsymbol{y})),$$

where $d_j = 1$ or -1. Note that $d_j = \deg(u_h(x; \boldsymbol{y}), \partial B_\rho(y^{(j)}))$. We easily check that

$$\text{(2.7)} \quad \mathrm{div}(\nabla\Theta) = 0, \quad x \in \Omega, x \ne y^{(j)} (1 \le j \le m), \quad \frac{\partial \Theta}{\partial \nu} = 0, \quad x \in \partial\Omega,$$

$$\text{(2.8)} \quad \Delta u_h + |\nabla u_h|^2 u_h = 0, \quad x \in \Omega, x \ne y^{(j)} (1 \le j \le m), \quad \frac{\partial u_h}{\partial \nu} = 0, \quad x \in \partial\Omega.$$

Hence u_h is a harmonic map from $\Omega \setminus \{y^{(1)}, \cdots, y^{(m)}\}$ into $S^1 \subset \mathbb{C}$, having singularity at $x = y^{(j)}, j = 1, \ldots, m$.

We introduce the conjugate harmonic function $S(x,p)$ to $\varphi(x;p)$. The Cauchy-Riemann equation implies

$$\text{(2.9)} \qquad \nabla_x S(x,p)^\perp = \nabla_x \varphi(x;p).$$

Then a solution to the elliptic equation

$$\Delta_x S = 0, \quad x \in \Omega, \qquad S = -\log|x-p|, \quad x \in \partial\Omega \tag{2.10}$$

gives the conjugate harmonic function. Indeed it follows from the facts that the domain is contractible and that the solution $S = S(x,p)$ to (2.10) admits the boundary condition

$$\frac{\partial}{\partial \nu_x} S^{\perp} = -\langle \tau, \nabla_x S \rangle = \frac{\langle \tau, (x-p) \rangle}{|x-p|^2} = -\frac{\langle \nu, (x-p)^{\perp} \rangle}{|x-p|^2} = \frac{\partial}{\partial \nu_x} \varphi,$$

where τ denotes the unit tangential vector on $\partial\Omega$.

Set

$$H(x,p) := \mathrm{Arg}(x-p) + \varphi(x;p). \tag{2.11}$$

Then we have the following lemma:

LEMMA 2.1. *Given $p \in \Omega$, define*

$$G(x,p) := \log|x-p| + S(x,p), \qquad x \neq p, \ x \in \Omega. \tag{2.12}$$

Then

$$\nabla_x G(x,p)^{\perp} = \nabla_x H(x,p), \quad x \neq p, \ x \in \Omega, \tag{2.13}$$

and with an appropriate additive constant to $G(x,p)$,

$$\begin{cases} \Delta_x G = 0, & x \in \Omega \setminus \{p\}, \\ G = 0, & x \in \partial\Omega, \\ G(x,p) \sim \log|x-p| + O(1), & x \approx p, x \neq p \end{cases} \tag{2.14}$$

hold, namely $G(x,p)$ is the Green function of the Laplacian Δ with Dirichlet (zero) boundary condition.

The next proposition is the main result of the present section.

PROPOSITION 2.2. *Let ρ be a positive number satisfying*

$$\rho < \min\{\frac{1}{2} \min_{1 \leq j \leq m} \mathrm{dist}(\partial\Omega, y^{(j)}), \ \frac{1}{3} \min_{1 \leq j,k \leq m, j \neq k} |y^{(j)} - y^{(k)}|\}.$$

Set $\Omega_\rho := \Omega \setminus (\cup_{i=1}^{m} \overline{B_\rho(y^{(j)})})$ and

$$E_\rho(u_h) := \frac{1}{2} \int_{\Omega_\rho} |\nabla_x u_h(x; \boldsymbol{y})|^2 dx. \tag{2.15}$$

Then

$$E_\rho(u_h) = -m\pi \log \rho + V(\boldsymbol{y}) + O(\rho) \tag{2.16}$$

where

$$V(\boldsymbol{y}) := -\pi \sum_{j=1}^{m} S(y^{(j)}, y^{(j)}) - \pi \sum_{k=1}^{m} \sum_{j \neq k} d_j d_k G(y^{(j)}, y^{(k)}) \tag{2.17}$$

To prove this proposition we prepare two lemmas.

LEMMA 2.3.

(i) *$E_\rho(u_h)$ in* (2.15) *satisfies*

$$(2.18)\quad \begin{aligned} E_\rho(u_h) = & -\frac{1}{2}\sum_{k=1}^{m}\left\{\int_{\partial B_\rho(y^{(k)})} G(x,y^{(k)})\frac{\partial}{\partial \nu_x}G(x,y^{(k)})d\sigma \right. \\ & \left. +\sum_{j,\ell}{}' d_j d_\ell \int_{\partial B_\rho(y^{(k)})} G(x,y^{(j)})\frac{\partial}{\partial \nu_x}G(x,y^{(\ell)})d\sigma \right\}, \end{aligned}$$

where $\sum_{j,\ell}'$ *denotes the summation over indices,* $1 \le j,\ell \le m, j \ne k$ *or* $\ell \ne k$.

(ii) *Define*

$$(2.19)\qquad F_\rho^{(k)} := \frac{1}{2}\int_{B_\rho(y^{(k)})}\left(|\nabla u_h|^2 - \frac{1}{|x-y^{(k)}|^2}\right)dx.$$

Then

$$(2.20)\qquad F_\rho^{(k)} = -\pi S(y^{(k)},y^{(k)}) - \pi\sum_{j\ne k} d_k d_j G(y^{(j)},y^{(k)})$$

$$+\frac{1}{2}\left\{\int_{\partial B_\rho(y^{(k)})} S(x,y^{(k)})\frac{\partial}{\partial \nu_x}\log|x-y^{(k)}|d\sigma + G(x,y^{(k)})\frac{\partial}{\partial \nu_x}S(x,y^{(k)})d\sigma\right\}$$

$$+\sum_{j,\ell}{}' d_j d_\ell \int_{\partial B_\rho(y^{(k)})} G(x,y^{(j)})\frac{\partial}{\partial \nu_x}G(x,y^{(\ell)})d\sigma,$$

where $\sum_{j,k}'$ *is as in* (i).

LEMMA 2.4. *The identity*

$$(2.21)\qquad E_\rho(u_h) + \sum_{k=1}^{m} F_\rho^{(k)} = -m\pi\log\rho + V(\boldsymbol{y})$$

holds.

Before proving Lemmas 2.3, 2.4, we prove Proposition 2.2 with the aid of Lemma 2.4.
Proof of Proposition 2.2 : We see that there is a positive constant such that

$$\left||\nabla u_h|^2 - \frac{1}{|x-y^{(k)}|^2}\right| \le \frac{C_1}{|x-y^{(k)}|}, \qquad x \in B_\rho(y^{(k)}),$$

where C_1 is independent of ρ. Therefore we obtain

$$\int_{B_\rho(y^{(k)})}\left||\nabla u_h|^2 - \frac{1}{|x-y^{(k)}|^2}\right| \le 2\pi C_1\int_0^\rho \frac{1}{r}rdr = 2\pi C_1\rho$$

from which

$$|F_\rho^{(k)}| \le C\rho$$

follows. Hence we obtain the assertion of the proposition by Lemma 2.4. □

Now we prove Lemmas 2.3 and 2.4 in the rest of this section. We first prove Lemma 2.4.

Proof of Lemma 2.4 : We can verify

$$\begin{aligned}
&\int_{\partial B_\rho(y^{(k)})} G(x,y^{(k)})\frac{\partial}{\partial\nu_x}G(x,y^{(k)})d\sigma \\
&= \int_{\partial B_\rho(y^{(k)})}\left(\log|x-y^{(k)}|\frac{\partial}{\partial\nu_x}\log|x-y^{(k)}| + S(x,y^{(k)})\frac{\partial}{\partial\nu_x}\log|x-y^{(k)}|\right)d\sigma \\
&\qquad + \int_{\partial B_\rho(y^{(k)})} G(x,y^{(k)})\frac{\partial}{\partial\nu_x}S(x,y^{(k)})d\sigma, \\
&= 2\pi\log\rho + \int_{\partial B_\rho(y^{(k)})} S(x,y^{(k)})\frac{\partial}{\partial\nu_x}\log|x-y^{(k)}|d\sigma \\
&\qquad + \int_{\partial B_\rho(y^{(k)})} G(x,y^{(k)})\frac{\partial}{\partial\nu_x}S(x,y^{(k)})d\sigma.
\end{aligned}$$

Here we computed

$$\int_{\partial B_\rho(y^{(k)})}\log|x-y^{(k)}|\frac{\partial}{\partial\nu_x}\log|x-y^{(k)}|d\sigma = \int_0^{2\pi}\frac{\log\rho}{\rho}\rho d\theta = 2\pi\log\rho.$$

Using (2.18),(2.19) and the above indentity, we obtain (2.21). □

Proof of Lemma 2.3: We first prove (2.18). Put

$$\psi^{(j)} := d_j(\mathrm{Arg}(x-y^{(j)}) + \varphi(x,y^{(j)})).$$

Then

$$E_\rho(u_h) = \frac{1}{2}\int_{\Omega_\rho}\left|\sum_{j=1}^m \nabla_x\psi^{(j)}\right|^2 dx = \frac{1}{2}\int_{\Omega_\rho}\sum_{j=1}^m\sum_{\ell=1}^m \nabla_x\psi^{(j)}\cdot\nabla_x\psi^{(\ell)}dx$$

By Lemma 2.1 we have

$$\nabla_x\psi^{(j)}\cdot\nabla_x\psi^{(\ell)} = d_jd_\ell\nabla_xG(x,y^{(j)})\cdot\nabla_xG(x,y^{(\ell)}),$$

therefore

$$\begin{aligned}
(2.22)\quad E_\rho(u_h) &= \frac{1}{2}\sum_{j=1}^m\sum_{\ell=1}^m d_jd_\ell\int_{\Omega_\rho}\nabla_xG(x,y^{(j)})\cdot\nabla_xG(x,y^{(\ell)})dx \\
&= \frac{1}{2}\sum_{j=1}^m\sum_{\ell=1}^m d_jd_\ell\left(-\sum_{k=1}^m\int_{\partial B_\rho(y^{(k)})}G(x,y^{(j)})\frac{\partial}{\partial\nu_x}G(x,y^{(\ell)})d\sigma\right) \\
&= -\frac{1}{2}\sum_{k=1}^m\sum_{j=1}^m\sum_{\ell=1}^m\int_{\partial B_\rho(y^{(k)})}G(x,y^{(j)})\frac{\partial}{\partial\nu_x}G(x,y^{(\ell)})d\sigma
\end{aligned}$$

which agrees with (2.18).

Next we prove (2.20). We put

$$
(2.23)\quad \begin{cases} F_\rho^{(k)} = J_1^{(k)} + J_2^{(k)}, \\ J_1^{(k)} := \dfrac{1}{2}\displaystyle\int_{B_\rho(y^{(k)})} \{2\nabla_x \mathrm{Arg}(x-y^{(k)})\cdot \nabla\varphi(x;y^{(k)}) + |\nabla_x\varphi(x;y^{(k)})|^2\}dx, \\ J_2^{(k)} := \dfrac{1}{2}\displaystyle\int_{B_\rho(y^{(k)})} (|\nabla_x\Theta|^2 - |\nabla_x\psi^{(k)}|^2)dx \end{cases}
$$

In terms of (2.2) and (2.9) we can write

$$
\begin{aligned}
J_1^{(k)} &= \frac{1}{2}\int_{B_\rho(y^{(k)})} \{2\nabla_x \log|x-y^{(k)}|\cdot\nabla_x S(x,y^{(k)}) + |\nabla_x S(x,y^{(k)})|^2\}dx \\
&= \frac{1}{2}\int_{B_\rho(y^{(k)})} \{\nabla_x \log|x-y^{(k)}|\cdot\nabla_x S(x,y^{(k)}) + \nabla_x G(x,y^{(k)})\cdot\nabla_x S(x,y^{(k)})\}dx.
\end{aligned}
$$

We easily check

$$
(2.24)\quad \begin{aligned} J_1^{(k)} = -\pi S(y^{(k)},y^{(k)}) + \frac{1}{2}\Bigg\{&\int_{\partial B_\rho(y^{(k)})} S(x,y^{(k)})\frac{\partial}{\partial\nu_x}\log|x-y^{(k)}|d\sigma \\ &+\int_{\partial B_\rho(y^{(k)})} G(x,y^{(k)})\frac{\partial}{\partial\nu_x}S(x,y^{(k)})d\sigma\Bigg\}. \end{aligned}
$$

Next we compute $J_2^{(k)}$. Let

$$
\Psi^{(k)} := \sum_{j\neq k}\psi^{(j)}.
$$

Then

$$
(2.25)\quad \begin{aligned}
J_2^{(k)} &= \frac{1}{2}\int_{B_\rho(y^{(k)})} (2\nabla_x\psi^{(k)}\cdot\nabla_x\Psi^{(k)} + |\nabla_x\Psi^{(k)}|^2)dx \\
&= \frac{1}{2}\int_{B_\rho(y^{(k)})} \sum_{j\neq k} 2d_k d_j \nabla_x G(x,y^{(k)})\cdot\nabla_x G(x,y^{(j)})dx \\
&\qquad + \frac{1}{2}\int_{B_\rho(y^{(k)})}\sum_{j\neq k}\sum_{\ell\neq k} d_j d_\ell \nabla_x G(x,y^{(j)})\cdot\nabla_x G(x,y^{(\ell)})dx
\end{aligned}
$$

Note that

$$
(2.26)\quad \begin{aligned} &\int_{B_\rho(y^{(k)})} \nabla_x G(x,y^{(k)})\cdot\nabla_x G(x,y^{(j)})dx \\ &= -2\pi G(y^{(k)},y^{(j)}) + \int_{\partial B_\rho(y^{(k)})} \frac{\partial}{\partial\nu_x}G(x,y^{(k)})G(x,y^{(j)})d\sigma, \end{aligned}
$$

$$
(2.27)\quad \int_{B_\rho(y^{(k)})} \nabla_x G(x,y^{(k)})\cdot\nabla_x G(x,y^{(j)})dx = \int_{\partial B_\rho(y^{(k)})} G(x,y^{(k)})\frac{\partial}{\partial\nu_x}G(x,y^{(j)})d\sigma,
$$

and for $j, \ell \neq k$,

$$\int_{B_\rho(y^{(k)})} \nabla_x G(x, y^{(j)}) \cdot \nabla_x G(x, y^{(\ell)}) dx = \int_{\partial B_\rho(y^{(k)})} G(x, y^{(j)}) \frac{\partial}{\partial \nu_x} G(x, y^{(\ell)}) d\sigma \tag{2.28}$$

holds. Applying (2.26), (2.27) and (2.28) to (2.25) yields

$$\begin{aligned} J_2^{(k)} &= -\pi \sum_{j \neq k} d_k d_j G(y^{(j)}, y^{(k)}) \\ &\quad + \frac{1}{2} \sum_{j,\ell}{}' d_j d_\ell \int_{\partial B_\rho(y^{(k)})} G(x, y^{(j)}) \frac{\partial}{\partial \nu_x} G(x, y^{(\ell)}) d\sigma \end{aligned} \tag{2.29}$$

Adding (2.24) to (2.29), we obtain (2.20). □

REMARK 2.5. Recall that the function $S(x) := S(x, x)$ is called a Robin function. Hence in the expressions of (2.17) and (2.20) we can use this Robin function.

3. Energy of the symmetric vortex solution. We let $f_\infty(s)$ be a unique solution to

$$\begin{cases} \dfrac{d^2}{ds^2} f + \dfrac{1}{s} \dfrac{d}{ds} f - \dfrac{1}{s^2} f + (1 - f^2) f = 0, \quad 0 < s < \infty, \\ f(0) = 0, \quad f(\infty) = 1, \quad 0 < f(s) < 1 \quad (0 < s < \infty). \end{cases} \tag{3.1}$$

It has the asymptotics such that for a sufficiently large number $R_1 > 0$,

$$\begin{cases} f_\infty(s) = 1 - \dfrac{1}{2s^2} + O(1/s^4), \qquad s > R_1, \\ f_\infty(s) = a_1 s + O(s^3), \qquad s < 1/R_1, \end{cases} \tag{3.2}$$

where a_1 is a positive constant (see [5] or [18]). Define

$$f_\epsilon(r) := f_\infty(r/\epsilon). \tag{3.3}$$

Then $\tilde{u}_\epsilon = f_\epsilon(r) e^{i\theta}$ gives a symmetric vortex solution to

$$\Delta u + \frac{1}{\epsilon^2}(1 - |u|^2) u = 0, \qquad x \in \mathbb{R}^2.$$

Note that by (3.2)

$$\begin{cases} f_\epsilon(r) = 1 - \dfrac{\epsilon^2}{2r^2} + O(\epsilon^4/r^4), \\ f_\epsilon'(r) = \dfrac{\epsilon^2}{r^3} + O(\epsilon^4/r^5), \qquad\qquad r > \epsilon R_1, \\ f_\epsilon''(r) = \dfrac{\epsilon^2}{r^4} + O(\epsilon^4/r^6) \end{cases} \tag{3.4}$$

($'=d/dr$, $''=d^2/dr^2$). Given $\rho>0$, define

$$I_\epsilon(\rho):=\frac{1}{2}\int_{B_\rho(0)}\left\{|\nabla\tilde{u}_\epsilon|^2+\frac{1}{2}(1-|\tilde{u}_\epsilon|^2)^2\right\}dx.$$

Then we easily verify

$$I_\epsilon(\rho)=\pi\int_0^\rho\{|f_\epsilon'|^2+\frac{1}{r^2}f_\epsilon^2+\frac{1}{\epsilon^2}(1-f_\epsilon^2)^2\}rdr. \tag{3.5}$$

PROPOSITION 3.1. *Let R_1 be as in (3.2) and let ρ satisfy*

$$\epsilon R_1<\rho<1.$$

Then $I_\epsilon(\rho)$ of (3.5) admits

$$I_\epsilon(\rho)=\pi\log(1/\epsilon)+\pi\log\rho+C_0+O(\epsilon^2/\rho^2), \tag{3.6}$$

where $C_0=C_0(R_1)$ is a constant independent of ϵ and ρ.

Proof. Since f_ϵ satisfies

$$f_\epsilon''+\frac{1}{r}f_\epsilon'-\frac{1}{r^2}f_\epsilon+\frac{1}{\epsilon^2}(1-f_\epsilon^2)f_\epsilon=0,\quad f_\epsilon(0)=0, \tag{3.7}$$

we obtain

$$\int_0^\rho(|f_\epsilon'|^2+\frac{1}{r^2}f_\epsilon^2)rdr=\frac{1}{\epsilon^2}\int_0^\rho(1-f_\epsilon^2)f_\epsilon^2rdr+O(\epsilon^2/\rho^2), \tag{3.8}$$

where we used

$$[rf_\epsilon'f_\epsilon]_{r=0}^\rho=\rho f_\epsilon'(\rho)f_\epsilon(\rho)=O(\epsilon^2/\rho^2)$$

(recall (3.4)). By virtue of (3.8)

$$\begin{aligned}I_\epsilon(\rho)\;&=\pi\int_0^\rho\{\frac{1}{\epsilon^2}(1-f_\epsilon^2)f_\epsilon^2+\frac{1}{2\epsilon^2}(1-f_\epsilon^2)^2\}rdr+O(\epsilon^2/\rho^2)\\&=\frac{\pi}{2\epsilon^2}\int_0^\rho(1-f_\epsilon^2)(1+f_\epsilon^2)rdr+O(\epsilon^2/\rho^2).\end{aligned}$$

Put

$$\begin{cases}I_\epsilon(\rho)=I_\epsilon^{(1)}+I_\epsilon^{(2)}+O(\epsilon^2/\rho^2),\\[2mm] I^{(1)}:=\dfrac{\pi}{2\epsilon^2}\displaystyle\int_0^{\epsilon R_1}(1-f_\epsilon^2)(1+f_\epsilon^2)rdr,\\[2mm] I^{(2)}:=\dfrac{\pi}{2\epsilon^2}\displaystyle\int_{\epsilon R_1}^{\rho}(1-f_\epsilon^2)(1+f_\epsilon^2)rdr.\end{cases} \tag{3.9}$$

We compute $I^{(1)}$ and $I^{(2)}$. By (3.3)

$$I^{(1)}=\frac{\pi}{2}\int_0^{R_1}(1-\{f_\infty(s)\}^2)(1+\{f_\infty(s)\}^2)sds \tag{3.10}$$

which is independent of ϵ and ρ. On the other hand we can compute

$$\begin{aligned} I^{(2)} &= \pi \int_{\epsilon R_1}^{\rho} \{\frac{1}{\epsilon^2}(1-f_\epsilon^2)(1+f_\epsilon^2)/2\} r dr \\ &= \pi \int_{\epsilon R_1}^{\rho} \frac{1}{\epsilon^2}(1-f_\epsilon)(1+f_\epsilon)(1-\epsilon^2/2r^2+O(\epsilon^4/r^4)) r dr \\ &= \pi \int_{\epsilon R_1}^{\rho} \frac{1}{r} dr + \pi \int_{\epsilon R_1}^{\rho} O(\epsilon^2/r^3) dr. \end{aligned}$$

Thus

$$I^{(2)} = \pi \log \rho - \pi \log \epsilon - \pi \log R_1 + O(1/R_1) + O(\epsilon^2/\rho^2). \tag{3.11}$$

Using (3.10) and (3.11) in (3.9), we obtain the desired (3.6).

4. An approximate solution. Given ρ_0, set

$$Y := \{\boldsymbol{y} \in \hat{\Omega}^m : |y^{(j)} - y^{(k)}| \geq 2\rho_0,\ j \neq k, \quad \mathrm{dist}(y^{(j)}, \partial\Omega) \geq \frac{3}{2}\rho_0\}. \tag{4.1}$$

We take ρ_0 small so that the set Y is nonempty. Throughout this section, we assume that $\rho(\epsilon)$ is a positive function satisfying

$$\lim_{\epsilon\to 0} \rho(\epsilon) = 0, \quad \lim_{\epsilon\to 0} \frac{\epsilon}{\rho(\epsilon)} = 0. \tag{4.2}$$

We note that Propositions 2.2, 3.1 in the previous two sections still hold for putting $\rho = 2\rho(\epsilon)$.

Let χ be a function in $C^\infty(\mathbb{R})$ satisfying

$$\chi(s) = \begin{cases} 1, & s \leq 0, \\ 0, & s \geq 1 \end{cases}$$

and define

$$\chi_\epsilon(r) := \chi(r/\rho(\epsilon) - 1).$$

Then

$$\chi_\epsilon(r) = \begin{cases} 1, & r \leq \rho(\epsilon), \\ 0, & 2\rho(\epsilon) \leq r. \end{cases}$$

Set

$$\Omega_{\rho(\epsilon)} = \Omega \setminus \cup_{j=1}^m \overline{B_{\rho(\epsilon)}(y^{(j)})}, \qquad \Omega_{2\rho(\epsilon)} = \Omega \setminus \cup_{j=1}^m \overline{B_{2\rho(\epsilon)}(y^{(j)})} \tag{4.3}$$

and for $\boldsymbol{y} \in Y$ define a positive function with zeros at $y = y^{(j)}$, $1 \leq j \leq m$, as follows:

$$w_\epsilon(x;\boldsymbol{y}) := \begin{cases} g_\epsilon := 1 - \frac{\epsilon^2}{2}|\nabla\Theta|^2, & x \in \Omega_{2\rho(\epsilon)}, \\ g_\epsilon + \chi_\epsilon^{(j)}(f_\epsilon^{(j)} - g_\epsilon), & \rho(\epsilon) \leq |x - y^{(j)}| \leq 2\rho(\epsilon), \quad 1 \leq j \leq m, \\ f_\epsilon^{(j)}, & x \in B_{\rho(\epsilon)}(y^{(j)}), \quad 1 \leq j \leq m, \end{cases} \tag{4.4}$$

where $\Theta = \Theta(x; \boldsymbol{y})$ is defined by (2.5) and

$$f_\epsilon^{(j)}(x) := f_\epsilon(|x - y^{(j)}|), \qquad \chi_\epsilon^{(j)} := \chi_\epsilon(|x - y^{(j)}|). \tag{4.5}$$

We write

$$\mathcal{F}(u) := \Delta u + \frac{1}{\epsilon^2}(1 - |u|^2)u. \tag{4.6}$$

With $u_h = u_h(x; \boldsymbol{y})$ in (2.6), define

$$u_\epsilon(x; \boldsymbol{y}) := w_\epsilon(x; \boldsymbol{y}) u_h(x; \boldsymbol{y}). \tag{4.7}$$

Then

$$\mathcal{F}(u_\epsilon) = [\Delta w_\epsilon + \{-|\nabla\Theta|^2 + \frac{1}{\epsilon^2}(1 - w_\epsilon^2)\} w_\epsilon + 2i \nabla w_\epsilon \cdot \nabla\Theta] u_h. \tag{4.8}$$

Here we used (2.8) and

$$\nabla u_h = i(\nabla\Theta)u_h, \quad |\nabla u_h|^2 = |\nabla\Theta|^2, \qquad x \neq y^{(j)} \quad (1 \leq j \leq m).$$

LEMMA 4.1. *There exist a constant $C_1 > 0$ and a small number $\epsilon_1 > 0$ such that for $\epsilon \in (0, \epsilon_1)$,*

$$|\mathcal{F}(u_\epsilon)| \leq \begin{cases} C_1 \epsilon^2 \left\{ 1 + \displaystyle\sum_{j=1}^m \frac{1}{|x - y^{(j)}|^4} \right\}, & x \in \Omega_{2\rho(\epsilon)}, \\ C_1 \left\{ \dfrac{1}{|x - y^{(j)}|} + \dfrac{\epsilon^2}{|x - y^{(j)}|^4} \right\}, & \rho(\epsilon) \leq |x - y^{(j)}| \leq 2\rho(\epsilon), \ 1 \leq j \leq m, \\ C_1 \left\{ \dfrac{f_\epsilon^{(j)}}{|x - y^{(j)}|} + |f_\epsilon'(|x - y^{(j)}|)| \right\}, & x \in B_{\rho(\epsilon)}(y^{(j)}), \ 1 \leq j \leq m. \end{cases} \tag{4.9}$$

Before proving this lemma, we estimate L^1-norm of $\mathcal{F}(u_\epsilon)$. Since

$$\int_0^{\rho(\epsilon)} |f_\epsilon'(r)| r dr = \epsilon \int_0^{\rho(\epsilon)/\epsilon} \frac{d}{ds} f_\infty(s) s ds = O(\rho(\epsilon)),$$

and

$$\int_0^{\rho(\epsilon)} \frac{f_\epsilon(r)}{r} r dr \leq \rho(\epsilon)$$

hold, by (4.9) of Lemma 4.1 we easily check that there is a constant $C_1 > 0$ such that

$$\int_{\Omega_{\rho(\epsilon)}} |\mathcal{F}(u_\epsilon)| dx \leq C_1 \epsilon^2 / (\rho(\epsilon))^2,$$

and

$$\int_{B_{\rho(\epsilon)}(y^{(j)})} |\mathcal{F}(u_\epsilon)| dx \leq C_1 \rho(\epsilon),$$

for each j. Moreover we can similarly estimate the derivative of $\mathcal{F}(u_\epsilon)$ with respect to $y_p^{(k)}$ to obtain

$$\int_\Omega \left| \frac{\partial}{\partial y_p^{(k)}} \mathcal{F}(u_\epsilon) \right| dx \le C_1 \max\{\log(1/\epsilon), \epsilon^2/\rho(\epsilon)^3\}.$$

Hence we have

LEMMA 4.2. *Let ϵ_1 be a number as in Lemma 4.1. Then there is a constant C_1 such that for $\epsilon \in (0, \epsilon_1)$,*

$$\int_\Omega |\mathcal{F}(u_\epsilon)| dx \le C_1 \max\{\rho(\epsilon), \epsilon^2/\rho(\epsilon)^2\}, \tag{4.10}$$

$$\int_\Omega \left| \frac{\partial}{\partial y_p^{(k)}} \mathcal{F}(u_\epsilon) \right| dx \le C_1 \max\{\log(1/\epsilon), \epsilon^2/\rho(\epsilon)^3\}. \tag{4.11}$$

Proof of Lemma 4.1 : Henceforth we often simply write ρ for $\rho(\epsilon)$ if there is no confusion.

First we check

$$|\nabla\Theta|^2 \le |\sum_{j=1}^m d_j \nabla_x (\mathrm{Arg}(x - y^{(j)}) + \varphi(x; y^{(j)}))|^2$$

$$\le C_1 \left\{ 1 + \sum_{j=1}^m \frac{1}{|x - y^{(j)}|^2} \right\}, \tag{4.12}$$

$$|\Delta|\nabla\Theta|^2| \le C_1 \left\{ 1 + \sum_{j=1}^m \frac{1}{|x - y^{(j)}|^4} \right\}, \tag{4.13}$$

$$\left| |\nabla\Theta|^2 - \frac{1}{|x - y^{(j)}|^2} \right| \le C_1 \left\{ 1 + \sum_{j=1}^m \frac{1}{|x - y^{(j)}|} \right\}, \qquad x \in B_\rho(y^{(j)}). \tag{4.14}$$

For $\rho \le |x - y^{(j)}| \le 2\rho$, by (3.4)

$$f_\epsilon^{(j)} - g_\epsilon = \frac{\epsilon^2}{2}(|\nabla\Theta|^2 - 1/|x - y^{(j)}|^2) + O(\epsilon^4/|x - y^{(j)}|^4),$$

thus

$$|f_\epsilon^{(j)} - g_\epsilon| \le \frac{C_1 \epsilon^2}{|x - y^{(j)}|}, \quad \rho \le |x - y^{(j)}| \le 2\rho. \tag{4.15}$$

We also see that

$$-|\nabla\Theta|^2 + \frac{1}{\epsilon^2}(1 - g_\epsilon^2) = -\frac{\epsilon^2}{4}|\nabla\Theta|^2, \quad x \in \Omega_{2\rho},$$

and

$$\begin{aligned}
&-|\nabla\Theta|^2 + \frac{1}{\epsilon^2}(1-w_\epsilon^2) \\
&= -|\nabla\Theta|^2 + \frac{1}{\epsilon^2}(1-g_\epsilon^2) - \chi_\epsilon^{(j)}(1+g_\epsilon)(f_\epsilon^{(j)}-g_\epsilon)/\epsilon^2 \\
&\quad + \chi_\epsilon^{(j)}(1-g_\epsilon)(f_\epsilon^{(j)}-g_\epsilon)/\epsilon^2 - (\chi_\epsilon^{(j)})^2(f_\epsilon^{(j)}-g_\epsilon)^2/\epsilon^2, \quad \rho \le |x-y^{(j)}| \le 2\rho,
\end{aligned}$$

from which

$$(4.16)\quad \left| -|\nabla\Theta|^2 + \frac{1}{\epsilon^2}(1-w_\epsilon^2) \right| \le \begin{cases} C_1\epsilon^2 \left\{ 1 + \sum_{j=1}^{m} \frac{1}{|x-y^{(j)}|^4} \right\}, x \in \Omega_{2\rho}, \\ C_1 \left\{ \frac{1}{|x-y^{(j)}|} + \frac{\epsilon^2}{|x-y^{(j)}|^4} \right\}, \\ \qquad\qquad \rho \le |x-y^{(j)}| \le 2\rho \end{cases}$$

follows. By (4.13) and (4.16), we obtain

$$(4.17)\quad |\Delta w_\epsilon + \{-|\nabla\Theta|^2 + \frac{1}{\epsilon^2}(1-w_\epsilon^2)\}w_\epsilon| \le C_1\epsilon^2 \left\{ 1 + \sum_{j=1}^{m} \frac{1}{|x-y^{(j)}|^4} \right\}, \quad x \in \Omega_{2\rho}.$$

Moreover combining (4.16) with

$$\begin{aligned}
|\Delta w_\epsilon| &\le |\Delta g_\epsilon| + |\Delta\chi_\epsilon^{(j)}||f_\epsilon^{(j)}-g_\epsilon| + 2|\nabla\chi_\epsilon^{(j)}||\nabla(f_\epsilon^{(j)}-g_\epsilon)| + |\Delta(f_\epsilon^{(j)}-g_\epsilon)| \\
&\le \frac{C_1\epsilon^2}{|x-y^{(j)}|^4} + \frac{C_1\epsilon^2}{\rho^2|x-y^{(j)}|} + \frac{C_1\epsilon^2}{\rho|x-y^{(j)}|^2} + \frac{C_1\epsilon^2}{|x-y^{(j)}|^3} \\
&\le \frac{C_1\epsilon^2}{|x-y^{(j)}|^4}, \qquad \rho \le |x-y^{(j)}| \le 2\rho,
\end{aligned}$$

we get

$$(4.18)\quad |\Delta w_\epsilon + \{-|\nabla\Theta|^2 + \frac{1}{\epsilon^2}(1-w_\epsilon^2)\}w_\epsilon| \le C_1 \left\{ \frac{1}{|x-y^{(j)}|} + \frac{\epsilon^2}{|x-y^{(j)}|^4} \right\},$$

for $\rho \le |x-y^{(j)}| \le 2\rho$.

On the other hand we can compute

$$\nabla w_\epsilon \cdot \nabla\Theta = \begin{cases} \nabla g_\epsilon \cdot \nabla\Theta = -\frac{\epsilon^2}{2}\nabla(|\nabla\Theta|^2)\cdot\nabla\Theta, \quad x \in \Omega_{2\rho}, \\ \{\nabla g_\epsilon + (f_\epsilon^{(j)}-g_\epsilon)\nabla\chi_\epsilon^{(j)} + \chi_\epsilon^{(j)}\nabla(f_\epsilon^{(j)}-g_\epsilon)\}\cdot\nabla\Theta, \\ \qquad\qquad \rho \le |x-y^{(j)}| \le 2\rho. \end{cases}$$

Thus we obtain

$$(4.19)\qquad |\nabla w_\epsilon \cdot \nabla\Theta| \le \epsilon^2 C_1 \left\{ 1 + \sum_{j=1}^{m} \frac{1}{|x-y^{(j)}|^4} \right\}, \qquad x \in \Omega_{2\rho}.$$

Furthermore, with the aid of

$$\nabla_x |x - y^{(j)}| \cdot \mathrm{Arg}_x(x - y^{(j)}) = 0$$

we can show

$$|\nabla \chi_\epsilon^{(j)} \cdot \nabla \Theta| \le C_1/\rho.$$

Hence we have

$$\begin{aligned} |\nabla w_\epsilon \cdot \nabla \Theta| &\le C_1 \left\{ \frac{\epsilon^2}{|x - y^{(j)}|^4} + \frac{\epsilon^2}{\rho |x - y^{(j)}|} + \frac{\epsilon^2}{|x - y^{(j)}|^3} \right\} \\ &\le \frac{C_1 \epsilon^2}{|x - y^{(j)}|^4}, \qquad \rho \le |x - y^{(j)}| \le 2\rho. \end{aligned} \tag{4.20}$$

Finally we compute

$$\begin{aligned} \mathcal{F}(u_\epsilon) &= \{\Delta f_\epsilon^{(j)} - |\nabla \Theta|^2 f_\epsilon^{(j)} + \frac{1}{\epsilon^2}(1 - (f_\epsilon^{(j)})^2) f_\epsilon^{(j)} + 2i \nabla f_\epsilon^{(j)} \cdot \nabla \Theta\} e^{i\Theta} \\ &= - \left\{ \left(|\nabla \Theta|^2 - \frac{1}{|x - y^{(j)}|^2} \right) f_\epsilon^{(j)} + 2i \nabla f_\epsilon^{(j)} \cdot \nabla \Theta \right\} e^{i\Theta}, \quad x \in B_\rho(y^{(j)}). \end{aligned}$$

Since

$$\nabla f_\epsilon^{(j)} \cdot \nabla \Theta = f_\epsilon'(|x - y^{(j)}|) \frac{x - y^{(j)}}{|x - y^{(j)}|} \cdot \left\{ \nabla \Theta - \frac{(x - y^{(j)})^\perp}{|x - y^{(j)}|^2} \right\},$$

we obtain

$$|\mathcal{F}(u_\epsilon)| \le C_1 \{ f_\epsilon^{(j)}/|x - y^{(j)}| + |f_\epsilon'(|x - y^{(j)}|)| \}, \qquad x \in B_\rho(y^{(j)}). \tag{4.21}$$

In terms of (4.17) (4.18), (4.19), (4.20) and (4.21) we get to the desired estimate (4.9). □

Next we show some estimates for derivatives of the approximate solution for later arguments. Let $\langle u, v \rangle$ denotes

$$\langle u, v \rangle_{L^2} = \mathrm{Re} \int_\Omega u(x) \overline{v(x)} dx \tag{4.22}$$

for $u, v \in L^2(\Omega; \mathbb{C})$. Recall that $\mathbb{C}$ is identified with $\mathbb{R}^2$, therefore this is the inner product in $L^2(\Omega; \mathbb{R}^2)$.

LEMMA 4.3.

$$\int_\Omega \left| \frac{\partial u_\epsilon}{\partial y_p^{(k)}} \right| dx = O(1), \tag{4.23}$$

$$\left\langle \frac{\partial u_\epsilon}{\partial y_p^{(k)}}, \frac{\partial u_\epsilon}{\partial y_q^{(j)}} \right\rangle_{L^2} = \begin{cases} \pi \log(1/\epsilon) + O(1), & k = j \textit{ and } p = q, \\ O(1), & k \ne j \textit{ or } p \ne q, \end{cases} \tag{4.24}$$

and

$$\int_\Omega \left| \frac{\partial^2 u_\epsilon}{\partial y_p^{(k)} y_q^{(j)}} \right| dx = O(\log(1/\epsilon)), \quad 1 \le k, j \le m, \quad 1 \le p, q \le 2. \tag{4.25}$$

Proof. The estimate (4.23) is easily verified, therefore we prove (4.24). First we write

$$\left\langle \frac{\partial u_\epsilon}{\partial y_p^{(k)}}, \frac{\partial u_\epsilon}{\partial y_q^{(j)}} \right\rangle_{L^2} = \int_\Omega \left(\frac{\partial w_\epsilon}{\partial y_p^{(k)}} \frac{\partial w_\epsilon}{\partial y_q^{(j)}} + w_\epsilon^2 \frac{\partial \Theta}{\partial y_p^{(k)}} \frac{\partial \Theta}{\partial y_q^{(j)}} \right) dx$$

Since

$$\frac{\partial w_\epsilon}{\partial y_p^{(k)}} = \begin{cases} -\dfrac{\epsilon^2}{2} \dfrac{\partial}{\partial y_p^{(k)}} |\nabla \Theta|^2, & x \in \Omega_{2\rho}, \\[2ex] -\dfrac{\epsilon^2}{2} \dfrac{\partial}{\partial y_p^{(k)}} |\nabla \Theta|^2 + \dfrac{\partial}{\partial y_p^{(k)}} \chi_\epsilon^{(j)} (f_\epsilon^{(j)} - g_\epsilon) & \\[2ex] + \chi_\epsilon^{(j)} \dfrac{\partial}{\partial y_p^{(k)}} (f_\epsilon^{(j)} - g_\epsilon), & \rho \le |x - y^{(j)}| \le 2\rho, \\[2ex] \dfrac{\partial}{\partial y_p^{(k)}} f_\epsilon^{(j)}, & x \in B_\rho(y^{(j)}), \end{cases}$$

we obtain

$$\int_\Omega \left| \frac{\partial w_\epsilon}{\partial y_p^{(k)}} \right|^2 dx \le C_1 \int_{\Omega_\rho} \frac{\epsilon^4}{|x - y^{(j)}|^6} + \int_{B_\rho(y^{(k)})} |f_\epsilon'(|x - y^{(j)}|)|^2 dx + O(1).$$

Therefore from a simple computation we can see

$$\int_\Omega \left| \frac{\partial w_\epsilon}{\partial y_p^{(k)}} \right|^2 dx = O(1). \tag{4.26}$$

We also easily check

$$\int_\Omega \frac{\partial w_\epsilon}{\partial y_p^{(k)}} \frac{\partial w_\epsilon}{\partial y_q^{(j)}} dx = O(1). \tag{4.27}$$

Next we compute

$$\int_\Omega w_\epsilon^2 \frac{\partial \Theta}{\partial y_p^{(k)}} \frac{\partial \Theta}{\partial y_q^{(j)}} dx.$$

First consider the case for $k = j$ and $p = q = 1$. Then

$$\frac{\partial \Theta}{\partial y_1^{(k)}} = d_k \left\{ -\frac{(x_2 - y_2^{(k)})}{|x - y^{(k)}|^2} + \frac{\partial}{\partial y_1^{(k)}} \varphi(x; y^{(k)}) \right\},$$

by which

$$\int_{\Omega_\rho} w_\epsilon^2 \left|\frac{\partial\Theta}{\partial y_1^{(k)}}\right|^2 dx = \int_0^{2\pi}\int_\rho^{\rho_0} \frac{r^2\sin^2\theta}{r^4} r dr d\theta + O(1)$$

$$= \pi(\log\rho_0 - \log\rho) + O(1),$$

and

$$\int_{B_\rho(y^{(k)})} (f_\epsilon^{(k)})^2 \left|\frac{\partial\Theta}{\partial y_1^{(k)}}\right|^2 dx = \int_0^{2\pi}\int_0^{\rho} (f_\epsilon(r))^2 \frac{\sin^2\theta}{r} dr d\theta + O(1)$$

$$= \pi\int_0^{\rho/\epsilon} \frac{(f_\infty(s))^2}{s} ds + O(1)$$

$$= \pi(\log\rho - \log\epsilon) + O(1).$$

Thus recalling that $\rho = \rho(\epsilon)$ satisfying (4.2) and ρ_0 is a fixed number, we obtain

$$\int_\Omega w_\epsilon^2 \left|\frac{\partial\Theta}{\partial y_1^{(k)}}\right|^2 dx = \pi\log(1/\epsilon) + O(1). \tag{4.28}$$

It follows from (4.26) and (4.28) that

$$\int_\Omega \left|\frac{\partial u_\epsilon}{\partial y_1^{(k)}}\right|^2 dx = \pi\log(1/\epsilon) + O(1). \tag{4.29}$$

Similarly we obtain the same estimate for $k = j$ and $p = q = 2$.

Next we consider the case for $k = j$ and $p \neq q$. For simplicity, we set $p = 1, q = 2$. Then

$$\frac{\partial\Theta}{\partial y_1^{(k)}}\frac{\partial\Theta}{\partial y_2^{(k)}} = \frac{-(x_2 - y_2^{(k)})}{|x-y^{(k)}|^2}\frac{(x_1 - y_1^{(k)})}{|x-y^{(k)}|^2} + O(1/|x-y^{(k)}|), \qquad x \approx y^{(k)}.$$

Because of

$$\int_{B_\rho} (f_\epsilon^{(k)})^2 \frac{-(x_2 - y_2^{(k)})}{|x-y^{(k)}|^2}\frac{(x_1 - y_1^{(k)})}{|x-y^{(k)}|^2} dx = \int_0^{2\pi}\int_0^\rho \frac{-\sin\theta\cos\theta}{r} f_\epsilon^2 dr d\theta = 0,$$

one can easily check

$$\int_\Omega w_\epsilon^2 \frac{\partial\Theta}{\partial y_1^{(k)}}\frac{\partial\Theta}{\partial y_2^{(k)}} = O(1). \tag{4.30}$$

As for the remaining cases, $k \neq j$, we easily verify

$$\frac{\partial\Theta}{\partial y_p^{(k)}}\frac{\partial\Theta}{\partial y_q^{(j)}} = O(1/|x-y^{(j)}|), \qquad x \approx y^{(j)},$$

from which

$$\int_\Omega w_\epsilon^2 \frac{\partial\Theta}{\partial y_p^{(k)}}\frac{\partial\Theta}{\partial y_q^{(j)}} = O(1)$$

follows. This concludes the proof of (4.24).

To prove (4.25), we repeat the similar computations done for (4.24). We leave it to the readers. □

5. Energy for the approximate solution. The following lemma involves the closeness between the energy of u_ϵ and the renormalized energy $V(\boldsymbol{y})$:

LEMMA 5.1. *Let $E_\epsilon(u)$ be the energy defined by (1.3). Assume (4.2). Then for small $\epsilon > 0$, $E_\epsilon(u_\epsilon(\cdot;\boldsymbol{y}))$ is C^1 in $\boldsymbol{y}$ and*

$$E_\epsilon(u_\epsilon) = m\pi \log(1/\epsilon) + C_0 + V(\boldsymbol{y}) + O(\rho(\epsilon)) + O(\epsilon^2/\rho(\epsilon)^2). \tag{5.1}$$

Moreovery if $\epsilon^2/\rho(\epsilon)^3 \to 0$ as $\epsilon \to 0$, then

$$\frac{\partial}{\partial y_p^{(j)}} E_\epsilon(u_\epsilon) = \frac{\partial}{\partial y_p^{(j)}} V(\boldsymbol{y}) + O(\rho(\epsilon)) + O(\epsilon^2/\rho(\epsilon)^3). \tag{5.2}$$

holds.

Proof. Since the proof of the former part is simple, we focus on (5.2). For simplicity of notation we simply write ρ for $\rho(\epsilon)$ again. Set

$$e_\epsilon(u_\epsilon) := |\nabla w_\epsilon|^2 + w_\epsilon^2 |\nabla u_h|^2 + \frac{1}{2\epsilon^2}(1 - w_\epsilon^2)^2.$$

Then

$$\begin{aligned}\frac{\partial}{\partial y_p^{(j)}} E_\epsilon(u_\epsilon) &= \frac{1}{2}\int_\Omega \frac{\partial}{\partial y_p^{(j)}} e_\epsilon(u_\epsilon)dx \\ &= \frac{1}{2}\int_{\Omega_\rho} \frac{\partial}{\partial y_p^{(j)}} e_\epsilon(u_\epsilon)dx + \frac{1}{2}\sum_{k=1}^m \int_{B_\rho(y^{(k)})} \frac{\partial}{\partial y_p^{(j)}} e_\epsilon(u_\epsilon)dx.\end{aligned}$$

We write

$$\begin{aligned}& e_\epsilon(u_\epsilon) = |\nabla u_h|^2 + W_\epsilon, \qquad x \in \Omega_\rho \\ & W_\epsilon := |\nabla w_\epsilon|^2 + (1 - w_\epsilon^2)\{-|\nabla u_h|^2 + \frac{1}{2\epsilon^2}(1 - w_\epsilon^2)\}, \quad x \in \Omega_\rho,\end{aligned} \tag{5.3}$$

and

$$\begin{aligned} e_\epsilon(u_\epsilon) &= |\nabla f_\epsilon^{(k)}|^2 + (f_\epsilon^{(k)})^2 |\nabla u_h|^2 + \frac{1}{2\epsilon^2}(1 - (f_\epsilon^{(k)})^2))^2 \\ &= e_\epsilon(f_\epsilon^{(k)} e^{i\mathrm{Arg}(x - y^{(k)})}) + |\nabla u_h|^2 - \frac{1}{|x - y^{(k)}|^2} + R_\epsilon^{(k)}, \quad x \in B_\rho(y^{(k)}),\end{aligned} \tag{5.4}$$

where we put

$$R_\epsilon^{(k)} := ((f_\epsilon^{(k)})^2 - 1)\left\{|\nabla u_h|^2 - \frac{1}{|x - y^{(k)}|^2}\right\}. \tag{5.5}$$

Note that

$$e_\epsilon(f_\epsilon^{(k)} e^{i\mathrm{Arg}(x - y^{(k)})}) = |\nabla f_\epsilon^{(k)}|^2 + (f_\epsilon^{(k)})^2/|x - y^{(k)}|^2 + \frac{1}{2\epsilon^2}(1 - (f_\epsilon^{(k)})^2)^2.$$

If we prove

$$\int_{B_\rho(y^{(k)})} \frac{\partial}{\partial y_p^{(j)}} e_\epsilon(f_\epsilon^{(k)} e^{i\mathrm{Arg}(x-y^{(k)})})dx = 0, \tag{5.6}$$

$$\left|\sum_{k=1}^{m} \int_{B_\rho(y^{(k)})} \frac{\partial}{\partial y_p^{(j)}} R_\epsilon^{(k)} dx\right| = O(\epsilon^2/\rho^2) + O(\rho), \tag{5.7}$$

$$\left|\int_{\Omega_\rho} \frac{\partial}{\partial y_p^{(j)}} W_\epsilon dx\right| = O(\epsilon^2/\rho^3), \tag{5.8}$$

and

$$\begin{aligned} &\frac{1}{2}\int_{\Omega_\rho} \frac{\partial}{\partial y_p^{(j)}} |\nabla u_h|^2 dx + \frac{1}{2}\sum_{k=1}^{m} \int_{B_\rho(y^{(k)})} \frac{\partial}{\partial y_p^{(j)}} \left\{|\nabla u_h|^2 - \frac{1}{|x-y^{(k)}|^2}\right\} dx \\ &= \frac{\partial}{\partial y_p^{(j)}} V(\boldsymbol{y}) + O(\rho), \end{aligned} \tag{5.9}$$

then it is clear that the desired (5.2) follows. Henceforth we fix $p = 1$. For $p = 2$ the same computation works, so omit it.

First consider (5.6). It is trivial that it holds for $k \neq j$. For $k = j$ it follows from

$$\begin{aligned} &\int_{B_\rho(y^{(j)})} \frac{\partial}{\partial y_1^{(j)}} e_\epsilon(f_\epsilon^{(j)} e^{i\mathrm{Arg}(x-y^{(j)})})dx \\ &= \int_{B_\rho(y^{(j)})} -\frac{\partial}{\partial x_1^{(j)}} e_\epsilon(f_\epsilon^{(j)} e^{i\mathrm{Arg}(x-y^{(j)})})dx \\ &= \int_0^{2\pi} - \{|f_\epsilon'(\rho)|^2 + f_\epsilon(\rho)^2/\rho^2 + \frac{1}{2\epsilon^2}(1 - f_\epsilon(\rho)^2)^2\}\rho\cos\theta d\theta = 0. \end{aligned}$$

Second we prove (5.7). For $k \neq j$,

$$\int_{B_\rho(y^{(k)})} \left|\frac{\partial R_\epsilon^{(k)}}{\partial y_1^{(j)}}\right| dx \le \int_{B_\rho(y^{(k)})} \frac{C_1}{|x-y^{(k)}|} dx = O(\rho).$$

On the other hand for $k = j$ we need a more careful computation. Let

$$\begin{aligned} \tilde{R}_\epsilon^{(j)} &:= 2((f_\epsilon^{(j)})^2 - 1)\sum_{\ell\neq j} \nabla_x \log|x - y^{(j)}| \cdot \nabla_x S(x, y^{(\ell)}), \\ \hat{R}_\epsilon^{(j)} &:= R_\epsilon^{(j)} - \tilde{R}_\epsilon^{(j)}. \end{aligned} \tag{5.10}$$

Then we easily see

$$\frac{\partial}{\partial y_1^{(j)}} \tilde{R}_\epsilon^{(j)} = -\frac{\partial}{\partial x_1^{(j)}} \tilde{R}_\epsilon^{(j)} + 2((f_\epsilon^{(j)})^2 - 1)\sum_{\ell\neq j} \nabla_x \log|x - y^{(j)}| \cdot \nabla_x \frac{\partial}{\partial x_1^{(j)}} S(x, y^{(\ell)}).$$

With the aid of this identity we compute

$$\int_{B_\rho(y^{(j)})} \frac{\partial}{\partial y_1^{(j)}} R_\epsilon^{(j)} dx \tag{5.11}$$

$$
\begin{aligned}
&= \int_{B_\rho(y^{(j)})} -\frac{\partial}{\partial x_1^{(j)}} \tilde{R}_\epsilon^{(j)} dx + O(\rho) \\
&= 2\int_0^{2\pi} \left\{ (f_\epsilon(\rho))^2 - 1) \sum_{\ell \neq j} (\cos\theta, \sin\theta) \cdot \nabla_x S(y^{(j)} + \rho e^{i\theta}) \right\} d\theta + O(\rho) \\
&= O(\epsilon^2/\rho^2) + O(\rho)
\end{aligned}
$$

(we used (3.4)).

As for (5.8), we verify that leading terms of W_ϵ are as follows:

$$
W_\epsilon = C_1 \sum_{k=1}^{m} \left\{ \frac{\epsilon^4}{|x - y^{(k)}|^6} + \frac{\epsilon^2}{|x - y^{(k)}|^4} \right\} + \cdots, \qquad x \in \Omega_\rho.
$$

Thus

$$
\left| \frac{\partial}{\partial y_1^{(j)}} W_\epsilon \right| \le \frac{C_1 \epsilon^2}{|x - y^{(j)}|^5},
$$

which yields (5.8).

Finally we prove (5.9). Recall

$$
|\nabla u_h|^2 = \sum_{j=1}^{m} \sum_{\ell=1}^{m} \nabla_x G(x, y^{(j)}) \cdot \nabla_x G(x, y^{(\ell)})
$$

(see (2.22)). We obtain

$$
\begin{aligned}
\text{(5.12)} \qquad \frac{\partial}{\partial y_1^{(j)}} |\nabla u_h|^2 &= 2\nabla_x \frac{\partial}{\partial y_1^{(j)}} G(x, y^{(j)}) \cdot \nabla_x G(x, y^{(j)}) \\
&\quad + 2\sum_{\ell \neq j} d_j d_\ell \nabla_x \frac{\partial}{\partial y_1^{(j)}} G(x, y^{(j)}) \cdot \nabla_x G(x, y^{(\ell)}).
\end{aligned}
$$

We use the similar computation done in Lemma 2.3. In fact

$$
\begin{aligned}
\text{(5.13)} \qquad & \frac{1}{2} \int_{\Omega_\rho} \frac{\partial}{\partial y_1^{(j)}} |\nabla u_h|^2 dx \\
&= -\sum_{k=1}^{m} \int_{\partial B_\rho(y^{(k)})} \frac{\partial}{\partial y_1^{(j)}} G(x, y^{(j)}) \frac{\partial}{\partial \nu_x} G(x, y^{(j)}) d\sigma_x \\
&\quad - \sum_{\ell \neq j} d_j d_\ell \sum_{k=1}^{m} \int_{\partial B_\rho(y^{(k)})} \frac{\partial}{\partial y_1^{(j)}} G(x, y^{(j)}) \frac{\partial}{\partial \nu_x} G(x, y^{(\ell)}) d\sigma_x,
\end{aligned}
$$

where we noticed

$$
\frac{\partial}{\partial y_1^{(j)}} G(x, y^{(j)}) = 0, \qquad x \in \partial\Omega.
$$

Moreover since

$$
\int_{\partial B_\rho(y^{(k)})} \frac{\partial}{\partial y_1^{(j)}} \log|x - y^{(j)}| \frac{\partial}{\partial \nu_x} \log|x - y^{(j)}| d\sigma_x = -\int_0^{2\pi} \frac{\cos\theta}{\rho} d\theta = 0,
$$

we have

$$(5.14)\quad \frac{1}{2}\int_{\Omega_\rho} \frac{\partial}{\partial y_1^{(j)}}|\nabla u_h|^2 dx$$

$$= -\sum_{k=1}^{m}\int_{\partial B_\rho(y^{(k)})}\left\{\frac{\partial}{\partial y_1^{(j)}}\log|x-y^{(j)}|\frac{\partial}{\partial \nu_x}S(x,y^{(j)}) + \frac{\partial}{\partial y_1^{(j)}}S(x,y^{(j)})\frac{\partial}{\partial \nu_x}G(x,y^{(j)})\right\}d\sigma_x$$

$$-\sum_{k=1}^{m}\sum_{\ell\neq j} d_j d_\ell \int_{\partial B_\rho(y^{(k)})}\frac{\partial}{\partial y_1^{(j)}}G(x,y^{(j)})\frac{\partial}{\partial \nu_x}G(x,y^{(\ell)})d\sigma_x.$$

Similarly we can compute

$$(5.15)\quad \frac{1}{2}\int_{B_\rho(y^{(j)})}\frac{\partial}{\partial y_1^{(j)}}\left\{|\nabla u_h|^2 - \frac{1}{|x-y^{(j)}|^2}\right\}dx = -2\pi\frac{\partial}{\partial y_1^{(j)}}S(y^{(j)},y^{(j)})$$

$$+\int_{\partial B_\rho(y^{(k)})}\left\{\frac{\partial}{\partial y_1^{(j)}}\log|x-y^{(j)}|\frac{\partial}{\partial \nu_x}S(x,y^{(j)}) + \frac{\partial}{\partial y_1^{(j)}}S(x,y^{(j)})\frac{\partial}{\partial \nu_x}G(x,y^{(j)})\right\}d\sigma_x$$

$$+\sum_{\ell\neq j} d_j d_\ell \int_{|x-y^{(j)}|=\rho}\frac{\partial}{\partial y_1^{(j)}}G(x,y^{(j)})\frac{\partial}{\partial \nu_x}G(x,y^{(\ell)})d\sigma_x.$$

On the other hand for $k \neq j$, because of

$$\frac{\partial}{\partial y_1^{(j)}}\left\{|\nabla u_h|^2 - \frac{1}{|x-y^{(j)}|^2}\right\}$$

$$= \frac{\partial}{\partial y_1^{(j)}}\left\{|\nabla_x G(x,y^{(j)})|^2 + 2\sum_{\ell\neq j} d_j d_\ell \nabla_x G(x,y^{(j)})\cdot\nabla_x G(x,y^{(\ell)})\right\}$$

and considering $|\nabla_x G(x,y^{(j)})|^2$ is smooth in $B_\rho(y^{(k)})$, one can check

$$(5.16)\quad \frac{1}{2}\int_{B_\rho(y^{(k)})}\frac{\partial}{\partial y_1^{(k)}}\left\{|\nabla u_h|^2 - \frac{1}{|x-y^{(j)}|^2}\right\}dx$$

$$= \sum_{\ell\neq j}\int_{B_\rho(y^{(k)})} d_j d_\ell \nabla_x \frac{\partial}{\partial y_1^{(j)}}G(x,y^{(j)})\cdot\nabla_x G(x,y^{(\ell)})dx + O(\rho)$$

$$= \sum_{\ell\neq j}\int_{\partial B_\rho(y^{(k)})} d_j d_\ell \frac{\partial}{\partial y_1^{(j)}}G(x,y^{(j)})\frac{\partial}{\partial \nu_x}G(x,y^{(\ell)})d\sigma_x$$

$$-2\pi d_j d_k \frac{\partial}{\partial y_1^{(j)}}G(y^{(k)},y^{(j)}) + O(\rho).$$

Combining (5.15) with (5.16) yields

$$(5.17)\quad \frac{1}{2}\sum_{k=1}^{m}\int_{B_\rho(y^{(k)})}\frac{\partial}{\partial y_1^{(k)}}\left\{|\nabla u_h|^2 - \frac{1}{|x-y^{(j)}|^2}\right\}dx$$

$$= \int_{\partial B_\rho(y^{(k)})}\left\{\frac{\partial}{\partial y_1^{(j)}}\log|x-y^{(j)}|\frac{\partial}{\partial \nu_x}S(x,y^{(j)}) + \frac{\partial}{\partial y_1^{(j)}}S(x,y^{(j)})\frac{\partial}{\partial \nu_x}G(x,y^{(j)})\right\}d\sigma_x$$

$$+\sum_{k=1}^{m}\sum_{\ell\neq j}\int_{\partial B_\rho(y^{(k)})} d_j d_\ell \frac{\partial}{\partial y_1^{(j)}}G(x,y^{(j)})\cdot\nabla_x G(x,y^{(\ell)})dx$$

$$-2\pi\frac{\partial}{\partial y_1^{(j)}}S(y^{(j)},y^{(j)})-2\pi\sum_{k\neq j}d_jd_k\frac{\partial}{\partial y_1^{(j)}}G(y^{(k)},y^{(j)})+O(\rho)$$

Since

$$\frac{\partial}{\partial y_1^{(j)}}V(\boldsymbol{y})=-2\pi\frac{\partial}{\partial y_1^{(j)}}S(y^{(j)},y^{(j)})-2\pi\sum_{k\neq j}d_jd_k\frac{\partial}{\partial y_1^{(j)}}G(y^{(k)},y^{(j)}),$$

adding (5.14) to (5.17), we obtain the desired result (5.9). □

6. Dynamics for the vortex solution. Given small $\rho_0>0$, we let

$$Y_0:=\{\boldsymbol{y}\in\hat{\Omega}^m\ :\ |y^{(j)}-y^{(k)}|>\frac{5}{2}\rho_0,\quad \mathrm{dist}(y^{(j)},\partial\Omega)>2\rho_0\}\subset Y \tag{6.1}$$

(Y is defined by (4.1)) and

$$\mathcal{M}_\epsilon:=\{u=u_\epsilon(\cdot,\boldsymbol{y})\ :\ \boldsymbol{y}\in Y_0\}. \tag{6.2}$$

We denote a Banach space of continuous functions by

$$X:=C^0(\overline{\Omega};\mathbb{C})\quad\text{with norm}\quad \|u\|_0:=\sup_{x\in\overline{\Omega}}|u(x)|. \tag{6.3}$$

Then $\mathcal{M}_\epsilon$ is a $2m$-dimensional submanifold of X. We discuss the dynamics of the solution to (1.1) in a neighborhood of the manifold $\mathcal{M}_\epsilon$. We derive the motion law projected on $\mathcal{M}_\epsilon$. To carry out it, we borrow the idea from §2 of [7]. We, however, need a more careful consideration because our approximate solution is not so nice as in [7], where their approximate solution has exponentially small error.

Define an operator $K_\epsilon:\overline{Y_0}\times X\longmapsto\mathbb{R}^{2m}$ by

$$\begin{aligned}K_\epsilon(\boldsymbol{y},u)&=(K_1^{(1)},K_2^{(1)},\cdots,K_1^{(j)},K_2^{(j)},\cdots,K_1^{(m)},K_2^{(m)}),\\ K_q^{(j)}=K_q^{(j)}(\boldsymbol{y},u)&:=\left\langle u-u_\epsilon(\cdot,\boldsymbol{y}),\frac{\partial u_\epsilon}{\partial y_q^{(j)}}(\cdot,\boldsymbol{y})\right\rangle_{L^2},\quad q=1,2.\end{aligned} \tag{6.4}$$

Note that

$$K_\epsilon(\boldsymbol{y},u_\epsilon(\cdot,\boldsymbol{y}))=0.$$

We first show

LEMMA 6.1. *$K_\epsilon(\boldsymbol{y},u)$ is a C^1 function in $(\boldsymbol{y},u)$ and the derivative with respect to $\boldsymbol{y}$ at $(\boldsymbol{y},u)=(\boldsymbol{y},u_\epsilon(\cdot;\boldsymbol{y}))$ can be written as*

$$\frac{\partial}{\partial\boldsymbol{y}}K_\epsilon(\boldsymbol{y},u_\epsilon(\cdot;\boldsymbol{y}))=-\pi\log(1/\epsilon)I_\epsilon,\qquad I_\epsilon=I_{2m}+O(1/\log(1/\epsilon)), \tag{6.5}$$

where I_{2m} is the $2m\times 2m$-identity matrix. Moreover for any $\boldsymbol{y}_0\in\overline{Y_0}$,

$$\left|\frac{\partial}{\partial\boldsymbol{y}}K_\epsilon(\boldsymbol{y},u)-\frac{\partial}{\partial\boldsymbol{y}}K_\epsilon(\boldsymbol{y}_0,u_\epsilon(\cdot,\boldsymbol{y}_0))\right|\le C_1\{1+\log(1/\epsilon)\|u-u_\epsilon(\cdot,\boldsymbol{y})\|_0\} \tag{6.6}$$

holds.

Proof. It is easy to see that $K_\epsilon(\boldsymbol{y}, u)$ is C^1. Since

$$\frac{\partial K_q^{(j)}}{\partial y_p^{(k)}}(\boldsymbol{y}, u_\epsilon(\cdot;\boldsymbol{y})) = -\left\langle \frac{\partial u_\epsilon}{\partial y_p^{(k)}}(\cdot,\boldsymbol{y}), \frac{\partial u_\epsilon}{\partial y_q^{(j)}}(\cdot,\boldsymbol{y}) \right\rangle_{L^2},$$

(6.5) immediately follows from (4.24) of Lemma 4.3. To prove (6.6), we use (4.25) in addition to (4.24). In fact

$$\frac{\partial K_q^{(j)}}{\partial y_p^{(k)}}(\boldsymbol{y}, u) = -\left\langle \frac{\partial u_\epsilon}{\partial y_p^{(k)}}(\cdot,\boldsymbol{y}), \frac{\partial u_\epsilon}{\partial y_q^{(j)}}(\cdot,\boldsymbol{y}) \right\rangle_{L^2} + \left\langle u - u_\epsilon(\cdot,\boldsymbol{y}), \frac{\partial^2 u_\epsilon}{\partial y_p^{(k)}\partial y_q^{(j)}}(\cdot,\boldsymbol{y}) \right\rangle_{L^2}.$$

Noticing that (4.24) holds for $\boldsymbol{y}$ and $\boldsymbol{y}_0$, we easily get to (6.6) by (4.24) and (4.25). □

Next for arbitrarily given $\boldsymbol{y}_0 \in \overline{Y_0}$, we define

$$\text{(6.7)}\quad \mathcal{H}_\epsilon(\boldsymbol{y}, u) := \boldsymbol{y} - A_\epsilon(\boldsymbol{y}_0)^{-1}K_\epsilon(\boldsymbol{y}, u), \qquad A_\epsilon(\boldsymbol{y}_0) := \frac{\partial}{\partial \boldsymbol{y}}K_\epsilon(\boldsymbol{y}_0, u_\epsilon(\cdot,\boldsymbol{y}_0)).$$

Put

$$\text{(6.8)}\quad V_\eta(\boldsymbol{y}_0) := \{\boldsymbol{y} \in Y : |\boldsymbol{y} - \boldsymbol{y}_0| \le \eta\}, \quad X_\delta(\boldsymbol{y}_0) := \{u \in X : \|u - u_\epsilon(\cdot,\boldsymbol{y}_0)\|_0 \le \delta\}.$$

Then $\mathcal{H}_\epsilon : V_\eta(\boldsymbol{y}_0) \times X_\delta(\boldsymbol{y}_0) \to \mathbb{R}^{2m}$ and

$$\mathcal{H}_\epsilon(\boldsymbol{y}, u) = \boldsymbol{y} \Leftrightarrow K_\epsilon(\boldsymbol{y}, u) = 0, \qquad \mathcal{H}_\epsilon(\boldsymbol{y}_0, u_\epsilon(\cdot,\boldsymbol{y}_0)) = \boldsymbol{y}_0$$

hold. We prove that $\mathcal{H}_\epsilon$ is a contraction mapping.

LEMMA 6.2. *There exist small $\delta_1 > 0$ and $\epsilon_1 > 0$ such that if $\delta \in (0, \delta_1)$ and $\epsilon \in (0, \epsilon_1)$, for any $\boldsymbol{y}_0 \in \overline{Y_0}$ there exists an $\eta = \eta(\boldsymbol{y}_0, \epsilon) > 0$ for which the map $\mathcal{H}_\epsilon$ of (6.7) carries $V_\eta(\boldsymbol{y}_0) \times X_\delta(\boldsymbol{y}_0)$ into $V_\eta(\boldsymbol{y}_0)$ and*

$$\text{(6.9)}\quad |\mathcal{H}_\epsilon(\boldsymbol{y}_1, u) - \mathcal{H}_\epsilon(\boldsymbol{y}_2, u)| \le \frac{1}{2}|\boldsymbol{y}_1 - \boldsymbol{y}_2|, \qquad \boldsymbol{y}_j \in V_\eta(\boldsymbol{y}_0) \ \ (j = 1, 2).$$

Proof. By the definition of $\mathcal{H}_\epsilon$ we can compute

$$\begin{aligned}\text{(6.10)}\quad & |\mathcal{H}_\epsilon(\boldsymbol{y}_1, u) - \mathcal{H}_\epsilon(\boldsymbol{y}_2, u)| \\ &= |A_\epsilon(\boldsymbol{y}_0)^{-1}\{A_\epsilon(\boldsymbol{y}_0)(\boldsymbol{y}_1 - \boldsymbol{y}_2) - (K_\epsilon(\boldsymbol{y}_1, u) - K_\epsilon(\boldsymbol{y}_2, u))\}| \\ &\le \frac{|I_\epsilon^{-1}|_{op}}{\pi \log(1/\epsilon)} \left| A_\epsilon(\boldsymbol{y}_0) - \int_0^1 \frac{\partial}{\partial \boldsymbol{y}}K(\boldsymbol{y}_2 + t(\boldsymbol{y}_1 - \boldsymbol{y}_2), u)dt \right|_{op} |\boldsymbol{y}_1 - \boldsymbol{y}_2|\end{aligned}$$

where $|\cdot|_{op}$ denotes the operater norm for linear maps from $\mathbb{R}^{2m}$ into itself. We estimate

$$\begin{aligned}\text{(6.11)}\quad & \left| A_\epsilon(\boldsymbol{y}_0) - \int_0^1 \frac{\partial}{\partial \boldsymbol{y}}K(\boldsymbol{y}_2 + t(\boldsymbol{y}_1 - \boldsymbol{y}_2), u)dt \right|_{op} \\ &= \int_0^1 \left| \frac{\partial}{\partial \boldsymbol{y}}K(\boldsymbol{y}_0, u_\epsilon(\cdot,\boldsymbol{y}_0)) - \frac{\partial}{\partial \boldsymbol{y}}K(\boldsymbol{y}_2 + t(\boldsymbol{y}_1 - \boldsymbol{y}_2), u) \right|_{op} dt \\ &\le C_1\{1 + (\log\frac{1}{\epsilon})\|u - u_\epsilon(\cdot,\tilde{\boldsymbol{y}})\|_0\} \\ &\le C_1[1 + (\log\frac{1}{\epsilon})\{\|u - u_\epsilon(\cdot,\boldsymbol{y}_0)\|_0 + \|u_\epsilon(\cdot,\boldsymbol{y}_0) - u_\epsilon(\cdot,\tilde{\boldsymbol{y}})\|_0\}].\end{aligned}$$

where we put $\tilde{y} = y_2 + t(y_1 - y_2)$ and used (6.6) of Lemma 6.1. Let $\delta_1 > 0$ be a number satisfying $\delta_1 < \pi/6$ and take $\eta = \eta(y_0, \epsilon)$ so that

$$\|u_\epsilon(\cdot, y_0) - u_\epsilon(\cdot, y)\|_0 \le \frac{\pi}{6}$$

holds for $|y - y_0| < \eta(y_0, \epsilon)$. Then inserting (6.11) into (6.10) yields (6.9).

To prove

$$\mathcal{H}_\epsilon : V_\eta(y_0) \times X_\delta(y_0) \longmapsto V_\eta(y_0),$$

we notice

$$|\mathcal{H}_\epsilon(y, u) - y_0| = |\mathcal{H}_\epsilon(y, u) - \mathcal{H}_\epsilon(y_0, u_\epsilon(\cdot; y_0))| \le \frac{1}{2}|y - y_0| \le \frac{1}{2}\eta.$$

We concluded the proof. □

By the above lemma, we can use the uniform contraction mapping principle (see Chap.1, 1.2.6 of [11]) to obtain the following lemma:

LEMMA 6.3. *For each $u \in X_\delta(y_0)$ there is a fixed point of $\mathcal{H}_\epsilon$, $y = y_\epsilon(u) \in V_\eta(y_0)$ such that $y_\epsilon(u)$ is a C^1 function satisfying $y_\epsilon(u_\epsilon(y_0)) = y_0$, where $\eta(= \eta(y_0, \epsilon))$ is as in Lemma* 6.2.

We can extend the function $y_\epsilon(u)$ to the one defined in a neighborhood of $\mathcal{M}_\epsilon$.

LEMMA 6.4. *Let Y be as in* (4.1) *and δ_1 be a number as in Lemma* 6.2. *For any $\delta \in (0, \delta_1)$, define*

$$\mathcal{U}_\delta(\mathcal{M}_\epsilon) := \{u \in X \ : \ \min_{y \in \overline{Y_0}} \|u - u_\epsilon(\cdot; y)\|_0 < \delta\}, \tag{6.12}$$

where $\mathcal{M}_\epsilon$ is defined by (6.2). *Then there is a C^1 mapping $y_\epsilon : \mathcal{U}_\delta(\mathcal{M}_\epsilon) \to Y$ such that $K_\epsilon(y_\epsilon(u), u) = 0$. Moreover for $u \in \mathcal{U}_\delta(\mathcal{M}_\epsilon)$, $y_\epsilon(u)$ minimizes $\|u - u_\epsilon(\cdot; y)\|_{L^2(\Omega)}$.*

Proof. By virtue of the compactness of $\overline{Y_0}$ and that δ_1 can be taken uniformly for $y_0 \in \overline{Y_0}$, the former assertion of the lemma immediately follows. To prove the latter part, put

$$\alpha(\xi) := \|u - u_\epsilon(\cdot; y_\epsilon(u) + \xi)\|^2_{L^2(\Omega)}.$$

Then one can easily verify

$$\frac{\partial}{\partial \xi}\alpha(0) = K_\epsilon(y_\epsilon(u), u) = 0,$$

and

$$\begin{aligned}\frac{\partial^2}{\partial \xi_p^{(k)} \partial \xi_q^{(j)}}\alpha(0) \;&= -\left\langle \frac{\partial u_\epsilon}{\partial y_p^{(k)}}(\cdot, y_\epsilon(u)), \frac{\partial u_\epsilon}{\partial y_q^{(j)}}(\cdot, y_\epsilon(u)) \right\rangle_{L^2} \\ &\quad + \left\langle u - u_\epsilon(\cdot, y_\epsilon(u)), \frac{\partial^2 u_\epsilon}{\partial y_p^{(k)} \partial y_q^{(j)}}(\cdot, y_\epsilon(u)) \right\rangle_{L^2}.\end{aligned}$$

By Lemma 4.3 there is a small $\delta_1 > 0$ such that if $\delta \in (0, \delta_1)$, every principal curvature of $\alpha(\boldsymbol{\xi})$ at $\boldsymbol{\xi} = 0$ is positive for any small $\epsilon > 0$. This proved the lemma. □

We have the main result on the dynamics of vortex solutions.

THEOREM 6.5. *Assume that $\rho(\epsilon)$ is a positive function satisfying*

$$\lim_{\epsilon\to+0} \rho(\epsilon) = 0, \qquad \lim_{\epsilon\to+0} \frac{\epsilon}{\rho(\epsilon)^{3/2}} = 0, \tag{6.13}$$

and that $u_\epsilon(x; \boldsymbol{y}), \boldsymbol{y} \in Y$ are the functions defined by (4.7), *where Y is as in* (4.1). *Let $\boldsymbol{y}_\epsilon : \mathcal{U}_\delta(\mathcal{M}_\epsilon) \to Y$ be a C^1 function appearing in Lemma* 6.4. *If for a positive $T > 0$, a solution $u^\epsilon(x,t)$ of* (1.1) *in $\mathcal{U}_\delta(\mathcal{M}_\epsilon)$ satisfies*

$$\|u^\epsilon(\cdot,t) - u_\epsilon(\cdot; \boldsymbol{y}_\epsilon(u^\epsilon(\cdot,t)))\|_0 = o(\epsilon), \qquad t \in [0,T], \tag{6.14}$$

then there is a $\epsilon_1 > 0$ such that $\boldsymbol{y} = \boldsymbol{y}_\epsilon(u^\epsilon(\cdot,t))$ admits

$$\log(1/\epsilon)\frac{d}{dt}\boldsymbol{y} = -\frac{1}{\pi}\nabla V(\boldsymbol{y}) + o(1). \tag{6.15}$$

Proof. For simplicity of notation, we drop ϵ in the solution $u^\epsilon(x,t)$ and write $\boldsymbol{y}(t) = \boldsymbol{y}_\epsilon(u(\cdot,t))$. We also simply write $\langle\cdot,\cdot\rangle$ for $\langle\cdot,\cdot\rangle_{L^2}$. Lemma 6.4 tells that

$$\mathcal{K}_\epsilon(\boldsymbol{y}(t), u(\cdot,t)) = 0,$$

which implies

$$\left\langle u(\cdot,t) - u_\epsilon(\cdot; \boldsymbol{y}(t)), \frac{\partial}{\partial y_q^{(j)}} u_\epsilon(\cdot; \boldsymbol{y}(t)) \right\rangle = 0, \qquad q = 1,2, \quad 1 \le j \le m. \tag{6.16}$$

Differentiating (6.16) with respect to t yields

$$\begin{aligned}
&\left\langle u_t - \sum_{p,k} \frac{\partial u_\epsilon}{\partial y_p^{(k)}} \dot{y}_p^{(k)}, \frac{\partial u_\epsilon}{\partial y_q^{(j)}} \right\rangle + \left\langle u(\cdot,t) - u_\epsilon(\cdot; \boldsymbol{y}(t)), \sum_{p,k} \frac{\partial^2 u_\epsilon}{\partial y_p^{(k)} \partial y_q^{(j)}} \dot{y}_p^{(k)} \right\rangle \\
&= \left\langle \mathcal{F}(u(\cdot,t)), \frac{\partial u_\epsilon}{\partial y_q^{(j)}} \right\rangle \\
&\quad - \sum_{p,k} \left\{ \left\langle \frac{\partial u_\epsilon}{\partial y_p^{(k)}}, \frac{\partial u_\epsilon}{\partial y_q^{(j)}} \right\rangle - \left\langle u(\cdot,t) - u_\epsilon(\cdot; \boldsymbol{y}(t)), \frac{\partial^2 u_\epsilon}{\partial y_p^{(k)} \partial y_q^{(j)}} \right\rangle \right\} \dot{y}_p^{(k)} = 0,
\end{aligned}$$

where $\dot{y} = dy/dt$. By putting

$$v_\epsilon(x,t) := u(x,t) - u_\epsilon(x; \boldsymbol{y}(t)),$$

we obtain

$$\begin{aligned}
&\sum_{p,k} \left\{ \left\langle \frac{\partial u_\epsilon}{\partial y_p^{(k)}}, \frac{\partial u_\epsilon}{\partial y_q^{(j)}} \right\rangle - \left\langle v_\epsilon(\cdot,t), \frac{\partial^2 u_\epsilon}{\partial y_p^{(k)} \partial y_q^{(j)}} \right\rangle \right\} \dot{y}_p^{(k)} \\
&= \left\langle \mathcal{F}(u_\epsilon + v_\epsilon(\cdot,t)), \frac{\partial u_\epsilon}{\partial y_q^{(j)}} \right\rangle
\end{aligned} \tag{6.17}$$

$(u_\epsilon = u_\epsilon(\cdot; \boldsymbol{y}(t)))$. We write

$$
\begin{aligned}
\mathcal{F}(u_\epsilon + v_\epsilon) &= \mathcal{F}(u_\epsilon) + \mathcal{L}(v_\epsilon) + N(v_\epsilon) \\
\mathcal{L}(v_\epsilon) &:= \Delta v_\epsilon + \frac{1}{\epsilon^2}(1 - |u_\epsilon|^2)v_\epsilon - \frac{2}{\epsilon^2}\mathrm{Re}(u_\epsilon \overline{v_\epsilon})u_\epsilon, \\
N(v_\epsilon) &:= \frac{1}{\epsilon^2}\{|v_\epsilon|^2 u_\epsilon + 2\mathrm{Re}(u_\epsilon \overline{v_\epsilon})v_\epsilon + |v_\epsilon|^2 v_\epsilon\}.
\end{aligned}
\tag{6.18}
$$

We can easily verify by integration by part that

$$
\begin{aligned}
\left\langle \mathcal{F}(u_\epsilon), \frac{\partial u_\epsilon}{\partial y_q^{(j)}} \right\rangle &= -\frac{1}{2}\int_\Omega \frac{\partial}{\partial y_q^{(j)}}\{|\nabla u_\epsilon|^2 + \frac{1}{2}(1 - |u_\epsilon|^2)^2\}dx + O(\epsilon^2) \\
&= -\frac{\partial}{\partial y_q^{(j)}} E_\epsilon(u_\epsilon) + O(\epsilon^2),
\end{aligned}
\tag{6.19}
$$

$$
\left\langle \mathcal{L}(v_\epsilon), \frac{\partial u_\epsilon}{\partial y_q^{(j)}} \right\rangle = \int_{\partial\Omega} \frac{\partial v_\epsilon}{\partial \nu}\frac{\partial u_\epsilon}{\partial y_q^{(j)}} d\sigma - \int_{\partial\Omega} v_\epsilon \frac{\partial}{\partial \nu}\frac{\partial u_\epsilon}{\partial y_q^{(j)}} d\sigma + \left\langle v_\epsilon, \mathcal{L}(\partial u_\epsilon / \partial y_q^{(j)}) \right\rangle.
$$

Recall

$$
\frac{\partial v_\epsilon}{\partial \nu} = -\frac{\partial u_\epsilon}{\partial \nu} = O(\epsilon^2), \qquad x \in \partial\Omega.
$$

Moreover applying Lemmas 4.2 and 5.1, we obtain

$$
\left| \langle v_\epsilon, \mathcal{L}(\partial u_\epsilon / \partial y_q^{(j)}) \rangle \right| = \left| \left\langle v_\epsilon, \frac{\partial}{\partial y_q^{(j)}} \mathcal{F}(u_\epsilon) \right\rangle \right| \leq C_1 \max\{\log(1/\epsilon), \epsilon^2/\rho^2\} \|v_\epsilon\|_0,
$$

and

$$
\left| \left\langle N(v_\epsilon), \frac{\partial u_\epsilon}{\partial y_q^{(j)}} \right\rangle \right| \leq C_1 \frac{1}{\epsilon^2} \|v_\epsilon\|_0^2.
$$

Hence from the assumption (6.14)

$$
\left\langle \mathcal{F}(u_\epsilon + v_\epsilon), \frac{\partial u_\epsilon}{\partial y_q^{(j)}} \right\rangle = -\frac{\partial}{\partial y_q^{(j)}} E_\epsilon(u_\epsilon) + o(1)
\tag{6.20}
$$

follows.

We apply Lemma 4.3 to the left hand side of (6.17) and use (5.2) of Lemma 5.1 in (6.20) to obtain

$$
\log(1/\epsilon)\dot{y}_q^{(j)} = -\frac{1}{\pi}\frac{\partial}{\partial y_q^{(j)}} V(\boldsymbol{y}) + o(1),
\tag{6.21}
$$

which is the desired equation. □

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