A NOTE TO THE REGULARITY OF SOLUTIONS FOR THE EVOLUTION P-LAPLACIAN EQUATIONS

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In this note we consider the following Cauchy problem:

\begin{equation}
\begin{cases}
\frac{\partial u}{\partial t} = \text{div}(|\nabla u|^{p-2}\nabla u) & (x,t) \in Q_T = R^N \times (0,T) \\
u(x,0) = u_0(x) & x \in R^N.
\end{cases}
\end{equation}

where $p > 2$, $u_0 \in L^1_{\text{loc}}(R^N)$ satisfies that there exist constants $r > 0$ and $\rho_0 > 0$ such that

\begin{equation}
||u_0||_{r,\rho_0} \equiv \sup_{x_0 \in R^N} \int_{B_\rho(x_0)} |u_0(x)| dx < \infty, \quad B_\rho(x_0) = \{|x - x_0| < \rho\}.
\end{equation}

It is well known that there exists a solution $u \in C^0_{\text{loc}}(Q_T) \cap L^\infty_{\text{loc}}(Q_T), \quad \nabla u \in C^{1,\frac{\alpha}{p}}_{\text{loc}}(Q_T)$ to (1) (see [C],[DF],[DH]). The proofs of $\nabla u \in C^{1,\frac{\alpha}{p}}_{\text{loc}}(Q_T)$ are very complex and difficult. In this note we use another approach to prove the Hölder continuity of $\nabla u$. We prove $u_t \in L^\infty_{\text{loc}}(Q_T), \quad \nabla u \in C^{1,\frac{\alpha}{1+p}}_{\text{loc}}(Q_T)$, where the Hölder index to $t$ is greater than $\beta$.

**DEFINITION.** A function $u(x,t)$ defined in $Q_T$ is called a weak solution of (1), if $u \in C^0_{\text{loc}}(Q_T) \cap L^p(0,T; W^{1,p}_{\text{loc}}(R^N)) \cap L^\infty(Q_T)$, $\alpha \in (0,1)$ and for any $\phi(x,t) \in C^1(Q_T)$, $\phi = 0$ if $|x|$ large enough,

\begin{equation}
\int_{R^N} u(x,t)\phi(x,t) dx + \int_0^t \int_{R^N} [-u\phi_t + |\nabla u|^{p-2}\nabla u \cdot \nabla \phi] dx dt = \int_{R^N} u_0(x)\phi(x,0) dx.
\end{equation}

We obtain the following result.

**THEOREM 1.** Let $u_0 \geq 0$, $||u_0(x)||_{r,\rho_0} < \infty$ for some $\rho_0 > 0$. Then there exist constants $C > 0$ and $\beta \in (0,1)$ such that the solution of (1) satisfies

\begin{equation}
\sup_{R^N} |u_t(x,t)| \leq Ct^{-\frac{N(p-1)+p}{\alpha}} ||u_0||_{r,\rho_0}^{\frac{\alpha}{p}}, \quad \nabla u \in C^{1,\frac{\alpha}{1+p}}_{\text{loc}}(Q_T),
\end{equation}

where $\kappa = N(p-2) + p$.

**Proof.** Let $u$ be the solution of (1). According to [WZYL], for $\forall \delta \in (0,T)$, $u(x,t + \delta)$ is the limit of the solutions of the following boundary value problems

\begin{equation}
\begin{cases}
\frac{\partial v}{\partial t} = \text{div}(|\nabla v|^{p-2}\nabla v) & (x,t) \in B_n \times (0,T - \delta) \\
v(x,t) = u(x,\delta) & (x,t) \in \partial B_n \times (0,T - \delta) \\
v(x,0) = u(x,\delta) & x \in B_n
\end{cases}
\end{equation}

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where $B_n = \{ |x| < n \}$. Let $v_n$ be the solution of (5). Set
\[ w_n(x,t) = \lambda^\gamma v_n(x,\lambda t), \quad \lambda > 1, \quad \gamma = \frac{1}{p - 2}. \]
Then $w_n$ satisfies
\[
\begin{aligned}
\frac{\partial w}{\partial t} &= \text{div}(|\nabla w|^{p-2}\nabla w) \\
(x,t) &\in B_n \times (0, \frac{T}{\lambda}) \\
w(x,0) &= \lambda^\gamma u(x,\delta) \\
w(x,t) &\in B_n.
\end{aligned}
\]
Set $g_n = w_n - v_n$. By Comparison principle $g_n \geq 0$ and
\[
\begin{align*}
\int_{B_n} g_n(x,t)\phi(x,t)dx &= \int_0^t \int_{B_n} g_n\phi_t dxdt + \int_0^t \int_{B_n} (|\nabla w_n|^{p-2}\nabla w_n - |\nabla v_n|^{p-2}\nabla v_n) \cdot \nabla \phi dxdt \\
&- \int_{B_n} (\lambda^\gamma - 1)u(x,\delta)\phi(x,0)dx \\
\phi &\in C^1(B_n \times (0,T)) \quad \phi = 0 \text{ near } \partial B_n.
\end{align*}
\]
where $\phi \in C^1(B_n \times (0,T)) \quad \phi = 0 \text{ near } \partial B_n$. Notice that (see [DH])
\[
\|u(x,\delta)\|_{L^\infty(\mathbb{R}^N)} \leq C\delta^{-\frac{\alpha}{2}}\|u_0\|_{L^\alpha(\mathbb{R}^N)}^{\beta}.
\]
In (7), we take
\[
\phi = (g_n - k) + \quad k = (\lambda^\gamma - 1)\|u(x,\delta)\|_{L^\infty(\mathbb{R}^N)}.
\]
Using Steklov averaging process, we get
\[
\begin{align*}
\int_{B_n} (g_n - k)^2 dx &= +2\int_0^t \int_{B_n \cap \{w > k\}} (|\nabla w_n|^{p-2}\nabla w_n - |\nabla v_n|^{p-2}\nabla v_n)(\nabla w_n - \nabla v_n)dxdt = 0.
\end{align*}
\]
This implies $g_n \leq k$ a.e. on $B_n \times (0, \frac{T}{\lambda})$. Thus
\[
0 \leq \lambda^\gamma v_n(x,\lambda t) - v_n(x,t) \leq (\lambda^\gamma - 1)\|u(x,\delta)\|_{L^\infty(\mathbb{R}^N)}.
\]
Divided (8) by $\lambda - 1$ and let $\lambda \to 1^+$, we get
\[
|\lambda^\gamma v_n(x,t) + tv_n(x,t)| \leq \gamma\|u(x,\delta)\|_{L^\infty(\mathbb{R}^N)}.
\]
This inequality implies
\[
|u_t(x,t + \delta)| \leq \frac{C\|u(x,\delta)\|_{L^\infty(\mathbb{R}^N)}}{t}.
\]
Let $\delta \to t$, we get the first estimate of (4).
We now prove the second estimate of (4). Notice that for fixed $t \in (0,T)$ $u(x,t)$ is a solution of the following elliptic equations
\[
\text{div}(|\nabla u|^{p-2}\nabla u) = u_t(x,t) \quad x \in \mathbb{R}^N.
\]
By [T], there exist constants $\beta \in (0,1)$, $C > 0$ dependent only on $|u_t|_{L^\infty}$, $|u|_{L^\infty}$ such that
\[
|\nabla u(x_1,t) - \nabla u(x_2,t)| \leq C|x_1 - x_2|^\beta.
\]

We now prove that $\nabla u$ is Hölder continuous to $t$. For convenience, we assume that $u$ is a smooth solution, otherwise by uniqueness of solution we can consider the regularized problem. Take the $x_j$-derivative in (1) to obtain

$$\frac{\partial u_{x_j}}{\partial t} = (\text{div}(|\nabla u|^{p-2}\nabla u))_{x_j}$$

Let $x_0 \in \mathbb{R}^N$, $0 < t_1 \leq t_2$, $\Delta t = t_2 - t_1$, $B(\Delta t) = B(\Delta t) \times (t_1, t_2)$. Integrating (10) over $B(\Delta t) \times (t_1, t_2)$ and by integrating by parts, we get

$$\int_{B(\Delta t)} (u_{x_j}(x, t_2) - u_{x_j}(x, t_1)) \, dx$$

$$= \int_{t_1}^{t_2} \int_{\partial B(\Delta t)} (\text{div}(|\nabla u|^{p-2}\nabla u))_{x_j} \, d\sigma \, dt$$

$$= \int_{t_1}^{t_2} \int_{\partial B(\Delta t)} \nabla u \cdot \nu_j \, d\sigma \, dt$$

where $\nu = (\nu_1, \nu_2, ..., \nu_N)$ is the unit outward normal vector of $\partial B(\Delta t)$. By the mean value theorem, there exists $x^* \in B(\Delta t)$ such that

$$|u_{x_j}(x^*, t_2) - u_{x_j}(x^*, t_1)| \leq C(\Delta t)^{1-\delta}.$$ 

Combining (9) and (12) and taking $\delta = \frac{1}{1+\beta}$, we get

$$|u_{x_j}(x_0, t_2) - u_{x_j}(x_0, t_1)| \leq |u_{x_j}(x_0, t_2) - u_{x_j}(x^*, t_2)|$$

$$+ |u_{x_j}(x^*, t_2) - u_{x_j}(x^*, t_1)| + |u_{x_j}(x^*, t_1) - u_{x_j}(x_0, t_1)| \leq C(\Delta t)^{\frac{\beta}{1+\beta}}.$$ 

Therefore $u_{x_j} \in C^{1, \frac{\beta}{1+\beta}}_{loc} (\mathbb{R}^N)$ and Theorem 1 is proved.

**Remark 1.** If the initial value $u_0$ is bounded, Theorem 1 holds for $u_0$ of variable sign. In fact if $u$ is a solution of (1), by the uniqueness of solution $v = u + \|u_0\|_{L^\infty(\mathbb{R}^N)}$ is a nonnegative solution of (1) with initial value $u_0 + \|u_0\|_{L^\infty(\mathbb{R}^N)}$. Thus Theorem 1 holds for $v$, so does $u$.

**Remark 2.** For the first boundary value problem, similar theorem holds.

Consider the following problem

$$\left\{ \begin{array}{ll}
\frac{\partial u}{\partial t} = \text{div}(|\nabla u|^2 \nabla u) & (x, t) \in \Omega \times (0, T) \\
u(x, t) = \psi(x, t) & (x, t) \in \partial \Omega \times (0, T) \\
u(x, 0) = u_0(x) & x \in \mathbb{R}^N
\end{array} \right.$$ 

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded region.

**Theorem 2.** Let $u_0 \in L^\infty(\mathbb{R}^N)$, $\psi, \psi_t \in L^\infty(\partial \Omega \times (0, T))$. Then the solution $u$ of (13) satisfies

$$|u_t(x, t)| \leq \frac{C}{t}, \quad \nabla u \in C^{1, \frac{\beta}{1+\beta}}_{loc} (\Omega \times (0, T)).$$
Proof. Without loss of generality, we assume \( u \geq 0 \), and \( u \) large enough, otherwise replace \( u \) by \( u + C \), \( C > \|u\|_{L^\infty} \). Set
\[
v(x,t) = \lambda^\gamma u(x, \lambda t), \quad \lambda > 1, \quad \gamma = \frac{1}{p-2}.
\]
Then \( v \) is the solution of (13) with
\[
v(x,t) = \lambda^\gamma \psi(x, \lambda t) \quad (x, t) \in \partial \Omega \times (0, T), \quad v(x, 0) = \lambda^\gamma u_0(x) \quad x \in \mathbb{R}^N.
\]
Notice that if \( \lambda - 1 \) is small enough, \( \psi \) large enough, we have
\[
\lambda^\gamma \psi(x, \lambda t) - \psi(x, t) = (\lambda^\gamma - 1)\psi(x, \lambda t) + \psi_t(x, \xi)(\lambda - 1)t
\]
\[
= (\lambda - 1)(\frac{\lambda^\gamma - 1}{\lambda - 1} \psi(x, \lambda t) + t\psi_t(x, \xi)) > 0.
\]
By comparison principle \( \lambda^\gamma u(x, \lambda t) \geq u(x, t) \). Hence similar to the proof in Theorem 1, we can prove Theorem 2.

REFERENCES


