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CONVERGENCE ANALYSIS OF RELAXATION SCHEMES FOR CONSERVATION LAWS WITH STIFF SOURCE TERMS

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Abstract. We analyze the convergence for relaxation approximation applied to conservation laws with stiff source terms. We suppose that the source term $q(u)$ is dissipative. Semi-implicit relaxing schemes are investigated and the corresponding stability theory is established. In particular, we proved that the numerical solution of a first-order relaxing scheme is uniformlly l^∞, l^1 and TV-stable, in the sense that they can be bounded by a constant independent of the the relaxation parameter ϵ and the the Lipschitz constant of the stiff source term, and time step Δt. Concergence of the relaxing scheme is then established. The results obtained for the first-order relaxing scheme can be extended to MUSCL relaxing schemes.

1. Introduction. We consider the following Cauchy problem

$$\begin{cases} u_t + f(u)_x = q(u) & x \in \mathbf{R}, \quad t > 0\,, \\ u(x,0) = u_0(x) & x \in \mathbf{R}\,, \end{cases} \tag{1.1}$$

where $f \in C^1(\mathbb{R})$, $f(0) = 0$ and $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$. The conservation law (1.1) is stiff if the time scale introduced by the source term q is small compared with the characteristic speed f' and some other appropriate length scale. It is observed that a realistic assumption on the source term is $q'(u) \leq 0$ for all $u \in R$. It is indeed the case for many practical problems. e.g. in the model of combustion [4, 9], water waves in presence of the friction force in the bottom [24]. This assumption is also used in many theoretical papers, for example, Chalabi [2], Chen, Levermore and Liu [3], Tang [19] and Schroll and Winther [14]. In the sense of Chen, Levermore and Liu [3], $q' < 0$ means the dissipativity of the source term. Furthermore, as usual, we assume that $u = 0$ is an equilibrium. Hence, throughout this paper we assume that

$$q(0) = 0, \quad -K \leq q'(u) \leq 0, \quad \text{for some constant } K \gg 1 \tag{1.2}$$

We want to approximate the global weak entropy solution of the Cauchy problem (1.1) by relaxation schemes. The system (1.1) can be related to a singular perturbation problem:

$$\begin{cases} u_t^\varepsilon + v_x^\varepsilon = q(u^\varepsilon) \\ v_t^\varepsilon + \mathsf{a} u_x^\varepsilon = -\dfrac{1}{\varepsilon}\Big(v^\varepsilon - f(u^\varepsilon)\Big), & \varepsilon > 0\,. \end{cases} \tag{1.3}$$

The relaxation limit for 2×2 nonlinear systems of conservation laws (without the source term) was first studied by Liu [8], who justified some nonlinear stability criteria for diffusion waves, expansion waves and traveling waves. A general mathematical framework was analyzed for the nonlinear systems by Chen, Levermore and Liu [3]. The presence of relaxation mechanisms is widespread in both the continuum mechanics as well as the kinetic theory contexts. Relaxation is known to provide a subtle

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dissipative mechanism for discontinuities against the destabilizing effect of nonlinear response [8]. The relaxation models can be loosely interpreted as discrete velocity kinetic equations. The relaxation parameter, ε, plays the role of the mean free path and the system models the macroscopic conservation law. In that sense they are a discrete velocity analogue of the kinetic equations introduced by Perthame and Tadmor [13] and Lions et al. [10].

On the numerical side, relaxation schemes proposed by Jin and Xin [5] are a class of nonoscillatory numerical schemes for systems of conservation laws. They provide a new way of approximating the solutions of the nonlinear conservation laws. The computational results that are available, see e.g. Jin and Xin [5] as well as Aregba-Driollet and Natalini[1], indicate that the relaxation schemes obtained in the limit $\varepsilon \longrightarrow 0$ provide a promising class of schemes. The main advantages of these schemes are that they require neither the computation of the Jacobians of fluxes for the conservation laws nor the use of Riemann-solvers. This important property is shared by other schemes such as the high resolution central schemes introduced by Nessyahu and Tadmor [12].

For homogeneous conservation laws, there have been many recent studies concerning the asymptotic convergence of the relaxation systems to the corresponding equilibrium conservation laws as the rate of the relaxation tends to zero. Katsoulakis and Tzavaras [6] introduced a class of relaxation systems, the contractive relaxation systems, and established an $\mathcal{O}(\sqrt{\varepsilon})$ error bound in the case that the equilibrium equation is a scalar multi-dimensional one. Kurganov and Tadmor [7] studied convergence and error estimates for a class of relaxation systems, including (1.3) with $q \equiv 0$, and concluded an $\mathcal{O}(\varepsilon)$ order of convergence for scalar convex conversation laws. The novelty of their approach is the use of a weak Lip'-measure of the error, which allows them to obtain sharp error estimates. For the relaxation system (1.3) with $q \equiv 0$, Natalini [11] proved that the solutions to the relaxation system converges strongly to the unique entropy solution of the corresponding conservation laws as $\varepsilon \longrightarrow 0$. Based on a general framework developed in [20, 16], the $\mathcal{O}(\varepsilon)$ rate of convergence in L^1 of Teng [22] and pointwisely away from the shock discontinuity of Tadmor and Tang [17] are established for (1.3) with $q \equiv 0$ in the case when the equilibrium solutions are piecewise smooth. The convergence theory for the relaxing scheme (2.1) and scheme (2.6) with $g \equiv 0$ can be found in [1, 18, 23, 25].

In this research, we wish to analyze a fully-discretized semi-implicit scheme for approximating the relaxation system (1.3). It is the semi-implicit treatment that makes the CFL condition independent of the Lipschitz constant of the stiff source term. We show that the solutions of the numerical scheme, $(u_j^{n,\varepsilon}, v_j^{n,\varepsilon})$, are l^∞, l^1 and TV-bounded by a constant independent of the relaxation parameter ε and the Lipschitz constant of the stiff source term q. Then it can be shown that $(u_j^{n,\varepsilon}, v_j^{n,\varepsilon})$ converge to the solution of the corresponding relaxed scheme. Due to the limitation of space, the convergence rate of the numerical schemes will not be investigated in this work. It will be reported elsewhere.

2. Numerical schemes. We choose a time step Δt, a spatial mesh size Δx, a parameter a which will be related to the characteristic speed of the conservation law and a small relaxation parameter $\varepsilon > 0$. For these we define the mesh ratio $\lambda = \Delta t/\Delta x$, the CFL parameter $\mu = \sqrt{\mathsf{a}}\lambda \in (0,1)$ and the scale parameter $k = \Delta t/\varepsilon$. The mesh is given by the points $(j\Delta x, n\Delta t)$ for $j \in \mathbb{Z}$, $u \in \mathbb{N}_0$. The approximatate solution takes the discrete values $u_j^{n,\varepsilon}$ at the mesh points. Furthermore, relaxing schemes involve the discrete relaxation fluxes $v_j^{n,\varepsilon}$. For stiff problems, most of the numerical methods are semi-implicit, which are related to operator splitting (explicit method for the homogeneous conservation laws and implicit method for the stiff ODEs). Theoretical analysis for the semi-implicit methods have been given by Chalabi [2], Schroll and Winther [14], Tang [19]. In this work we will concentrate on a semi-implicit relaxing scheme:

$$
\begin{aligned}
& u_j^{n+1,\varepsilon} - u_j^{n,\varepsilon} + \frac{\lambda}{2}\left(v_{j+1}^{n,\varepsilon} - v_{j-1}^{n,\varepsilon}\right) \\
& \qquad - \frac{\mu}{2}\left(u_{j+1}^{n,\varepsilon} - 2u_j^{n,\varepsilon} + u_{j-1}^{n,\varepsilon}\right) = q\left(u_j^{n+1,\varepsilon}\right)\Delta t \\
& v_j^{n+1,\varepsilon} - v_j^{n,\varepsilon} + \frac{\mathsf{a}\lambda}{2}\left(u_{j+1}^{n,\varepsilon} - u_{j-1}^{n,\varepsilon}\right) \\
& \qquad - \frac{\mu}{2}\left(v_{j+1}^{n,\varepsilon} - 2v_j^{n,\varepsilon} + v_{j-1}^{n,\varepsilon}\right) = -k\left(v_j^{n+1,\varepsilon} - f\left(u_j^{n+1,\varepsilon}\right)\right)
\end{aligned}
\tag{2.1}
$$

The discrete initial data are given by averaging the initial data u_0 over mesh cells $I_j = \left(\left(j-\frac{1}{2}\right)\Delta x, \left(j+\frac{1}{2}\right)\Delta x\right)$, i.e. taking

$$
u_j^{0,\varepsilon} = \frac{1}{\Delta x}\int_{I_j} u_0(x)\,dx, \quad \text{and setting } v_j^{0,\varepsilon} = f(u_j^{0,\varepsilon}) \tag{2.2}
$$

In this paper we will show that the solution, $(u_j^{n,\varepsilon}, v_j^{n,\varepsilon})$, of the relaxing scheme (2.1) converge to the solution, (u_j^n, v_j^n), of the relaxed scheme

$$
\begin{cases}
v_j^n = f\left(u_j^n\right) \\
u_j^{n+1} - u_j^n + \frac{\lambda}{2}\left(v_{j+1}^n - v_{j-1}^n\right) - \frac{\mu}{2}\left(u_{j+1}^{n+1} - 2u_j^n + u_{j-1}^n\right) = q\left(u_j^{n+1}\right)\Delta t
\end{cases}
\tag{2.3}
$$

with initial data

$$
u_j^0 = \frac{1}{\Delta x}\int_{I_j} u_0(x)\,dx, \tag{2.4}
$$

The relaxed scheme (2.3) is a consistent, conservative and monotone scheme approximating the conservation law (1.1).

The main results to be shown in this work are the following: If the constant a in the relaxing scheme (2.1) satisfies

$$
\max_{|\xi|\le \mathbf{M}} f'(\xi)^2 \le \mathsf{a}, \qquad \text{where} \quad \mathbf{M} := 4\|\mathrm{u}_0\|_{\mathrm{BV}}
$$

then there exists a constant C *independent* of the relaxation parameter ε and the Lipschitz constant of the stiff source q such that

$$
\begin{aligned}
TV(u^{n,\varepsilon}) &\le C, & TV(v^{n,\varepsilon}) &\le C, \\
\|u^{n,\varepsilon}\|_{l^\infty} &\le C, & \|v^{n,\varepsilon}\|_{l^\infty} &\le C, \\
\|u^{n,\varepsilon}\|_{l^1} &\le C, & \|v^{n,\varepsilon}\|_{l^1} &\le C.
\end{aligned}
$$

We will further show that there exists a constant $C(K)$ *independent* of the relaxation parameter ε and time step Δt (but may depend on the Lipschitz constant of the stiff source q) such that

$$
\begin{aligned}
\sum_j \left|u_j^{n+1,\varepsilon} - u_j^n\right|\Delta x &\le C(K)\Delta t, \\
\sum_j \left|v_j^{n+1,\varepsilon} - v_j^n\right|\Delta x &\le C(K)\Delta t.
\end{aligned}
$$

With the above estimates, we can show that the solutions of the relaxing scheme (2.1) converge to the solutions of the relaxed scheme (2.3). Then the piewise constant function $u_\Delta(x,t)$ constructed by the solution, u_j^n, of the relaxed scheme (2.3) converges to the entropy solution of the Cauchy problem (1.1).

A class of more accurate schemes, the MUSCL relaxing schemes, were proposed by Jin and Xin [5]:

$$
\begin{aligned}
\frac{u_j^{n+1,\epsilon}-u_j^{n,\epsilon}}{\Delta t} &+ \frac{1}{2\Delta x}(v_{j+1}^{n,\epsilon}-v_{j-1}^{n,\epsilon}) - \frac{\sqrt{\mathsf{a}}}{2\Delta x}(u_{j+1}^{n,\epsilon}-2u_j^{n,\epsilon}+u_{j-1}^{n,\epsilon}) \\
&+ \frac{1-\beta}{4}\left[(\sigma_j^{+,\epsilon}-\sigma_{j-1}^{+,\epsilon})-(\sigma_{j+1}^{-,\epsilon}-\sigma_j^{-,\epsilon})\right] = q\left(u_j^{n+1,\epsilon}\right), \\
\frac{v_j^{n+1,\epsilon}-v_j^{n,\epsilon}}{\Delta t} &+ \frac{\mathsf{a}}{2\Delta x}(u_{j+1}^{n,\epsilon}-u_{j-1}^{n,\epsilon}) - \frac{\sqrt{\mathsf{a}}}{2\Delta x}(v_{j+1}^{n,\epsilon}-2v_j^{n,\epsilon}+v_{j-1}^{n,\epsilon}) \\
&+ \frac{\sqrt{\mathsf{a}}(1-\beta)}{4}\left[(\sigma_j^{+,\epsilon}-\sigma_{j-1}^{+,\epsilon})+(\sigma_{j+1}^{-,\epsilon}-\sigma_j^{-,\epsilon})\right] \\
&= -\frac{1}{\epsilon}(v_j^{n+1,\epsilon}-f(u_j^{n+1,\epsilon}))
\end{aligned}
\tag{2.5}
$$

where $\sigma_j^{\pm,\epsilon}$ and $\theta_j^{\pm,\epsilon}$ are defined by

$$
\begin{aligned}
\sigma_j^{\pm,\epsilon} &= \frac{1}{\Delta x}\Delta_+(v_j^{n,\epsilon}\pm\sqrt{\mathsf{a}}u_j^{n,\epsilon})\phi(\theta_j^{\pm,\epsilon}) \\
\theta_j^{\pm,\epsilon} &= \frac{\Delta_-(v_j^{n,\epsilon}\pm\sqrt{\mathsf{a}}u_j^{n,\epsilon})}{\Delta_+(v_j^{n,\epsilon}\pm\sqrt{\mathsf{a}}u_j^{n,\epsilon})}
\end{aligned}
$$

and $\beta=\mu=\sqrt{\mathsf{a}}\Delta t/\Delta x$, and $\Delta_\pm u_j = \mp(u_j-u_{j\pm1})$. The corresponding relaxed scheme as $\varepsilon\to 0$ limit of (2.5) is as follows

$$
\begin{aligned}
& v_j^n = f(u_j^n) \triangleq f_j^n, \\
& \frac{u_j^{n+1}-u_j^n}{\Delta t} + \frac{1}{2\Delta x}(v_{j+1}^n-v_{j-1}^n) - \frac{\sqrt{\mathsf{a}}}{2\Delta x}(u_{j+1}^n-2u_j^n+u_{j-1}^n) \\
& \quad +\frac{1-\beta}{4}\left[(\sigma_j^+-\sigma_{j-1}^+)-(\sigma_{j+1}^--\sigma_j^-)\right] = q\left(u_j^{n+1}\right).
\end{aligned}
\tag{2.6}
$$

To gurantee the entropy consistency of the relaxed scheme (2.6) the following slightly stronger conditions are proposed in Tang et al. [21]:

$$\sqrt{\mathsf{a}}\frac{\Delta t}{\Delta x} = \mu < 1; \qquad \text{CFL condition} \tag{2.7}$$

$$\sup_u |f'(u)| \le \frac{1}{\alpha}\sqrt{\mathsf{a}}; \qquad \text{subcharacteristic condition} \tag{2.8}$$

$$0 \le \frac{\phi(\theta)}{\theta} \le X, \qquad 0 \le \phi(\theta) \le X, \qquad \text{limiter function condition.} \tag{2.9}$$

The parameters α and X in the conditions (2.7)-(2.9) are required to satisfy

$$\alpha > 1, \quad 0 < X < 2, \quad 1-\frac{1}{\alpha} \ge X(1-\mu). \tag{2.10}$$

In [21], it has been shown that under the assumptions (2.7)-(2.9) the second-order relaxed scheme (2.6) with $q \equiv 0$ satisfies the cell entropy inequalities. As a consequence, the L^1 convergence rate $\mathcal{O}(\sqrt{\Delta t})$ for the relaxed scheme is established.

In this work, we will restrict our attention to the study of the relaxing scheme (2.1) and its corresponding relaxed scheme (2.3). We wish to point out that the results obtained in this work can be extended to the MUSCL relaxing schemes (2.5)-(2.6).

3. Stability Properties of the Relaxing Scheme. This section is devoted to establishing the l^∞-stability, the bound of the total variation, and the l^1-stability of the numerical solution for the relaxing scheme (2.1) with initial data (2.2). To begin with, we take the Riemann invariants

$$\begin{pmatrix} R_{1,j}^{n,\varepsilon} \\ R_{2,j}^{n,\varepsilon} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(u_j^n - \frac{v_j^{n,\varepsilon}}{\sqrt{\mathsf{a}}}\right) \\ \frac{1}{2}\left(u_j^n + \frac{v_j^{n,\varepsilon}}{\sqrt{\mathsf{a}}}\right) \end{pmatrix}. \tag{3.1}$$

It follows from the above equations that

$$u_j^{n,\varepsilon} = R_{1,j}^{n,\varepsilon} + R_{2,j}^{n,\varepsilon}, \quad v_j^{n,\varepsilon} = \sqrt{\mathsf{a}}\left(R_{2,j}^{n,\varepsilon} - R_{1,j}^{n,\varepsilon}\right). \tag{3.2}$$

Then the relaxing scheme (2.1) can be rewritten as

$$\begin{cases} R_{1,j}^{n+1,\varepsilon} = R_{1,j}^{n+\frac{1}{2},\varepsilon} + \frac{k}{2}\left[R_{2,j}^{n+1,\varepsilon} - R_{1,j}^{n+1,\varepsilon} - \frac{1}{\sqrt{\mathsf{a}}}f\left(R_{1,j}^{n+1,\varepsilon} + R_{2,j}^{n+1,\varepsilon}\right)\right] \\ \qquad + \frac{\Delta t}{2}q\left(R_{1,j}^{n+1,\varepsilon} + R_{2,j}^{n+1,\varepsilon}\right) \\ R_{2,j}^{n+1,\varepsilon} = R_{2,j}^{n+\frac{1}{2},\varepsilon} - \frac{k}{2}\left[R_{2,j}^{n+1,\varepsilon} - R_{1,j}^{n+1,\varepsilon} - \frac{1}{\sqrt{\mathsf{a}}}f\left(R_{1,j}^{n+1,\varepsilon} + R_{2,j}^{n+1,\varepsilon}\right)\right] \\ \qquad + \frac{\Delta t}{2}q\left(R_{1,j}^{n+1,\varepsilon} + R_{2,j}^{n+1,\varepsilon}\right) \end{cases} \tag{3.3}$$

with

$$\begin{pmatrix} R_{1,j}^{n+\frac{1}{2},\varepsilon} \\ R_{2,j}^{n+\frac{1}{2},\varepsilon} \end{pmatrix} = \begin{pmatrix} (1-\mu)R_{1,j}^{n,\varepsilon} + \mu R_{1,j+1}^{n,\varepsilon} \\ (1-\mu)R_{2,j}^{n,\varepsilon} + \mu R_{2,j-1}^{n,\varepsilon} \end{pmatrix}. \tag{3.4}$$

It follows from $u_0 \in BV(\mathbb{R})$ that there exists a constant $\mathbf{M}_1$ such that

$$\|u_0\|_{BV} \le \mathbf{M}_1 . \tag{3.5}$$

By the definition of the initial data $u_j^{0,\varepsilon}$,

$$TV(u^0) = \sum_j \left|u_{j+1}^0 - u_j^0\right| \le \mathbf{M}_1 . \tag{3.6}$$

We first assume that

$$\max_{|\xi| \le \|u_0\|_\infty} f'(\xi)^2 \le \mathsf{a}, \tag{3.7}$$

where the constant a will be determined latter. It follows from (3.6)-(3.7) and the definition of the initial data $u_j^{0,\varepsilon}$ that

$$
\begin{aligned}
TV(R_1^0, R_2^0) &:= \sum_j \left(\left| R_{1,j+1}^{0,\varepsilon} - R_{1,j}^{0,\varepsilon} \right| + \left| R_{2,j+1}^{0,\varepsilon} - R_{2,j}^{0,\varepsilon} \right| \right) \\
&\leq \sum_j \left(\left| u_{j+1}^{0,\varepsilon} - u_j^{0,\varepsilon} \right| + \frac{1}{\sqrt{\mathsf{a}}} \max_{|\xi| \leq \|u_0\|_\infty} |f'(\xi)| \left| u_{j+1}^{0,\varepsilon} - u_j^{0,\varepsilon} \right| \right) \\
&\leq 2\mathbf{M}_1 .
\end{aligned} \tag{3.8}
$$

Next, we choose the parameter a satisfying the following subcharacteristic condition (cf. Liu [8])

$$
\max_{|\xi| \leq \mathbf{M}} f'(\xi)^2 \leq \mathsf{a} \tag{3.9}
$$

where $\mathbf{M} = 4\mathbf{M}_1 = 4\|u_0\|_{BV}$. Since

$$
|u_0(x)| \leq \|u_0\|_{BV} + u_0(-\infty) \leq \mathbf{M}_1 < \mathbf{M},
$$

the condition (3.9) also includes our earlier assumption (3.7). In the remaining of this section, we will need the following facts:

$$
u_0(-\infty) = 0 \quad \text{and} \quad v_0(-\infty) = f(u_0(-\infty)) = 0 . \tag{3.10}
$$

They are justified by assumptions $u_0 \in L^1(R) \cap BV(R)$ and $f(0) = 0$.

LEMMA 3.1. *Under the subcharacteristic condition* (3.9), *the relaxing scheme* (3.3) *is TVD (total variation diminishing), i.e.*

$$
\begin{aligned}
TV\left(R_1^{n+1,\varepsilon}, R_2^{n+1,\varepsilon}\right) &:= \sum_j \left(\left| R_{1,j+1}^{n+1,\varepsilon} - R_{1,j}^{n+1,\varepsilon} \right| + \left| R_{2,j+1}^{n+1,\varepsilon} - R_{2,j}^{n+1,\varepsilon} \right| \right) \\
&\leq TV\left(R_1^{n,\varepsilon}, R_2^{n,\varepsilon}\right) \leq \frac{1}{2}\mathbf{M} .
\end{aligned} \tag{3.11}
$$

Proof. We prove the lemma by induction. We need to prove the following: If

$$
TV\left(R_1^{n,\varepsilon}, R_2^{n,\varepsilon}\right) \leq \frac{1}{2}\mathbf{M}, \tag{3.12}
$$

$$
\sup_{j \in \mathbb{Z}} \left| u_j^{n,\varepsilon} \right| \leq \mathbf{M}, \tag{3.13}
$$

then the following estimates will hold:

$$
TV\left(R_1^{n+1,\varepsilon}, R_2^{n+1,\varepsilon}\right) \leq TV\left(R_1^{n,\varepsilon}, R_2^{n,\varepsilon}\right) \leq \frac{1}{2}\mathbf{M}, \tag{3.14}
$$

$$
\sup_{j \in \mathbb{Z}} \left| u_j^{n+1,\varepsilon} \right| \leq \mathbf{M} . \tag{3.15}
$$

We first observe that the assumption (3.12) with $n = 0$ is true due to the estimate (3.8), and the assumption (3.13) with $n = 0$ is true due to (3.6) and (3.10). Adding the two equations of (3.3) gives

$$
u_j^{n+1,\varepsilon} = (1-\mu)\left(R_{1,j}^{n,\varepsilon} + R_{2,j}^{n,\varepsilon}\right) + \mu\left(R_{2,j-1}^{n,\varepsilon} + R_{1,j+1}^{n,\varepsilon}\right) + q(u_j^{n+1,\varepsilon})\Delta t \tag{3.16}
$$

where the equations (3.1) and (3.4) are used. The induction assumption (3.12) yields

$$\begin{aligned} |R_{1,j}^{n,\varepsilon}| + |R_{2,j}^{n,\varepsilon}| &\leq TV\,(R_1^{n,\varepsilon}, R_2^{n,\varepsilon}) + \lim_{j\to-\infty} |R_{1,j}^{n,\varepsilon}| + \lim_{j\to-\infty} |R_{2,j}^{n,\varepsilon}| \\ &\leq TV\,(R_1^{n,\varepsilon}, R_2^{n,\varepsilon}) \leq \frac{1}{2}\mathbf{M}, \end{aligned} \tag{3.17}$$

where we have used the facts that

$$\lim_{j\to-\infty} |R_{i,j}^{n,\varepsilon}| = \lim_{j\to\infty} \left|R_{i,j}^{0,\varepsilon}\right| = \lim_{x\to-\infty} \frac{1}{2}\left(u_0(x) + (-1)^i \frac{f(u_0(\cdot))}{\sqrt{\mathsf{a}}}\right) = 0, \quad i = 1,2.$$

It follows from the above two results, i.e. (3.16) and (3.17) that

$$\left|u_j^{n+1,\varepsilon} - q\left(u_j^{n+1,\varepsilon}\right)\Delta t\right| \leq \mathbf{M}.$$

Then it follows from the assumption $q(0) = 0$ and $q'(u) \leq 0$ for $u \in \mathbf{R}$ that

$$\left|u_j^{n+1,\varepsilon}\right| = \left(1 - q'\left(\xi_j^{n+1,\varepsilon}\right)\Delta t\right)^{-1} \mathbf{M} \leq \mathbf{M}, \quad j \in \mathbb{Z}, \tag{3.18}$$

which verifies (3.15). Set $\overline{R}_{i,j} = R_{i,j+1} - R_{i,j}, \quad i = 1,2$. Subtracting $(3.3)_j$ from $(3.3)_{j+1}$ gives

$$\begin{cases} \left(1 + \frac{k}{2}\left(1 + \frac{f'(\zeta)}{\sqrt{\mathsf{a}}}\right) - q'(\eta)\frac{\Delta t}{2}\right)\overline{R}_{1,j}^{n+1,\varepsilon} \\ \quad - \left(\frac{k}{2}\left(1 - \frac{f'(\zeta)}{\sqrt{\mathsf{a}}}\right) + q'(\eta)\frac{\Delta t}{2}\right)\overline{R}_{2,j}^{n+1,\varepsilon} = \overline{R}_{1,j}^{n+\frac{1}{2},\varepsilon}, \\ - \left(\frac{k}{2}\left(1 + \frac{f'(\zeta)}{\sqrt{\mathsf{a}}}\right) + q'(\eta)\frac{\Delta t}{2}\right)\overline{R}_{1,j}^{n+1,\varepsilon} \\ \quad + \left(1 + \frac{k}{2}\left(1 - \frac{f'(\zeta)}{\sqrt{\mathsf{a}}}\right) - q'(\eta)\frac{\Delta t}{2}\right)\overline{R}_{2,j}^{n+1,\varepsilon} = \overline{R}_{2,j}^{n+\frac{1}{2},\varepsilon} \end{cases} \tag{3.19}$$

where ζ is an intermediate value between $u_{j+1}^{n+1,\varepsilon}$ and $u_j^{n+1,\varepsilon}$, which and (3.18) yield that $|\zeta| \leq \mathbf{M}$. Thus the subcharacteristic condition (3.9) can be applied to obtain

$$\begin{aligned} \alpha &= \frac{k}{2}\left(1 - \frac{f'(\zeta)}{\sqrt{\mathsf{a}}}\right) \geq 0, \\ \gamma &= \frac{k}{2}\left(1 + \frac{f'(\zeta)}{\sqrt{\mathsf{a}}}\right) \geq 0. \end{aligned}$$

Solving the equation (3.19) gives

$$\begin{aligned} \overline{R}_{1,j}^{n+1,\varepsilon} &= \frac{1}{\mathcal{A}}\left[(1 + \alpha - q'(\eta)\Delta t/2)\,\overline{R}_{1,j}^{n+\frac{1}{2},\varepsilon} + (\alpha + q'(\eta)\Delta t/2)\,\overline{R}_{2,j}^{n+\frac{1}{2},\varepsilon}\right], \\ \overline{R}_{2,j}^{n+1,\varepsilon} &= \frac{1}{\mathcal{A}}\left[(\gamma + q'(\eta)\Delta t/2)\,\overline{R}_{1,j}^{n+\frac{1}{2},\varepsilon} + (1 + \gamma - q'(\eta)\Delta t/2)\,\overline{R}_{2,j}^{n+\frac{1}{2},\varepsilon}\right], \end{aligned}$$

where $\mathcal{A} = (1+k)(1 - q'(\eta)\Delta t)$. Using the above equations gives

$$\begin{aligned} \left|\overline{R}_{1,j}^{n+1,\varepsilon}\right| + \left|\overline{R}_{2,j}^{n+1,\varepsilon}\right| \leq \frac{1}{\mathcal{A}}\Big\{ &[1 + \alpha - q'(\eta)\Delta t/2 + |\gamma + q'(\eta)\Delta t/2|]\left|\overline{R}_{1,j}^{n+\frac{1}{2},\varepsilon}\right| \\ &+ [1 + \gamma - q'(\eta) + |\alpha + q'(\eta)\Delta t/2|]\left|\overline{R}_{2,j}^{n+\frac{1}{2},\varepsilon}\right|\Big\} \end{aligned} \tag{3.20}$$

where we have used the facts that$\alpha \geq 0$, $\gamma \geq 0$ and $q'(\eta) \leq 0$. Observe that

$$\begin{aligned} &1+\alpha-q'(\eta)\Delta t/2+|\gamma+q'(\eta)\Delta t/2| \\ &\leq \max(1+k, 1+\alpha-\gamma-q'(\eta)\Delta t) \\ &\leq (1+k)(1-q'(\eta)\Delta t)=\mathcal{A}, \end{aligned}$$

and

$$\begin{aligned} &1+\gamma-q'(\eta)\Delta t/2+|\alpha+q'(\eta)\Delta t/2| \\ &\leq \max(1+k, 1+\gamma-\alpha-q'(\eta)\Delta t) \leq \mathcal{A}. \end{aligned}$$

The above results, together with (3.20), yield

$$\begin{aligned} \left|\overline{R}^{n+1,\varepsilon}_{1,j}\right|+\left|\overline{R}^{n+1,\varepsilon}_{2,j}\right| &\leq \left|\overline{R}^{n+\frac{1}{2},\varepsilon}_{1,j}\right|+\left|\overline{R}^{n+\frac{1}{2},\varepsilon}_{2,j}\right| \\ &\leq \mu\left|\overline{R}^{n,\varepsilon}_{2,j-1}\right|+(1-\mu)\left(\left|\overline{R}^{n,\varepsilon}_{1,j}\right|+|\overline{R}^{n,\varepsilon}_{2,j}|\right)+\mu\left|\overline{R}^{n,\varepsilon}_{1,j+1}\right|. \end{aligned} \tag{3.21}$$

Summation of (3.21) over j gives (3.14). This finishes the induction and the proof of this lemma is thereby complete. □

Having the above lemma, we are now ready to state and prove the following theorem on the uniform boundedness of the relaxing solutions $(u_j^{n,\varepsilon}, v_j^{n,\varepsilon})$.

THEOREM 3.1. *Under the subcharacteristic condition* (3.9), *the numerical solutions* $(u_j^{n,\varepsilon}, v_j^{n,\varepsilon})$ *of the relaxing scheme* (2.1) *satisfy the following estimates:*

- TV-stability:

$$\begin{aligned} TV(u^{n,\varepsilon}) &= \sum_j |u^{n,\varepsilon}_{j+1}-u^{n,\varepsilon}_j| \leq \frac{1}{2}\mathrm{M}, \\ TV(v^{n,\varepsilon}) &= \sum_j |v^{n,\varepsilon}_{j+1}-v^{n,\varepsilon}_j| \leq \frac{\sqrt{\mathsf{a}}}{2}\mathrm{M}; \end{aligned} \tag{3.22}$$

- l^∞-stability:

$$\sup_j |u_j^{n,\varepsilon}| \leq \frac{1}{2}\mathrm{M}, \qquad \sup_j |v_j^{n,\varepsilon}| \leq \frac{1}{2}\sqrt{\mathsf{a}}\mathrm{M}; \tag{3.23}$$

- l^1-stability:

$$\sum_j |u_j^{n,\varepsilon}|\Delta x \leq 2\|u_0\|_{L^1}, \qquad \sum_j |v_j^{n,\varepsilon}|\Delta x \leq 2\sqrt{\mathsf{a}}\|u_0\|_{L^1}. \tag{3.24}$$

Proof. The TV-stability (3.22) follows directly from the transformation (3.2) and Lemma 3.1. The TV-bounds and the facts $u_j^{n,\varepsilon} \to 0$ and $v_j^{n,\varepsilon} \to 0$ as $j \to \infty$ lead to the l^∞ estimates of the numerical solutions. We now need to prove the l^1-stability (3.24). Using the assumptions $f(0) = q(0) = 0$ and the mean value theorem for $f(u^{n+1,\varepsilon})$ and $q(u^{n+1,\varepsilon})$ in the scheme (3.3) we obtain

$$\begin{aligned} (1+\gamma-q'(\eta)\Delta t/2)\, R^{n+1,\varepsilon}_{1,j}-(\alpha+q'(\eta)\Delta t/2)\, R^{n+1,\varepsilon}_{2,j} &= R^{n+\frac{1}{2},\varepsilon}_{1,j}, \\ -(\gamma+q'(\eta)\Delta t/2)\, R^{n+1,\varepsilon}_{1,j}+(1+\alpha-q'(\eta)\Delta t/2)\, R^{n+1,\varepsilon}_{2,j} &= R^{n+\frac{1}{2},\varepsilon}_{2,j}, \end{aligned}$$

with

$$\alpha = \frac{k}{2}\left(1 - \frac{f'(\xi)}{\sqrt{\mathsf{a}}}\right) \geq 0\,,$$

$$\gamma = \frac{k}{2}\left(1 + \frac{f'(\xi)}{\sqrt{\mathsf{a}}}\right) \geq 0\,, \tag{3.25}$$

where ξ and η are intermediate values between $u_j^{n+1,\varepsilon}$ and 0. Similar to the proof for (3.11) we can obtain from the above equations that

$$\begin{aligned}
&\sum_j \left(|R_{1,j}^{n,\varepsilon}| + |R_{2,j}^{n,\varepsilon}|\right)\Delta x \\
&\leq \sum_j \left(|R_{1,j}^{n-1,\varepsilon}| + |R_{2,j}^{n-1,\varepsilon}|\right)\Delta x \\
&\leq \sum_j \left(|R_{1,j}^{0,\varepsilon}| + |R_{2,j}^{0,\varepsilon}|\right)\Delta x\,.
\end{aligned}$$

By using the relation $u_j^{n,\varepsilon} = R_{1,j}^{n,\varepsilon} + R_{2,j}^{n,\varepsilon}$, we have

$$\begin{aligned}
\sum_j |u_j^{n,\varepsilon}|\Delta x &\leq \sum_j \left(|R_{1,j}^{n,\varepsilon}| + |R_{2,j}^{n,\varepsilon}|\right)\Delta x \\
&\leq \sum_j \left(|R_{1,j}^{0,\varepsilon}| + |R_{2,j}^{0.\varepsilon}|\right)\Delta x \\
&\leq \sum_j \left(|u_j^{0,\varepsilon}| + \left|\frac{f(u_j^{0,\varepsilon})}{\sqrt{\mathsf{a}}}\right|\right)\Delta x \\
&\leq \sum_j \left(1 + \frac{|f'(\xi)|}{\sqrt{\mathsf{a}}}\right)|u_j^{0,\varepsilon}|\Delta x \leq 2\|u_0\|_{L^1}\,.
\end{aligned}$$

Similarly, using $v_j^{n,\varepsilon} = \sqrt{\mathsf{a}}(R_{1,j}^{n,\varepsilon} - R_{2,j}^{n,\varepsilon})$ leads to the second equation of (3.24). □

4. Convergence Analysis. In this section, we will discuss the convergence of the relaxing scheme. In order to carry out the convergence analysis, the continuity of the numerical solution in time and the difference between $v^{n,\varepsilon}$ and $f(u^{n,\varepsilon})$ need to be investigated.

LEMMA 4.1. *Under the subcharacteristic condition* (3.9), *the solutions of the relaxing scheme* (3.3) *satisfy:*

$$\begin{aligned}
&\sum_j \left(\left|R_{1,j}^{n+1,\varepsilon} - R_{1,j}^{n,\varepsilon}\right| + \left|R_{2,j}^{n+1,\varepsilon} - R_{2,j}^{n,\varepsilon}\right|\right)\Delta x \\
&\qquad \leq \left\{2K\|u_0\|_{L^1} + (K\mu + 3\sqrt{\mathsf{a}})\mathrm{M}\right\}\Delta t, \qquad n \in \mathbb{N}_0.
\end{aligned} \tag{4.1}$$

Proof. Set $\widetilde{R}_{i,j}^{n+1,\varepsilon} = R_{i,j}^{n+1,\varepsilon} - R_{i,j}^{n,\varepsilon}, \quad i = 1, 2$. Substracting $(3.3)^n$ from $(3.3)^{n+1}$

gives

$$(4.2)\qquad \begin{cases} \left(1+\gamma-q'(\eta)\frac{\Delta t}{2}\right)\widetilde{R}_{1,j}^{n+1,\varepsilon}-\left(\alpha+q'(\eta)\frac{\Delta t}{2}\right)\widetilde{R}_{2,j}^{n+1,\varepsilon} \\ =\widetilde{R}_{1,j}^{n+\frac{1}{2},\varepsilon}=\mu\widetilde{R}_{1,j+1}^{n,\varepsilon}+(1-\mu)\widetilde{R}_{1,j}^{n,\varepsilon}, \\ -\left(\gamma+q'(\eta)\frac{\Delta t}{2}\right)\widetilde{R}_{1,j}^{n+1,\varepsilon}+\left(1+\alpha-q'(\eta)\frac{\Delta t}{2}\right)\widetilde{R}_{2,j}^{n+1,\varepsilon} \\ =\widetilde{R}_{2,j}^{n+\frac{1}{2},\varepsilon}=\mu\widetilde{R}_{2,j-1}^{n,\varepsilon}+(1-\mu)\widetilde{R}_{2,j}^{n,\varepsilon}. \end{cases}$$

where

$$\alpha=\frac{k}{2}\left(1-\frac{f'(\xi)}{\sqrt{a}}\right),\quad \gamma=\frac{k}{2}\left(1+\frac{f'(\xi)}{\sqrt{a}}\right),$$

and ξ is an intermediate value between $u_j^{n+1,\varepsilon}$ and u_j^n. Therefore $|\xi|\le M/2$ by (3.23). Thus the subcharacteristic condition (3.9) implies $\alpha\ge 0,\gamma\ge 0$. Using the techniques similar to those used in the last section we can show that

$$(4.3)\qquad I^{n+1}:=\sum_j\left(\left|\widetilde{R}_{1,j}^{n+1,\varepsilon}\right|+\left|\widetilde{R}_{2,j}^{n+1,\varepsilon}\right|\right)\le\sum_j\left(\left|\widetilde{R}_{1,j}^{n,\varepsilon}\right|+\left|\widetilde{R}_{2,j}^{n,\varepsilon}\right|\right).$$

Now we need to estimate I^1. Using (3.3) and (3.4), i.e. the relaxing scheme for $R_1^{n,\varepsilon}$ and $R_2^{n,\varepsilon}$, with $n=0$ we obtain

$$(4.4)\qquad \begin{aligned} I^1 &\le \sum_j\mu\left(\left|R_{1,j+1}^{0,\varepsilon}-R_{1,j}^{0,\varepsilon}\right|+\left|R_{2,j}^{0,\varepsilon}-R_{2,j-1}^{0,\varepsilon}\right|\right)\\ &\quad+\sum_j k\left|\left(R_{2,j}^{1,\varepsilon}-R_{1,j}^{1,\varepsilon}\right)-f(R_{1,j}^{1,\varepsilon}+R_{2,j}^{1,\varepsilon})/\sqrt{a}\right|\\ &\quad+\sum_j\left|g\left(R_{1,j}^{1,\varepsilon}+R_{2,j}^{1,\varepsilon}\right)\Delta t\right|\\ &= I_1^1+I_2^1+I_3^1. \end{aligned}$$

The initial condition $v_j^{0,\varepsilon}=f(u_j^{0,\varepsilon})$ in (2.2) is equivalent to

$$(R_{2,j}^{0,\varepsilon}-R_{1,j}^{0,\varepsilon})-f(R_{1,j}^{0,\varepsilon}+R_{2,j}^{0,\varepsilon})/\sqrt{a}=0.$$

Using the above identity we obtain

$$\begin{aligned} &k\left[\left(R_{2,j}^{1,\varepsilon}-R_{1,j}^{1,\varepsilon}\right)-f(R_{1,j}^{1,\varepsilon}+R_{2,j}^{1,\varepsilon})/\sqrt{a}\right]\\ &=k\left[\left(R_{2,j}^{1,\varepsilon}-R_{1,j}^{1,\varepsilon}\right)-f(R_{1,j}^{1,\varepsilon}+R_{2,j}^{1,\varepsilon})/\sqrt{a}-\left(R_{2,j}^{0,\varepsilon}-R_{1,j}^{0,\varepsilon}\right)+f(R_{1,j}^{0,\varepsilon}+R_{2,j}^{0,\varepsilon})/\sqrt{a}\right]\\ &=-2\gamma_j\widetilde{R}_{1,j}^{1,\varepsilon}+2\alpha_j\widetilde{R}_{2,j}^{1,\varepsilon}, \end{aligned}$$

with

$$\alpha_j=\frac{k}{2}\left(1-\frac{f'(\xi_j)}{\sqrt{a}}\right),\quad \gamma_j=\frac{k}{2}\left(1+\frac{f'(\xi_j)}{\sqrt{a}}\right),$$

where ξ_j is an intermediate value between $u_j^{1,\varepsilon}$ and $u_j^{0,\varepsilon}$. Therefore, it follows from the subcharacteristic condition (3.9) that

$$0 \le \alpha_j, \quad \gamma_j \le k, \quad |\alpha_j - \gamma_j| \le k.$$

Using the relaxing scheme (3.3) with $n = 0$, we obtain

$$\begin{aligned} & k\Big[\left(R_{2,j}^{1,\varepsilon} - R_{1,j}^{1,\varepsilon}\right) - f\left(R_{2,j}^{1,\varepsilon} + R_{1,j}^{1,\varepsilon}\right)/\sqrt{\mathsf{a}}\Big] \\ = & -2\mu\left(\gamma_j(R_{1,j+1}^{0,\varepsilon} - R_{1,j}^{0,\varepsilon}) + \alpha_j(R_{2,j}^{0,\varepsilon} - R_{2,j-1}^{0,\varepsilon})\right) \\ & -k(\gamma_j + \alpha_j)\Big[\left(R_{2,j}^{1,\varepsilon} - R_{1,j}^{1,\varepsilon}\right) - f\left(R_{2,j}^{1,\varepsilon} + R_{1,j}^{1,\varepsilon}\right)/\sqrt{\mathsf{a}}\Big] \\ & +(\alpha_j - \gamma_j)q\left(R_{1,j}^{1,\varepsilon} + R_{2,j}^{1,\varepsilon}\right)\Delta t\,, \end{aligned}$$

which gives that

$$\begin{aligned} & k\Big|\left(R_{2,j}^{1,\varepsilon} - R_{1,j}^{1,\varepsilon}\right) f\left(R_{2,j}^{1,\varepsilon} + R_{1,j}^{1,\varepsilon}\right)/\sqrt{\mathsf{a}}\Big| \\ \le & \frac{2\mu k}{1+k}\left(\left|R_{1,j+1}^{0,\varepsilon} - R_{2,j}^{0,\varepsilon}\right| + \left|R_{2,j}^{0,\varepsilon} - R_{2,j-1}^{0,\varepsilon}\right|\right) + \frac{k}{1+k}\left|q\left(R_{1,j}^{1,\varepsilon} + R_{2,j}^{1,\varepsilon}\right)\right|\Delta t \end{aligned} \tag{4.5}$$

Summation of the inequality in (4.5) over j gives

$$I_2^1 \le \frac{k}{1+k}\left(2I_1^1 + I_3^1\right) \tag{4.6}$$

Using the definition of the discrete initial data (2.2), the assumption on the source term of $-K \le q'(u) \le 0$ and the relaxing scheme on $u_j^{n,\varepsilon}$ (2.1) with $n = 0$ we obtain

$$u_j^{1,\varepsilon}\left(1 - \Delta t g'(\xi)\right) = u_j^{0,\varepsilon} + \mu\left(R_{1,j+1}^{0,\varepsilon} - R_{1,j}^{0,\varepsilon}\right) - \mu\left(R_{2,j}^{0,\varepsilon} - R_{2,j-1}^{0,\varepsilon}\right)$$

where ξ is an intermediate value between 0 and $u_j^{1,\varepsilon}$. Therefore,

$$\left|u_j^{n+1,\varepsilon}\right| \le \left|u_j^{n,\varepsilon}\right| + \mu\left(\left|R_{1,j+1}^{n,\varepsilon} - R_{1,j}^{n,\varepsilon}\right| + \left|R_{2,j}^{n,\varepsilon} - R_{2,j-1}^{n,\varepsilon}\right|\right)$$

and

$$\begin{aligned} I_3^1 &= \lambda\sum_j\left|q(u_j^{1,\varepsilon})\right|\Delta x \\ &\le \lambda K\sum_j\left|u_j^{1,\varepsilon}\right|\Delta x \\ &\le \lambda K\Big(\sum_j|u_j^{0,\varepsilon}|\Delta x + \mu\Delta x\sum_j\left|R_{1,j+1}^{0,\varepsilon} - R_{1,j}^{0,\varepsilon}\right| + \left|R_{2,j}^{0,\varepsilon} - R_{2,j-1}^{0,\varepsilon}\right|\Big) \\ &\le \lambda K\sum_j\left|\int_{I_j} u_0(x)dx\right| + \lambda K\mu\Delta x TV\left(R_1^{0,\varepsilon}, R_2^{0,\varepsilon}\right) \\ &\le \lambda K\left(\|u_0\|_{L^1} + \mu\Delta x\mathbf{M}/2\right) \end{aligned} \tag{4.7}$$

Substitute (4.6) and (4.7) into (4.4) gives

$$\begin{aligned} I^1 &\le I_1^1 + \frac{k}{1+k}\left(2I_1^1 + I_3^1\right) + I_3^1 \\ &\le 2\lambda K\|u_0\|_{L^1} + (\lambda K\mu + 3\mu)\mathbf{M}\,, \end{aligned} \tag{4.8}$$

provided that $\Delta x \leq 1$. Then the estimate (4.1) follows from (4.3) and (4.8). □

By the transformation (3.2), the following corollary is an immediate consequence of Lemma 4.1.

COROLLARY 4.1. *Assume that the subcharacteristic condition* (3.9) *holds. Then the solutions of the relaxing scheme* (2.1) *satisfy*

$$\sum_j \left|u_j^{n+1,\varepsilon} - u_j^{n,\varepsilon}\right|\Delta x \leq \left\{2K\|u_0\|_{L^1} + (K\mu + 3\sqrt{\mathrm{a}})\mathbf{M}\right\}\Delta t, \tag{4.9}$$

$$\sum_j \left|v_j^{n+1,\varepsilon} - v_j^{n,\varepsilon}\right|\Delta x \leq \sqrt{\mathrm{a}}\left\{2K\|u_0\|_{L^1} + (K\mu + 3\sqrt{\mathrm{a}})\mathbf{M}\right\}\Delta t. \tag{4.10}$$

Next, we consider the difference between $v^{n,\varepsilon}$ and $f(u^{n,\varepsilon})$. The following result will be useful in the convergence analysis for ε tends to zero.

LEMMA 4.2. *Assume that the subcharacteristic condition* (3.9) *holds. Then the solution of the relaxing scheme* (2.1) *with initial data* (2.2) *satisfy*

$$\begin{aligned} \|v^{n,\varepsilon} - f(u^{n,\varepsilon})\|_1 &:= \sum_j \left|v_j^{n,\varepsilon} - f\left(u_j^{n,\varepsilon}\right)\right|\Delta x \\ &\leq \sqrt{\mathrm{a}}\left(2K\|u_0\|_{L^1} + (K\mu + 4\sqrt{\mathrm{a}})\mathbf{M}\right)\varepsilon. \end{aligned} \tag{4.11}$$

Proof. It follows from the second equation of the scheme (2.1) that

$$\begin{aligned} &k\sum_j \left|v_j^{n+1,\varepsilon} - f\left(u_j^{n+1,\varepsilon}\right)\right|\Delta x \\ &\leq \sum_j \left|v_j^{n+1,\varepsilon} - v_j^n\right|\Delta x + \frac{\sqrt{\mathrm{a}}\mu}{2}\sum_j \left|u_{j+1}^n - u_{j-1}^n\right|\Delta x + \frac{\mu}{2}\sum_j \left|v_{j+1}^n - 2v_j^n + v_{j-1}^n\right|\Delta x. \end{aligned}$$

Note $k = \Delta t/\varepsilon$ and $\mu \in (0,1)$. Then the desired estimate (4.11) follows from the BV-boundedness of $u^{n,\varepsilon}$ and $v^{n,\varepsilon}$ and Corollary 4.1. □

We are now ready to state and to prove the following main theorem of this section.

THEOREM 4.1. *Under the subcharacteristic condition* (3.9), *the solutions of the relaxing scheme* (3.3) *converge to the solutions of the relaxed scheme* (2.3) *as* ε *tends to zero for fixed* Δt. *Furthermore, the solutions of the relaxed scheme* (2.3) *satisfy the following estimates*

$$\|u^n\|_{l^\infty} \leq \frac{1}{2}\mathbf{M}, \qquad \|v^n\|_{l^\infty} \leq \frac{1}{2}\sqrt{\mathrm{a}}\mathbf{M}, \tag{4.12}$$

$$TV(u^n) \leq \frac{1}{2}\mathbf{M}, \qquad TV(v^n) \leq \frac{1}{2}\sqrt{\mathrm{a}}\mathbf{M}, \tag{4.13}$$

$$\sum_j \left|u_j^{n+1} - u_j^n\right|\Delta x \leq \left\{2K\|u_0\|_{L^1} + (K\mu + 3\sqrt{\mathrm{a}})\mathbf{M}\right\}\Delta t, \tag{4.14}$$

$$\sum_j \left|v_j^{n+1} - v_j^n\right|\Delta t \leq \sqrt{\mathrm{a}}\left\{2K\|u_0\|_{L^1} + (K\mu + 3\sqrt{\mathrm{a}})\mathbf{M}\right\}\Delta t, \tag{4.15}$$

for all nonnegative integer n.

Proof. Define the linear interpolant of the relaxing solutions:

$$(u^{n,\varepsilon}(x), v^{n,\varepsilon}(x)) = \sum_j \left(u_j^{n,\varepsilon}, v_j^{n,\varepsilon}\right) \chi_{[x_j-\Delta x/2, x_j+\Delta x/2)}(x),$$

where $\chi_{[a,b)}$ is the characteristic function on the interval $[a,b)$. It follows from Theorem 3.1 that $(u^{n,\varepsilon}(\bullet), v^{n,\varepsilon}(\bullet))$, $n \in \mathbb{N}_0$ are bounded piecewise constant functions of bounded variation uniformly with respect to n and ε. By Helley's theorem for each fixed n and a standard diagonal process, there exists a subsequence $(u^{n,\varepsilon_i}(x), v^{n,\varepsilon_i}(x))$ such that $(u^{n,\varepsilon_i}(x), v^{n,\varepsilon_i}(x))$ converges to a piecewise constant function

$$(u^n(x), v^n(x)) = \sum_j (u_j^n, v_j^n)\chi_{[x_j-\Delta x/2, x_j+\Delta x/2)}(x)$$

pointwisely for $n \in \mathbb{N}_0$ as $\varepsilon_i \to 0$. Therefore $(u_j^{n,\varepsilon_i}, v_j^{n,\varepsilon_i})$ converges to (u_j^n, v_j^n) as $\varepsilon_i \to 0$ for $j \in \mathbb{Z}$, $n \in \mathbb{N}_0$. Furthermore, it follows from (4.11) in Lemma 4.2 that

$$\sum_j \left|v_j^{n,\varepsilon_i} - f\left(u_j^{n,\varepsilon_i}\right)\right| \Delta x \le \sqrt{\mathrm{a}}\left(2K\|u_0\|_1 + (K\mu + 4\sqrt{\mathrm{a}})\mathrm{M}\right)\varepsilon_i .$$

Then by letting $\varepsilon_i \to 0$ we obtain

$$\sum_j \left|v_j^n - f\left(u_j^n\right)\right| \Delta x = 0\,,$$

which implies that

$$v_j^n = f(u_j^n) \quad \text{for } j \in \mathbb{Z},\ n \in \mathbb{N}_0. \tag{4.16}$$

Then taking the limit $\varepsilon_i \to 0$ in the first equation of the relaxing scheme (2.1) we obtain

$$u_j^{n+1} - u_j^n + \frac{\lambda}{2}\left(v_{j+1}^n - v_{j-1}^n\right) - \frac{\mu}{2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) = q\left(u_j^{n+1}\right)\Delta t. \tag{4.17}$$

The above two equations are exactly the relaxed scheme (2.3). The estimates (4.12)-(4.14) for the relaxed solutions follow from the results in Theorem 3.1 and Corollary 4.1 . □

REMARK 4.1. Note that (u_j^n, v_j^n) is uniquely determine by the relaxed scheme (2.3) and initial data (2.4). So the whole sequence $\left(u_j^{n,\varepsilon}, v_j^{n,\varepsilon}\right)$ converges to (u_j^n, v_j^n).

REMARK 4.2. Consider the piecewise constant function

$$u_\Delta(x,t) = \sum_j \sum_n u_j^n \chi_{[x_j-\Delta x/2, x_j+\Delta x/2)}(x)\chi_{[t_n, t_{n+1})}(t)\,,$$

for $-\infty < x < \infty$, $0 \le t < \infty$. Using the estimates for u_j^n in Theorem 4.1 and standard arguments of Helley's theorem, see e.g. Chapter 17 Smoller [15], we can show that the solution $u_\Delta(x,t)$ given by the relaxed scheme (2.3) converges to the entropy solution of conservation law (1.1).

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