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$C^{1,1}$ ESTIMATES FOR ELLIPTIC EQUATIONS WITH PARTIAL AND PIECEWISE CONTINUOUS COEFFICIENTS*

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Abstract. In the paper, we establish existence, uniqueness and optimal $C^{1,1}$ regularity of L^p-viscosity solutions of Dirichlet problem of linear elliptic equations with partially and piecewise Hölder coefficients. For piecewise Hölder coefficients, our $C^{1,1}$ estimates are independent of the distance between interfaces of discontinuity of the coefficients.

Key words. Discontinuous coefficients, uniqueness, $C^{1,1}$ estimates.

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1. Introduction. Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with boundary $\partial\Omega$ of class C^2. Consider the following second order elliptic differential operator

$$Lu := \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u, \quad a_{ij}(x) = a_{ji}(x),$$

where $x = (x_1, \cdots, x_n) \in \Omega$, and $\{a_{ij}\}$ is a Lebesgue measurable matrix-valued function on Ω, which satisfies the uniformly elliptic condition

$$\lambda|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \text{for any } \xi \in \mathbb{R}^n, \tag{1.1}$$

with $0 < \lambda \leq \Lambda < \infty$.

A fundamental question in elliptic equation theory is when the Dirichlet problem

$$Lu = f \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega \tag{1.2}$$

is uniquely solvable. From the classical Schauder theory [10], (1.2) has a unique classical solution provided the given boundary condition and coefficients are smooth. If $\{a_{ij}\}$ is only continuous, the L^p-Schauder theory or $W^{2,p}$ theory implies the unique solvability of (1.2). In [5], Chiarenza, Frasca and Longo discovered that the L^p-Schauder theory still holds if $\{a_{ij}\}$ belongs to the Sarason class VMO. If $\Omega = \mathbb{R}^n$, Krylov and D. Kim [13], [14] can reduce $\{a_{ij}\}$ to be of VMO only in partial variables. Most recently, Dong and D. Kim in [8] removed the condition $\Omega = \mathbb{R}^n$ and obtained L^p-Schauder theory for a general class of elliptic and parabolic equations of higher order.

For $\alpha \in (0, 1]$, the partial Hölder semi-norm of function u with respect to partial variables $x' = (x_1, \cdots, x_{n-1})$ in Ω is defined as

$$[u]_{C^{\alpha}_{x'}(\Omega)} = \sup_{t_* < x_n < t^*} \sup_{\substack{(x', x_n), (y', x_n) \in \Omega \\ x' \neq y'}} \frac{|u(x', x_n) - u(y', x_n)|}{|x' - y'|^{\alpha}},$$

where $t^* = \sup\{t : \{x_n = t\} \cap \Omega \neq \emptyset\}$ and $t_* = \inf\{t : \{x_n = t\} \cap \Omega \neq \emptyset\}$. We say a function $u \in L^\infty(\Omega)$ belongs to $C^{\alpha}_{x'}(\Omega)$ if $[u]_{C^{\alpha}_{x'}(\Omega)} < \infty$. Similarly, we define

$$[u]_{C^{k,\alpha}_{x'}(\Omega)} = [D^k_{x'} u]_{C^{\alpha}_{x'}(\Omega)},$$

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where $D_{x'} = (D_1, \cdots, D_{x_{n-1}})$ and k is a nonnegative integer. For $x = (x', x_n), \hat{x} = (\hat{x}', \hat{x}_n)$, let

$$\beta(x, \hat{x}) = \sum_{i,j} |a_{ij}(x', x_n) - a_{ij}(\hat{x}', x_n)|$$

and

$$\omega_{a_{ij}}(r) = \sup_{\hat{x}\in\Omega} \left(\frac{1}{|B_r(\hat{x}) \cap \Omega|} \int_{B_r(\hat{x})\cap\Omega} \beta(x, \hat{x})^n \, \mathrm{d}x \right)^{1/n}$$

We say $\{a_{ij}\}$ is uniformly continuous with respect to x' in Ω in L^n sense or $a_{ij} \in C_{x',L^n}(\Omega)$ if $\omega_{a_{ij}}(r) \to 0$ as $r \to 0$.

Next, we recall the definition of L^p-viscosity solution in [3].

Definition 1.1. *For $f \in L^p_{loc}(\Omega)$, $p > n/2$, a function $u \in C(\Omega)$ is an L^p-viscosity subsolution (supersolution) of $Lu = f$ in Ω if for any $\phi \in W^{2,p}_{loc}(\Omega)$ touching u from above (resp. below) at point $\hat{x}$ locally one has*

$$ess \limsup_{x\to\hat{x}} (a_{ij}(x)D_{ij}\phi(x) - f(x)) \geq 0$$
$$(ess \liminf_{x\to\hat{x}} (a_{ij}(x)D_{ij}\phi(x) - f(x)) \leq 0).$$

We say $u \in C(\Omega)$ is an L^p-viscosity of $Lu = f$ if u is both an L^p-viscosity subsolution and supersolution.

Note that the test functions in Definition 1.1 are of class $W^{2,p}_{loc}$ not C^2. When a_{ij} and f are continuous, viscosity solutions defined in [6] are equivalent to L^p-viscosity solutions, see [3] for more details.

Theorem 1.1. *Suppose that $a_{ij} \in C_{x',L^n}(\Omega)$ satisfies ellipticity condition (1.1) and $f \in L^p(\Omega)$ for some $p > n$. Then for every boundary value $\varphi \in C(\partial\Omega)$, there exists a unique L^p-viscosity solution $u \in W^{2,p}_{loc}(\Omega) \cap C(\overline{\Omega})$ of the Dirichlet problem (1.2). Moreover, for any $\Omega' \subset\subset \Omega$, we have*

$$\|u\|_{W^{2,p}(\Omega')} \leq C\{\|\varphi\|_{L^\infty(\partial\Omega)} + \|f\|_{L^p(\Omega)}\}, \tag{1.3}$$

where C is a positive constant depending only on $n, \lambda, \Lambda, p, \Omega$, $dist(\Omega', \Omega)$ and $\omega_{a_{ij}}(r)$.

If $a_{ij}, f \in C^\alpha_{x'}(\Omega)$, then $u \in C^{1,1}_{loc}(\Omega)$ and

$$\begin{aligned} &\|u\|_{C^{1,1}(\Omega')} + \|D_{x'}u\|_{C^{1,\alpha}(\Omega')} \\ &\qquad \leq C\{\|\varphi\|_{L^\infty(\partial\Omega)} + \|f\|_{L^\infty(\Omega)} + [f]_{C^\alpha_{x'}(\Omega)}\}, \end{aligned} \tag{1.4}$$

where C is a positive constant depending only on $n, \lambda, \Lambda, \Omega$, $dist(\Omega', \Omega)$ and $[a_{ij}]_{C^\alpha_{x'}(\Omega)}$.

When a_{ij} are constants, estimate (1.4) was obtained by Dong and S. Kim [9], Tian and Wang [21] independently. They also considered the right hand function f satisfying general partially Hölder conditions. Moreover, [9] established a version of Schauder estimates in partial variables.

The $C^{1,1}$ regularity in Theorem 1.1 and the following two theorems is optimal. Indeed, it is easy to verify that

$$u = x_1^2 - 3\chi_{\{x_2>0\}}x_2^2 - 5\chi_{\{x_2<0\}}x_2^2$$

is an L^p-viscosity solution (strong solution actually) of equation

$$A(x)D_{11}u + D_{22}u = 0 \quad \text{in } B_1(0),$$

where χ is the characteristic function and $A(x) = 3\chi_{\{x_2>0\}} + 5\chi_{\{x_2<0\}}$. However, u does not belong to $C^2(B_1)$.

THEOREM 1.2. *Let $\mathcal{M} \subset \mathbb{R}^n$ be an $(n-1)$-dimensional embedded (but not necessarily connected or compact) $C^{1,\alpha}$ hypersurface for some $\alpha \in (0,1)$. Suppose that $\mathcal{M} \cap \Omega \neq \emptyset$ and for any point $x \in \Omega$ there exists a positive constant r, depending on x, such that a_{ij} and f are uniformly Hölder continuous on every connected component of $B_r(x) \setminus \mathcal{M}$ but might be discontinuous cross $\mathcal{M}$. Then for every boundary value $\varphi \in C(\partial\Omega)$, there exists a unique L^p-viscosity solution $u \in C^{1,1}_{loc}(\Omega) \cap C(\overline{\Omega})$ of the Dirichlet problem (1.2).*

In view of the Krylov-Safonov's Hölder estimates (Chapter 9 of [10]), another natural way to define solutions of (1.2) is by approximating. Solutions defined by such a way are usually called *good solutions*. It follows from [3] that good solutions are L^p-viscosity solutions. Cerutti, Escauriaza and Fabes [4] established the uniqueness of good solution of (1.2) if a_{ij} are continuous except at a countable set of points having at most one accumulation point. Safonov [20] pointed out that if the Hausdorff dimension of discontinuous set is less than some positive constant $\delta = \delta(n, \lambda, \Lambda)$, then there is uniqueness, too. However, one cannot expect to extend Safonov's result to merely measurable a_{ij} because of the counterexample of Nadirashvili [19].

Elliptic equations (systems) of divergence form with piecewise Hölder coefficients have been studied extensively. A problem arising from composite material asks: Are interior L^∞ bounds for gradients of $W^{1,2}$ weak solutions independent of the distances between the surfaces of discontinuity of the coefficients? Li and Vogelius [16], Li and Nirenberg [15] gave a positive answer for elliptic equations and systems respectively. For nondivergent elliptic equations, we consider the following model: Let Ω_1, Ω_2 be two disjoint subdomains of Ω with $\text{dist}(\Omega_1, \Omega_2) = \varepsilon > 0$ and $\partial\Omega_1, \partial\Omega_2 \in C^{1,\beta}$ for some $\beta \in (0,1)$. Denote $\Omega_3 := \Omega \setminus (\overline{\Omega_1 \cup \Omega_2})$. Suppose that a_{ij} and f are uniformly Hölder continuous with exponent α in each subdomain, namely,

$$a_{ij}|_{\Omega_i}, f|_{\Omega_i} \in C^\alpha(\Omega_i), \quad i = 1, 2, 3.$$

Suppose also that a_{ij} satisfies elliptic condition (1.1). Then we have

THEOREM 1.3. *Assume the above conditions, and let u be an L^p-viscosity solution of $a_{ij}D_{ij}u = f$ in Ω. Then $u \in C^{1,1}_{loc}(\Omega)$ and for any $\Omega' \subset\subset \Omega$,*

$$0 < \alpha' \leq \min\{\alpha, \frac{\beta}{n(\beta+1)}\},$$

we have

$$\|u\|_{C^{1,1}(\Omega')} \leq C\{\|u\|_{L^\infty(\Omega)} + \sum_{m=1}^{3} \|f\|_{C^{\alpha'}(\Omega_m)}\}, \tag{1.5}$$

where C is a positive constant depending only on $n, \lambda, \Lambda, \|a_{ij}\|_{C^{\alpha'}(\Omega_m)}$, $dist(\Omega', \partial\Omega)$ and $C^{1,\beta}$ norms of the Ω_m but independent of ε.

Note that in Theorem 1.2 and Theorem 1.3, we only assume $\mathcal{M}$ and $\partial\Omega_1$ and $\partial\Omega_2$ are $C^{1,\alpha}$. Therefore, we cannot flat them to prove those theorems. As in [16] and [15], Theorem 1.3 also holds for multiple subdomains cases.

Remark 1.1. *By Theorem 1.3, we can weaken the condition of* $\mathcal{M}$ *in Theorem 1.2. For example,* $\mathcal{M} = B_{1/2}(\frac{1}{2}e_1) \cup B_{1/2}(-\frac{1}{2}e_1) \subset \Omega$*, where* $e_1 = (1, 0, \cdots, 0)$*. Indeed, we can approximate* $\mathcal{M}$ *by* $\mathcal{M}_\varepsilon = B_{1/2}((\frac{1}{2}+\varepsilon)e_1) \cup B_{1/2}(-(\frac{1}{2}+\varepsilon)e_1) \subset \Omega$ *for some small positive* ε*. Note that* $\mathcal{M}$ *is even not Lipschitz continuous.*

The paper is organized as follows: In section 2, we establish the existence and uniqueness of L^p-viscosity solution to the Dirichlet problem of good equations. In section 3, we prove several perturbation results. In section 4, the main theorems are proved.

Note: Recently, the estimate (1.4) is also obtained by Hongjie Dong [7] with partial Dini continuous coefficients. His method is completely different from ours. We would like to thank him for sending us the paper and some helpful comments.

2. Uniqueness and regularity for viscosity solutions of good equations. We begin with the following strong maximum principle.

Theorem 2.1. *Let* $u \in C(\overline{\Omega})$ *be an* L^p*-viscosity subsolution of* $Lu = 0$ *in* Ω*. If* u *attains its maximum in the interior of* Ω*, then* $u \equiv constant$*.*

Proof. Suppose that u attains its maximum M in the interior of Ω and is not identical to M. It follows that there exist some point $\hat{x} \in \Omega$ and positive constants r, R such that

$$\mathcal{T} = \{x \in \mathbb{R}^n : r < |x - \hat{x}| < R\} \subset \Omega, \quad \sup_{|x-\hat{x}|=r} u(x) < M$$

and $u(x_0) = M$ for some $x_0 \in \mathcal{T}$. Consider

$$v(x) = u(x) + \exp\{-\frac{\gamma}{2}|x - \hat{x}|^2\}, \quad x \in \mathcal{T},$$

where $\gamma > 0$ is large so that

$$\sup_{|x-\hat{x}|=r} v(x) < M.$$

For any point x on the outer boundary of $\mathcal{T}$, i.e., $|x - \hat{x}| = R$, we have

$$\begin{aligned} v(x) &= u(x) + \exp\{-\frac{\gamma}{2}|x - \hat{x}|^2\} \\ &\le u(x_0) + \exp\{-\frac{\gamma}{2}|x - \hat{x}|^2\} \\ &< v(x_0). \end{aligned}$$

It follows that $v(x)$ must reach its maximum at some point $\overline{x} \in \mathcal{T}$. By the definition of $v(x)$,

$$u(x) \le u(\overline{x}) + \exp\{-\frac{\gamma}{2}|\overline{x} - \hat{x}|^2\} - \exp\{-\frac{\gamma}{2}|x - \hat{x}|^2\} =: \phi(x).$$

Then $\phi \in C^2(\mathcal{T})$ touches u at $\overline{x}$ from above. By the definition of L^p-viscosity subsolution

$$\begin{aligned} 0 &\le ess \limsup_{x \to \overline{x}} a_{ij}(x) D_{ij}\phi(x) \\ &= ess \limsup_{x \to \overline{x}} \gamma \exp\{-\frac{\gamma}{2}|x - \hat{x}|^2\}(\sum_i a_{ii} - \gamma a_{ij}(x_i - \hat{x}_i)(x_j - \hat{x}_j)) \\ &\le ess \limsup_{x \to \overline{x}} \gamma \exp\{-\frac{\gamma}{2}|x - \hat{x}|^2\}(n\Lambda - \gamma\lambda r^2) \\ &< 0, \end{aligned}$$

provided $\gamma > \frac{n\Lambda}{\lambda r^2}$. The proof is completed. □

The following comparison principle was proved in [3] for fully nonlinear elliptic equations. Now it can be derived directly from Theorem 2.1.

Corollary 2.1. *Let $p > n/2$ and $f \in L^p(\Omega)$. Assume that $u, v \in C(\overline{\Omega})$ are two L^p-viscosity solutions of $Lu = f$ in Ω and $u = v$ on $\partial\Omega$. If $v \in W^{2,p}_{loc}(\Omega)$, then $u = v$ in Ω.*

Set $\mathcal{A}(\lambda, \Lambda)$ as the class of measurable matrix-valued functions $\{a_{ij}(x)\}$ satisfying (1.1).

Theorem 2.2. *Suppose $A_{ij} = A_{ij}(x_n) \in \mathcal{A}(\lambda, \Lambda)$ for $x_n \in (-1, 1)$ and $F = F(x_n) \in L^\infty((-1, 1))$. Let $B_1 = B_1(0) \subset \mathbb{R}^n$ be the unit ball centered at the origin and $\varphi \in C(\partial B_1)$. Then there exists a unique L^p-viscosity solution $u \in C(\overline{B}_1)$ of*

$$A_{ij}D_{ij}u = F \quad \text{in } B_1, \ u = \varphi \quad \text{on } \partial B_1. \tag{2.1}$$

Moreover, $u \in C^{1,1}_{loc}(B_1) \cap C^\infty_{x'}(B_1)$ and

$$\sum_{k=0}^{l} \|D^k_{x'} u\|_{C^{1,1}(\overline{B}_{1/2})} \le C\big\{\|u\|_{L^\infty(B_1)} + \|F\|_{L^\infty(B_1)}\big\}, \tag{2.2}$$

where C is a positive constant depending only on n, λ, Λ and l.

Our argument is based on the following boundary $C^{1,\alpha}$ estimate without assuming oscillation condition on the coefficients of L. It was initially established by L. Wang [22] for parabolic equations, while the elliptic version can be found in [17], see [18] for a proof.

Lemma 2.1. *Let $B_1^+ = B_1 \cap \{x_n > 0\}$. Suppose $u \in C^2(B_1^+)$ is a solution of $Lu = f \in L^n(B_1^+)$ in B_1^+ and $u(x', 0) = \varphi(x')$ is differentiable at the origin*

$$|\varphi(x') - \varphi(0) - D_{x'}\varphi(0)x'| \le N|x'|^{1+\alpha},$$

where $x' = (x_1, \cdots, x_{n-1})$ and $\alpha \in (0, 1)$. Assume

$$\|u\|_{L^\infty(B_1)} + \|f\|_{L^n(B_1)} \le 1.$$

Then there exist constants B, C and $\gamma = \gamma(\alpha)$ such that

$$|B| \le CN$$

and

$$|u(x) - u(0) - D_{x'}\varphi(0)x' - Bx_n| \le CN|x|^{1+\gamma},$$

where $C > 0$ depends only n, λ, Λ.

In fact, Lemma 2.1 holds for viscosity solutions.

Proof of Theorem 2.2. Existence. Choose smooth $A^\varepsilon_{ij}(x_n) \in \mathcal{A}(\lambda, \Lambda)$ such that $A^\varepsilon_{ij}(x_n) \to A_{ij}(x_n)$ *a.e.* in $[-1, 1]$ as $\varepsilon \to 0$. Analogously, find smooth functions $F^\varepsilon(x_n)$ and $\varphi^\varepsilon(x)$ such that $F^\varepsilon(x_n) \to F(x_n)$ *a.e.* in $[-1, 1]$ and $\varphi^\varepsilon \to \varphi$ uniformly on

∂B_1 as $\varepsilon \to 0$. According to the standard linear elliptic equation theory there exists a unique smooth solution u^ε of approximating Dirichlet problem

$$A^\varepsilon_{ij} D_{ij} u^\varepsilon = F^\varepsilon \text{ in } B_1, \quad u^\varepsilon = \varphi^\varepsilon \text{ on } \partial B_1.$$

Because of the Krylov-Safonov estimates for linear elliptic equations, there exists a function u^0 and a subsequence of u^ε, which will be still denoted as u^ε, such that $u^\varepsilon \to u^0$ locally uniformly as $\varepsilon \to 0$. By Theorem 3.8 of [3], $u^0 \in C(\overline{B}_1)$ is an L^p-viscosity solution to (2.1).

Regularity. As for the approximating solutions u^ε, we have

$$\|u^\varepsilon\|_{C^\alpha(\overline{B}_{7/8})} \le C\big\{\|u^\varepsilon\|_{L^\infty(B_1)} + \|F^\varepsilon\|_{L^n(B_1)}\big\}, \tag{2.3}$$

where $\alpha \in (0,1)$ and C depend only on n, λ, Λ.

Next, we intend to derive uniformly higher order derivatives estimates for u^ε independent of ε.

Let e_τ $(1 \le \tau < n)$ be the τ-th coordinate direction. Since A^ε_{ij} and F^ε depend only on x_n, for $\beta \in (0,1], 0 < |h| \le 1/16$,

$$w^\varepsilon_\beta(x) := \frac{1}{|h|^\beta}(u^\varepsilon(x - he_\tau) - u^\varepsilon(x))$$

satisfies

$$A^\varepsilon_{ij}(x) D_{ij} w^\varepsilon_\beta = 0 \quad \text{in } B_{7/8}. \tag{2.4}$$

Making use of (2.3) and arguing as Corollary 5.7 of [2], we have

$$\|D_{x'} u^\varepsilon\|_{C^\alpha(\overline{B}_{3/4})} \le C(n, \lambda, \Lambda)\big\{\|u^\varepsilon\|_{L^\infty(B_1)} + \|F^\varepsilon\|_{L^n(B_1)}\big\}.$$

Note that $D_{x_\tau} u^\varepsilon$ are classical solutions of (2.4). By bootstrapping we obtain

$$\sum_{k \le l} \|D^k_{x'} u^\varepsilon\|_{C^\alpha(\overline{B}_{1/2})} \le C\big\{\|u^\varepsilon\|_{L^\infty(B_1)} + \|F^\varepsilon\|_{L^n(B_1)}\big\}, \tag{2.5}$$

where $C > 0$ depends only on n, λ, Λ and $l(\ge 2)$. Thanks to Lemma 2.1, we have

$$\|D_n D_{x'} u^\varepsilon\|_{L^\infty(\overline{B}_{1/4})} \le C(n, \lambda, \Lambda)\big\{\|D^3_{x'} u^\varepsilon\|_{L^\infty(\overline{B}_{1/2})}\big\}. \tag{2.6}$$

Making use of equations and combining (2.5) and (2.6), the uniform L^∞ bound for $D_{nn} u^\varepsilon$ follows. By covering argument and letting $\varepsilon \to 0$, $u^0 \in C^{1,1}_{loc}(B_1) \cap C^\infty_{x'}(B_1)$ and estimate (2.2) holds for u^0.

Uniqueness. Since $u^0 \in C^{1,1}_{loc}(B_1)$, by Corollary 2.1 $u = u^0$ if and only if u is an L^p-viscosity solution of (2.1). □

Let

$$D_m = \{x \in \mathbb{R}^n : \ell_{m-1} < x_n < \ell_m\}, \quad m = 1, \cdots, \kappa,$$

where ℓ_m are increasing constants lying between -1 and 1 with $\ell_0 = -1$, $\ell_\kappa = 1$. Let $A_{ij}(x) = A^m_{ij}$ in D_m, where $\{A^m_{ij}\} \in \mathcal{A}(\lambda, \Lambda)$ are constant matrices. Similarly, F is a constant function in each D_m but may vary with D_m. As a special case of Theorem 2.2, we have

COROLLARY 2.2. *Assume A_{ij}, f are piecewise constants as above. Suppose $\varphi \in C(\partial B_1)$. Then there exists a unique L^p-viscosity solution u of*

$$A_{ij}D_{ij}u = F \quad in\ B_1,\ u = \varphi \quad on\ \partial B_1. \tag{2.7}$$

Moreover, $u \in C^{1,1}_{loc}(B_1) \cap C^\infty(\overline{D}_m \cap B_1)$ and

$$\|u\|_{C^{1,1}(B_{1/2})} + \|u\|_{C^k(\overline{D}_m \cap B_{1/2})} \le C\big\{\|u\|_{L^\infty(B_1)} + \|F\|_{L^\infty(B_1)}\big\}, \tag{2.8}$$

where C is a positive constant depending only on n, λ, Λ, k.

Note that the estimate (2.8) is independent of the width $\ell_m - \ell_{m-1}$ of stripe D_m and κ.

Proof of Corollary 2.2. In view of Theorem 2.2, there exists a unique L^p-viscosity solution $u \in C^{1,1}_{loc}(B_1) \cap C^\infty_{x'}(B_1)$ of Dirichlet problem (2.7). Since A_{ij} and F are constants in D_m, we have

$$A_{ij}D_{ij}D_n u = 0 \quad \text{in } D_m. \tag{2.9}$$

Note that $D_n u$ is C^∞ on $\{x_n = \ell_m\}$ and $\{x_n = \ell_{m-1}\}$. Hence, it follows from standard elliptic equations theory that $u \in C^\infty(\overline{D}_m \cap B_1)$. By estimate (2.2),

$$\|D^2_{x'}D_n u\|_{L^\infty(\overline{D}_m \cap B_{3/4})} + \|D_{x'}D^2_n u\|_{L^\infty(\overline{D}_m \cap B_{3/4})} \le C\{\|u\|_{L^\infty(B_1)} + \|F\|_{L^\infty(B_1)}\}.$$

Making use of equation (2.9), we therefore obtain

$$\|D^3_n u\|_{L^\infty(\overline{D}_m \cap B_{3/4})} \le C\{\|u\|_{L^\infty(B_1)} + \|F\|_{L^\infty(B_1)}\}.$$

In combination, we conclude that

$$\|D^3 u\|_{L^\infty(\overline{D}_m \cap B_{3/4})} \le C\{\|u\|_{L^\infty(B_1)} + \|F\|_{L^\infty(B_1)}\}.$$

By bootstrapping, the estimate (2.8) follows. □

3. Perturbation results. Throughout this section we may assume all the L^p-viscosity solutions are smooth, but the estimates we shall derive are independent of the smoothness of solutions. For convenience, we say a constant is universal means that it depends only on dimension n and ellipticity constants λ, Λ.

LEMMA 3.1 (Approximation lemma). *Let $a_{ij}(x) \in \mathcal{A}(\lambda, \Lambda)$ be measurable functions in B_1 and $f \in L^n(B_1)$. Suppose $u \in C(B_1)$ is an L^p-viscosity solution of*

$$a_{ij}D_{ij}u = f \quad in\ B_1$$

with $\|u\|_{C(B_1)} \le 1$. Assume there exist functions $A_{ij} \in \mathcal{A}(\lambda, \Lambda)$ and F depending only on the variable x_n such that

$$\|a_{ij} - A_{ij}\|_{L^n(B_{3/4})} \le \varepsilon$$

where $\varepsilon \in (0, 1/16)$, then there exists a function $v \in C(\overline{B}_{3/4})$ with

$$A_{ij}D_{ij}v = F \quad in\ B_{3/4}$$

in L^p-viscosity solution sense such that

$$\|u-v\|_{C(\overline{B}_{1/2})} \leq C\{(1+\|F\|_{L^\infty(B_1)})\varepsilon^\gamma + \|f-F\|_{L^n(B_1)}\},$$

where $\gamma < 1$ and C are positive universal constants.

Proof. By Theorem 2.2, there exists a unique $v \in C(\overline{B}_{3/4}) \cap C^{1,1}_{loc}(B_1)$ solving

$$A_{ij}D_{ij}v = F \text{ in } B_{3/4}, \quad v = u \text{ on } \partial B_{3/4}.$$

According to the Krylov-Safonov estimate, there exist universal constants $\alpha \in (0,1)$ and $C > 0$ such that

$$\|u\|_{C^\alpha(\overline{B}_{3/4})} \leq C\{1 + \|f\|_{L^n(B_1)}\}.$$

It follows from global Hölder estimates ([2], Proposition 4.13) that

$$\begin{aligned} \|v\|_{C^{\alpha/2}(\overline{B}_{3/4})} &\leq C\{\|u\|_{C^\alpha(\overline{B}_{3/4})} + \|F\|_{L^\infty(B_1)}\} \\ &\leq C\{1 + \|f\|_{L^n(B_1)} + \|F\|_{L^\infty(B_1)}\} =: CK. \end{aligned} \tag{3.1}$$

Since $u - v = 0$ on $\partial B_{3/4}$, we have

$$\|u-v\|_{L^\infty(\partial B_{3/4-\delta})} \leq \delta^{\alpha/2}\|u-v\|_{C^{\alpha/2}(\overline{B}_{3/4})} \leq CK\delta^{\alpha/2}, \tag{3.2}$$

where $0 < \delta < 1/4$. Next, we claim that

$$\|D^2v\|_{L^\infty(B_{3/4-\delta})} \leq CK\delta^{\alpha/2-2}. \tag{3.3}$$

Indeed, for any fixed $\overline{x} \in B_{3/4-\delta}$ define

$$w(x) = \frac{v(\overline{x}+\delta x) - v(\overline{x})}{\delta^{\alpha/2}}, \quad x \in B_1.$$

It follows from estimate (3.1) that $|w(x)| \leq CK$ for all $x \in B_1$. Note that $w(x)$ satisfies

$$A_{ij}(\overline{x}+\delta x)D_{ij}w(x) = \delta^{2-\alpha/2}F(\overline{x}+\delta x) \quad \text{for } \ x \in B_1$$

and from the $C^{1,1}$ estimates in Theorem 2.2 we conclude that

$$|D^2w(0)| \leq CK,$$

or

$$|D^2v(\overline{x})| \leq CK\delta^{\alpha/2-2}.$$

Therefore, the claim follows. Note that $u - v$ is an L^p-viscosity solution of

$$a_{ij}D_{ij}(u-v) = f - F - (a_{ij} - A_{ij})D_{ij}v \quad \text{for } x \in B_{3/4}.$$

By Alexandroff maximum principle and estimates (3.2) and (3.3),

$$\begin{aligned} \|u-v\|_{L^\infty(B_{3/4-\delta})} &\leq \|u-v\|_{L^\infty(\partial B_{3/4-\delta})} + C\|f-F\|_{L^n(B_{3/4-\delta})} \\ &\quad + C\|D^2v\|_{L^\infty(B_{3/4-\delta})}\|(a_{ij}-A_{ij})\|_{L^n(B_{3/4-\delta})} \\ &\leq CK(\delta^{\alpha/2} + \delta^{\alpha/2-2}\varepsilon) + C\|f-F\|_{L^n(B_1)}. \end{aligned}$$

Take $\delta = \varepsilon^{1/2} < 1/4$ and then $\gamma = \alpha/4$. Since

$$\|f\|_{L^n(B_1)} \le |B_1| \|F\|_{L^\infty(B_1)} + \|f - F\|_{L^n(B_1)},$$

the proof is completed. □

PROPOSITION 3.1. *Suppose u is an L^p-viscosity solution of*

$$a_{ij} D_{ij} u = f \quad \text{in } B_1.$$

Let A_{ij}, F be as in Lemma 3.1. For $\alpha \in (0,1)$, suppose that

$$\Big(\frac{1}{|B_r|} \int_{B_r} |f - F|^n \, \mathrm{d}x \Big)^{1/n} \le C_1 r^{\alpha - 1} \quad \text{for any } 0 < r \le 1.$$

Then there exists a $\theta > 0$ depending only on n, λ, Λ and α, such that if

$$\Big(\frac{1}{|B_r|} \int_{B_r} |a_{ij} - A_{ij}|^n \, \mathrm{d}x \Big)^{1/n} \le \theta \quad \text{for any } 0 < r \le 1,$$

then u is $C^{1,\alpha}$ at 0; that is, there is an affine function l such that

$$\begin{aligned} &\|u - l\|_{L^\infty(B_r)} \le N r^{1+\alpha} \quad \text{for any } 0 < r < 1, \\ &|l(0)| + |Dl(0)| \le N \end{aligned}$$

and

$$N \le C\{\|u\|_{L^\infty(B_1)} + \|F\|_{L^\infty(B_1)} + C_1\},$$

where $C > 0$ depends only on n, λ, Λ and α.

Proof. From Lemma 3.1, we can follow the proof of Theorem 2 of [1], which improves Cordes-Nirenberg's $C^{1,\alpha}$ estimate. We only need to point out that L^p-viscosity solutions of the approximating equation

$$A_{ij} D_{ij} v = F \quad \text{in } B_1$$

have $C^{1,1}$ estimates. That is exactly Theorem 2.2. □

Similarly, based on Lemma 3.1 and $C^{1,1}$ estimates of good equations the following $W^{2,p}$ estimates hold. We refer Chapter 7 of [2] for a proof.

PROPOSITION 3.2. *Suppose u is an L^p-viscosity solution of*

$$a_{ij} D_{ij} u = f \quad \text{in } B_1.$$

Then for any $p \in (n, \infty)$ there exist positive constants θ, C depending only on n, λ, Λ and p such that if

$$\Big(|B_r(\hat{x})|^{-1} \int_{B_r(\hat{x})} \beta(x, \hat{x})^n \, \mathrm{d}x \Big)^{1/n} \le \theta$$

for any ball $B_r(\hat{x}) \subset B_1$, where $\beta(x, \hat{x})$ is as in Theorem 1.1, we have

$$\|u\|_{W^{2,p}(B_{1/2})} \le C\{\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)}\}.$$

Definition 3.1. *For $\beta > 0$ and function G, define*

$$\|G\|_{Y^{n,\beta}}(0) = \sup_{0<r<1} \frac{1}{r^\beta} \Big(\fint_{B_r} |G|^n \,\mathrm{d}x \Big)^{1/n},$$

where $\fint_{B_r} \cdot \,\mathrm{d}x = \frac{1}{|B_r|} \int_{B_r} \cdot \,\mathrm{d}x$.

Proposition 3.3. *Let $a_{ij}(x) \in \mathcal{A}(\lambda, \Lambda)$ defined in B_1 and $f \in L^n(B_1)$. Suppose that u is an L^p-viscosity solution of*

$$a_{ij} D_{ij} u = f \quad \text{in } B_1$$

with $\|u\|_{L^\infty(B_1)} \le 1$. Let $A_{ij} \in \mathcal{A}(\lambda, \Lambda)$ and F be functions depending only on the variable x_n. For any $\alpha \in (0,1)$, there exist constants $\theta \in (0, 1/16)$ and N depending only on $n, \lambda, \Lambda, \alpha$ such that if

$$\|a_{ij} - A_{ij}\|_{Y^{n,\alpha}}(0) + \|f - F\|_{Y^{n,\alpha}}(0) \le \theta$$

and

$$\|F\|_{L^\infty(B_1)} \le 1,$$

then we have

$$|D^2 u(0)| \le N,$$

and for $x \in B_{1/4}$

$$|D_{x'} u(x) - D_{x'} u(0) - D D_{x'} u(0) \cdot x| \le N |x|^{1+\alpha},$$

where $D D_{x'} u(0) \cdot x = (D D_{x_1} u(0) \cdot x, \cdots, D D_{x_{n-1}} u(0) \cdot x)$

Proof. Note that we can assume $F \equiv 0$. Indeed, let $w \in C(B_1)$ be a solution of

$$\begin{aligned} A_{ij} D_{ij} w &= F \quad \text{in } B_1, \\ w &= u \quad \text{on } \partial B_1. \end{aligned}$$

Then w is bounded and has the estimates in Theorem 2.2. Let $\tilde{u} = u - w$ and $\tilde{f} = (A_{ij} - a_{ij}) D_{ij} w + f - F$. We have

$$\begin{aligned} a_{ij} D_{ij} \tilde{u} &= \tilde{f} \quad \text{in } B_1, \\ \tilde{u} &= 0 \quad \text{on } \partial B_1. \end{aligned}$$

Hence, we only need to establish Proposition 3.3 for $\tilde{u}(\frac{1}{2}x)$ for $x \in B_1$.

We will inductively find a sequence of functions $w_k \in C(\frac{3}{4^{k+1}} B_1)$, $k = 0, 1, \cdots$, such that for all k,

$$A_{ij} D_{ij} w_k = 0 \quad \text{in } \frac{3}{4^{k+1}} B_1, \tag{3.4}$$

$$\begin{aligned} |w_k(x)| &\le C 4^{-(k+1)(2+\alpha)} && \text{for } x \in \frac{3}{4^{k+1}} B_1, \\ |D w_k(x)| &\le C' 4^{-(k+1)(1+\alpha)} && \text{for } x \in \frac{2}{4^{k+1}} B_1, \\ |D^2 w_k(x)| &\le C' 4^{-(k+1)\alpha} && \text{for } x \in \frac{2}{4^{k+1}} B_1, \\ |D_{x'} D^2 w_k(x)| &\le C' 4^{(k+1)(1-\alpha)} && \text{for } x \in \frac{2}{4^{k+1}} B_1, \end{aligned} \tag{3.5}$$

$$|u(x)-\sum_{l=0}^{k} w_l(x)| \le 4^{-(k+1)(2+\alpha)} \quad \text{for } x\in \frac{2}{4^{k+1}}B_1, \tag{3.6}$$

and

$$|Du(x)-\sum_{l=0}^{k} Dw_l(x)| \le C''4^{-(k+1)(1+\alpha)} \quad \text{for } x\in \frac{1}{4^{k+1}}B_1, \tag{3.7}$$

where C, C' and C'' are positive universal constants.

It follows from Lemma 3.1 that there exists a function $w_0 \in C(\frac{3}{4}B_1)$ satisfying

$$A_{ij}D_{ij}w_0 = 0 \quad \text{in } \frac{3}{4}B_1$$

and a constant C_0 depending only on n, λ and Λ such that

$$\|u-w_0\|_{L^\infty(B_{1/2})} \le C_0\theta^\gamma \le 4^{-(2+\alpha)}$$

for small θ. From the proof of Lemma 3.1, we see that $w_0 = u$ on $\partial B_{3/4}$. By Alexandroff maximum principle, there exists a universal constant C_1 such that

$$\|w_0\|_{L^\infty(B_{3/4})} \le C_1\|u\|_{L^\infty(B_1)} \le C_1'(1+\theta)4^{-(2+\alpha)},$$

where $C_1' = C_1 4^3$. By Theorem 2.2, there exists a universal constant C_2

$$\|Dw_0\|_{L^\infty(B_{1/2})} + \|D^2w_0\|_{L^\infty(B_{1/2})} + \|D_{x'}D^2w_0\|_{L^\infty(B_{1/2})} \le C_2\|w_0\|_{L^\infty(B_{3/4})}.$$

Note that

$$a_{ij}D_{ij}(u-w_0) = (A_{ij}-a_{ij})D_{ij}w_0 + f \quad \text{in } B_{1/2}.$$

It follows from Proposition 3.1 that

$$\begin{aligned}
&\|D(u-w_0)\|_{L^\infty(B_{1/4})}\\
&\quad \le C_3\big\{\|u-w_0\|_{L^\infty(B_{1/2})} + \|(A_{ij}-a_{ij})D_{ij}w_0\|_{L^n(B_{1/2})} + \|f\|_{L^n(B_{1/2})}\big\}\\
&\quad \le C_3\big\{\|u-w_0\|_{L^\infty(B_{1/2})} + \theta\|D_{ij}w_0\|_{L^\infty(B_{1/2})} + \theta\big\}\\
&\quad \le C_3(1+\theta(1+\theta)C_2C_1' + \theta^{1-\gamma})4^{-(2+\alpha)},
\end{aligned}$$

where C_3 is a positive universal constant. Thus (3.7) for w_0 follows. We should keep in mind that the constants $C_0, C_1', C_2, C_3 \ge 1$ are all universal and would not change in the following arguments.

We have proved (3.5) – (3.7) for $k=0$. Assume they hold up to $k \ge 0$. We prove for $k+1$. Set

$$W(x) = \Big(u - \sum_{l=0}^{k} w_l\Big)(4^{-(k+1)}x),$$

$$a_{ij}^{k+1}(x) = a_{ij}(4^{-(k+1)}x), \quad A_{ij}^{k+1}(x) = A_{ij}(4^{-(k+1)}x)$$

and

$$h_{k+1}(x) = 4^{-2(k+1)}\Big((A_{ij}^{k+1} - a_{ij}^{k+1})\sum_{l=0}^{k} D_{ij}^2 w_l(4^{-(k+1)}x) + f(4^{-(k+1)}x)\Big).$$

Then W solves

$$a_{ij}^{k+1} D_{ij} W = h_{k+1} \quad \text{in } B_1.$$

By direct computations we obtain

$$\|a_{ij}^{k+1} - A_{ij}^{k+1}\|_{L^n(B_1)} \le 4^{-(k+1)\alpha}\theta, \quad \|f(4^{-(k+1)}x)\|_{L^n(B_1)} \le 4^{-(k+1)\alpha}\theta.$$

According to the induction hypothesis, we have

$$\sum_{l=0}^{k} D_{ij}^2 w_l(4^{-(k+1)}x) \le C' \sum_{l=0}^{\infty} 4^{-l\alpha} \quad \text{for } x \in B_1,$$

and

$$\|W\|_{L^\infty(B_1)} \le 4^{-(k+1)(2+\alpha)}.$$

Therefore,

$$\|h_{k+1}\|_{L^n(B_1)} \le C'\theta 4^{-(k+2)(2+\alpha)}(1 + \sum_{l=0}^{\infty} 4^{-l\alpha}).$$

Using Lemma 3.1, we may find a function $v_{k+1} \in C(B_{3/4})$ satisfying

$$A_{ij}^{k+1} D_{ij} v_{k+1} = 0 \text{ in } B_{3/4}, \quad v_{k+1} = W \text{ on } \partial B_{3/4}$$

such that

$$\begin{aligned}\|W - v_{k+1}\|_{L^\infty(B_{1/2})} &\le C_1\big\{\|W\|_{L^\infty(B_1)}(4^{-(k+1)\alpha}\theta)^\gamma + \|h_{k+1}\|_{L^n(B_1)}\big\} \\ &\le 4^{-(k+2)(2+\alpha)}\end{aligned}$$

provided θ small. By Alexandroff maximum principle,

$$\begin{aligned}\|v_{k+1}\|_{L^\infty(B_{3/4})} &\le C_1' 4^{-(2+\alpha)}(\|W\|_{L^\infty(B_1)} + \|h_{k+1}\|_{L^n(B_1)}) \\ &\le C_1'(1 + 4^{-1}\theta C'(1 + \sum_{l=0}^{\infty} 4^{-l\alpha}))4^{-(k+2)(2+\alpha)}.\end{aligned}$$

By Theorem 2.2, we have

$$\begin{aligned}&\|Dv_{k+1}(x)\|_{L^\infty(B_{1/2})} + \|D^2 v_{k+1}(x)\|_{L^\infty(B_{1/2})} \\ &\qquad + \|D_{x'} D^2 v_{k+1}(x)\|_{L^\infty(B_{1/2})} \le C_2 \|v_{k+1}\|_{L^\infty(B_{3/4})}.\end{aligned}$$

Note that

$$a_{ij} D_{ij}(W - v_{k+1}) = h_{k+1} + (A_{ij} - a_{ij}) D_{ij} v_{k+1} \quad \text{in } B_{1/2}.$$

It follows from Proposition 3.1 that

$$\|D(W-v_{k+1})\|_{L^\infty(B_{1/4})}\le C''4^{-(k+2)(2+\alpha)}.$$

From the process above, we see that the constants C, C', C'' and θ can be chosen as follows:

$$C=2C_1',\quad C'=2C_1'C_2,\quad C''=10C'C_3$$
$$C_0\theta^\gamma\le\frac{1}{2}4^{-(2+\alpha)}\text{ and }C'\theta\sum_{l=0}^{\infty}4^{-l\alpha}\le\frac{1}{2}.$$

Let

$$w_{k+1}(x)=v(4^{k+1}x)\quad\text{for }x\in\frac{3}{4^{k+2}}B_1.$$

We see that (3.4) – (3.7) hold for $k+1$.

By (3.5) and (3.7), for $4^{-(k+2)}\le|x|<4^{-(k+1)}$,

$$\begin{aligned}
&|Du(x)-\sum_{l=0}^{\infty}Dw_l(0)|\\
&\le|Du(x)-\sum_{l=0}^{k}Dw_l(x)|+|\sum_{l=0}^{k}Dw_l(x)-\sum_{l=0}^{k}Dw_l(0)|+|\sum_{l=k+1}^{\infty}Dw_l(0)|\\
&\le 16C''|x|^{1+\alpha}+C'\sum_{l=0}^{k}4^{-(l+1)\alpha}|x|+C'\sum_{l=k+1}^{\infty}4^{-(l+1)(1+\alpha)}\\
&\le N|x|.
\end{aligned}$$

It follows that

$$Du(0)=\sum_{l=0}^{\infty}Dw_l(0)$$

and

$$|D^2u(0)|\le N.$$

Furthermore, for $4^{-(k+2)}\le|x|<4^{-(k+1)}$,

$$\begin{aligned}
&|D_{x'}u(x)-\sum_{l=0}^{\infty}D_{x'}w_l(0)-\sum_{l=0}^{\infty}DD_{x'}w_l(0)\cdot x|\\
&\le|D_{x'}u(x)-\sum_{l=0}^{k}D_{x'}w_l(x)|+\sum_{l=0}^{k}|D_{x'}w_l(x)-D_{x'}w_l(0)-DD_{x'}w_l(0)\cdot x|\\
&\quad+\sum_{l=k+1}^{\infty}|D_{x'}w_l(0)|+\sum_{l=k+1}^{\infty}|DD_{x'}w_l(0)\cdot x|\\
&\le 16C''|x|^{1+\alpha}+C'|x|^2\sum_{l=0}^{k}4^{(l+1)(1-\alpha)}\\
&\quad+C'\sum_{l=k+1}^{\infty}4^{-(l+1)(1+\alpha)}+C'|x|\sum_{l=k+1}^{\infty}4^{-(l+1)\alpha}\\
&\le N|x|^{1+\alpha},
\end{aligned}$$

where we have used the elementary inequality

$$|x|^{1-\alpha}\sum_{l=0}^{k}4^{(l+1)(1-\alpha)} \le 4^{-(k+1)(1-\alpha)}\frac{4^{1-\alpha}(4^{(k+1)(1-\alpha)}-1)}{4^{1-\alpha}-1} \le C(\alpha).$$

Thus the proof is completed. □

4. Proof of the main theorems. In view of the perturbation results of last section, the proofs of main theorems reduce to verify various approximating conditions of coefficients and functions on the right hand side of the equations.

Proof of Theorem 1.1. The existence follows from the same lines we used to prove Theorem 2.2. As there we suppose u^0 is the limit of a sequence of classical solutions u^ε of Dirichlet problem

$$a_{ij}^\varepsilon D_{ij}u^\varepsilon = f^\varepsilon \quad \text{in } \Omega \text{ and } u^\varepsilon = \varphi^\varepsilon \quad \text{on } \partial\Omega.$$

It suffices to verify that u^0 is in $W^{2,p}_{loc}(\Omega)$ and $C^{1,1}_{loc}(\Omega)$ in the two cases respectively. We only verify the second one, and the first is similar.

For any fixed point $\overline{x} = (\overline{x}', \overline{x}_n) \in \Omega$, we can find $r > 0$ such that $B_{2r}(\overline{x}) \subset \Omega$. Let $A_{ij}^\varepsilon = a_{ij}^\varepsilon(\overline{x}', x_n)$ and $F^\varepsilon = f^\varepsilon(\overline{x}', x_n)$. Since $a_{ij}, f \in C^\alpha_{x'}$ for some $\alpha \in (0,1)$, for the θ in Proposition 3.3 one can find a small constant r_0 independent of ε such that

$$\|\overline{a}_{ij}^\varepsilon - \overline{A}_{ij}^\varepsilon\|_{Y^{n,\alpha}}(0) + \|\overline{f}^\varepsilon - \overline{F}^\varepsilon\|_{Y^{n,\alpha}}(0) \le \theta,$$

where

$$\overline{a}_{ij}^\varepsilon(x) = a_{ij}^\varepsilon(\overline{x} + r_0 r x), \quad \overline{A}_{ij}^\varepsilon(x) = A_{ij}^\varepsilon(\overline{x} + r_0 r x),$$

$$\overline{f}^\varepsilon(x) = f^\varepsilon(\overline{x} + r_0 r x), \quad \overline{F}^\varepsilon(x) = F^\varepsilon(\overline{x} + r_0 r x),$$

for $x \in B_1$. Then $\overline{u}^\varepsilon(x) = u^\varepsilon(\overline{x} + r_0 r x)$ is a classical solution of

$$\overline{a}_{ij}^\varepsilon D_{ij}\overline{u}^\varepsilon = \overline{f}^\varepsilon \quad \text{in } B_1.$$

By Proposition 3.3, we have

$$\begin{aligned}&|D^2\overline{u}^\varepsilon(0)| + \sup_{|x'|<1}\frac{|D^2\overline{u}^\varepsilon(x',0) - D^2\overline{u}^\varepsilon(0',0)|}{|x'|^\alpha}\\ &\le \frac{C}{r^2}\left\{\|\overline{u}^\varepsilon\|_{L^\infty(B_r(0))} + \|\overline{f}^\varepsilon\|_{L^\infty(B_r(0))} + [\overline{f}^\varepsilon]_{C^\alpha_{x'}(B_r(0))}\right\},\end{aligned}$$

where $C > 0$ depends only on n, λ, Λ and $[\overline{a}_{ij}^\varepsilon]_{C^\alpha_{x'}(B_r(0))}$. Rescaling back and by standard covering argument and then letting $\varepsilon \to 0$, we complete the verification and the estimate (1.4) follows. □

LEMMA 4.1. *Let $g(x') \in C^{1,\beta}(B_1')$, $\beta \in (0,1)$, satisfying $g(0) = 0$ and $Dg(0) = 0$, where B_1' is the $(n-1)$-dimensional ball centered at the origin. Set*

$$\Omega^+ = \{(x', x_n) \in B_1 : x_n > g(x')\} \text{ and } \Omega^- = B_1 \setminus \overline{\Omega^+}.$$

Let h be a function in $L^\infty(B_1)$ with $h|_{\Omega^+} \in C^\beta(\Omega^+)$ and $h|_{\Omega^-} \in C^\beta(\Omega^-)$, respectively. Set

$$\overline{h}(x) = \begin{cases} \lim\limits_{y\in\Omega^+, y\to 0} h(y) & \text{in } B_1^+, \\ \lim\limits_{y\in\Omega^-, y\to 0} h(y) & \text{in } B_1^-, \end{cases}$$

where $B_1^+ := B_1 \cap \{x_n > 0\}$ and $B_1^- := B_1 \cap \{x_n < 0\}$. Then

$$\|h - \overline{h}\|_{Y^{n,\beta/n}}(0) \le N,$$

where N is a positive constant depending only on $\|h\|_{C^\beta(\Omega^\pm)}$ and the $C^{1,\beta}$ norm of g.

Proof. Since $g(0) = 0$ and $Dg(0) = 0$, $|g(x')| \le C|x'|^{1+\beta}$. By direct calculations,

$$\begin{aligned} \int_{B_r^+} |h - \overline{h}|^n \,\mathrm{d}x &= \int_{B_r^+\cap\Omega^+} |h - \overline{h}|^n \,\mathrm{d}x + \int_{B_r^+\setminus\Omega^+} |h - \overline{h}|^n \,\mathrm{d}x \\ &\le C \int_{B_r^+\cap\Omega^+} |x|^{n\beta} \,\mathrm{d}x + C \int_{B_r'} |x'|^{1+\beta} \,\mathrm{d}x' \\ &\le Cr^{n+\beta}, \end{aligned}$$

where B_r^+ is the upper half ball. Analogously, we have

$$\int_{B_r^-} |h - \overline{h}|^n \,\mathrm{d}x \le Cr^{n+\beta}.$$

In combination, we complete the proof. ∎

By Proposition 3.3 and Lemma 4.1, some proper scaling yields

Corollary 4.1. *Let g and Ω^+, Ω^- as in Lemma 4.1. Suppose $a_{ij}|_{\Omega^\pm}, f|_{\Omega^\pm} \in C^\alpha(\Omega^\pm)$ and u is an L^p-viscosity solution of*

$$a_{ij}D_{ij}u = f \quad \text{in } B_1.$$

Then u is $C^{1,\gamma}$ at 0, with $\gamma = \frac{1}{n}\min\{\alpha, \beta\}$.

Proof of Theorem 1.3. By Corollary 4.1, u is $C^{1,\gamma}$ on the boundary of Ω_1 and Ω_2. Therefore, to prove Theorem 1.3, it suffices to show estimate (1.5) in Ω_k, $k = 1, 2, 3$. Let $\Omega' \subset\subset \Omega$. For any point $\overline{x} \in \Omega_k \cap \Omega'$, there exists a ball $B_c(\overline{x})$ centered at $\overline{x}$ with radius c and a positive integer κ such that $B_c(\overline{x}) \cap (\partial\Omega_1 \cup \partial\Omega_2)$ contains at most κ connected components, where c and κ are independent of $\overline{x}$ but depend on $n, dist(\Omega', \Omega)$ and $C^{1,\beta}$ modulus of $\partial\Omega_1 \cup \partial\Omega_2$, see [16]. To estimate D^2u at a point $\overline{x}$ in $\Omega_k \cap \Omega'$, we may assume $\overline{x}$ close to some Ω_k; otherwise it follows from standard interior estimates for elliptic equations. We take $\overline{x}$ as the origin. By suitable rotating and scaling, we may suppose that the components of $\partial\Omega_k$ contained in B_1 take the form

$$x_n = g_m(x') \quad \text{for } x' \in B_1',\ m = 1, \cdots, \kappa$$

with

$$-1 < g_1(x') < \cdots < g_\kappa(x') < 1$$

and $C^{1,\beta}(B_1')$, where $B_1' = \{x' : |x'| < 1\}$. Set $g_0(x') = -1$ and $g_{\kappa+1}(x') = 1$. Denote

$$\tilde{\Omega}_m = \{x \in B_1 : g_{m-1}(x') < x_n < g_m(x')\}, \quad 1 \le m \le \kappa + 1.$$

We may suppose that $g_{m_0-1}(0') < 0 < g_{m_0}(0')$, and closest point on $\partial\Omega_k$ to the origin is $(0', g_{m_0}(0'))$. So $\tilde{D} g_{m_0}(0') = 0$. Finally, we introduce

$$D_m = \{x \in \mathbb{R}^n : g_{m-1}(0') < x_n < g_m(0')\}, \quad m = 1, \cdots, \kappa + 1$$

and define

$$A_{ij} = \begin{cases} \lim_{y \in \tilde{\Omega}_m, y \to (0', g_{m-1}(0'))} a_{ij}(y) & \text{in } D_m, m > m_0, \\ a_{ij}(0), & \text{in } D_{m_0}, \\ \lim_{y \in \tilde{\Omega}_m, y \to (0', g_m(0'))} a_{ij}(y) & \text{in } D_m, m < m_0. \end{cases}$$

Analogously, we define F corresponding to f.

It turns out that the above definitions give a nice approximating property (see [16], Lemma 5.2).

LEMMA 4.2. *For* $0 < \alpha' \le \min\{\alpha, \frac{\beta}{n(\beta+1)}\}$*, there exists a positive constant* N *depending only* $n, \kappa, \beta, \lambda$ *and* Λ*, as well as* $\max_{1\le m\le\kappa} \|a_{ij}\|_{C^{\alpha'}(\tilde{\Omega}_m)}$, $\max_{1\le m\le\kappa} \|f\|_{C^{\alpha'}(\tilde{\Omega}_m)}$ *and* $\max_{1\le m\le\kappa} \|g_m\|_{C^{1,\beta}(\overline{B}_1')}$ *such that*

$$\|a_{ij} - A_{ij}\|_{Y^{n,\alpha'}}(0) + \|f - F\|_{Y^{n,\alpha'}}(0) \le N.$$

Then (1.5) follows from Proposition 3.3 and Lemma 4.2 by appropriate scaling. Therefore, we complete the proof of Theorem 1.3. □

REMARK 4.1. *Comparing with lemma 4.1, the exponent* α' *in lemma 4.2 can be slightly improved to* $\min\{\alpha, \frac{\beta}{n}\}$ *if constant* N *is permitted to depend on the distance between* $(x', g_m(x'))$ *and* $(x', g_{m-1}(x'))$.

REMARK 4.2. *Making use of the same approach of [16] and [15], one can establish*

$$\|u\|_{C^{2,\alpha'}(\overline{\Omega}_k \cap \Omega')} \le C\{\|u\|_{C^0(\Omega)} + \sum_{m=1}^{3} \|f\|_{C^{\alpha'}(\Omega_m)}\}, \quad k = 1, 2, 3 \tag{4.1}$$

where C *is a positive constant depending only* $n, \lambda, \Lambda, \|a_{ij}\|_{C^{\alpha'}(\Omega_m)}$, $dist(\Omega', \partial\Omega)$ *and* $C^{1,\beta}$ *norms of the* Ω_m *but independent of* ε.

Proof of Theorem 1.2. Arguing as proving Theorem 2.2, there exists an L^p-viscosity solution u of (1.2). Let $\mathcal{M}$ be as in Theorem 1.2. Since $\mathcal{M}$ is a $C^{1,\alpha}$ embedded $n-1$-dimensional hypersurface, for every point $x_0 \in \mathcal{M}$ there exists an $(n-1)$-dimensional locally tangent hyperplane $l(x)$ to $\mathcal{M}$ at x_0 (assuming $l(x) \subset \{x_n = 0\}$) and small ball $B_r(x_0)$ such that $\mathcal{M} = \{(x', g(x')) : |x'| < r\}$ in $B_r(x_0)$, where $g(x')$ is a smooth function. By Theorem 1.3, $u \in C^{1,1}_{loc}(B_r(x_0))$. The uniqueness follows from Corollary 2.1. Thus we complete the proof. □

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