METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 18, No. 4, pp. 391–416, December 2011

003

ON THE BOUSSINESQ SYSTEM: REGULARITY CRITERIA AND SINGULAR LIMITS*

LUIGI C. BERSELLI† AND STEFANO SPIRITO‡

Abstract. We consider the 3D Boussinesq system and we prove several criteria, not involving the density, for the continuation of smooth solutions. We give particular emphasis to the results in bounded domains, under various boundary conditions. The results we prove are partially known and we collect them in a unified framework, mainly with the perspective of understanding the stabilization/smoothing required by numerical methods (especially by large scales methods). In the final section, we also consider the vanishing viscosity/diffusivity limits, proving (locally-in-time) sharp singular limits for smooth solutions of the Cauchy problem.

Key words. Boussinesq equations, regularity criteria, vanishing viscosity/diffusivity limits.

AMS subject classifications. 35Q35 (35B65 76B03, 35B30).

1. Introduction.

When studying the motion of fluids which can undergo density/temperature/salinity variations, there are limited regimes in which, instead of the full density-dependent Navier-Stokes, some simpler equations known as Boussinesq (or also Oberbeck-Boussinesq) equations are accurate enough:

$$
\begin{aligned}
u_t + \nabla\cdot(u\otimes u) - \frac{1}{Re}\Delta u + \nabla\pi &= -\frac{1}{Fr^2}\rho' \mathrm{e}_3 && \text{in } \Omega\times(0,T),\\
\nabla\cdot u &= 0 && \text{in } \Omega\times(0,T),\\
\rho'_t + \nabla\cdot(\rho' u) - \frac{1}{Re\,Pr}\Delta\rho' &= 0 && \text{in } \Omega\times(0,T).
\end{aligned}
\tag{1.1}
$$

Here the domain Ω is a smooth bounded subset of $\mathbb{R}^3$, the unknowns (u, p, ρ') are velocity, pressure, and "salinity perturbation," respectively, and $\mathrm{e}_3 = (0, 0, 1)$ is the unit vector in the vertical direction. The non-dimensional non-negative parameters are the Reynolds number Re, the Prandtl number Pr, and the Froude number Fr. The Boussinesq system plays an important roles in the atmospheric sciences, see Majda [35] and also in geophysical modeling, see Cushman-Roisin and Beckers [23]. For a derivation of this system, with full details on the underlying assumptions, see Rajagopal, Růžička, and Srinivasa [43]. In particular, to a first approximation, the mixing phenomena can be described by means of this system and it is particularly relevant the study of the limits of solutions, as the parameters go to infinity. This has been addressed analytically by Chae [17] and Hou and Li [27] in the 2D case, this problem being raised by Moffat [37].

We started a numerical study of these equations to describe mixing, since it is one of the most important processes to understand and predict transport of pollutants as well as the details of thermohaline circulation, see [9, 11, 14]. In the numerical approach one is forced to use approximate models: Direct numerical simulations – despite the increase in the available computational power- are still not effective, since the scales in the ocean circulation cannot be all resolved simultaneously. Basin models

*Received July 30, 2011; accepted for publication May 24, 2012.
†Università degli Studi di Pisa, Dipartimento di Matematica Applicata "U. Dini", Via F. Buonarroti 1/c, I-56127 Pisa, Italy (berselli@dma.unipi.it).
‡Università dell'Aquila, Dipartimento di Matematica, via Vetoio 1, I-67010 Coppito (AQ), Italy (stefano.spirito@dm.univaq.it).

are configured for $\mathcal{O}(10^6 m)$ to $\mathcal{O}(10^7 m)$, regional or coastal models from $\mathcal{O}(10^3 m)$ to $\mathcal{O}(10^5 m)$, and they both require sub-grid-scale parametrization. However, there exist small-scale ocean flows which often play a significant role in an accurate representation of the large ocean scales. More precise motivations for the study of this physical problem and the requirement of suitable numerical methods to handle all scales are explained, for instance, in Özgökmen *et al.* [40, 41]. On the other hand, the 3D system (1.1) with viscosity represents a model involving most of the difficult features of the Navier-Stokes equations and also adding new interesting problems -especially in the sub-grid modeling– since a cascade mechanism *à la* Kolmogorov is not clear for this system. Our aim has been that of detecting the most suitable Large Eddy Simulation (LES) approach to the study of these equations, cf. [13]. One main fact coming from numerical experiments is that, when the viscosity is non-zero, smoothing of the equation for the balance of ρ' is not so relevant in order to have stability of the numerical solutions, cf. [41]. We prove this fact rigorously, by showing that all known regularity criteria for weak solution to the Navier-Stokes equations are still correct also for the Boussinesq equations. Some partial results are proved by Fan and Ozawa [24] and many others are also scattered through the literature.

We also observe that another motivation for the study of this system is that (in the limit of high Re) the 2D Boussinesq equations are also a reduced mathematical model for the 3D Euler equations, see Constantin [22]. Concerning the mathematical analysis early and more recent results for the Boussinesq system are those in [16, 18, 20, 28, 29, 33, 38], while recent results, especially concerning singular limits, are those in [17, 27, 39].

The goals of the present paper are: a) to make a general overview of the problem; b) to prove results in a bounded domain, under different boundary conditions. Part of the paper can be considered as a review, even if the treatment of the initial-boundary value problem seems original and requires precise way of dealing with the various boundary terms. We also point out that it is clear that essentially all results for the Cauchy problem in Sobolev spaces can be adapted to more general context of Besov spaces as in [25, 34, 42]. Many criteria can be improved by considering only two components of the velocity and so on. The interested reader can adapt the proofs (especially in the cases without boundaries) to many other related situations. On the other hand, the treatment of the bounded domain requires different techniques and sharp estimates for the boundary integrals.

In view of applications to numerical methods, the results we prove can be used for instance to detect classes of eddy viscosity models or α-models for the Boussinesq systems, since we stress what is the requested minimal amount of smoothing in the velocity, *and only in the velocity*, to have stable approximate models (cf. the recent results in Zhou and Fan [47]). Our effort is also devoted, since some of the results we prove are not completely new, to put the regularity criteria in a general framework, treating all cases in a unified way, and showing the relevant difference in the various cases.

We also observe that we will study mainly the problem without diffusivity. This is also relevant from the point of view of numerical experiments, because in many problems the parameter Pr can be extremely large. In order to capture the properties which are independent from this parameter we study the so called "viscous problem" (opposed to the "viscous-diffusive," cf. Eq. (2.1) *versus* (2.2)). The relevance of limiting values for the parameters justifies also the study of some singular limits in the last part of the paper.

Plan of the paper. In Section 2 we precisely state the problems we will study, in terms of equations and boundary conditions. Next, in Section 3 we prove the main result of the paper, that is a series of scaling invariant criteria involving only the velocity, for the continuation of smooth solutions. Finally in Section 4 we prove the sharp convergence of smooth solutions, in terms of both viscosity and diffusivity, towards solution of the so-called Euler Boussinesq equations.

2. Setting of the problem and main results. In order to simplify the notation we rewrite the Boussinesq system (1.1) in the following way (with the viscosity ν, the diffusivity κ, and renaming ρ' as ρ) and we call "viscous-diffusive system" the system with both ν and k positive,

$$\begin{aligned} u_t - \nu\Delta u + (u\cdot\nabla)\,u + \nabla\pi &= -\rho\,\mathrm{e}_3 && \text{in } \Omega\times(0,T),\\ \operatorname{div} u &= 0 && \text{in } \Omega\times(0,T),\\ \rho_t + (u\cdot\nabla)\,\rho &= k\Delta\rho && \text{in } \Omega\times(0,T), \end{aligned} \tag{2.1}$$

while we call "viscous system" that with $\nu > 0$ and the diffusivity $k = 0$

$$\begin{aligned} u_t - \nu\Delta u + (u\cdot\nabla)\,u + \nabla\pi &= -\rho\,\mathrm{e}_3 && \text{in } \Omega\times(0,T),\\ \operatorname{div} u &= 0, && \text{in } \Omega\times(0,T),\\ \rho_t + (u\cdot\nabla)\,\rho &= 0 && \text{in } \Omega\times(0,T). \end{aligned} \tag{2.2}$$

Both systems have to be supplemented with initial data (u_0, ρ_0) such that $\operatorname{div} u_0 = 0$.

Concerning the boundary conditions at $\Gamma = \partial\Omega$ we observe that a natural condition for both systems (hence for any choice of $\kappa, \nu \geq 0$) is the slip condition

$$u\cdot n = 0 \qquad \text{on } \Gamma\times(0,T). \tag{2.3}$$

Since we consider the problem with non-zero viscosity we couple the slip boundary condition on the velocity (and consequently we will prove results in two different cases) with one of the following two relevant conditions:

$$\text{a)} \qquad u\times n = 0 \qquad \text{on } \Gamma\times(0,T) \qquad \text{(Dirichlet)}, \tag{2.4a}$$

$$\text{b)} \qquad \omega\times n = 0 \qquad \text{on } \Gamma\times(0,T) \qquad \text{(Navier's type)}, \tag{2.4b}$$

where $\omega = \nabla\times u$ is the curl of the velocity, while n is the unit normal exterior vector on Γ.

Concerning the density ρ, since u is tangential, we can deal with the following conditions:

$$\begin{aligned} &\rho = 0 \qquad \text{on } \partial\Omega\times(0,T) \qquad \text{when } k>0,\\ &\text{no boundary conditions for } \rho \text{ when } k=0. \end{aligned}$$

For the viscous system (2.2), by applying the same techniques as for the Navier-Stokes equations one can prove the following result of local existence and uniqueness of strong solutions. In the sequel we will use the classical Lebesgue spaces $(L^p(\Omega), \|\,.\,\|_p)$ and $(L^p(\Gamma), \|\,.\,\|_{p,\Gamma})$, and the Sobolev spaces $(W^{k,p}(\Omega), \|\,.\,\|_{k,p})$ for $k \in \mathbb{R}$ (we do not distinguish between scalar and vector valued functions). We will use mainly the Hilbert space $H^k(\Omega) = W^{k,2}(\Omega)$ and we will denote by $(W^{s,p}(\Gamma), \|\,.\,\|_{s,p,\Gamma})$ the standard trace spaces on the boundary Γ. We denote by C generic constants which may change from line to line and independent of the solution.

THEOREM 2.1. *Let be given $\Omega \subset \mathbb{R}^3$ a bounded, smooth, and simply connected open set. Let $u_0, \rho_0 \in H^3(\Omega)$, with $\operatorname{div} u_0 = 0$ and u_0 satisfying the boundary conditions* (2.3)-(2.4a) *such that the compatibility condition*

$$-\nu P\Delta u_0 + P\Big[(u_0 \cdot \nabla)\, u_0)\Big] + P[\rho_0\, \mathrm{e}_3] = 0 \tag{2.5}$$

is satisfied, where P is the Leray L^2-projection operator on tangential and divergence-free vector fields. Then, there exists $T_0 = T_0(\|u_0\|_{3,2}, \|\rho\|_{3,2}, \nu) > 0$ and a unique solution for the system (2.2) *with boundary conditions* (2.3)-(2.4a) *such that*

$$u, \rho \in C([0, T_0]; H^3(\Omega)).$$

The same result holds true also with the boundary conditions (2.3)-(2.4b) *and in this case no compatibility conditions are required, apart that the initial velocity satisfies the imposed boundary conditions.*

Proof. The result can be proved by adapting the classical results for the Navier-Stokes as summarized in [26], with the techniques employed also in [20, 38]. Furthermore, continuity up to $t = 0$ can be proved by adding the necessary compatibility conditions, which disappear in the case $\nu = k = 0$.

We give just a sketch of the *a priori* estimates needed to prove the result by following an approximation approach similar to α-models. To this end we consider the following approximate system

$$\begin{aligned} v_t - \nu\Delta v + (v \cdot \nabla)\, v + \nabla q &= -\sigma\, \mathrm{e}_3 && \text{in } \Omega \times (0,T),\\ \operatorname{div} v &= 0 && \text{in } \Omega \times (0,T),\\ \sigma_t + (v_\epsilon \cdot \nabla)\, \sigma &= 0 && \text{in } \Omega \times (0,T),\\ v(0,x) &= u_0 && \text{in } \Omega,\\ \sigma(0,x) &= \rho_0 && \text{in } \Omega, \end{aligned} \tag{2.6}$$

where $v_\epsilon = (I + \epsilon A)^{-1} v$, and A is the Stokes operator with Dirichlet boundary conditions. In such a way (2.6) can be considered a variant of the Leray-α model. By testing with Av the first equation and we get immediately the well-known estimate

$$\frac{d}{dt}\|\nabla v\|_2^2 + \|Av\|_2^2 \leq C\big(\|\nabla v\|_2^6 + \|\sigma\|_2^2\big),$$

which implies that there exists a positive T_0 such that $v \in L^\infty(0, T_0; H^1) \cap L^2(0, T_0; H^2)$, whenever $\sigma \in L^2(0, T_0; L^2)$. This is immediately satisfied since the transport equation for σ is made by means of $v_\epsilon \in L^\infty(0, T_0; H^3)$ due to the properties of the Stokes operator. Hence, the streamlines are well-defined and $\|\sigma(t)\|_2 = \|\sigma_0\| = \|\rho_0\|$. This is enough for uniqueness by standard estimates for the strong solutions of the Navier-Stokes system. Next, one can prove uniform estimate in terms of ϵ by taking the time-derivative of the first equation and testing by v_t and then Av_t, together with the standard bootstrap technique summarized for instance in [26, §5]. Once uniform estimates in $L^\infty(0, T; H^3)$ are proved for both v and σ one can take the limit (even weak limit since the limit problem has unique solution) $\epsilon \to 0^+$ and $(u, \rho) := \lim_{\epsilon \to 0}(v, \sigma)$ is a solution of the original problem (2.2), which is $L^\infty(0, T_0; H^3) \cap C([\lambda, T_0]; H^3)$ for all $0 < \lambda \leq T_0$. Finally the compatibility condition (2.5) is exactly as [45, Eq. (3.26)], which implies the continuity up to $t = 0$.

In the case of the boundary conditions (2.3)-(2.4b) the result holds true more or less in the same way. Certain simplification, especially to prove boundedness of the velocity in $L^2(0,T_0;H^2(\Omega))$, can be obtained through the successive application of the curl operator, as in [6, 46], summarized also in [10]. □

The main results we will prove concerning sufficient conditions for regularity in arbitrary time-intervals $[0,T]$ are the following.

THEOREM 2.2. *Let (u,ρ) be a local smooth solution of the system* (2.2) *with boundary conditions as above (Only the criterion* (2.10) *necessarily requires Navier's conditions, while* (2.9) *in the case of Navier's conditions is proved only for a restricted range of exponents). If for some $T>0$ one of the following conditions holds true:*

$$\int_0^T \|u(s)\|_p^q\,ds<\infty \qquad \frac{2}{q}+\frac{3}{p}=1 \quad \text{for } p>3, \tag{2.7}$$

$$\int_0^T \|\nabla u(s)\|_p^q\,ds<\infty \qquad \frac{2}{q}+\frac{3}{p}=2 \quad \text{for } p>\frac{3}{2}, \tag{2.8}$$

$$\int_0^T \Big\|\pi(s)+\frac{|u(s)|^2}{2}\Big\|_p^q\,ds<\infty \qquad \frac{2}{q}+\frac{3}{p}=2 \quad \text{for } p>\frac{3}{2}, \tag{2.9}$$

$$\measuredangle\left(\frac{\omega(x,t)}{|\omega(x,t)|},\frac{\omega(y,t)}{|\omega(y,t)|}\right)\le c|x-y|^{1/2},\quad \forall\, x\neq y\in\Omega: \quad \omega\neq 0 \text{ and a.e. } t\in[0,T], \tag{2.10}$$

(where $\measuredangle(u,v)$ denotes the angle between two unit vectors, identified with the length of a geodesic connecting them on a spherical unit surface), then is possible to continue u and ρ as a regular solution up to $t=T$.

In the final section we will study also the sharp vanishing-viscosity/diffusivity limit for the Cauchy problem, that is the $\Omega=\mathbb{R}^3$ (but the same result holds true also in the space-periodic case). We consider the viscous-diffusive Boussinesq system with viscosity $\nu_n\ge 0$ and diffusivity $k_n\ge 0$

$$\begin{aligned} u^n_t-\nu_n\Delta u^n+(u^n\cdot\nabla)\,u^n+\nabla\pi^n&=-\rho^n \mathrm{e}_3 && \text{in } \mathbb{R}^3\times(0,T),\\ \operatorname{div} u^n&=0 && \text{in } \mathbb{R}^3\times(0,T),\\ \rho^n_t-k_n\Delta\rho^n+(u^n\cdot\nabla)\,\rho^n&=0 && \text{in } \mathbb{R}^3\times(0,T),\\ u^n(0,x)&=u^n_0 && \text{in } \mathbb{R}^3,\\ \rho^n(0,x)&=\rho^n_0 && \text{in } \mathbb{R}^3. \end{aligned} \tag{2.11}$$

Here, both ν_n and k_n are assumed to vanish as $n\to\infty$ and since we consider the Cauchy problem in the whole space or the periodic problem, no boundary condition are required. We will prove the strong convergence (uniformly in time, with values in the same space of the initial data) of the sequence (u^n,ρ^n) to (u,ρ), where (u,ρ) is the unique local smooth solution of the following Euler-Boussinesq system

$$\begin{aligned} u_t+(u\cdot\nabla)\,u+\nabla\pi&=-\rho\,\mathrm{e}_3 && \text{in } \mathbb{R}^3\times(0,T),\\ \rho_t+(u\cdot\nabla)\,\rho&=0 && \text{in } \mathbb{R}^3\times(0,T),\\ \operatorname{div} u&=0 && \text{in } \mathbb{R}^3\times(0,T),\\ u(0,x)&=u_0 && \text{in } \mathbb{R}^3,\\ \rho(0,x)&=\rho_0 && \text{in } \mathbb{R}^3. \end{aligned} \tag{2.12}$$

The precise statement of the theorem is the following.

THEOREM 2.3. *Let be given u_0^n, $\rho_0^n \in H^3(\mathbb{R}^3)$, with $\nabla \cdot u_0^n = 0$, such that $\|u_0^n - u_0\|_{3,2} + \|\rho_0^n - \rho_0\|_{3,2} \to 0$ as $n \to +\infty$. Let $T^* = T^*(\|u_0\|_{3,2}, \|\rho_0\|_{3,2}) > 0$ be the maximal time of existence of the unique smooth solution of the system* (2.12). *Then for any n there exists a unique smooth solution (u^n, ρ^n) of the system* (2.11) *such that*

$$(u^n, \rho^n) \in C([0, T^*[; H^3(\mathbb{R}^3)).$$

In addition, for any $T < T^*$ and for any couple of sub-sequences $\{k_n\}_n$, $\{\nu_n\}_n$ converging to zero (note that one of them can also be identically zero) it follows

$$\sup_{t\in[0,T]} \Big(\|u^n(t) - u(t)\|_{3,2} + \|\rho^n(t) - \rho(t)\|_{3,2} \Big) \overset{n\to\infty}{\to} 0. \tag{2.13}$$

3. Proof of the continuation criteria. In this section we prove Theorem 2.2 and to this end we recall some preliminary results which will be used in the proof. The first is the classical regularity result for the Stokes operator which dates back to Cattabriga for the Dirichlet problem. In the case of Navier's type slip-without-friction boundary conditions (2.3)-(2.4b) see for instance the recent results in Amrouche and Seloula [1].

LEMMA 3.1. *Let Ω be a bounded and simply connected domain in $\mathbb{R}^3$ with smooth boundary Γ and let $\mathbb{Z} \ni m \geq -1$ and $q \in]1, +\infty[$. For any $F \in W^{m,q}(\Omega)$ there exists a unique solution (u, p) of the following Stokes problem*

$$\begin{aligned} -\Delta u + \nabla \pi &= F && \text{in } \Omega, \\ \operatorname{div} u &= 0 && \text{in } \Omega, \end{aligned}$$

with u satisfying either the boundary conditions (2.3)-(2.4a) *or* (2.3)-(2.4b)*, such that $u \in W^{m+2,q}(\Omega)$ and $\pi \in W^{m+1,q}(\Omega)$. The solution satisfies the estimate*

$$\|u\|_{m+2,q} + \|\pi\|_{m+1,q} \leq C\|F\|_{m,q}, \tag{3.1}$$

for some C depending only on Ω and q.

In addition to the classical integration by parts, in some calculations we will also use the following Gauss-Green formula, where $\omega = \operatorname{curl} u$, cf. [5].

LEMMA 3.2. *Let u and ϕ be two smooth enough vector fields, tangential to the boundary Γ. Then it follows*

$$-\int_\Omega \Delta u \cdot \phi \, dx = \int_\Omega \nabla u \cdot \nabla \phi \, dx - \int_\Gamma (\omega \times n) \cdot \phi \, dS + \int_\Gamma \phi \cdot \nabla n \cdot u dS.$$

Moreover, assume that u is divergence-free and on Γ the slip conditions (2.3)-(2.4b) *hold true. Then*

$$-\frac{\partial \omega}{\partial n} \cdot \omega = (\epsilon_{1jk}\, \epsilon_{1\beta\gamma} + \epsilon_{2jk}\, \epsilon_{2\beta\gamma} + \epsilon_{3jk}\, \epsilon_{3\beta\gamma})\, \omega_j\, \omega_\beta\, \partial_k n_\gamma, \tag{3.2}$$

where ϵ_{ijk} is a totally anti-symmetric tensor. In particular, the latter identity implies that there exists $c = c(\Omega) > 0$ such that

$$-\int_\Omega \Delta \omega \cdot \omega \geq \int_\Omega |\nabla \omega|^2 - c \int_\Gamma |\omega|^2 \, dS.$$

In the sequel we will also use the regularity of the time-dependent Stokes problem, see for instance Sohr [44].

LEMMA 3.3. *Let Ω be a bounded smooth domain in $\mathbb{R}^3$. Let $1 < \alpha < \infty$, $1 < \beta < \infty$, and $0 < T < \infty$. Then, for every $f \in L^\alpha(0,T;L^\beta(\Omega))$ and for every $u_0 \in W^{2,\beta}(\Omega) \cap W_0^{1,\beta}(\Omega)$ with $\nabla \cdot u_0 = 0$ there exists a unique solution u of the time dependent Stokes system*

$$
\begin{cases}
u_t - \Delta u + \nabla \pi = f & \text{in } \Omega \times]0,T], \\
\operatorname{div} u = 0 & \text{in } \Omega \times]0,T], \\
u = 0 & \text{on } \Gamma \times]0,T], \\
u(0,x) = u_0(x) & \text{in } \Omega,
\end{cases}
$$

and it satisfies the estimate

$$
\int_0^T \|u_t(\tau)\|_\beta^\alpha \, d\tau + \int_0^T \|\Delta u(\tau)\|_\beta^\alpha \, d\tau + \int_0^T \|\nabla \pi(\tau)\|_\beta^\alpha \, d\tau \leq C(\alpha,\beta,\Omega,T) \left(\int_0^T \|f(\tau)\|_\beta^\alpha \, d\tau + \|u_0\|_{W^{2,\beta}}^\alpha \right),
$$

with a constant $C(\alpha,\beta,\Omega,T)$ independent of f and u_0.

3.1. Proof of the main estimate. In this section we prove the main estimate, that is the uniform-in-time boundedness of the L^2-norm of the gradient of velocity. Once this step is done (and for this one we need to use different tools, depending on the various conditions we assume) the continuation argument will follow easily by a bootstrap argument.

Since we deal with local smooth solutions all manipulations which we perform are completely justified, and we work in $[0,T[$ for some positive T. We suppose by contradiction that T is the maximal time of existence of a smooth solution starting from (u_0,ρ_0), hence that the H^3-norm is bounded in each closed sub-interval of $[0,T[$ and that

$$
\limsup_{t \to T^-} \|u(t)\|_{3,2} + \|\rho(t)\|_{3,2} = +\infty.
$$

We will show that under the hypotheses of Theorem 2.2 the H^3-norms of both velocity and density remain uniformly bounded, hence we can uniquely continue the solution beyond T, contradicting its maximality. For simplicity we set $\nu = 1$ and, by scaling, the reader can easily adapt the proof to different values of the viscosity.

First we observe that, by standard energy method we have

$$
u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \tag{3.3}
$$

$$
\rho \in L^\infty(0,T;L^p(\Omega)) \quad \text{for any } p \in [1,\infty], \tag{3.4}
$$

and our aim will be first that of increasing by one degree the known regularity of the velocity. We split this in different propositions, related with the different criteria we want to study. We start with the well-known Leray-Prodi-Serrin condition on the velocity.

Proposition 3.4. *Let (u,ρ) be a solution of the system* (2.2) *with boundary condition* (2.3)-(2.4a) *or* (2.3)-(2.4b). *If* (2.7) *holds then*

$$u \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)). \tag{3.5}$$

Proof. Let us consider first the case of Dirichlet boundary conditions (2.3)-(2.4a). Taking the L^2-inner product of the first equation of (2.2) with Au (where A is the Stokes operator associated with zero boundary conditions), and after an integration by parts we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2 + \|Au\|_2^2 \leq \int_\Omega |u|\,|\nabla u|\,|Au|\,dx + \int_\Omega |\rho|\,|Au|\,dx.$$

On the first term from the right-hand-side we use Hölder inequality with exponents p, 2, and $2p/(2-p)$ (for some $p>3$) and on the second one Cauchy-Schwartz inequality to get (with Young inequality)

$$\frac{d}{dt}\|\nabla u\|_2^2 + \|Au\|_2^2 \leq C\|u\|_p\|\nabla u\|_{\frac{2p}{p-2}}\|Au\|_2 + \|\rho\|_2^2.$$

With standard interpolation inequalities, the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, and the norm equivalence coming from Lemma 3.1 we get

$$\begin{aligned}\frac{d}{dt}\|\nabla u\|_2^2 + \|Au\|_2^2 &\leq C\|u\|_p\|\nabla u\|_2^{1-\frac{3}{p}}\|\nabla u\|_6^{\frac{3}{p}}\|Au\|_2 + \|\rho\|_2^2\\ &\leq C\|u\|_p\|\nabla u\|_2^{1-\frac{3}{p}}\|Au\|_2^{1+\frac{3}{p}} + \|\rho\|_2^2.\end{aligned}$$

By using Young inequality and the relation between q and p from (2.7) we get

$$\frac{d}{dt}\|\nabla u\|_2^2 + \|Au\|_2^2 \leq C\|u\|_p^q\|\nabla u\|_2^2 + \|\rho\|_2^2.$$

Now we can apply Gronwall lemma and from (2.7) and (3.4) we get (3.5).

The case of Navier's boundary conditions (2.3)-(2.4b) can be treated in a slightly different way. First, we consider the equation for the vorticity ω

$$\omega_t + (u\cdot\nabla)\,\omega - \Delta\omega = (\omega\cdot\nabla)\,u - \nabla\times(\rho\,\mathrm{e}_3). \tag{3.6}$$

By multiplying (3.6) by ω and by integrating by parts using Lemma 3.2, we observe that

$$\begin{aligned}\int_\Omega \nabla\times(\rho\,\mathrm{e}_3)\cdot\omega\,dx &= \int_\Gamma (\rho\,\mathrm{e}_3\times n)\cdot\omega\,dS + \int_\Omega \rho\,\mathrm{e}_3\cdot(\nabla\times\omega)\,dx\\ &= \int_\Gamma \rho\,\mathrm{e}_3\cdot(n\times\omega)\,dS + \int_\Omega \rho\,\mathrm{e}_3\cdot(\nabla\times\omega)\,dx,\end{aligned}$$

and the boundary term vanishes due to (2.4b). Next, by integrating by parts

$$\int_\Omega (\omega\cdot\nabla)\,u\cdot\omega\,dx = \int_\Gamma (\omega\cdot n)(\omega\cdot u)\,dS - \int_\Omega (\omega\cdot\nabla)\,\omega\cdot u\,dx,$$

and again the boundary term vanishes since $(u \cdot n)_{|\Gamma} = 0$ and $(\omega \times n)_{|\Gamma} = 0$ imply that $(\omega \cdot u)_{|\Gamma} = 0$. Hence, we are reduced to

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_2^2 + \|\nabla\omega\|_2^2 \leq \int_\Omega |u|\,|\omega|\,|\nabla\omega|\,dx + \int_\Omega |\rho|\,|\nabla\omega| + \int_\Gamma \left|\frac{\partial\omega}{\partial n}\cdot\omega\right|\,dS. \tag{3.7}$$

To handle the boundary integral we argue as follows, by using (3.2), trace interpolation inequalities, and Young inequality (again here is critical to use the Navier boundary conditions and especially Lemma 3.2)

$$\int_\Gamma \left|\frac{\partial\omega}{\partial n}\cdot\omega\right|\,dS \leq C\int_\Gamma |\omega|^2\,dS \leq C\|\omega\|_2^2 + \frac{1}{2}\|\nabla\omega\|_2^2.$$

Then, by using again Hölder inequality on the remaining term in (3.7) we get (by using also that we have a sort of Poincaré inequality for functions tangential to the boundary, see [32])

$$\int_\Omega |u|\,|\omega|\,|\nabla\omega|\,dx \leq \|u\|_p\,\|\omega\|_{\frac{2p}{p-2}}\|\nabla\omega\|_2 \leq C\|u\|_p\|\omega\|_2^{1-\frac{3}{p}}\|\nabla\omega\|_2^{1+\frac{3}{p}},$$

and consequently by Young inequality

$$\frac{d}{dt}\|\omega\|_2^2 + \|\nabla\omega\|_2^2 \leq C(1+\|u\|_p^q)\|\omega\|_2^2 + \|\rho\|_2^2.$$

By Gronwall lemma, we get $\omega \in L^\infty(0,T;L^2(\Omega))$. Then (3.5) follows by recalling the inequality (see [46])

$$\|\nabla u\|_2 \leq C(\Omega)\big(\|\nabla\times u\|_2 + \|\operatorname{div} u\|_2 + \|u\cdot n\|_{1/2,\Gamma}\big),$$

□

We pass now to the second criterion.

PROPOSITION 3.5. *Let* (u,ρ) *be a solution of the system* (2.2) *with boundary conditions* (2.3)-(2.4a) *or* (2.3)-(2.4b). *If* (2.8) *holds then*

$$u \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)).$$

Proof. We start by observing that classical regularity results of Navier-Stokes equations with $f = \rho$ implies that the local smooth solution satisfies $u_{tt} \in L^2(\lambda,T;L^2(\Omega))$, for all $\lambda > 0$, and some compatibility conditions are needed to reach $t = 0$ (hence this part of the proof works also if the initial datum does not satisfy the compatibility condition and the solution may be not continuous at time $t = 0$). By using this observation, it follows that all the calculation we are going to do over the time interval (λ, T) are not formal, but completely justified, since each term is well-defined and the boundary integrals (arising in the integration by parts) vanish. As in [8] we take the time derivative of first equation in (2.2), we take the L^2-inner product against u_t, and we use the continuity equation (to substitute ρ_t by $-(u\cdot\nabla)\,\rho$). We observe that $\int_\Omega (u\cdot\nabla)\,u_t\cdot u_t\,dx = 0$ and also

$$-\int_\Omega \rho_t \mathrm{e}_3\cdot u_t = \int_\Omega (u\cdot\nabla)\,\rho\,\mathrm{e}_3\cdot u_t\,dx = -\int_\Omega (u\cdot\nabla)\,u_t\cdot\rho\,\mathrm{e}_3\,dx + \int_\Gamma (u\cdot n)(\rho\,\mathrm{e}_3\cdot u_t)\,dS, \tag{3.8}$$

where in the integration by parts we used the fact that u is divergence-free. In addition, the boundary term vanishes since $(u \cdot n)_{|\Gamma} = 0$. We observe that the same latter calculations are correct also in the case of the Navier's type boundary conditions. We then obtain

$$\frac{1}{2}\frac{d}{dt}\|u_t\|_2^2 + \|\nabla u_t\|_2^2 \leq \left|\int_\Omega u_t \cdot \nabla u \cdot u_t \, dx\right| + \left|\int_\Omega (u \cdot \nabla)\, u_t \cdot \rho\, \mathrm{e}_3 \, dx\right|.$$

and by using Hölder inequality with conjugate exponents p, p' such that $p > 3/2$ we get

$$\frac{d}{dt}\|u_t\|_2^2 + \|\nabla u_t\|_2^2 \leq C\|u_t\|_{2p'}^2 \|\nabla u\|_p + C\|u\|_6 \|\nabla u_t\|_2 \|\rho\|_3.$$

We estimate the first term of the right-hand-side by using a Gagliardo-Nirenberg inequality and with Young inequality we get

$$\begin{aligned}\frac{d}{dt}\|u_t\|_2^2 + \|\nabla u_t\|_2^2 &\leq C\big(\|u_t\|_2^{\frac{2p-3}{p}} \|\nabla u_t\|_2^{\frac{3}{p}} \|\nabla u\|_p + \|\nabla u\|_2^2 \|\rho\|_3^2\big) + \frac{1}{4}\|\nabla u_t\|_2^2 \\ &\leq C\big(\|\nabla u\|_p^{\frac{2p}{2p-3}} \|u_t\|_2^2 + \|\nabla u\|_2^2 \|\rho\|_3^2\big) + \frac{1}{2}\|\nabla u_t\|_2^2.\end{aligned}$$

By choosing $\lambda > 0$ such that $u_t(\lambda, \cdot) \in L^2(\Omega)$ and this is always possible since this property holds true almost everywhere in $(0, T)$, with Gronwall inequality we obtain

$$u_t \in L^\infty(\lambda, T; L^2(\Omega)) \cap L^2(\lambda, T; H^1(\Omega)).$$

Next, by multiplying (2.2) with u and by integrating over Ω we have

$$\int_\Omega |\nabla u|^2 dx \leq \|u_t\|_2 \|u\|_2 + \|\rho\|_2 \|u\|_2 \qquad a.e.\ t \in [\lambda, T],$$

and we then obtain $u \in L^\infty(\lambda, T; H^1(\Omega))$. Since u is a local smooth solution, we have finally that

$$u \in L^\infty(0, T; H^1(\Omega)).$$

When we consider the Navier's boundary conditions (2.3)-(2.4b) the proof of the Proposition is quite similar, we have only to check that the boundary term in the integration by parts of the Laplacian remains under control. This holds true since we have

$$-\int_\Omega \Delta u_t u_t = \int_\Omega |\nabla u_t|^2 - \int_\Gamma \omega_t \times n \cdot u_t + \int_\Gamma u_t \cdot \nabla n \cdot u_t \, dS. \tag{3.9}$$

Observe then that on Γ we have $\omega_t \times n = (\omega \times n)_t = 0$ and the first boundary term in (3.9) vanishes. The other boundary integral can be estimated by the usual trace inequalities as

$$\left|\int_\Gamma u_t \cdot \nabla n \cdot u_t \, dS\right| \leq \frac{1}{2}\int_\Omega |\nabla u_t|^2 \, dx + C(\Omega)\int_\Omega |u_t|^2 \, dx.$$

The proof continues then as in the case of the Dirichlet boundary conditions. □

We pass now to the third criterion, concerning the Bernoulli's pressure.

Proposition 3.6. *Let (u, ρ) be a solution of the system* (2.2) *with the Dirichlet boundary conditions* (2.3)(2.4a). *If* (2.9) *holds then*

$$u \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)).$$

Proof. In the proof we adapt the techniques introduced in [12, 19, 30] to our setting. Let us observe that the result in the present form is not proved in any other reference also in the case $\rho = const$. The first step is to rewrite the momentum equation in (2.2) in the rotational form

$$u_t - \Delta u + \text{curl } u \times u + \nabla\left(\pi + \frac{|u|^2}{2}\right) = -\rho\, e_3. \tag{3.10}$$

First we consider the case $\frac{3}{2} < p \leq 3$. By taking the L^2-inner product of (3.10) with $u|u|$, after suitable integration by parts, we obtain

$$\frac{1}{3}\frac{d}{dt}\|u\|_3^3 + \int_\Omega |u|\,|\nabla u|^2\,dx + \frac{4}{9}\|\nabla|u|^{\frac{3}{2}}\|_2^2 \leq \frac{2}{3}\mathcal{I} + \int_\Omega |\rho|\,|u|^2\,dx, \tag{3.11}$$

where

$$\mathcal{I}(t) := \int_\Omega \left|\pi(t) + \frac{|u(t)|^2}{2}\right|\,|u(t)|^{\frac{1}{2}}\,|\nabla|u(t)|^{\frac{3}{2}}|\,dx. \tag{3.12}$$

Consider the sub-case $\frac{9}{4} \leq p \leq 3$. We have, for each $\epsilon > 0$,

$$\begin{aligned}
\mathcal{I} &\leq \left\|\pi + \frac{|u|^2}{2}\right\|_p \|u\|_{\frac{p}{p-2}}^{\frac{1}{2}} \|\nabla|u|^{\frac{3}{2}}\|_2 \\
&\leq \left\|\pi + \frac{|u|^2}{2}\right\|_p \|u\|_3^{\frac{4p-9}{2p}} \|u\|_9^{\frac{3(3-p)}{2p}} \|\nabla|u|^{\frac{3}{2}}\|_2 \\
&\leq \left\|\pi + \frac{|u|^2}{2}\right\|_p \|u\|_3^{\frac{4p-9}{2p}} \|\nabla|u|^{\frac{3}{2}}\|_2^{\frac{3}{p}} \\
&\leq C_\epsilon \left\|\pi + \frac{|u|^2}{2}\right\|_p^{\frac{2p}{2p-3}} \|u\|_3^{\frac{4p-9}{2p-3}} + \varepsilon\|\nabla|u|^{\frac{3}{2}}\|_2^2,
\end{aligned}$$

where we have used, in the order, the Hölder inequality with exponents $2, p, \frac{2p}{p-2}$, the interpolation inequality, the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$, and the Young inequality. Thus, choosing ϵ small enough, the following differential inequality holds:

$$\frac{1}{3}\frac{d}{dt}\|u\|_3^3 + \int_\Omega |u||\nabla u|^2\,dx + \frac{2}{9}\|\nabla|u|^{\frac{3}{2}}\|_2^2 \leq C\Big(\left\|\pi + \frac{|u|^2}{2}\right\|_p^q \|u\|_3^{3(1-\mu)} + \|\rho\|_3^3 + \|u\|_3^3\Big),$$

for a suitable $\mu \in [\frac{2}{3}, 1]$. By using Gronwall lemma we have

$$\|u(T)\|_3^3 + \int_0^T \|\nabla|u|^{\frac{3}{2}}(\tau)\|_2^2\,d\tau \leq C(T)\Big[\|u_0\|_3^3 + \Big(\int_0^T \|2\pi(\tau) + |u|^2(\tau)\|_p^q\,d\tau\Big)^{\frac{1}{1-\mu}} + \|\rho\|_{L^\infty(L^3)}\Big].$$

which implies that $|u|^{\frac{3}{2}} \in L^2(0,T; H_0^1(\Omega)) \hookrightarrow L^2(0,T; L^6(\Omega))$, and consequently

$$u \in L^3(0,T; L^9(\Omega)),$$

which is a special case of (2.7). Then, by the previous results, we have that $u \in L^\infty(0,T; H^1(\Omega))$, and this ends the proof in the case $9/4 \leq p \leq 3$.

Next, we consider the case $\frac{3}{2} < p < \frac{9}{4}$. We apply Hölder inequality in (3.12) with $p_1 = \frac{9}{4}$, $p_2 = 18$, $p_3 = 2$, interpolation inequalities, Poincaré inequality (observe that π has zero mean value, while $|u|$ vanishes at the boundary), and Young inequality with exponents $\frac{9-2p}{12-4p}$ and $\frac{9-2p}{2p-3}$ to get

$$
\begin{aligned}
\mathcal{I} &\leq \|2\pi + |u|^2\|_{\frac{9}{4}} \|\nabla |u|^{\frac{3}{2}}\|_2 \||u|^{\frac{1}{2}}\|_{18} \\
&\leq \|2\pi + |u|^2\|_{\frac{9}{4}} \|\nabla |u|^{\frac{3}{2}}\|_2^{\frac{4}{3}} \\
&\leq \|2\pi + |u|^2\|_p^{\frac{2p}{9-2p}} \|2\pi + |u|^2\|_{\frac{9}{2}}^{\frac{9-4p}{9-2p}} \|\nabla |u|^{\frac{3}{2}}\|_2^{\frac{4}{3}} \\
&\leq C\|2\pi + |u|^2\|_p^{\frac{2p}{9-2p}} (\|2\pi\|_{\frac{9}{2}}^{\frac{9-4p}{9-2p}} + \||u|^2\|_{\frac{9}{2}}^{\frac{9-4p}{9-2p}}) \|\nabla |u|^{\frac{3}{2}}\|_2^{\frac{4}{3}} \qquad (3.13)\\
&\leq C\|2\pi + |u|^2\|_p^{\frac{2p}{9-2p}} (\|\nabla\pi\|_{\frac{9}{5}}^{\frac{9-4p}{9-2p}} + \|\nabla|u|^2\|_{\frac{9}{5}}^{\frac{9-4p}{9-2p}}) \|\nabla |u|^{\frac{3}{2}}\|_2^{\frac{4}{3}} \\
&\leq C_\epsilon \|2\pi + |u|^2\|_p^{\frac{2p}{2p-3}} + \varepsilon (\|\nabla\pi\|_{\frac{9}{5}}^{\frac{9-4p}{12-4p}} + \|\nabla|u|^2\|_{\frac{9}{5}}^{\frac{9-4p}{12-4p}}) \|\nabla |u|^{\frac{3}{2}}\|_2^{\frac{9-2p}{9-3p}}.
\end{aligned}
$$

To handle the pressure we use now the mixed estimates of Lemma 3.3. We rewrite the equation for the velocity of the system (2.2) as

$$
\begin{aligned}
u_t - \Delta u + \nabla\pi &= -(u\cdot\nabla)\,u - \rho\,\mathrm{e}_3 && \text{in } \Omega\times(0,T),\\
\nabla\cdot u &= 0 && \text{in } \Omega\times(0,T),\\
u &= 0 && \text{on } \Gamma\times(0,T),\\
u(0,x) &= u_0 && \text{in } \Omega.
\end{aligned}
$$

We use, for $1 < \alpha < \infty$ and $\beta = 9/5$, the mixed estimate

$$
\|\nabla\pi\|_{L^\alpha(0,T;L^{9/5}(\Omega)} \leq C\big(\|(u\cdot\nabla)u\|_{L^\alpha(0,T;L^{9/5}(\Omega))} + \|u_0\|_{W^{2,9/5}(\Omega)} + \|\rho\|_{L^\alpha(0,T;L^{9/5}(\Omega))}\big),
$$

and, with Hölder inequality, we get

$$
\|(u\cdot\nabla)u\|_{9/5} \leq \||u|^{1/2}|\nabla u|\,|u|^{1/2}\|_{9/5} \leq \left(\int_\Omega |u|\,|\nabla u|^2\,dx\right)^{\frac{1}{2}} \|u\|_9^{\frac{1}{2}}.
$$

To estimate the $L^{9/5}$-norm involving $\nabla|u|^2$, we observe that $|\nabla|u|^2| \leq 2|u|\,|\nabla u|$ and we similarly obtain

$$
\|\nabla|u|^2\|_{9/5} \leq 2\left(\int_\Omega |u|\,|\nabla u|^2\,dx\right)^{\frac{1}{2}} \|u\|_9^{\frac{1}{2}}.
$$

Coming back to (3.13), if we integrate the last term $\big(\|\nabla\pi\|_{9/5}^{\frac{9-4p}{12-4p}} + \|\nabla|u|^2\|_{9/5}^{\frac{9-4p}{12-4p}}\big)\|\nabla|u|^{\frac{3}{2}}\|_2^{\frac{9-2p}{9-3p}}$ with respect to time on $(0,T)$, use the result for the Stokes system in Lemma 3.3, and collect the above estimates for $L^{9/5}(\Omega)$ terms we get (with application of the Hölder inequality and the inequality $\|u\|_9^{\frac{3}{2}} = \| \, |u|^{\frac{3}{2}} \|_6 \le C \|\nabla |u|^{\frac{3}{2}} \|_2)$

$$\int_0^T \big(\|\nabla\pi(\tau)\|_{9/5}^{\frac{9-4p}{12-4p}} + \|\nabla|u|^2(\tau)\|_{9/5}^{\frac{9-4p}{12-4p}}\big)\|\nabla|u|^{\frac{3}{2}}(\tau)\|_2^{\frac{9-2p}{9-3p}}\,d\tau$$
$$\le \left[\int_0^T \|\nabla\pi(\tau)\|_{9/5}^{\frac{3}{2}} + \|\nabla|u|^2(\tau)\|_{9/5}^{\frac{3}{2}}\,d\tau\right]^{\frac{9-4p}{18-6p}}\left[\int_0^T \|\nabla|u|^{\frac{3}{2}}(\tau)\|_2^2\,d\tau\right]^{\frac{9-2p}{18-6p}}$$
$$\le C\left[\int_0^T \big(\int_\Omega |u|\,|\nabla u|^2\,dx\big)^{\frac{3}{4}}\|\nabla|u|^{\frac{3}{2}}(\tau)\|_2^{1/2}\,d\tau + \|u_0\|_{W^{2,9/5}}^{\frac{3}{2}} + \|\rho\|_{L^{\frac{3}{2}}(L^{9/5})}^{\frac{3}{2}}\right]^{\frac{9-4p}{18-6p}}\cdot$$
$$\cdot\left[\int_0^T \|\nabla|u|^{\frac{3}{2}}(\tau)\|_2^2\,d\tau\right]^{\frac{9-2p}{18-6p}}.$$

Another application of Young inequality finally implies

$$\int_0^T \big(\|\nabla\pi(\tau)\|_{9/5}^{\frac{9-4p}{12-4p}} + \|\nabla|u|^2(\tau)\|_{9/5}^{\frac{9-4p}{12-4p}}\big)\|\nabla|u|^{\frac{3}{2}}(\tau)\|_2^{\frac{9-2p}{9-3p}}\,d\tau$$
$$\le C\left(\int_0^T\int_\Omega |u|\,|\nabla u|^2\,dxd\tau + \int_0^T \|\nabla|u|^{\frac{3}{2}}(\tau)\|_2^2\,d\tau + \|u_0\|_{W^{2,9/5}} + \|\rho\|_{L^\infty(L^{9/5})}\right).$$

By integrating (3.11) on $(0,T)$ and by using the latter estimate we obtain

$$\frac{1}{3}\|u(T)\|_3^3 + \int_0^T\int_\Omega |u|\,|\nabla u|^2\,dx + \frac{4}{9}\|\nabla|u|^{\frac{3}{2}}(\tau)\|_2^2\,d\tau$$
$$\le \frac{1}{3}\|u_0\|_3^3 + C_\epsilon\int_0^T \|2\pi + |u|^2\|_p^{\frac{2p}{2p-3}}\,d\tau +$$
$$+\,\epsilon\, C\left[\int_0^T\int_\Omega |u|\,|\nabla u|^2\,dxd\tau + \int_0^T \|\nabla|u|^{\frac{3}{2}}(\tau)\|_2^2\,d\tau + \|u_0\|_{W^{2,9/5}} + \|\rho\|_{L^\infty(L^{9/5})}\right].$$

By choosing $\epsilon > 0$ small enough, we can absorb the two terms form the right-hand side involving space derivatives of u into the left-hand side. By applying Gronwall lemma and by and using hypothesis (2.9) we obtain finally that $\int_0^T \|\nabla|u|^{\frac{3}{2}}\|_2^2\,d\tau < +\infty$. Hence, by the same argument as before $u \in L^\infty(0,T;H^1(\Omega))$.

Finally, we consider the case $p > 3$. In this case the proof is rather different. First, we multiply (2.2) by $Au = -P\Delta u$ and integrate over $(t_0,t) \subset [0,T)$ to get

$$\|\nabla u(t)\|_2^2 + \int_{t_0}^t \|P\Delta u(\tau)\|_2^2\,d\tau \le \|\nabla u(t_0)\|_2^2 + C\int_{t_0}^t \|(u(\tau)\cdot\nabla)u(\tau)\|_2^2\,d\tau + \|\rho\|_{L^\infty(t_0,t;L^2)}^2.$$

These calculations are possible, since u is also a weak solution and consequently $u \in H_0^1(\Omega)$ for almost all $t_0 \in (0,T)$. Hence, we obtain the *a-priori* estimate

$$\sup_{t_0<\sigma<t}\|\nabla u(\sigma)\|_2^2 \le \|\nabla u(t_0)\|_2^2 + C\int_0^t \||\nabla u(\tau)||u(\tau)|\|_2^2\,d\tau + \|\rho\|_{L^\infty(L^2)}^2. \tag{3.14}$$

Next, we derive the differential inequality satisfied by the L^4-norm of the velocity. Within this range of the exponent "p" we need to use a different (with respect to the previous cases) approach to the proof. The use of estimates involving the L^3-norm seems not possible, apart in the simpler case of the domain without boundaries. To get the L^4-estimates we multiply the equations (3.10) by $|u|^2 u$ and integrate by parts to obtain

$$\frac{1}{4}\frac{d}{dt}\|u\|_4^4 + \int_\Omega |u|^2|\nabla u|^2\,dx + \frac{1}{2}\,\|\nabla|u|^2\|_2^2 \leq C\mathcal{J} + \int_\Omega |\rho|\,|u|^3\,dx, \tag{3.15}$$

where

$$\mathcal{J}(t) := \int_\Omega (2\pi(t) + |u(t)|^2)|\nabla u(t)|\,|u(t)|^2\,dx.$$

By using Hölder and interpolation inequality and the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ we estimate the right-hand side as follows (with $p > 3$)

$$\begin{aligned}
\mathcal{J} &\leq \|\nabla u\|_2\,\|2\pi + |u|^2\|_p\,\|\,|u|^2\|_{\frac{2p}{p-2}} \\
&\leq C\|\nabla u\|_2\,\|2\pi + |u|^2\|_p\,\|u\|^2_{\frac{4p}{p-2}} \\
&\leq C\|\nabla u\|_2\,\|2\pi + |u|^2\|_p\,\|u\|_4^{\frac{2(p-3)}{p}}\,\||u|^2\|_6^{\frac{3}{p}} \\
&\leq C\|\nabla u\|_2\,\|2\pi + |u|^2\|_p\,\|u\|_4^{\frac{2(p-3)}{p}}\,\|\nabla|u|^2\|_2^{\frac{3}{p}}.
\end{aligned}$$

First, we observe that

$$u(t_0,\cdot) \in L^4(\Omega) \cap H_0^1(\Omega) \qquad \text{for almost all } t_0 > 0, \tag{3.16}$$

By using interpolation and Young's inequality and (3.14) we estimate, for $\sigma \in (t_0, t)$, the right-hand side as follows

$$\begin{aligned}
\mathcal{J}(\sigma) &\leq C\|2\pi(\sigma) + |u|^2(\sigma)\|_p^{\frac{2p}{2p-3}}\|\nabla u(\sigma)\|_2^2 + \frac{1}{4}\|\nabla|u(\sigma)|^2\|_2^2 + C\|2\pi(\sigma) + |u|^2(\sigma)\|_p^{\frac{2p}{2p-3}}\|u(\sigma)\|_4^4 \\
&\leq C\|2\pi(\sigma) + |u|^2(\sigma)\|_p^{\frac{2p}{2p-3}}\Big(\int_{t_0}^t\int_\Omega |\nabla u(\tau)||u(\tau)|^2\,dxd\tau + \|\nabla u(t_0)\|_2^2 + \|\rho\|^2_{L^\infty(L^2)}\Big) \\
&\qquad + \frac{1}{4}\|\nabla|u|^2\|_2^2 + C\|2\pi(\sigma) + |u|^2(\sigma)\|_p^{\frac{2p}{2p-3}}\|u(\sigma)\|_4^4.
\end{aligned}$$

We then integrate the inequality (3.15) not on the whole interval, but on $(t_0, t) \subseteq [0, T)$, with t_0 such that (3.16) holds true

$$\begin{aligned}
&\frac{1}{4}\|u(t)\|_4 + \frac{1}{2}\int_{t_0}^t \|\nabla|u|^2(\tau)\|_2^2\,d\tau + \int_0^t\int_\Omega |\nabla u(\tau)||u(\tau)|^2\,dxd\tau \leq \\
&\leq \frac{1}{4}\|u(t_0)\|_4 + C\|\nabla u(t_0)\|_2^2 + C_1\int_0^t\int_\Omega |\nabla u(\tau)||u(\tau)|^2\,dxd\tau \cdot \int_{t_0}^t \|(2\pi + |u|^2)(\tau)\|_p^{\frac{2p}{2p-3}}\,d\tau \\
&+ \frac{1}{4}\int_{t_0}^t \|\nabla|u|^2(\tau)\|_2^2\,d\tau + C_2\int_{t_0}^t \|(2\pi + |u|^2)(\tau)\|_p^{\frac{2p}{2p-3}}\|u(\tau)\|_4^4\,d\tau + \int_{t_0}^t\int_\Omega |\rho|\,|u|^3\,dx.
\end{aligned}$$

Since $\frac{2}{\frac{2p}{2p-3}} + \frac{3}{p} = 2$, by using (2.9) and the absolute continuity of the Lebesgue measure we can choose $t_0 \in (0, T)$, among the set satisfying (3.16), near enough to T such that

$$\int_{t_0}^T \|(2\pi + |u|^2)(\tau)\|_p^{\frac{2p}{2p-3}}\,d\tau \leq \frac{1}{2C_1}.$$

Hence, we can absorb the second and the third term from the right-hand side into the left-hand side and, after using Hölder inequality in the last term, we obtain by Gronwall inequality

$$\sup_{t_0<t<T} \|u(t)\|_4^4 \leq C\big(\|u(t_0)\|_4^4 + \|\nabla u(t_0)\|_2^2\big) \text{ Exp}\Big(\int_{t_0}^T \|(2\pi + |u|^2)(\tau)\|_p^{\frac{2p}{2p-3}} + \|\rho\|_{L^\infty(L^4)}^4 \, d\tau\Big),$$

which is finite and then implies directly again the requested regularity of u, being a sub-case of (2.7).

We observe now that, at least for $\frac{9}{4} < p \leq 3$ the proof can be done in the same way also in the case of Navier's boundary conditions. In fact, by using the same techniques, testing with $|u|u$, and integrating by parts with the Gauss-Green formula

$$-\int_\Omega \Delta u \, |u|u \, dx = \int_\Omega \nabla u \nabla(|u|u) \, dx - \int_\Gamma \omega \times n|u|u \, dS + \int_\Gamma |u|u \cdot \nabla n \cdot u \, ds$$

we get

$$\frac{1}{3}\frac{d}{dt}\|u\|_3^3 + \int_\Omega |u| \, |\nabla u|^2 \, dx + \frac{4}{9}\|\nabla |u|^{\frac{3}{2}}\|_2^2 \leq \frac{2}{3}\mathcal{I} + C\int_\Gamma |u|^3 \, dS + \int_\Omega |\rho| \, |u|^2 \, dx,$$

and, by the trace and Young inequality,

$$\begin{aligned}\int_\Gamma |u|^3 \, dS = \int_\Gamma ||u|^{\frac{3}{2}}|^2 \, dS = \||u|^{\frac{3}{2}}\|_{2,\Gamma}^2 &\leq C\||u|^{\frac{3}{2}}\|_2 \|\nabla |u|^{\frac{3}{2}}\|_2 \\ &\leq C\|u\|_3^{\frac{3}{2}} \|\nabla |u|^{\frac{3}{2}}\|_2 \leq C\|u\|_3^3 + \frac{2}{9}\|\nabla |u|^{\frac{3}{2}}\|_2^2.\end{aligned}$$

Hence, we obtain the differential inequality

$$\frac{1}{3}\frac{d}{dt}\|u\|_3^3 + \int_\Omega |u| \, |\nabla u|^2 \, dx + \frac{2}{9}\|\nabla |u|^{\frac{3}{2}}\|_2^2 \leq \frac{2}{3}\mathcal{I} + \int_\Omega |\rho| \, |u|^2 \, dx + C\|u\|_3^3,$$

and the proof continues as in the Dirichlet case. □

We now sketch the proof in the case of the condition on the direction of the vorticity. In this case one has to restrict to the Navier-type boundary conditions, since we need to use in a substantial way the equation for the vorticity.

Proposition 3.7. *Let* (u, ρ) *be a solution of the system* (2.2) *with boundary conditions* (2.3)-(2.4b). *If* (2.9) *holds then*

$$u \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)).$$

The proof of this proposition is straightforward once the following rather technical result (and whose proof is too long to be reproduced here) is taken for granted:

Lemma 3.8. *(cf. [5, Prop. 4.1]) Let us assume that* (2.10) *holds and that u is a smooth in* $[0, T)$. *Then, to each* $\varepsilon > 0$ *there corresponds a positive* $C_\varepsilon > 0$ *such that the following inequality holds:*

$$\left| \int_\Omega (\omega(x) \cdot \nabla u(x)) \cdot \omega(x) \, dx \right| \leq \varepsilon \|\nabla \omega\|_2^2 + C_\varepsilon(\|\omega\|_2^4 + \|\omega\|_2^3), \quad a.e. \ t \in [0, T[.$$

Proof. [Proof of Proposition 3.7] By multiplying (3.6) by ω, by integrating by parts and by using the above Lemma we get

$$\frac{d}{dt}\|\omega\|_2^2 + \|\nabla \omega\|_2^2 \leq C(1 + \|\omega\|_2^2)\|\omega\|_2^2 + \|\rho\|_2^2.$$

The energy estimate and Gronwall lemma imply that $\omega \in L^\infty(0, T; L^2(\Omega))$, ending the proof as in Proposition 3.4 □

3.2. The bootstrap argument. We now show how to prove the full regularity once the first step has been done in the different cases, and with the different boundary conditions.

The first result shows how to improve by "one degree" the spatial regularity of the velocity in Hilbert spaces.

PROPOSITION 3.9. *Let* (u, ρ) *be a smooth solution of the system* (2.2) *with boundary conditions* (2.3)-(2.4a) *or* (2.3)-(2.4b). *If*

$$u \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)),$$

then

$$u \in L^\infty(0, T; H^2(\Omega)).$$

Proof. As explained before, the calculations with the time derivative are justified in the domain $(\lambda, T) \times \Omega$ for some $\lambda > 0$ (even if without compatibility conditions). As in Proposition 3.5 by taking the time derivative of momentum equation in the system (2.2), then by taking the L^2-inner product with u_t, by using the continuity equation, and finally integrating by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_t\|_2^2 + \|\nabla u_t\|_2^2 \le \int_\Omega |u_t|^2 |\nabla u| \, dx + \int_\Omega |u| \, |\nabla u_t| \, |\rho| \, dx.$$

We remark that the various boundary terms vanish as in the proof of (3.8) from Proposition 3.5. By Hölder inequality we get

$$\frac{d}{dt}\|u_t\|_2^2 + \|\nabla u_t\|_2^2 \le C\Big(\|u_t\|_4^2 \|\nabla u\|_2 + \|u\|_4 \|\nabla u_t\|_2 \|\rho\|_4\Big).$$

We use now the standard (Ladyžhenskaya) interpolation inequality $\|f\|_4 \le C\|f\|_2^{1/4}\|\nabla f\|_2^{3/4}$ to estimate the L^4-norm and we get, using also the Young's inequality,

$$\begin{aligned}\frac{d}{dt}\|u_t\|_2^2 + \|\nabla u_t\|_2^2 &\le C\big(\|u_t\|_2^{\frac{1}{2}} \|\nabla u_t\|_2^{\frac{3}{2}} \|\nabla u\|_2 + \|\rho\|_4^4 + \|u\|_2 \|\nabla u\|_2^3\big) \\ &\le C\big(\|\nabla u\|_2^4 \|u_t\|_2^2 + \|\rho\|_4^4 + \|u\|_2 \|\nabla u\|_2^3\big).\end{aligned}$$

Finally, by using Gronwall lemma, and the uniform estimate on $\|\nabla u\|_2$ and $\|\rho\|_4$ we have

$$u_t \in L^\infty(\lambda, T; L^2(\Omega)) \cap L^2(\lambda, T; H^1_0(\Omega)). \tag{3.17}$$

To show now the further regularity of the second order derivatives, we use now Lemma 3.1. To this aim we rewrite the equations for the velocity from system (2.2) in the following way

$$\begin{aligned} -\Delta u + \nabla \pi &= -(u \cdot \nabla)\, u - u_t - \rho\, \mathrm{e}_3 && \text{in } \Omega \times (0, T), \\ \operatorname{div} u &= 0 && \text{in } \Omega \times (0, T), \end{aligned} \tag{3.18}$$

and we consider it as a stationary system, for each time $t \in [0, T]$, with the two different boundary conditions (it is crucial here that we obtained an estimate for u_t directly with energy estimates). From (3.1) we infer that, for all fixed $t \in (0, T]$

$$\|u\|_{2,2} \le C\Big(\|u_t\|_2 + \|\rho\|_2 + \|u \cdot \nabla u\|_2\Big).$$

From (3.4) and (3.17) we have that the last two term from the right-hand-side are bounded in $L^\infty(0,T;L^2(\Omega))$. By using Hölder inequality in the nonlinear term, the L^4-interpolation inequality, we get in $[\lambda, T]$,

$$\begin{aligned}
\|u\|_{2,2} &\le C(C + \|u\|_4\|\nabla u\|_4)\\
&\le C(C + \|u\|_4\|\nabla u\|_2^{\frac{1}{4}}\|u\|_{2,2}^{\frac{3}{4}})\\
&\le C(C + \|u\|_4^4\|\nabla u\|_2) + \epsilon\|u\|_{2,2}\\
&\le C(C + \|u\|_2\|\nabla u\|_2^4) + \epsilon\|u\|_{2,2}.
\end{aligned}$$

By absorbing the small ϵ-term in the left-hand-side and by (3.3) and (3.5) we have that

$$u \in L^\infty(\lambda, T; H^2(\Omega)). \tag{3.19}$$

Since u is a local smooth solution in $(0,T)$ we get finally $u \in L^\infty(0,T;H^2(\Omega))$. □

PROPOSITION 3.10. *Let (u,ρ) be a local strong solution of the system* (2.2) *with boundary conditions* (2.3)-(2.4a) *or* (2.3)-(2.4b). *If the condition* (3.19) *is satisfied, then*

$$\nabla u \in L^2(0,T;L^\infty(\Omega)).$$

Proof. We use again the regularity for the linear Stokes system. In fact, from Lemma 3.1 applied again to (3.18) we get, by using also the Poincaré inequality which is valid for functions tangential to the boundary,

$$\begin{aligned}
\|u\|_{2,6} &\le C\Big(\|u_t\|_6 + \|\rho\|_6 + \|u\cdot\nabla u\|_6\Big)\\
&\le C\Big(\|\nabla u_t\|_2 + \|\rho\|_6 + \|u\|_\infty\|\nabla u\|_6\Big)\\
&\le C\Big(\|\nabla u_t\|_2 + \|\rho\|_6 + \|u\|_{2,2}^2\Big).
\end{aligned}$$

By recalling (3.4), (3.17), and (3.19) we have $\|\nabla u_t\|_2 \in L^2$ and $\|\rho\|_6 + \|u\|_{2,2}^2 \in L^\infty$, we finally get $u \in L^2(\lambda, T; W^{2,6}(\Omega))$. By using Sobolev embedding we get

$$\nabla u \in L^2(\lambda, T; L^\infty(\Omega)),$$

and since u is a local smooth solution

$$\nabla u \in L^2(0,T;L^\infty(\Omega)). \tag{3.20}$$

□

The next step is to show improved regularity of the Lebesgue norms of the gradient of the density. Observe in fact that till this point we used only boundedness of ρ and no other information on its space (and also time) derivatives, than that coming from the basic energy estimate.

PROPOSITION 3.11. *Let (u,ρ) be a local strong solution of the system* (2.2) *in $[0,T[$ with boundary conditions* (2.3)-(2.4a) *or* (2.3)-(2.4b). *If the condition* (3.20) *holds then*

$$\nabla\rho \in L^\infty((0,T)\times\Omega).$$

Proof. By taking the gradient of the continuity equation in (2.2) we get the equation

$$\nabla \rho_t + (u \cdot \nabla)\, \nabla \rho = -(\nabla u)(\nabla \rho). \tag{3.21}$$

By multiplying (3.21) by $\nabla\rho|\nabla\rho|^{p-2}$ for $p > 3$ and by integrating by parts we obtain the following differential inequality

$$\frac{1}{p}\frac{d}{dt}\|\nabla\rho\|_p^p \leq \|\nabla u\|_\infty \|\nabla\rho\|_p^p.$$

Then, by integrating the above differential inequality with respect to time and by using (3.20) we get that there exists a constant C_1, independent of $p > 3$, such that

$$\sup_{t\in]0,T[} \|\nabla\rho\|_p \leq C_1.$$

Then, by sending p to $+\infty$, we end the proof of Proposition 3.11. □

We can now give the proof of the main result of this section.

Proof of Theorem 2.2. By previous propositions we have that if one the condition (2.7)– (2.10), holds true, then the following estimates are satisfied

$$u \in L^\infty(0, T; H^2(\Omega)), \tag{3.22a}$$

$$u_t \in L^\infty(\lambda, T; H^1(\Omega)), \tag{3.22b}$$

$$\nabla u \in L^2(0, T; L^\infty(\Omega)), \tag{3.22c}$$

$$\nabla\rho \in L^\infty((0, T) \times \Omega). \tag{3.22d}$$

To end the proof we need only to prove that these bounds imply that

$$u \in L^\infty(\lambda, T; H^3(\Omega)), \tag{3.23}$$

$$\rho \in L^\infty(\lambda, T; H^3(\Omega)). \tag{3.24}$$

Then, since (u, ρ) is a local smooth solution, the same bounds are true also near $t = 0$. Moreover, the fact that (u, ρ) can be extended to a global smooth solution of (2.2) follows then from a standard continuity argument.

First, we use Lemma 3.1 and the local existence theorem in order to prove (3.23). In fact, we observe that

$$\|(u \cdot \nabla)\, u\|_{1,2} \leq C\Big(\|u\|_\infty \|\nabla u\| + \|\nabla u\|_4^2 + \|u\|_\infty \|u\|_{2,2}\Big) \leq C\|u\|_{2,2}^2.$$

Next, by using (3.22a), we get

$$(u \cdot \nabla)\, u \in L^\infty(0, T; H^1(\Omega)).$$

Again the regularity for the stationary Stokes operator coming from Lemma 3.1, by using (3.22b) and (3.22d), and since the terms from the right-hand-side of (3.18) belong to $L^\infty(0, T; H^1)$, we get

$$u \in L^\infty(\lambda, T; H^3(\Omega)).$$

Since u is a local smooth solution we get (3.23).

Concerning the bound on the density let us consider a general differential operator D^α with α a multi-index such that $|\alpha| \leq 3$. Taking D^α of the density equation, multiplying by $D^\alpha\rho$ and integrating by parts (where only $u \cdot n = 0$ is needed to get rid of the term $\int_\Omega (u\cdot\nabla)\, D^\alpha\rho D^\alpha\rho\, dx$ which vanishes identically) and summing over $|\alpha| \leq 3$ we get the following inequality

$$\frac{d}{dt}\|\rho\|_{3,2}^2 \leq \|u\|_{3,2}\|\rho\|_{3,2}^2 + \|\nabla\rho\|_\infty\|u\|_{3,2}\|\rho\|_{3,2}.$$

By using Gronwall lemma and the known regularity from (3.22d) we get (3.24). □

4. Vanishing viscosity-diffusivity limit. In this section we consider the behavior of solutions as the viscosity and diffusivity both (and possibly independently) vanish. We adapt the proof of the sharp convergence results from [4, 36] for the singular limit Navier-Stokes → Euler to the Boussinesq system.

In order to avoid presence of boundary layers we consider the problem in the whole space, or equivalently in the periodic setting. We will also consider the special case of a domain with flat boundary and Navier's type conditions. Some further results for the problem in a bounded domain with Navier's type boundary conditions will appear in a forthcoming paper.

Before proving Theorem 2.3 we recall the well-known Kato-Ponce [31] commutator estimate.

LEMMA 4.1. *Let f and g in $\mathcal{S}(\mathbb{R}^3)$, $s > 0$, and $1 < p < \infty$, then*

$$\|J^s(fg) - fJ^s g\|_p \leq \|\nabla f\|_{p_1}\|g\|_{s-1,p_2} + \|J^s f\|_{p_3}\|g\|_{p_4},$$

and

$$\|J^s(fg)\|_p \leq \|f\|_{p_1}\|g\|_{s,p_2} + \|f\|_{s,p_3}\|g\|_{p_4},$$

where J^s is the pseudo-differential operator $(I-\Delta)^{\frac{s}{2}}$ and p_2 and p_3 are such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$.

We recall that convergence in norms weaker than $L^\infty(0,T;H^3)$ can be obtained by using uniform estimates and standard compactness results, together with uniqueness of the solution of the limit problem. Moreover getting strong convergence in $L^\infty(0,T;H^1)$ is again relatively easy (where the lost of two derivatives is due to the presence of the Laplacian), while arriving at the *sharp result* of convergence in the same space of the initial data requires more sophisticated tools. The main technical point of the proof is that of introducing a δ-regularized problem (whose solution is called u^δ) in order to estimate the difference of velocities (and in a symmetric way also of densities) as follows

$$\|u^n - u\|_{3,2} + \|\rho^n - \rho\|_{3,2} \leq \|u^n - u^\delta\|_{3,2} + \|u^\delta - u\|_{3,2} + \|\rho^n - \rho^\delta\|_{3,2} + \|\rho^\delta - \rho\|_{3,2},$$

where the four terms can be estimated in a sharper way using the improved regularity of u^δ. To this end we need a very precise dependence on δ (in fact norms stronger than H^3 blow-up as δ vanishes) of $\|u^\delta\|_{3,2} + \|\rho^\delta\|_{3,2}$ and also a suitable $L^\infty(\Omega)$ bound obtained by means of fractional estimates.

Proof of Theorem 2.3. The first item of Theorem 2.3 follows from the fact that there is no boundary and the H^3-norm of u_0^n and ρ_0^n are uniformly bounded. In fact, by following the same argument in Constantin [21] it is easy to show that there exists a time of existence $T > 0$ independent on n such that there exists both a unique solution (u^n, ρ^n) of (2.11) and a unique solution (u, ρ) of (2.12) in $C([0,T]; H^3(\mathbb{R}^3))$.

We use the same approach introduced in [3] with regularization of the problem (even if we regularize the data in a slightly different manner). The first step is the regularization of the initial datum of the system (2.12). For any $\delta > 0$ let u_0^δ be the unique solution of the following stationary Helmholtz-Stokes system

$$\begin{aligned} -\delta^2 \Delta u_0^\delta + u_0^\delta + \nabla q_0^\delta &= u_0 \qquad && \text{in } \mathbb{R}^3, \\ \nabla \cdot u_0^\delta &= 0 && \text{in } \mathbb{R}^3. \end{aligned}$$

Then, given $u_0 \in H^3$ we get a sequence of regularized initial data $\{u_0^\delta\}_{\delta>0}$ (observe that in absence of boundaries q_0^δ is a constant, hence the problem is the same as the Helmholtz one, cf. [15]) such that

$$\begin{aligned} &\|u_0^\delta\|_{3,2} \le \|u_0\|_{3,2}, \qquad \|u_0^\delta\|_{4,2} \le \frac{C\|u_0\|_{3,2}}{\delta}, \qquad \|u_0^\delta\|_{5,2} \le \frac{C\|u_0\|_{3,2}}{\delta^2}. \\ &\|u_0 - u_0^\delta\|_{s,2} \le C\delta^{3-s}, \qquad \text{for any } s \text{ in } [0,3), \end{aligned}$$

where the estimates can be easily proved by using the Fourier characterization of the differential operator, which acts as a differential filter (cf. also [13]). Concerning ρ_0, for any $\delta > 0$ we consider a sequence ρ_0^δ such that ρ_0^δ solves the following scalar Helmholtz system

$$-\delta^2 \Delta \rho_0^\delta + \rho_0^\delta = \rho_0 \qquad \text{in } \mathbb{R}^3,$$

and we get a sequence ρ_0^δ that satisfies (in term of δ) the same estimates of u_0^δ.

Let us consider the "regularized Euler-Boussinesq" equation with initial data u_0^δ

$$\begin{aligned} u_t^\delta + (u^\delta \cdot \nabla)\, u^\delta + \nabla \pi^\delta &= -\rho^\delta \mathrm{e}_3 \qquad && \text{in } \mathbb{R}^3 \times (0,T), \\ \operatorname{div} u^\delta &= 0 && \text{in } \mathbb{R}^3 \times (0,T), \\ \rho_t^\delta + (u^\delta \cdot \nabla)\, \rho^\delta &= 0 && \text{in } \mathbb{R}^3 \times (0,T), \\ u^\delta(0,x) &= u_0^\delta && \text{in } \mathbb{R}^3, \\ \rho^\delta(0,x) &= \rho_0^\delta && \text{in } \mathbb{R}^3. \end{aligned}$$

Since the H^3 norm of $(u_0^\delta, \rho_0^\delta)$ does not depend on δ, the life-span of (u^δ, ρ^δ) is bounded from below in the same way as that of (u, ρ). In fact, by the differential inequalities satisfied by (u, ρ), it follows that a unique solution in H^3 exists up to some time $T_0 \ge \frac{C}{\|u_0\|_{3,2} + \|\rho_0\|_{3,2}}$, hence the solution of the regularized problem exists is some interval bounded from below by $T_0^\delta \ge \frac{C}{\|u_0^\delta\|_{3,2} + \|\rho_0^\delta\|_{3,2}} \ge \frac{C}{\|u_0\|_{3,2} + \|\rho_0\|_{3,2}}$.

We work now in this common time interval $[0, T_0]$ and by standard energy estimates we get that (u^δ, ρ^δ) satisfies the following bounds

$$\|u^\delta\|_{3,2} \le C, \qquad \|u^\delta\|_{4,2} \le \frac{C}{\delta}, \qquad \|u_0^\delta\|_{5,2} \le \frac{C}{\delta^2}, \tag{4.1a}$$

$$\|\rho^\delta\|_{3,2} \le C, \qquad \|\rho^\delta\|_{4,2} \le \frac{C}{\delta}, \qquad \|\rho_0^\delta\|_{5,2} \le \frac{C}{\delta^2}, \tag{4.1b}$$

$$\|u_0 - u_0^\delta\|_{s,2} \le \delta^{3-s} \qquad \text{for any } s \in (0,3), \tag{4.1c}$$

$$\|\rho_0 - \rho_0^\delta\|_{s,2} \le \delta^{3-s} \qquad \text{for any } s \in (0,3), \tag{4.1d}$$

with constant independent of δ, uniformly in $[0, T_0]$.

In order to prove the convergence, we state the H^3-counterpart of the energy estimates for the difference between solutions corresponding to different viscosity/diffusivity/δ. In particular, we start by considering the difference between the solution of the Euler-Boussinesq and that of the regularized Euler-Boussinesq: We handle the equations for $w^\delta := u^\delta - u$ and $\theta^\delta := \rho^\delta - \rho$.

$$
\begin{aligned}
w_t^\delta + (w^\delta \cdot \nabla)\, u^\delta + (u \cdot \nabla)\, w^\delta + \nabla \pi^\delta &= -\theta^\delta\, \mathrm{e}_3 && \text{in } \mathbb{R}^3 \times (0, T_0),\\
\operatorname{div} w^\delta &= 0 && \text{in } \mathbb{R}^3 \times (0, T_0),\\
\theta_t^\delta + (w^\delta \cdot \nabla)\, \rho^\delta + (u \cdot \nabla)\, \theta^\delta &= 0 && \text{in } \mathbb{R}^3 \times (0, T_0).
\end{aligned}
$$

By taking the H^3-inner product of the first equation with w^δ and of the second equation with ρ^δ, by using Hölder inequality and Lemma 4.1 we get

$$
\begin{aligned}
&\frac{d}{dt}\|w^\delta\|_{3,2}^2 \le C(\|u\|_{3,2} + \|u^\delta\|_{3,2})\|w^\delta\|_{3,2}^2 + C\|u^\delta\|_{4,2}\|w^\delta\|_{3,2}\|w^\delta\|_\infty + \|\theta^\delta\|_{3,2}\|w^\delta\|_{3,2},\\
&\frac{d}{dt}\|\theta^\delta\|_{3,2}^2 \le C\|\rho^\delta\|_{4,2}\|\theta^\delta\|_{3,2}\|w^\delta\|_\infty + \|\theta^\delta\|_{3,2}\|w^\delta\|_{3,2}\|\rho^\delta\|_{3,2} + \|\theta^\delta\|_{3,2}^2\|u\|_{3,2},
\end{aligned}
$$

and by adding together the above inequalities

$$
\begin{aligned}
\frac{d}{dt}(\|w^\delta\|_{3,2}^2 + \|\theta^\delta\|_{3,2}^2) &\le C(\|u\|_{3,2} + \|u^\delta\|_{3,2} + \|u^\delta\|_{3,2} + 1)\big(\|w^\delta\|_{3,2}^2 + \|\theta^\delta\|_{3,2}^2\big)\\
&\quad + C(\|u^\delta\|_{4,2} + \|\rho^\delta\|_{4,2})\|w^\delta\|_\infty\big(\|w^\delta\|_{3,2}^2 + \|\theta^\delta\|_{3,2}^2\big)^{\frac{1}{2}}.
\end{aligned}
\tag{4.2}
$$

Moreover, by taking also the energy estimate at the level H^s, for some $3/2 < s < 2$ we get (by using the same tools) the inequality

$$
\frac{d}{dt}(\|w^\delta\|_{s,2}^2 + \|\theta^\delta\|_{s,2}^2) \le C(\|u\|_{s,2} + \|u^\delta\|_{s+1,2} + \|\rho^\delta\|_{s+1,2})(\|w^\delta\|_{s,2}^2 + \|\theta^\delta\|_{s,2}^2).
$$

Using the known H^3-bounds on u, ρ in $[0, T_0]$, it follows that there exists a constant C, depending only on T_0 and on $\|u_0\|_{H^3} + \|\rho_0\|_{H^3}$, but independent of the solution and of δ, such that

$$
\sup_{0<t<T_0} \|w^\delta(t)\|_{s,2}^2 + \|\theta^\delta(t)\|_{s,2}^2 \le C(\|w^\delta(0)\|_{s,2}^2 + \|\theta^\delta(0)\|_{s,2}^2).
$$

By using the properties of the regularization and Sobolev embedding into L^∞, this implies, for $3/2 < s < 2$

$$
\|w^\delta\|_{L^\infty(0,T_0;L^\infty)} + \|\theta^\delta\|_{L^\infty(0,T_0;L^\infty)} \le C\big(\|w^\delta\|_{L^\infty(0,T_0;H^s)} + \|\theta^\delta\|_{L^\infty(0,T_0;H^s)}\big) \le C\delta^{3-s}.
$$

By plugging this information in (4.2) and by using the bounds for u, u^δ, and ρ^δ (true because they are H^3-solutions in $[0, T_0]$) we get

$$
\frac{d}{dt}\big(\|w^\delta\|_{3,2}^2 + \|\theta^\delta\|_{3,2}^2\big)^{1/2} \le C\big(\|w^\delta\|_{3,2}^2 + \|\theta^\delta\|_{3,2}^2\big)^{1/2} + C\delta^{2-s}.
$$

This finally proves that

$$
\lim_{\delta\to 0} \sup_{t\in[0,T_0]} \|w^{n\delta}(t) - u(t)\|_{3,2} + \|\theta^{n\delta}(t) - \rho(t)\|_{3,2} = 0.
$$

Next, we consider the functions $w^{n\delta} := u^n - u^\delta$ and $\theta^{n\delta} := \rho^n - \rho^\delta$, which satisfy the following system

$$\begin{aligned} & w^{n\delta}_t - \nu_n \Delta w^{n\delta} - \nu_n \Delta u^\delta + \\ & \qquad + (w^{n\delta}\cdot\nabla)\, u^\delta + (u^n\cdot\nabla)\, w^{n\delta} + \nabla(\pi^n - \pi^\delta) = -\theta^{n\delta}\, \mathrm{e}_3 && \text{in } \mathbb{R}^3\times(0,T_0), \\ & \qquad \operatorname{div} w^{n\delta} = 0 && \text{in } \mathbb{R}^3\times(0,T_0), \\ & \theta^{n\delta}_t - k_n\Delta\theta^{n\delta} - k_n\Delta\rho^\delta + (u^n\cdot\nabla)\,\theta^{n\delta} + (w^{n\delta}\cdot\nabla)\,\rho^\delta = 0 && \text{in } \mathbb{R}^3\times(0,T_0). \end{aligned}$$

By using the same techniques as above we obtain the following estimate

$$\begin{aligned} & \frac{d}{dt}(\|w^{n\delta}\|^2_{3,2} + \|\theta^{n\delta}\|^2_{3,2}) + \nu_n\|\nabla w^{n\delta}\|^2_{3,2} + k_n\|\nabla\theta^{n\delta}\|^2_{3,2} \\ & \quad \le \nu_n\|u^\delta\|_{5,2}\|w^{n\delta}\|_{3,2} + k_n\|\rho^\delta\|_{5,2}\|\theta^{n\delta}\|_{3,2} \\ & \quad + C(\|u^n\|_{3,2} + \|u^\delta\|_{3,2} + \|u^\delta\|_{3,2} + 1)\big(\|w^{n\delta}\|^2_{3,2} + \|\theta^{n\delta}\|^2_{3,2}\big) \\ & \qquad + C(\|u^\delta\|_{4,2} + \|\rho^\delta\|_{4,2})\|w^{n\delta}\|_\infty\big(\|w^{n\delta}\|^2_{3,2} + \|\theta^{n\delta}\|^2_{3,2}\big)^{\frac{1}{2}}. \end{aligned}$$

and, from the upper bounds on the solution of the "regularized equations" in $[0,T_0]$, we get

$$\frac{d}{dt}(\|w^{n\delta}(t)\|^2_{3,2} + \|\theta^{n\delta}(t)\|^2_{3,2})^{\frac{1}{2}} \le C\Big[\frac{\nu_n + k_n}{\delta^2} + \frac{\|w^{n\delta}(t)\|_\infty}{\delta} + (\|w^{n\delta}(t)\|^2_{3,2} + \|\theta^{n\delta}(t)\|^2_{3,2})^{\frac{1}{2}}\Big],$$

for some C independent of δ (It depends on the norms of the initial data in H^3 and on T_0). In order to study the above differential inequalities we estimate the $L^\infty((0,T_0)\times\Omega)$ norm of $w^{n\delta}$ (and the same argument applies also to $\theta^{n\delta}$). By triangle inequality and Sobolev embedding we have

$$\begin{aligned} \frac{\|w^{n\delta}\|_\infty}{\delta} & \le \frac{\|u^n - u\|_\infty}{\delta} + \frac{\|u - u^\delta\|_\infty}{\delta} \le \frac{\|u^n - u\|_{2,2}}{\delta} + \frac{\|u^\delta - u\|_{s,2}}{\delta} \\ & = \frac{\|w^n\|_{2,2}}{\delta} + \frac{\|w^\delta\|_{s,2}}{\delta}, \end{aligned}$$

with $\frac{3}{2} < s < 2$. Let us consider first the term $\frac{\|u^\delta - u\|_{s,2}}{\delta}$ and from (4.1c) and (4.1d) we get

$$\frac{\|u^\delta - u\|_\infty}{\delta} \le \delta^{2-s},$$

hence this goes to zero, as δ vanishes.

It remains to estimate the term $\frac{\|u^n - u\|_{2,2}}{\delta}$. The functions $w^n := u^n - u$ and $\theta^n := \rho^n - \rho$ satisfy the following equations

$$\begin{aligned} & w^n_t - \nu_n\Delta w^n - \nu_n\Delta u + \\ & \qquad + (w^n\cdot\nabla)\, u + (u^n\cdot\nabla)\, w^n + \nabla(\pi^n - \pi^\delta) = -(\rho^n - \rho)\, \mathrm{e}_3 && \text{in } \mathbb{R}^3\times(0,T_0), \\ & \qquad \operatorname{div} w^n = 0 && \text{in } \mathbb{R}^3\times(0,T_0), \\ & \qquad \theta^n_t - k_n\Delta\theta^n - (u^n\cdot\nabla)\,\theta^n - (w^n\cdot\nabla)\,\rho = k_n\Delta\rho && \text{in } \mathbb{R}^3\times(0,T_0), \end{aligned}$$

and at this point we prove sub-optimal estimates. They are sub-optimal in the sense that they are in H^2, but with the improvement of explicit rates of convergence in terms of ν_n and k_n. We take the H^2-inner product of the first equation with w^n and the second equation with θ^n, and the loss of regularity makes this step easier than the previous ones. By using Lemma 4.1 with a suitable choice of p_1 and p_2 we get

$$\frac{d}{dt}(\|w^n\|_{2,2}^2 + \|\theta^n\|_{2,2}^2) \leq C\Big(\nu_n + k_n + (\|w^n\|_{2,2}^2 + \|\theta^n\|_{2,2}^2)\Big),$$

where as usual C depends on the H^3 norms of the solutions. By Gronwall lemma we get

$$\|w^n(t)\|_{2,2} + \|\theta^n(t)\|_{2,2} \leq \sqrt{\nu_n + k_n} + \|w^n(0)\|_{2,2} + \|\theta^n(0)\|_{2,2},$$

and finally

$$\sup_{t\in[0,T_0]} \Big(\|w^n(t)\|_\infty + \|\theta^n(t)\|_\infty\Big) \leq \big(\sqrt{\nu_n + k_n} + \|w^n(0)\|_{2,2} + \|\theta^n(0)\|_{2,2}\big).$$

By collecting all inequalities and by using Gronwall lemma we obtain

$$\begin{aligned}\|u^n - u\|_{3,2} + &\|\rho^n - \rho\|_{3,2} \leq \|w^{n\delta}\|_{3,2} + \|w^\delta\|_{3,2} + \|\theta^{n\delta}\|_{3,2} + \|\theta^\delta\|_{3,2} \\ &\leq C(T_0)\left(\frac{\nu_n + k_n}{\delta_n^2} + \frac{\sqrt{\nu_n + k_n}}{\delta_n} + \frac{\|w^n(0)\|_2 + \|\theta^n(0)\|_2}{\delta_n} + \delta_n^{2-s}\right) \\ &\quad + \|u_0^n - u_0\|_{3,2} + \|\rho_0^n - \rho_0\|_{3,2} + \|u_0^\delta - u\|_{3,2} + \|\rho_0^\delta - \rho\|_{3,2}.\end{aligned}$$

In order to obtain convergence, let us choose a sequence $\{\delta_n\}$ going to zero fast enough such that

$$\frac{\nu_n + k_n}{\delta_n^2} \to 0 \quad \text{and} \quad \frac{\|u_0^n - u_0\|_{2,2} + \|\rho_0^n - \rho_0\|_{2,2}}{\delta_n} \to 0, \qquad \text{as } n \to \infty,$$

and we get (2.13).

To complete the proof of Theorem 2.3 we have only to show that the convergence (2.13) holds in any closed sub-interval of $(0, T^*)$. Let us consider T_0 as new initial data and since

$$u^n(T_0) \to u(T_0) \quad \text{and} \quad \rho^n(T_0) \to \rho(T_0) \qquad \text{in } H^3(\mathbb{R}^3),$$

we can repeat the same argument as above and we can show that there exists a time $T_1 \geq \frac{C}{\|u(T_0)\|_{3,2} + \|\rho(T_0)\|_{3,2}}$ such that (2.13) holds in $[0, T_0 + T_1]$. We can iterate this process and define a sequence of positive numbers $\{T_k\}$ such that

1. Convergence (2.13) holds in $[0, \sum_{k=0}^N T_k]$, for each $N \geq 0$;
2. $T_k > \frac{C}{\|u(\sum_{i=0}^{k-1} T_i)\|_{3,2} + \|\rho(\sum_{i=0}^{k-1} T_i)\|_{3,2}}$.

Let us suppose that the series $\sum T_k = \overline{T} < T^*$, and that the sharp limit (2.13) holds only in $[0, \overline{T}]$ for some $\overline{T} < T^*$. If this happens the convergence of the series implies $T_i \to 0$. Due to the definition of T_i this will imply that $\lim_{k\to+\infty} \|u(\sum_{i=0}^k T_i)\|_{3,2} + \|\rho(\sum_{i=0}^k T_i)\|_{3,2} \to +\infty$, hence that

$$\limsup_{t\to\overline{T}} \|u(t)\|_{3,2} + \|\rho(t)\|_{3,2} = +\infty,$$

but this is a contradiction because $\overline{T} < T^*$, while T^* is the maximal time of existence of the solution starting from (u_0, ρ_0) at $t = 0$. ∎

4.1. Some remarks on the problem in a bounded domain. It is well-known that, in presence of Dirichlet boundary conditions (even for the Navier-Stokes equations with constant density) one cannot expect to have convergence of solutions towards those of the Euler equations, due to the creation of a *boundary layer*. On the other hand, recent results show that the convergence can take place in various topologies and in various physical situations, under boundary conditions different from the Dirichlet ones. In this section we show how this applies to the Boussinesq equations, at least in the simplest case: A domain with flat boundary and Navier's type slip-without-friction boundary conditions. We consider then the following initial boundary value problem in $\Omega=\mathbb{R}^3_+:=\{x\in\mathbb{R}^3:\ x_3>0\}$

$$
\begin{aligned}
&u_t-\nu_n\Delta u+(u\cdot\nabla)\,u+\nabla\pi=-\rho\,\mathrm{e}_3 &&\text{in } \mathbb{R}^3_+\times(0,T),\\
&\operatorname{div}u=0 &&\text{in } \mathbb{R}^3_+\times(0,T),\\
&\rho_t+(u\cdot\nabla)\,\rho=k_n\Delta\rho &&\text{in } \mathbb{R}^3_+\times(0,T),\\
&u^3=0 &&\text{on } \{x_3=0\}\times(0,T),\\
&\partial_3u^1=\partial_3u^2=0 &&\text{on } \{x_3=0\}\times(0,T),\\
&\partial_3\rho=0 &&\text{on } \{x_3=0\}\times(0,T).
\end{aligned}
\tag{4.3}
$$

This corresponds to set the slip-without-friction conditions (2.3)-(2.4b) for the velocity, while the Neumann ones for the density, when specialized to the half-space. In this case we will be able to pass to the limit in both ν and k and this fact is rather natural, since the Navier conditions (2.3)-(2.4b) represent an intermediate path between having no boundaries and having a solid boundary (in fact they have been used also for free-boundary problems). To better understand why the Navier conditions allow to simplify the problem, one can observe in the half-space case, it is possible to extend the solution (u,π,ρ) to the whole space by an even reflection (with respect to $\{x_3=0\}$) of u_1, u_2, p, and ρ, while extending in an odd way u_3. In this way one obtains a new triple $(\widetilde{u},\widetilde{p},\widetilde{\rho})$ which is a solution the Boussinesq equations in $\mathbb{R}^3$, hence on it we can apply exactly the same tools as before. This is well-known for the Navier-Stokes equations, see also Bae and Jin [2]. With this trick the problem is reduced to the Cauchy one and this is tractable with the standard tools. Very roughly speaking, results known for the Cauchy problem can be proved also in a flat domain, under Navier boundary conditions. A precise discussion of this, with the study of all the necessary compatibility conditions for the Navier-Stokes equations is performed in [7]. With this tool and by observing that the reflected functions $\widetilde{u}_0$, $\widetilde{\rho}_0\in H^3(\mathbb{R}^3)$ we can finally state the following result.

THEOREM 4.2. *Let be given $u_0,\rho_0\in H^3(\mathbb{R}^3_+)$ such that*

$$u^3_0=\partial_3u^1=\partial_3u^2=\partial_3\rho=0 \qquad \text{on } \{x_3=0\}.$$

Let $T^*=T^*(\|u_0\|_{3,2},\|\rho_0\|_{3,2})>0$ be the time of existence of the unique smooth solution of the system (2.12)*, starting with the same initial datum (u_0,ρ_0). Then, for any $n\in\mathbb{N}$ there exists a unique smooth solution (u^n,ρ^n) of the system* (4.3) *such that*

$$(u^n,\rho^n)\in C([0,T^*[;H^3(\mathbb{R}^3_+)).$$

In addition, for any sub-sequence $\{k_n\}_n$, $\{\nu_n\}_n$ converging to zero (note that one of them can also be identically zero) it follows, for all $T<T^*$,

$$\sup_{t\in[0,T]}\Big(\|u^n(t)-u(t)\|_{3,2}+\|\rho^n(t)-\rho(t)\|_{3,2}\Big)\overset{n\to\infty}{\to}0.$$

REFERENCES

[1] C. Amrouche and N.E.H. Seloula, *Stokes equations and elliptic systems with nonstandard boundary conditions*, C. R. Acad. Sci. Paris Series I, 349:11-12 (2011), pp. 703–708.

[2] H.-O. Bae and B. J. Jin, *Regularity for the Navier-Stokes equations with slip boundary condition*, Proc. Amer. Math. Soc., 136:7 (2008), pp. 2439–2443.

[3] H. Beirão da Veiga, *Perturbation theorems for linear hyperbolic mixed problems and applications to the compressible Euler equations*, Comm. Pure Appl. Math., 46:2 (1993), pp. 221–259.

[4] H. Beirão da Veiga, *On the sharp vanishing viscosity limit of viscous incompressible fluid flows*, in "New Directions in Mathematical Fluid Mechanics", Adv. Math. Fluid Mech., pp. 113–122. Birkhäuser, Basel, 2010.

[5] H. Beirão da Veiga and L. C. Berselli, *Navier-Stokes equations: Green's matrices, vorticity direction, and regularity up to the boundary*, J. Differential Equations, 246:2 (2009), pp. 597–628.

[6] H. Beirão da Veiga and F. Crispo, *Sharp inviscid limit results under Navier type boundary conditions. An L^p theory*, J. Math. Fluid Mech., 12 (2010), pp. 397–411.

[7] H. Beirão da Veiga, F. Crispo, and C. R. Grisanti, *Reducing slip boundary value problems from the half to the whole space. Applications to inviscid limits and to non-Newtonian fluids*, J. Math. Anal. Appl., 377:1 (2011), pp. 216–227.

[8] L. C. Berselli, *On a regularity criterion for the solutions to the 3D Navier-Stokes equations, Differential Integral Equations*, 15:9 (2002), pp. 1129–1137.

[9] L. C. Berselli, *Analysis of a large eddy simulation model based on anisotropic filtering*, J. Math. Anal. Appl., 386:1 (2011), pp. 149–170.

[10] L. C. Berselli, *Some results on the Navier-Stokes equations with Navier boundary conditions*, 2010. Riv. Mat. Univ. Parma., 1 (2010), pp. 1–75. Lecture notes of a course given at SISSA/ISAS, Trieste, Sep. 2009.

[11] L. C. Berselli, P. Fischer, T. Iliescu, and T. Özgökmen, *Horizontal Large Eddy Simulation of stratified mixing in a lock-exchange system*, J. Sci. Comput., 49:1 (2011), pp. 3–20.

[12] L. C. Berselli and G. P. Galdi, *Regularity criteria involving the pressure for the weak solutions to the Navier-Stokes equations*, Proc. Amer. Math. Soc., 130:12 (2002), pp. 3585–3595.

[13] L. C. Berselli, T. Iliescu, and W. J. Layton, *Mathematics of Large Eddy Simulation of turbulent flows*, Scientific Computation. Springer-Verlag, Berlin, 2006.

[14] L. C. Berselli, T. Iliescu, and T. Özgökmen, *Horizontal approximate deconvolution for stratified flows: Analysis and computations*, in "Quality and reliability of large-eddy simulations II", J. Meyers, B. J. Geurts, P. Sagaut, and M. V. Salvetti eds., ERCOFTAC Series, pp. 399–410. Springer, Berlin, 2011.

[15] L. C. Berselli and R. Lewandowski, *Convergence of approximate deconvolution models to the mean Navier-Stokes Equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 29 (2012), pp. 171–198.

[16] J. R. Cannon and E. DiBenedetto, *The initial value problem for the Boussinesq equations with data in L^p*, in "Approximation methods for Navier-Stokes problems (Proc. Sympos., Univ. Paderborn, Paderborn, 1979)", volume 771 of Lecture Notes in Math., pp. 129–144. Springer, Berlin, 1980.

[17] D. Chae, *Global regularity for the 2D Boussinesq equations with partial viscosity terms*, Adv. Math., 203:2 (2006), pp. 497–513.

[18] D. Chae, S.-K. Kim, and H.-S. Nam, *Local existence and blow-up criterion of Hölder continuous solutions of the Boussinesq equations*, Nagoya Math. J., 155 (1999), pp. 55–80.

[19] D. Chae and J. Lee, *Regularity criterion in terms of pressure for the Navier-Stokes equations*, Nonlinear Anal. T.M.A., 46:5 (2001), pp. 727–735.

[20] D. Chae and H.-S. Nam, *Local existence and blow-up criterion for the Boussinesq equations*, Proc. Roy. Soc. Edinburgh Sect. A, 127:5 (1997), pp. 935–946.

[21] P. Constantin, *Note on loss of regularity for solutions of the 3-D incompressible Euler and related equations*, Comm. Math. Phys., 104:2 (1986), pp. 311–326.

[22] P. Constantin, *Euler and Navier-Stokes equations*, Publ. Mat., 52:2 (2008), pp. 235–265.

[23] B. Cushman-Roisin and J.-M. Beckers, *Introduction to Geophysical Fluid Dynamics*, Series in International Geophysics, 101. Academic Press, 2nd edition, 2011.

[24] J. Fan and T. Ozawa, *Regularity criteria for the 3D density-dependent Boussinesq equations*, Nonlinearity, 22:3 (2009), pp. 553–568.

[25] J. Fan and Y. Zhou, *A note on regularity criterion for the 3D Boussinesq system with partial viscosity*, Appl. Math. Lett., 22:5 (2009), pp. 802–805.

[26] G. P. Galdi, *An introduction to the Navier-Stokes initial-boundary value problem*, in "Fundamental directions in mathematical fluid mechanics", Adv. Math. Fluid Mech., pp. 1–70. Birkhäuser, Basel, 2000.
[27] T. Y. Hou and C. Li, *Global well-posedness of the viscous Boussinesq equations*, Discrete Contin. Dyn. Syst., 12:1 (2005), pp. 1–12.
[28] N. Ishimura and H. Morimoto, *Remarks on the blow-up criterion for the 3-D Boussinesq equations*, Math. Models Methods Appl. Sci., 9:9 (1999), pp. 1323–1332.
[29] Y. Kagei, *On weak solutions of nonstationary Boussinesq equations*, Differential Integral Equations, 6:3 (1993), pp. 587–611.
[30] K. Kang and J. Lee, *On regularity criteria in conjunction with the pressure of Navier-Stokes equations*, Int. Math. Res. Not., pages Art. ID 80762, 25, 2006.
[31] T. Kato and G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure Appl. Math., 41:7 (1988), pp. 891–907.
[32] H. Kozono and T. Yanagisawa, *L^r-variational inequality for vector fields and the Helmholtz-Weyl decomposition in bounded domains*, Indiana Univ. Math. J., 58:4 (2009), pp. 1853–1920.
[33] M.-J. Lai, R. Pan, and K. Zhao, *Initial boundary value problem for two-dimensional viscous Boussinesq equations*, Arch. Ration. Mech. Anal., 199:3 (2011), pp. 739–760.
[34] X. Liu, M. Wang, and Z. Zhang, *Local well-posedness and blowup criterion of the Boussinesq equations in critical Besov spaces*, J. Math. Fluid Mech., 12:2 (2010), pp. 280–292.
[35] A. J. Majda, *Introduction to PDEs and waves for the atmosphere and ocean*, volume 9 of "Courant Lecture Notes in Mathematics", Courant Institute of Mathematical Sciences, New York, 2003.
[36] N. Masmoudi, *Remarks about the inviscid limit of the Navier-Stokes system*, Comm. Math. Phys., 270:3 (2007), pp. 777–788.
[37] H. K. Moffatt, *Some remarks on topological fluid mechanics*, in "An Introduction to the Geometry and Topology of Fluid Flows": R.L. Ricca Ed., pp. 3–10, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.
[38] H. Morimoto, *Nonstationary Boussinesq equations*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 39:1 (1992), pp. 61–75.
[39] K. Ohkitani, *Comparison between the Boussinesq and coupled Euler equations in two dimensions*, Sūrikaisekikenkyūsho Kōkyūroku, 1234 (2001), pp. 127–145. Tosio Kato's method and principle for evolution equations in mathematical physics (Sapporo, 2001).
[40] T. Özgökmen, T. Iliescu, and P. Fischer, *Large eddy simulation of stratified mixing in a three-dimensional lock-exchange system*, Ocean Modelling, 26 (2009), pp. 134–155.
[41] T. Özgökmen, T. Iliescu, P. Fischer, A. Srinivasan, and J. Duan, *Large eddy simulation of stratified mixing in two-dimensional dam-break problem in a rectangular enclosed domain*, Ocean Modelling, 16 (2007), pp. 106–140.
[42] H. Qiu, Y. Du, and Z. Yao, *A blow-up criterion for 3D Boussinesq equations in Besov spaces*, Nonlinear Anal., 73:3 (2010), pp. 806–815.
[43] K. R. Rajagopal, M. Růžička, and A. R. Srinivasa, *On the Oberbeck-Boussinesq approximation*, Math. Models Methods Appl. Sci., 6:8 (1996), pp. 1157–1167.
[44] H. Sohr, *The Navier-Stokes equations*, Birkhäuser Advanced Texts: Basel Textbooks. Birkhäuser Verlag, Basel, 2001.
[45] R. Temam, *Behaviour at time $t = 0$ of the solutions of semilinear evolution equations*, J. Differential Equations, 43:1 (1982), pp. 73–92.
[46] Y. Xiao and Z. Xin, *On the vanishing viscosity limit for the 3D Navier-Stokes equations with a slip boundary condition*, Comm. Pure Appl. Math., 60:7 (2007), pp. 1027–1055.
[47] Y. Zhou and J. Fan, *On the Cauchy problems for certain Boussinesq-α equations*, Proc. Roy. Soc. Edinburgh Sect. A, 140:2 (2010), pp. 319–327.