METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 19, No. 1, pp. 043–076, March 2012

003

GLOBAL ENTROPY SOLUTIONS OF THE GENERAL NONLINEAR HYPERBOLIC BALANCE LAWS WITH TIME-EVOLUTION FLUX AND SOURCE*

SHIH-WEI CHOU†, JOHN M. HONG‡, AND YING-CHIN SU§

Abstract. In this paper we investigate the initial and initial-boundary value problems for strictly hyperbolic balance laws with time-evolution of flux and source. Such nonlinear balance laws arise in, for instance, gas dynamics equations in time-dependent ducts and nozzles, shallow water equations, lanes-changing model in traffic flow and Einstein's field equations in a spherically symmetric space-time. To account for the time dependence of flux and source, we introduce the perturbed Riemann and boundary Riemann problems. Such Riemann problems have unique solutions within elementary waves and an additional family of waves. Based on the work of [12, 13], a new version of Glimm scheme is introduced and its stability is established by modified interaction estimates. Finally, the existence of global entropy solutions is achieved by showing the consistency of scheme, the weak convergence of source term and the entropy inequalities.

Key words. Initial value problem, initial-boundary value problem, hyperbolic conservation laws, nonlinear balance laws, entropy solutions, Riemann problem, perturbed Riemann problem, perturbed boundary Riemann problem, wave interaction estimates, generalized Glimm scheme.

AMS subject classifications. 35L60, 35L65, 35L67.

1. Introduction.

In this paper we are concern with the initial and initial-boundary value problems for strictly hyperbolic balance laws with time-evolution of flux and source:

$$\text{(IVP)}\quad u_t + f(a(x,t),u)_x = a_x g(a(x,t),a_x,a_t,u), \quad (x,t)\in\mathbb{R}\times\mathbb{R}^+, \tag{1.1a}$$
$$u(x,0) = u_0(x)\in\Omega, \quad -\infty < x < \infty, \tag{1.1b}$$

$$\text{(IBVP)}\quad u_t + f(a(x,t),u)_x = a_x g(a(x,t),a_x,a_t,u), \quad (x,t)\in\mathbb{R}^+\times\mathbb{R}^+, \tag{1.2a}$$
$$u(x,0) = u_0(x)\in\Omega, \quad x\geq 0, \tag{1.2b}$$
$$u_1(0,t) = u_{1B}(t), \quad t>0, \tag{1.2c}$$

where Ω is a ball of radius r in $\mathbb{R}^2$, $u=(u_1,u_2)$ is the unknown, $f=(f_1,f_2)$ and $g=(g_1,g_2)$ are smooth functions of their variables, and $u_{1B}(t)$, $u_0(x)\in L^\infty\cap BV$. In addition, $a(x,t)\in\mathbb{R}$ is assumed to be a given smooth function defined in $\mathbb{R}\times[0,\infty)$. The initial-boundary condition of (1.2a) is of Dirichlet type. Throughout this paper, we impose the following conditions.

(A_1) The total variations of $a(0,\cdot)$, $a(\cdot,t)$ and $a_t(\cdot,t)$ are sufficiently small for every $t\geq 0$.

(A_2) (i) Each component of $R_0(a,u)$ is non-zero for all (x,t) and $u\in\Omega$ where

$$R_0(a,u) := (D_u f)^{-1}(g-f_a)(a,u), \tag{1.3}$$

*Received August 15, 2011; accepted for publication July 17, 2012. Work partially supported by the National Science Council of Taiwan and Center for Theoretical Sciences, Mathematical Division, NCU.

†Department of Mathematics, National Central University, Taoyuan 32001, Taiwan (962401002@cc.ncu.edu.tw).

‡Department of Mathematics, National Central University, Taoyuan 32001, Taiwan (jhong@math.ncu.edu.tw).

§Department of Mathematics, Fu Jen Catholic University, New Taipei 24205, Taiwan (ycsu@math.fju.edu.tw).

(ii) $\dfrac{\partial f_1}{\partial u_2}$ is non-zero for all $a \in \mathbb{R}$ and $u \in \Omega$ in IBVP (1.2).

(A_3) Our system is strictly hyperbolic and the eigenvalues $\lambda_1(a,u)$ and $\lambda_2(a,u)$ of $D_u f$ satisfy

$$\lambda_1(a,u) < 0 < \lambda_2(a,u) \quad \text{in } \mathbb{R} \times \Omega.$$

This condition implies that the boundary in IBVP (1.2) is non-characteristic. We assume further that each characteristic field of (1.1a) is genuinely nonlinear, more precisely, for $i = 1, 2$,

$$\nabla \lambda_i(a,u) \cdot R_i(a,u) = 1 \quad \text{in } \mathbb{R} \times \Omega.$$

Here R_i is the normalized right eigenvector corresponding to λ_i.

One famous example for applications is the Euler equations of compressible isentropic gas dynamics

$$\begin{aligned} \partial_t(a\rho) + \partial_x(a\rho v) &= 0, \\ \partial_t(a\rho v) + \partial_x\big(a\rho v^2 + ap\big) &= a_x p, \end{aligned} \tag{1.4}$$

where $a = a(x,t) > 0$ denote the spatial-time dependent cross section of a duct, and ρ, v and $p = p(\rho)$ are the density, the velocity and the pressure of gas, respectively. Here we ignore the effect of gas flow through the duct, so that a is given. The others can be found in the shallow water equations with time-variation of riverbed elevation [32], the traffic flow model with lanes-changing intensity [19] and the Einstein's field equations for a spherically symmetric space-time [11]

$$\begin{aligned} u_t + f(u,A,B)_x &= \bar{g}(u,A,B,A_x,B_x,B_t,x), \\ A_x &= h^0(u,A,B,x), \\ B_x &= h^1(u,A,B,x), \end{aligned} \tag{1.5}$$

where u is related to the stress energy tensor T, and A and B are the elements of metric in the standard Schwarzschild coordinates. To treat (1.1a) as a warm up model of (1.5), we need to solve the second and the third equations of (1.5) for A and B in advance, then writing $\bar{g}$ as $A_x g$ (or $B_x g$) when A_x (or B_x) is non-zero.

In this paper, we will establish the existence of global entropy solutions by using a Glimm-type method. Our results can be extended to the general $n \times n$ systems in the same analysis. Therefore, we only consider the 2×2 case here.

We review some results on the subject and clarify the motivation of the study. The Riemann problem for $n \times n$ strictly hyperbolic conservation laws

$$u_t + f(u)_x = 0, \tag{1.6}$$

was first studied by Lax [21]. The solution is obtained by resolving the jump discontinuity of initial data into fans of elementary waves (including rarefaction waves, shocks and contact discontinuities), that is, the solution is self-similar and consists of at most n+1 constant states separated by these waves. The existence of BV solutions to a general Cauchy problem with initial data of small total variation is established by Glimm [9]. The solution is attained as the $\Delta x \downarrow 0$ limit of a family of approximate solutions $u_{\Delta x}(x,t)$ which are constructed in time steps of length $\Delta t = O(\Delta x)$ by the following procedure: Assuming that $u_{\Delta x}(x,t)$ has already determined on $\mathbb{R} \times [0, n\Delta t)$, one constructs initial data $u_{\Delta x}(x, n\Delta t)$ as a random step function approximation to $u_{\Delta x}(x, n\Delta t^-)$ and then obtains $u_{\Delta x}(x,t)$ in the next time strip $\mathbb{R} \times [n\Delta t, (n+1)\Delta t)$ by Lax's method. A random choice of initial data in each time step ensures the scheme is consistent, that is, the limit of $\{u_{\Delta x}(x,t)\}$ is a weak solution of (1.6). The convergence of the family $\{u_{\Delta x}(x,t)\}$, or a subfamily thereof, is secured by an a priori bound, independent of Δx, on the total variation, which is induced by approximate conservation laws that govern elementary wave interactions and is established with the help of Glimm functional. This scheme is established in the genuinely nonlinear case and generalized to the linearly degenerate case in the paper [27] referenced by Liu. The initial-boundary value problem for (1.6) with the shape of different boundaries was first studied by Goodman [10]. The author proved the global existence of weak solutions when initial and boundary data satisfy the so-called smallness and non-degeneracy conditions.

For the Cauchy problem of quasi-linear hyperbolic system

$$u_t + f(x,u)_x = g(x,u), \tag{1.7}$$

the global existence and asymptotic behavior of solutions were first established by Liu [26] by a steady state scheme. For further results on the initial-boundary value problems for (1.6) and (1.7), we refer to [1, 2, 3, 7, 8, 25, 28].

For the general strictly hyperbolic system

$$u_t + f(u,x,t)_x = g(u,x,t), \tag{1.8}$$

the Cauchy problem was studied by Dafermos-Hsiao [5] and Hong-LeFloch [13]. In [5], the authors use Glimm's scheme along with the method of fractional steps to construct a BV weak solution. An additional dissipativity condition on the flux and the source is imposed, that is, assume that there exists a constant $b > 0$ such that

$$R^{\mu\mu}(x,t) - \sum_{\kappa \neq \mu} \left| R^{\kappa\mu}(x,t) \right| \geq b, \quad \mu = 1, \cdots, n, \tag{1.9}$$

for every $(x,t) \in (-\infty, \infty) \times [0, \infty)$, where $R(x,t) = (R^{\kappa\mu}(x,t)) := -(r^{-1} g_u r)(0,x,t)$ and r is the $n \times n$ matrix consisting of normalized right eigenvectors of $D_u f$. On the other hand, an alternative version of Glimm scheme for (1.8) was introduced in [13] whose Riemann solutions were constructed by the techniques of asymptotic expansion to the classical Riemann solutions and the frozen variables (x,t) in f. Through the detailed wave interaction estimates, the global existence of entropy solutions for (1.8) was established provided that the L^1 norms of $\frac{\partial^2 A}{\partial t \partial u}$, $\frac{\partial^2 A}{\partial x \partial u}$, q and $\frac{\partial q}{\partial u}$ are sufficiently small where $A := D_u f$ and $q(t,x,u) := g(t,x,u) - \frac{\partial f}{\partial x}(t,x,u)$.

For the systems of nonlinear balance laws in the degenerate form

$$\begin{aligned} a_t &= 0, \\ u_t + f(a(x), u)_x &= a' g(a(x), u), \end{aligned} \tag{1.10}$$

the general $n \times n$ strictly hyperbolic case was first studied by LeFloch [23]. In [23], the addition of the extra equation $a_t = 0$ allows us to consider (1.10) as a non-conservative system so that the results in [5] can be applied. On the other hand, the generalized Glimm method for (1.10) was given in [12] that the residual only converges weakly in L^1. The Riemann solutions in [12] are "weaker than weak" because, due to the re-scaling of the source by discontinuities, the Riemann solutions do not solve the equations even weakly, yet the Glimm scheme is a valid method and converges. The 2×2 resonant systems in the form of (1.10) was first studied by Isaacson and Temple [18]. The method in [18] showed that incorporating the source term as a wave gave sharp time independent bounds for solutions of the initial value problem, while the operator-splitting method gave only time dependent bounds in this non-strictly hyperbolic setting. Recently, this framework was extended to quasi-linear wave equations [4, 14, 29],

$$u_{tt} - \big(p(\rho(x), u_x)\big)_x = \rho(x)h(\rho(x), u, u_x), \tag{1.11}$$

with applications to shallow water wave and the deformation of rubbery materials. In [29], ρ is considered as a more general form $\rho = \rho(x,t)$, and the result of global existence was obtained under a dissipative condition. The initial-boundary value problem for (1.10) was also studied in [15]. Some problems with regard to non-strictly hyperbolic and non-conservative systems, we refer to [6, 16, 17, 20, 22, 24, 30].

In this paper, we investigate the possibility that time dependent sources can be treated like source free equations by incorporating an additional family of waves, and to utilize this in the Glimm scheme. Since the time dependence of a is allowed, the framework in [12, 18] can only be carried out locally. To apply the method, the Riemann problems must account not only for the x-dependence of the source, but also for the time dependence. This then requires an approximate Riemann solver to account for time dependence. Since no total variation bound on the source in time is assumed, Glimm's functional in [9] may fail to be non-increasing in time. This will lead to the instability of Glimm method. Moreover, for approximate solutions $\{a^\varepsilon_{\Delta x}, u^\varepsilon_{\Delta x}\}$ generated by Glimm's scheme, it cannot be expected that $(\partial_x a^\varepsilon_{\Delta x})\cdot g(a^\varepsilon_{\Delta x}, \partial_x a^\varepsilon_{\Delta x}, \partial_t a^\varepsilon_{\Delta x}, u^\varepsilon_{\Delta x})$ does converge to $a_x g(a, a_x, a_t, u)$ in the weak sense.

In order to overcome difficulties above, the steps in the paper are thus as follows: (1) "weaker than weak" solutions of the Riemann and boundary Riemann problems are established that account for the leading order effects of time dependence in the source; (2) a modified Glimm-type interaction estimate and a boundary interaction estimate are obtained; (3) a Glimm-type argument is developed to prove time independent total variation estimates for the approximate solutions. At this stage, an additional assumption on $a(x,t)$ is necessary to obtain the total variation bounds of approximate solutions. (4) Finally, the weak convergence of the residual and

(1.12)
$$\iint_{t>0} \{\partial_x a^\varepsilon_{\Delta x} \cdot g(a^\varepsilon_{\Delta x}, \bar{a}_x, b^\varepsilon, u^\varepsilon_{\Delta x}) - a_x g(a, a_x, a_t, u)\}\, \phi dxdt \to 0 \quad \text{as } \varepsilon, \Delta x \to 0$$

for $\phi \in C^1_c$ are established to prove the Glimm method converges to a weak solution (modulo the usual subsequences). Here, $\bar{a}_x$ and b^ε are given in Section 5. Here we point out that condition (A_2) gives the existence and generic structure of the standing wave discontinuities for the Riemann and boundary Riemann problems [12]. Comparing to the results in [5, 12, 29], the contribution of this paper is that the global existence result can be extended to more general flux and source without the dissipative condition.

We now give the definitions of weak and entropy solutions to the problems (1.1) and (1.2), and state the main theorem.

Definition 1.1. *Let* $E := [0,\infty) \times [0,\infty)$ *and* $E_* := (-\infty,\infty) \times [0,\infty)$. *We say that a bounded measurable function* u *is a weak solution to (1.2) if* u *satisfies*

$R_\phi(a,u)$=0 for all $\phi \in C_c^1(E)$ where

$$R_\phi(a,u) := \iint_E u\phi_t + f(a,u)\phi_x + a_x g(a,a_x,a_t,u)\phi dxdt + \int_0^\infty u_0(x)\phi(x,0)dx$$

$$+\int_0^\infty f(a,u)(0,t)\phi(0,t)dt. \tag{1.13}$$

Similarly, u is a weak solution to (1.1) if for all $\phi \in C_c^1(E_*)$,

$$\iint_{E_*} u\phi_t + f(a,u)\phi_x + a_x g(a,a_x,a_t,u)\phi dxdt + \int_{-\infty}^\infty u_0(x)\phi(x,0)dx = 0. \tag{1.14}$$

Definition 1.2. *Let $\mathcal{U}$ be a convex subset of $\mathbb{R}^2$, and let $U : \mathcal{U} \to \mathbb{R}$ and $F : \mathbb{R}\times\mathcal{U} \to \mathbb{R}$. We say that $(U(u), F(a,u))$ is an entropy pair of (1.1a) if U is convex on $\mathcal{U}$, and U, F satisfy*

$$D_uF = (D_uU)(D_uf) \quad \text{on } \mathbb{R}\times\mathcal{U}. \tag{1.15}$$

Furthermore, u is called an entropy solution if u is the weak solution of (1.2) and satisfies

$$\iint_E \big[U(u)\phi_t + F(a,u)\phi_x + a_x\big(D_uU\big(g - f_a\big) + F_a\big)\phi\big]dxdt$$

$$+\int_0^\infty U(u_0(x))\phi(x,0)dx + \int_0^\infty F(a,u)(0,t)\phi(0,t)dt \geq 0 \tag{1.16}$$

for every entropy pair (U,F) and positive $\phi \in C_c^1(E)$. The definition for the entropy solutions of (1.1) can be given analogously.

Main Theorem. *Consider initial value problem (1.1) and initial-boundary value problem (1.2), where a, f and g satisfy conditions $(A_1)-(A_3)$. For a domain Σ, we define*

$$\omega(\Sigma) := \|a_t\|_{L^1(\Sigma)} + \|a_{xt}\|_{L^1(\Sigma)}.$$

Assume that $\omega(E)$, $\omega(E_*)$ and the total variations of u_0 and u_{1B} are sufficiently small, where E, E_* are in Definition 1.1. Let $\{\bar{u}^\varepsilon_{\theta,\Delta x}\}$ and $\{u^\varepsilon_{\theta,\Delta x}\}$ be respectively the sequence of approximate solutions for (1.1) and (1.2) by the generalized Glimm scheme. Then there exists a null set $N \subset \Phi$ and a sequence $\{\Delta x_i\} \to 0$ such that if $\theta \in \Phi\setminus N$, then $\bar{u}(x,t)$, $u(x,t)$ are respectively the entropy solutions to (1.1) and (1.2), where

$$\bar{u}(x,t) := \lim_{\Delta x_i,\varepsilon\to 0} \bar{u}^\varepsilon_{\theta,\Delta x_i}, \quad u(x,t) := \lim_{\Delta x_i,\varepsilon\to 0} u^\varepsilon_{\theta,\Delta x_i}.$$

This paper is organized as follows. In Section 2, to account for time dependence, we introduce a perturbed Riemann problem by asymptotic expansions of the flux and the source. Its approximate solution is constructed within elementary waves and an additional family of waves by the modified Lax method. In Section 3, we extend the results of Section 2 and [15] to the boundary Riemann problem. At the same time, the residual of approximate solutions in each grid is estimated for the consistency of our scheme. In Section 4, a generalized Glimm scheme (GGS for short) with boundary condition will be described. Moreover, the generalized versions of interaction estimates are also obtained. In Section 5, the main theorem is proved by the consistency of the GGS, the weak convergence of the source term and the entropy inequalities for the solution.

2. Generalized Riemann solver and its error estimate. In this section we give a generalized Riemann solver for (1.1a). To start, for given $(x_0,t_0)\in\mathbb{R}^+\times[0,\infty)$ and δ, $\kappa>0$ sufficiently small, we consider the following Riemann problem:

$$\text{(2.1)}\quad \begin{cases} u_t+f(a,u)_x=a_x g(a,a_x,a_t,u), & (x,t)\in(x_0-\kappa,x_0+\kappa)\times(t_0,t_0+\delta),\\ u(x,t_0)=\begin{cases} u_L, & x_0-\kappa\le x<x_0,\\ u_R, & x_0<x\le x_0+\kappa,\end{cases} \end{cases}$$

where a, f and g satisfy conditions $(A_1)-(A_3)$, and u_L and $u_R\in\Omega$ are two nearby constant states. When $a=a(x)$ and $g=g(a,u)$, by the results of [12], (2.1) can be treated like the source free equations by incorporating a linear degenerate wave field. Based on this observation, we utilize linear expansions of a, a_x and a_t in time $t=t_0$ to achieve a *perturbed Riemann problem (PRP)*, which is derived as follows.

First, we approximate a, a_x and a_t in $(x_0-\kappa,x_0+\kappa)\times(t_0,t_0+\delta)$ by

$$\text{(2.2)}\quad a(x,t)\approx\bar a(x)+\delta b(x),\quad a_x(x,t)\approx\bar a'(x)+\delta b'(x),\quad a_t(x,t)\approx b(x)+\delta c(x),$$

respectively, where

$$\text{(2.3)}\qquad \bar a(x):=a(x,t_0),\quad b(x):=a_t(x,t_0),\quad c(x):=a_{tt}(x,t_0).$$

Then, f and g are approximated by

$$\text{(2.4)}\qquad f(a,u)\approx f(\bar a,u)+\delta b(x)f_a(\bar a,u),$$

$$\text{(2.5)}\quad g(a,a_x,a_t,u)\approx\big(g+\delta b(x)g_a+\delta b'(x)g_{a_x}+\delta c(x)g_{a_t}\big)(\bar a(x),\bar a'(x),b(x),u),$$

respectively, where $g_{a_x}:=\frac{\partial g}{\partial a_x}$ and $g_{a_t}:=\frac{\partial g}{\partial a_t}$. By (2.2), (2.4), (2.5) and omitting $O(\delta^2)$ terms, the system in (2.1) is linearized to

$$\text{(2.6)}\quad u_t+\big[(f+\delta b(x)f_a)(\bar a,u)\big]_x=\bar a'\big[g+\delta(b(x)g_a+b'(x)g_{a_x}+c(x)g_{a_t})\big]+\delta b'(x)g,$$

where g, g_a, g_{a_x} and g_{a_t} are all evaluated at $(\bar a(x),\bar a'(x),b(x),u)$. Next, for $0<\varepsilon\ll1$, the initial data $u(x,t_0)$ together with $\bar a(x)$ and $b(x)$ are approximated by

$$\text{(2.7)}\qquad u^\varepsilon(x,t_0):=\begin{cases} u_L, & x_0-\kappa\le x<x_0-\varepsilon\kappa,\\ \psi^\varepsilon(x), & |x-x_0|\le\varepsilon\kappa,\\ u_R, & x_0+\varepsilon\kappa<x\le x_0+\kappa,\end{cases}$$

$$\text{(2.8)}\qquad (\bar a^\varepsilon(x),b^\varepsilon(x)):=\begin{cases} (a_L,b_L), & x_0-\kappa\le x<x_0-\varepsilon\kappa,\\ (\eta^\varepsilon(x),a_t(x,t_0)), & |x-x_0|\le\varepsilon\kappa,\\ (a_R,b_R), & x_0+\varepsilon\kappa<x\le x_0+\kappa,\end{cases}$$

where (a_L,b_L) and (a_R,b_R) are constant states, $\psi^\varepsilon(x)$ is a monotone function connecting u_L at $x=x_0-\varepsilon\kappa$ and u_R at $x=x_0+\varepsilon\kappa$, respectively, and $\eta^\varepsilon(x)$ is also a monotone function connecting a_L at $x=x_0-\varepsilon\kappa$ and a_R at $x=x_0+\varepsilon\kappa$, respectively. We then obtain the following *perturbed Riemann problem centered at* (x_0,t_0) *for* $0<\varepsilon\ll1$:

(2.9)

$$(P_\varepsilon):\begin{cases} u^\varepsilon_t+\big[(f+\delta b^\varepsilon(x)f_a)(\bar a^\varepsilon,u^\varepsilon)\big]_x=(\bar a^\varepsilon)'\big[g+\delta(b^\varepsilon(x)g_a+(b^\varepsilon)'(x)g_{a_x}+c(x)g_{a_t})\big]\\ \qquad\qquad +\delta(b^\varepsilon)'(x)g,\\ u^\varepsilon(x,t_0)=\begin{cases} u_L, & x_0-\kappa\le x<x_0-\varepsilon\kappa,\\ \psi^\varepsilon(x), & |x-x_0|\le\varepsilon\kappa,\\ u_R, & x_0+\varepsilon\kappa<x\le x_0+\kappa,\end{cases}\end{cases}$$

where g, g_a, g_{a_x} and g_{a_t} are all evaluated at $(\bar{a}^\varepsilon(x), \bar{a}'(x), b^\varepsilon(x), u^\varepsilon)$, and $\bar{a}^\varepsilon(x)$ and $b^\varepsilon(x)$ are given in (2.8). The *limiting perturbed Riemann problem*, denoted by (P_0), is given by taking the limit $\varepsilon \to 0$ to (P_ε). The system in (P_ε) is still genuinely nonlinear for small $\delta > 0$. To see this, we define

$$f^\delta(a, b, u) := f(a, u) + \delta b f_a(a, u). \tag{2.10}$$

Let $\{\lambda_i^\delta(a, b, u)\}_{i=1}^2$ be the eigenvalues of the Jacobian matrix $D_u f^\delta$ and let $\{R_i^\delta(a, b, u)\}_{i=1}^2$ be the corresponding right eigenvectors. Then there exist bounded functions $\{k_i(a, b, u)\}_{i=1}^2$ and $\{K_i(a, b, u)\}_{i=1}^2$ such that

$$\lambda_i^\delta = \lambda_i(a, u) + \delta k_i, \quad R_i^\delta = R_i(a, u) + \delta K_i, \quad i = 1, 2. \tag{2.11}$$

By (2.11) and condition (A_3), we have for sufficiently small δ

$$\begin{aligned}(\nabla\lambda_i^\delta \cdot R_i^\delta)(a, b, u) &= (\nabla\lambda_i \cdot R_i)(a, u) + \delta\big(\nabla\lambda_i \cdot K_i + \nabla k_i \cdot (R_i + \delta K_i)\big)(a, b, u) \\ &= (\nabla\lambda_i \cdot R_i)(a, u) + O(\delta) > 0, \quad i = 1, 2,\end{aligned} \tag{2.12}$$

which implies that the characteristic fields of (2.9) are genuinely nonlinear.

Now, we construct approximate solutions to (2.9) by modified Lax's method. Let

$$\begin{aligned}&D_L^\varepsilon := [x_0 - \kappa, x_0 - \varepsilon\kappa) \times [t_0, t_0 + \delta], \quad D_R^\varepsilon := (x_0 + \varepsilon\kappa, x_0 + \kappa] \times [t_0, t_0 + \delta], \\ &D_M^\varepsilon := [x_0 - \varepsilon\kappa, x_0 + \varepsilon\kappa] \times (t_0, t_0 + \delta], \quad \Gamma_\varepsilon := [x_0 - \varepsilon\kappa, x_0 + \varepsilon\kappa] \times \{t_0\}.\end{aligned} \tag{2.13}$$

Since $\bar{a}^\varepsilon$ and b^ε are constants in D_L^ε and D_R^ε, (2.9) is reduced to

$$\begin{cases} u_t^\varepsilon + f^\delta(a_L, b_L, u^\varepsilon)_x = 0, \quad \text{on } (D_L^\varepsilon)^o, \\ u_t^\varepsilon + f^\delta(a_R, b_R, u^\varepsilon)_x = 0, \quad \text{on } (D_R^\varepsilon)^o, \\ u^\varepsilon(x, t_0) = \begin{cases} u_L, & x_0 - \kappa \le x < x_0 - \varepsilon\kappa, \\ u_R, & x_0 + \varepsilon\kappa < x \le x_0 + \kappa, \end{cases} \end{cases} \tag{2.14}$$

where $(D_L^\varepsilon)^o$ and $(D_R^\varepsilon)^o$ denote the interiors of D_L^ε and D_R^ε, respectively. Owing to the genuinely nonlinearity of (2.14), either a rarefaction wave or a shock wave is adopted as the solution in every characteristic field. Let ψ_j^δ, $j = 1, 2$, denote the jth wave curve combined with either a rarefaction wave or a shock of the jth family. Then, by (2.11) and the results of [21], ψ_1^δ is parametrized as

$$\begin{aligned}&\psi_1^\delta(\eta_1; a_L, b_L, u_L) \\ &= u_L + \eta_1 R_1^\delta(a_L, b_L, u_L) + \frac{\eta_1^2}{2}(R_1^\delta \cdot \nabla R_1^\delta)(a_L, b_L, u_L) + O(|\eta_1|^3) \\ &= u_L + \eta_1 R_1(a_L, u_L) + \frac{\eta_1^2}{2}\big(R_1 \cdot \nabla R_1\big)(a_L, u_L) \\ &\quad + \Big(\eta_1\delta K_1 + \frac{\eta_1^2}{2}\delta K_1 \cdot \nabla R_1 + \frac{\eta_1^2}{2}\delta R_1 \cdot \nabla K_1 + \frac{\eta_1^2}{2}\delta^2 K_1 \cdot \nabla K_1\Big)(a_L, b_L, u_L) \\ &\quad + O(|\eta_1|^3),\end{aligned} \tag{2.15}$$

where η_1 is the (signed) wave strength of ψ_1^δ. If ψ_1^δ is the rarefaction wave, then η_1 is a function of $\frac{x - x_0 + \varepsilon\kappa}{t - t_0}$ and $\eta_1 > 0$. If ψ_1^δ is the shock, then $\eta_1 < 0$. In addition, the Rankine-Hugoniot condition of (2.14) gives

$$s_1^\delta[\psi_1^\delta] = [f^\delta(a_L, b_L, \psi_1^\delta)] = [f(a_L, \psi_1^\delta)] + [\delta b_L f_a(a_L, \psi_1^\delta)], \tag{2.16}$$

where s_1^δ is the speed of the shock-front and $[\cdot]$ denotes the difference of states across the shock. Notice that ψ_1^δ is independent of ε and the limit

$$\begin{aligned}\psi_1(\eta_1; a_L, u_L) &:= \lim_{\varepsilon,\delta\to 0} \psi_1^\delta(\eta_1; a_L, b_L, u_L)\\ &= u_L + \eta_1 R_1(a_L, u_L) + \frac{\eta_1^2}{2}(R_1\cdot\nabla R_1)(a_L, u_L) + O(|\eta_1|^3)\end{aligned} \tag{2.17}$$

is the 1-wave solution of $u_t + f(a_L, u)_x = 0$. Similarly, the curves of 2-rarefaction wave and 2-shock in D_R^ε can be proved as in (2.15) except that a_L and b_L are replaced by a_R and b_R.

It remains to construct approximate solutions of (2.9) in D_M^ε. Since $\bar{a}^\varepsilon = \eta^\varepsilon(x)$ is only x-dependent, we augment (2.9) by adding $\bar{a}_t^\varepsilon = 0$. By the framework of [12], an additional family of waves can be treated as waves of the zero characteristic field. To achieve the target, we can further approximate $b^\varepsilon(x)$ as

$$b^\varepsilon(x) \approx b^\varepsilon(x_0) + (x - x_0)(b^\varepsilon)'(x_0), \tag{2.18}$$

and apply this to (2.9). Let u_t^ε vanish in (2.9) and omit $O(1)\delta|x - x_0|$ terms, then we obtain a time-independent approximate solution (standing wave) $u_s^\varepsilon(x)$ in D_M^ε which satisfies the following ordinary differential equations:

$$\begin{aligned}\big[(f + \delta b^\varepsilon(x_0) f_a)(\eta^\varepsilon, u_s^\varepsilon)\big]_x &= (\eta^\varepsilon)'\big\{g + \delta\big(b^\varepsilon(x_0)g_a + (b^\varepsilon)'(x_0)g_{a_x} + c(x_0)g_{a_t}\big)\big\}\\ &\quad + \delta(b^\varepsilon)'(x_0)(g - f_a),\end{aligned} \tag{2.19}$$

where g, g_a, g_{a_x} and g_{a_t} are evaluated at $(\eta^\varepsilon(x), \bar{a}'(x), b^\varepsilon(x), u_s^\varepsilon)$, and $\bar{a}(x)$ is given in (2.3). Since $\eta^\varepsilon(x)$ is monotone, by the inverse function theorem, x can be expressed as a function of η^ε and there exists a nonzero continuous function q such that $q(\eta^\varepsilon) = (\frac{d\eta^\varepsilon}{dx})^{-1}$. Applying the re-scaling to (2.19), we would have the following initial value problem:

$$\begin{cases} \dfrac{du_s^\varepsilon}{d\eta^\varepsilon} = \widetilde{R}_0(\eta^\varepsilon, u_s^\varepsilon, x_0, t_0; \eta_1) + \delta\widetilde{L}(\eta^\varepsilon, u_s^\varepsilon, x_0, t_0; \eta_1),\\ u_s^\varepsilon(a_L) = u_1,\end{cases} \tag{2.20}$$

where

$$\widetilde{R}_0 := \big(D_u f + \delta b^\varepsilon(x_0) D_u f_a\big)^{-1}\cdot(N_1 + N_2), \tag{2.21}$$

$$\widetilde{L} := \big(D_u f + \delta b^\varepsilon(x_0) D_u f_a\big)^{-1}\cdot(b^\varepsilon)'(x_0)q(\eta^\varepsilon)\big(g - f_a\big), \tag{2.22}$$

$$N_1 := g - f_a + \delta b^\varepsilon(x_0)(g_a - f_{aa}), \quad N_2 := \delta\big((b^\varepsilon)'(x_0)g_{a_x} + c(x_0)g_{a_t}\big), \tag{2.23}$$

and the initial data u_1 satisfies

$$u_1 = u_L + \eta_1 R_1^\delta(a_L, b_L, u_L) + O(|\eta_1|^2).$$

It is noticed that $D_u f + \delta b^\varepsilon(x_0) D_u f_a$ is non-singular for small δ, and that $\delta\widetilde{L}$ in (2.20) is of order $\varepsilon\delta\kappa$ by the facts that $q(\eta^\varepsilon) = (\frac{d\eta^\varepsilon}{dx})^{-1} = O(\varepsilon\kappa)$ and $\widetilde{L}$ is bounded.

LEMMA 2.1. *For sufficiently small $\delta > 0$, there exist vectors J_1, J_2 and a matrix J_* evaluated at $(\eta^\varepsilon, u_s^\varepsilon, x_0, t_0)$ such that*

$$\widetilde{R}_0(\eta^\varepsilon, u_s^\varepsilon, x_0, t_0; \eta_1) = R_0(\eta^\varepsilon, u_s^\varepsilon) + \delta(J_1 + J_2), \tag{2.24}$$

where R_0 is given in (A_2),

$$J_1{:=}{-}b^\varepsilon(x_0)(D_uf)^{-1}J_*\big(N_1+N_2\big),$$
$$J_2{:=}(D_uf)^{-1}\big\{b^\varepsilon(x_0)(g_a-f_{aa})+(b^\varepsilon)'(x_0)g_{a_x}+c(x_0)g_{a_t}\big\},$$
$$J_*{:=}\sum_{i=0}^{\infty}(-1)^i\delta^ib^\varepsilon(x_0)^i\big[(D_uf_a)(D_uf)^{-1}\big]^{i+1},$$

and N_1, N_2 are given in (2.23).

Proof. First, by the facts that

$$D_uf+\delta b^\varepsilon(x_0)D_uf_a=\big[I+\delta b^\varepsilon(x_0)(D_uf_a)(D_uf)^{-1}\big](D_uf),$$

and $I+\delta b^\varepsilon(x_0)(D_uf_a)(D_uf)^{-1}$ is invertible for small $\delta>0$, we have

$$\big(D_uf+\delta b^\varepsilon(x_0)D_uf_a\big)^{-1}=(D_uf)^{-1}\big(I+\delta b^\varepsilon(x_0)(D_uf_a)(D_uf)^{-1}\big)^{-1}. \tag{2.25}$$

Next, since $b^\varepsilon(x_0)(D_uf_a)(D_uf)^{-1}$ is uniformly bounded and $\delta^ib^\varepsilon(x_0)^i[(D_uf_a)(D_uf)^{-1}]^i$ approaches zero matrix as δ tends to zero,

$$\big(I+\delta b^\varepsilon(x_0)(D_uf_a)(D_uf)^{-1}\big)^{-1}=I-\delta b^\varepsilon(x_0)J_*, \tag{2.26}$$

where J_* is given in the lemma. Then, by applying (2.25) and (2.26) to (2.21), we obtain (2.24). We complete the proof. □

Based on Lemma 2.1, we are ready to parametrize u_s^ε. According to the existence and uniqueness theorem of ordinary differential equations, the solution $(\eta^\varepsilon,u_s^\varepsilon)$ of (2.20) is a perturbation of the integral curve of $(1,R_0(\eta^\varepsilon,u_s^\varepsilon))$, which starts at (a_L,u_1). More precisely, u_s^ε in D_M^ε can be parametrized as

$$\begin{aligned}u_s^\varepsilon(\eta_0;x_0,t_0,u_1)&=u_1+\eta_0R_0(a_L,u_1)+\frac{\eta_0^2}{2}(R_0\cdot\nabla R_0)(a_L,u_1)\\&\quad+\eta_0\delta K_0+O(\delta|\eta_0|^2+|\eta_0|^3),\end{aligned} \tag{2.27}$$

where

$$K_0:=(J_1+J_2+\widetilde{L}), \tag{2.28}$$

which depends on a_L, u_1, $a_t(x_0,t_0)$, $a_{tx}(x_0,t_0)$ and $a_{tt}(x_0,t_0)$. Note that, by (A_2) and Lemma 2.1, the system in (2.20) has no equilibrium in D_M^ε for sufficiently small $\delta>0$. Hence, the total variation of u_s^ε in D_M^ε can be controlled due to the monotonicity of η^ε. Also, η_0 equals to the total variation of $\bar a^\varepsilon$.

By the previous analysis we have the following theorem.

THEOREM 2.2. *For the PRP* (2.9), *there exists a smooth parameter of states that can be connected to u_L on the right by a smooth standing wave given by* (2.20).

For the time-independent source, the standing waves are scale invariant and can be re-scaled into discontinuities. We then have

DEFINITION 2.3. *An admissible solution u_s is called the standing wave discontinuity of* (P_0) *if u_s is the $\varepsilon\downarrow 0$ limit of smooth standing waves $\{u_s^\varepsilon\}$.*

To give a description of our generalized Riemann solver for (P_0), we look for 2 intermediate states u_1 and u_2 such that u_L,u_1,u_2,u_R are separated by shocks, rarefaction waves or the standing wave discontinuity, which is given in the following:

DEFINITION 2.4. *Given $(x_0, t_0) \in \mathbb{R}^+ \times [0, \infty)$ and κ, $\delta > 0$ sufficiently small, we say that $u(x, t)$ is the generalized Riemann solver for (P_0) if the following conditions hold:*

(a) there exists two states (a_L, u_1), (a_R, u_2) which satisfy $u_2 = u_s(\eta_0; x_0, t_0, u_1)$ for the wave strength $\eta_0 := a_R - a_L$;

(b) in region $[x_0 - \kappa, x_0) \times [t_0, t_0 + \delta]$, $u(x, t)$ coincides with the 1-wave to the homogeneous PRP (2.9) with initial values a_L, u_L, u_1; while in the region $(x_0, x_0 - \kappa] \times [t_0, t_0 + \delta]$, $u(x, t)$ coincides with the 2-wave to (2.9) with initial values a_R, u_2, u_R.

The next theorem establishes existence and uniqueness for such Riemann solver.

THEOREM 2.5. *Given $(x_0, t_0) \in \mathbb{R}^+ \times [0, \infty)$. Assume that conditions $(A_1)-(A_3)$ hold and that u_L, $u_R \in \Omega$ with $|u_L - u_R|$ sufficiently small. Then there exists a neighborhood $N \subset \Omega$ such that if u_L, $u_R \in N$, then there exists a unique Riemann solver of (P_0) centered at (x_0, t_0) in the sense of Definition 2.4.*

Proof. In the previous analysis, we augment the system in (2.9) by adding $\bar{a}^\varepsilon_t = 0$. So, let $U^\varepsilon := (\bar{a}^\varepsilon, u^\varepsilon)^T$ and define

$$R^*_0 := (1, R_0)^T, \quad R^*_1 := (0, R_1)^T, \quad R^*_2 := (0, R_2)^T, \tag{2.29}$$

and

$$K^*_i := (0, K_i)^T, \quad H^*_i(U^\varepsilon, \Lambda^i) := R^*_i(U^\varepsilon) + \delta K^*_i(\Lambda^i), \quad i = 0, 1, 2, \tag{2.30}$$

where K_1, K_2 are given in (2.11), K_0 is given in (2.28), and the parameter Λ^i describes the point at which K^*_i (or K_i) is evaluated. According to (2.15) and (2.27), for any constant state $u^\varepsilon \in \Omega$, there is a set of C^2-mappings $\{\widetilde{T}^\delta_i\}^2_{i=0}$ such that $\widetilde{T}^\delta_i(\sigma_i; U^\varepsilon)$, $i = 1, 2$, can be connected to U^ε on the right by either an i-shock or an i-rarefaction wave with the wave strength σ_i, and $\widetilde{T}^\delta_0(\sigma_0; U^\varepsilon, x_0, t_0)$ can be connected to U^ε on the right by a standing wave (zero-wave) of strength σ_0. Define the composite mapping

$$\widetilde{T}^\delta(\sigma; U^\varepsilon) := \widetilde{T}^\delta_2(\sigma_2; \widetilde{T}^\delta_0(\sigma_0; \widetilde{T}^\delta_1(\sigma_1; U^\varepsilon), x_0, t_0)), \tag{2.31}$$

where $\sigma := (\sigma_1, \sigma_0, \sigma_2)$. By direct calculation we obtain

$$\widetilde{T}^\delta(\sigma; U_L) = U_L + \sum_{j=0}^{2} \sigma_j H^*_j(U_L, \Lambda^j_L) + O(\delta|\sigma|^2) + O(|\sigma|^2), \tag{2.32}$$

where $U_L := (a_L, u_L)$, $\Lambda^1_L = \Lambda^2_L = (a_L, b_L, u_L)$ and $\Lambda^0_L = (U_L, a_t(x_0, t_0), a_{tx}(x_0, t_0), a_{tt}(x_0, t_0))$. It is sufficient to show that, for every small $\delta > 0$, there exists some $\eta := (\eta_1, \eta_0, \eta_2)$ such that

$$\widetilde{T}^\delta(\eta; U_L) = U_R, \tag{2.33}$$

where $U_R := (a_R, u_R)$.

Define

$$T^\delta(\sigma; U^\varepsilon) := \widetilde{T}^\delta(\sigma; U^\varepsilon) - U^\varepsilon.$$

Then we have $T^\delta(0, 0, 0; U_L) = 0$. In addition, the Jacobian matrix $D_\sigma T^\delta$ at $\sigma = (0, 0, 0)$ can be calculated by

$$\begin{aligned} D_\sigma T^\delta(0, 0, 0; U_L) &= \left[H^*_1(U_L, \Lambda^1_L), H^*_0(U_L, \Lambda^0_L), H^*_2(U_L, \Lambda^2_L)\right] \\ &= R^*(U_L) + \delta K^*(\Lambda_L), \end{aligned} \tag{2.34}$$

where

$$R^* := [R_1^*, R_0^*, R_2^*], \quad K^*(\Lambda_L) := [K_1^*(\Lambda_L^1), K_0^*(\Lambda_L^0), K_2^*(\Lambda_L^2)], \tag{2.35}$$

and R_1^*, R_0^* and R_2^* are given in (2.29). Since $\{R_0^*, R_1^*, R_2^*\}$ is linearly independent for all $u \in \Omega$, there exists a sufficiently small $\delta^* > 0$ such that

$$\det(D_\sigma T^\delta(0,0,0;U_L)) = \det(R^*(U_L) + \delta K^*(\Lambda_L)) \neq 0 \tag{2.36}$$

for any $\delta \in (0, \delta^*)$. Therefore, by the inverse function theorem, there exists a unique $\eta = (\eta_1, \eta_0, \eta_2)$ such that (2.33) holds. The above argument is independent of ε. By taking the limit $\varepsilon \to 0$ and using the standing wave discontinuity as the zero-wave, we complete the proof. □

In the rest of this section, we estimate the residual of approximate solution of (2.9) on $D_{\kappa\delta} := [x_0 - \kappa, x_0 + \kappa] \times [t_0, t_0 + \delta]$. Given $\phi \in C_c^1(E)$, the residual R_ϕ is defined by

$$R_\phi(\bar{a}^\varepsilon, u^\varepsilon, D_{\kappa\delta}) := \iint_{D_{\kappa\delta}} u^\varepsilon \phi_t + f(\bar{a}^\varepsilon, u^\varepsilon)\phi_x + (\bar{a}^\varepsilon)' g(\bar{a}^\varepsilon, \bar{a}', b^\varepsilon, u^\varepsilon)\phi dx dt. \tag{2.37}$$

In Section 5, we will see that the estimate of R_ϕ is crucial to obtain the consistency of the GGS.

Theorem 2.6. *Let u^ε be the approximate solution to* (2.9) *constructed by the modified Lax method for* $0 < \varepsilon \ll 1$. *Then*

$$\begin{aligned} R_\phi(\bar{a}^\varepsilon, u^\varepsilon, D_{\kappa\delta}) = &\int_{x_0-\kappa}^{x_0+\kappa} u^\varepsilon(x, t_0+\delta)\phi(x, t_0+\delta)dx - \int_{x_0-\kappa}^{x_0+\kappa} u^\varepsilon(x, t_0)\phi(x, t_0)dx \\ &+ \int_{t_0}^{t_0+\delta} \big(f + \delta b_R f_a\big)(a_R, u^\varepsilon(x_0+\kappa, t))\phi(x_0+\kappa, t)dt \\ &- \int_{t_0}^{t_0+\delta} \big(f + \delta b_L f_a\big)(a_L, u^\varepsilon(x_0-\kappa, t))\phi(x_0-\kappa, t)dt \\ &+ O(\kappa^3) + O(\kappa^2)\|\phi\|_\infty \big(osc.\{\eta^\varepsilon \ in\ D_{\kappa\delta}\} + osc.\{a_t(\cdot, t_0)\ in\ D_{\kappa\delta}\}\big) \\ &+ O(\kappa)\varepsilon\|\phi\|_\infty \cdot osc.\{u^\varepsilon \ in\ D_{\kappa\delta}\}, \end{aligned} \tag{2.38}$$

where osc.$\{u\}$ denotes the oscillation of u and $\phi \in C_c^1(E)$.

Proof. First, by the construction of u^ε, it is easy to see that

$$R_\phi(\bar{a}^\varepsilon, u^\varepsilon, D_{\kappa\delta}) = R_\phi(a_L, u^\varepsilon, D_L^\varepsilon) + R_\phi(\eta^\varepsilon, u_s^\varepsilon, D_M^\varepsilon) + R_\phi(a_R, u^\varepsilon, D_R^\varepsilon), \tag{2.39}$$

where D_L^ε, D_M^ε and D_R^ε are given in (2.13). Next, we estimate R_ϕ in D_L^ε and D_R^ε, respectively. Applying the divergence theorem to R_ϕ together with $b^\varepsilon = b_L$ and the

Rankine-Hugoniot condition (2.17) for shock, we obtain

$$\begin{aligned}
&R_\phi(a_L,u^\varepsilon,D_L^\varepsilon)\\
&=\iint_{D_L^\varepsilon}\big\{\big(u^\varepsilon\phi\big)_t+\big((f+\delta b_L f_a)(a_L,u^\varepsilon)\phi\big)_x\big\}dxdt-\iint_{D_L^\varepsilon}\delta b_L f_a(a_L,u^\varepsilon)\phi_x dxdt\\
&=\int_{x_0-\kappa}^{x_0-\varepsilon\kappa}u^\varepsilon(x,t_0+\delta)\phi(x,t_0+\delta)dx-\int_{x_0-\kappa}^{x_0-\varepsilon\kappa}u^\varepsilon(x,t_0)\phi(x,t_0)dx\\
&\quad+\int_{t_0}^{t_0+\delta}\big(f+\delta b_L f_a\big)(a_L,u^\varepsilon(x_0-\varepsilon\kappa,t))\phi(x_0-\varepsilon\kappa,t)dt\\
&\quad-\int_{t_0}^{t_0+\delta}\big(f+\delta b_L f_a\big)(a_L,u^\varepsilon(x_0-\kappa,t))\phi(x_0-\kappa,t)dt+O(\kappa^3).
\end{aligned}\tag{2.40}$$

Similarly, we have

$$\begin{aligned}
R_\phi(a_R,u^\varepsilon,D_R^\varepsilon)=&\int_{x_0+\varepsilon\kappa}^{x_0+\kappa}u^\varepsilon(x,t_0+\delta)\phi(x,t_0+\delta)dx-\int_{x_0+\varepsilon\kappa}^{x_0+\kappa}u^\varepsilon(x,t_0)\phi(x,t_0)dx\\
&-\int_{t_0}^{t_0+\delta}\big(f+\delta b_R f_a\big)(a_R,u^\varepsilon(x_0+\varepsilon\kappa,t))\phi(x_0+\varepsilon\kappa,t)dt\\
&+\int_{t_0}^{t_0+\delta}\big(f+\delta b_R f_a\big)(a_R,u^\varepsilon(x_0+\kappa,t))\phi(x_0+\kappa,t)dt+O(\kappa^3).
\end{aligned}\tag{2.41}$$

Finally, we calculate $R_\phi(\eta^\varepsilon,u_s^\varepsilon,D_M^\varepsilon)$. According to (2.19) and applying integration by parts to R_ϕ, we obtain

$$\begin{aligned}
R_\phi(\eta^\varepsilon,u_s^\varepsilon,D_M^\varepsilon)=&\int_{x_0-\varepsilon\kappa}^{x_0+\varepsilon\kappa}u_s^\varepsilon(x,t_0+\delta)\phi(x,t_0+\delta)dx-\int_{x_0-\varepsilon\kappa}^{x_0+\varepsilon\kappa}u^\varepsilon(x,t_0)\phi(x,t_0)dx\\
&+\int_{t_0}^{t_0+\delta}\big(f+\delta b(x_0)f_a\big)(a_R,u^\varepsilon(x_0+\varepsilon\kappa,t))\phi(x_0+\varepsilon\kappa,t)dt\\
&-\int_{t_0}^{t_0+\delta}\big(f+\delta b(x_0)f_a\big)(a_L,u^\varepsilon(x_0-\varepsilon\kappa,t))\phi(x_0-\varepsilon\kappa,t)dt\\
&+E_1(\eta^\varepsilon,u_s^\varepsilon,(x_0,t_0))+E_0(u_s^\varepsilon,(x_0,t_0)),
\end{aligned}\tag{2.42}$$

where

$$\begin{aligned}
E_1(\eta^\varepsilon,u_s^\varepsilon,(x_0,t_0)):=&-\iint_{D_\varepsilon}\delta b(x_0)f_a(\eta^\varepsilon,u_s^\varepsilon)\phi_x dxdt\\
&-\iint_{D_\varepsilon}(\eta^\varepsilon)'\delta[b(x_0)g_a+b'(x_0)g_{a_x}+c(x_0)g_{a_t}]\phi dxdt\\
&-\iint_{D_\varepsilon}\delta b'(x_0)(g-f_a)\phi dxdt,
\end{aligned}$$

$$E_0(u_s^\varepsilon,(x_0,t_0)):=\int_{x_0-\varepsilon\kappa}^{x_0+\varepsilon\kappa}\big(u^\varepsilon(x,t_0)-u_s^\varepsilon(x,t_0^+)\big)\phi(x,t_0)dx.$$

The terms E_0 and E_1 can be estimated by

$$E_1(\eta^\varepsilon, \psi_s^\varepsilon, (x_0, t_0)) = O(\kappa^3) + O(\kappa^2)\|\phi\|_\infty \cdot osc.\{\eta^\varepsilon \text{ in } D_{\kappa\delta}\}. \tag{2.43}$$

$$E_0(u_s^\varepsilon, (x_0, t_0)) = O(\kappa)\varepsilon\|\phi\|_\infty \cdot osc.\{u^\varepsilon \text{ in } D_{\kappa\delta}\}. \tag{2.44}$$

In addition, we see that

$$\begin{aligned} &\int_{t_0}^{t_0+\delta} \delta(b(x_0) - b_R) f_a(a_R, u^\varepsilon(x_0 + \varepsilon\kappa, t))\phi(x_0 + \varepsilon\kappa, t)dt \\ &\quad - \int_{t_0}^{t_0+\delta} \delta(b(x_0) - b_L) f_a(a_L, u^\varepsilon(x_0 - \varepsilon\kappa, t))\phi(x_0 - \varepsilon\kappa, t)dt \\ &= O(\kappa^2)\|\phi\|_\infty \cdot osc.\{a_t(\cdot, t_0) \ in \ D_{\kappa\delta}\}. \end{aligned} \tag{2.45}$$

Therefore, by (2.39)−(2.45), we obtain (2.38). The proof is complete. □

3. Generalized boundary Riemann solver and its error estimate. In this section we study the following *boundary Riemann problem* near the boundary:

$$\begin{cases} u_t + f(a, u)_x = a_x g(a, a_x, a_t, u), & (x, t) \in (0, \kappa) \times (t_0, t_0 + \delta), \\ (a, u)(x, t_0) = (a_R, u_R), & 0 < x < \kappa, \\ (a, u_1)(0, t) = (a_B, u_{1B}), & t_0 < t < t_0 + \delta, \end{cases} \tag{3.1}$$

where $0 < \kappa$, $\delta \ll 1$, and (a_R, u_R) and (a_B, u_{1B}) are two constant states. By the same analysis used in (2.2)−(2.8), we consider the corresponding *perturbed boundary Riemann problem (PBRP)* for $0 < \varepsilon \ll 1$:

$$(P_\varepsilon^B): \begin{cases} u_t^\varepsilon + \big[(f + \delta b^\varepsilon(x) f_a)(\bar{a}^\varepsilon, u^\varepsilon)\big]_x = (\bar{a}^\varepsilon)'\big[g + \delta(b^\varepsilon(x) g_a + (b^\varepsilon)'(x) g_{a_x} \\ \qquad\qquad + c(x) g_{a_t})\big] + \delta(b^\varepsilon)'(x) g, \\ u^\varepsilon(x, t_0) = \begin{cases} \psi_B^\varepsilon(x), & 0 \le x \le \varepsilon\kappa, \\ u_R, & \varepsilon\kappa < x \le \kappa, \end{cases} \\ (a, u_1)(0, t) = (a_B, u_{1B}), \quad t_0 \le t < t_0 + \delta, \end{cases} \tag{3.2}$$

where g, g_a, g_{a_x} and g_{a_t} are all evaluated at $(\bar{a}^\varepsilon, \bar{a}', b^\varepsilon, u^\varepsilon)$ with $\bar{a}' = a_x(x, t_0)$, $c(x) := a_{tt}(x, t_0)$,

$$(\bar{a}^\varepsilon(x), b^\varepsilon(x)) := \begin{cases} (\eta_B^\varepsilon(x), a_t(x, t_0)), & 0 \le x \le \varepsilon\kappa, \\ (a_R, b_R), & \varepsilon\kappa < x \le \kappa, \end{cases} \tag{3.3}$$

$\eta_B^\varepsilon(x)$ is a monotone function connecting a_B at $x = 0$ and a_R at $x = \varepsilon\kappa$, respectively, and $\psi_B^\varepsilon(x)$ is a monotone function connecting boundary data (u_{1B}, u_{2B}) at $x = 0$ and u_R at $x = \varepsilon\kappa$, respectively. It is remarked that u_{2B} will be decided by the modified Lax method. The *limiting perturbed boundary Riemann problem*, which is denoted by (P_0^B), is defined by taking the limit $\varepsilon \to 0$ to (P_ε^B). Our objective is to extend the results in Section 2 to (P_ε^B) and (P_0^B). Then, in view of Definition 2.4, a generalized Riemann solution to (P_0^B) can be given by

$$u(x, t) := \lim_{\varepsilon \to 0} u^\varepsilon(x, t),$$

where $u^\varepsilon(x, t)$ is the approximate solution to (P_ε^B).

We recall that the second characteristic field of (3.2) is genuinely nonlinear. In the region $\{\varepsilon\kappa < x < \kappa, t_0 \le t < t_0 + \delta\}$, a elementary wave curve is adopted on account of the absence of the source term. For given $U_2 := (a_R, u_2)$ and a sufficiently small parameter σ_2, there exists a C^2 mapping $\widetilde{T}_2^\delta$ such that

$$\widetilde{T}_2^\delta(\sigma_2; U_2) = \big(a_R, u_2 + \sigma_2 R_2(U_2) + \sigma_2 \delta K_2(\Lambda_2^2) + O(|\sigma_2|^2)\big) = U_R, \tag{3.4}$$

where $U_R := (a_R, u_R)$, R_2 and K_2 are given in (2.11), and $\Lambda_2^2 := (a_R, b_R, u_2)$. In other words, U_R can be connected by a 2-wave curve starting at U_2 with the (signed) wave strength σ_2. For the zero characteristic field, we construct a time-independent approximate solution. Followed by the results in Section 2, for given $U_B := (a_B, u_{1B}, u_{2B})$ nearby U_2, there exists a mapping $\widetilde{T}_0^\delta$ such that

$$\widetilde{T}_0^\delta(\sigma_0; U_B, t_0) = \big(a_B, u_B + \sigma_0 R_0(U_B) + \sigma_0 \delta K_0(\Lambda_B^0) + O(|\sigma_0|^2)\big) = U_2, \tag{3.5}$$

where $\sigma_0 = a_R - a_B$, K_0 is given in (2.28), and $\Lambda_B^0 := (a_B, a_t(0,t_0), a_{tx}(0,t_0), a_{tt}(0,t_0), u_B)$, that is, U_2 can be connected by a zero-wave curve starting at U_B with the (signed) wave strength σ_0. Here $\widetilde{T}_0^\delta$ is at least a C^2 mapping of σ_0 and U_B, and σ_0 equals to the variation of $\bar{a}^\varepsilon$ near the boundary. Define the composite $\widetilde{T}_B$ of $\widetilde{T}_2^\delta$ and $\widetilde{T}_0^\delta$ by

$$\widetilde{T}_B(\sigma_0, \sigma_2; U_B) := \widetilde{T}_2^\delta(\sigma_2; \widetilde{T}_0^\delta(\sigma_0; U_B, t_0)). \tag{3.6}$$

Then, by (3.4)$-$(3.6), we obtain

$$\widetilde{T}_B(\sigma_0, \sigma_2; U_B) = U_B + \sum_{j=0,2} \sigma_j H_j^*(U_B, \Lambda_B^j) + O(|\sigma_0| + |\sigma_2|)^2 = U_R, \tag{3.7}$$

where H_j^*, $j = 0, 2$, are given in (2.30), and $\Lambda_B^2 := (a_B, b_B, u_B)$ with $b_B := a_t(0, t_0)$. Clearly, by (3.7), we have $\widetilde{T}_B(0,0;U_B) = U_B$, and

$$\frac{\partial \widetilde{T}_B}{\partial \sigma_0}\Big|_{(0,0;U_B)} = R_0^*(U_B) + \delta K_0^*(\Lambda_B^0), \quad \frac{\partial \widetilde{T}_B}{\partial \sigma_2}\Big|_{(0,0;U_B)} = R_2^*(U_B) + \delta K_2^*(\Lambda_B^2). \tag{3.8}$$

Now, we consider the mapping T_B given by

$$T_B(\sigma_0, \sigma_2) := [\widetilde{T}_B(\sigma_0, \sigma_2; U_B)]_{1,2} - [U_B]_{1,2}, \tag{3.9}$$

where $[U]_{1,2}$ denotes the 2-vector consisting of the first two components of U. Then, by (3.7)$-$(3.9), we have $T_B(0,0) = (0,0)$ and the Jacobian matrix

$$DT_B(0,0) = \begin{bmatrix} 1 & 0 \\ [R_0(U_B) + \delta K_0(\Lambda_B^0)]_1 & [R_2(U_B) + \delta K_2(\Lambda_B^2)]_1 \end{bmatrix}. \tag{3.10}$$

To apply Lax's method, we need the following lemma.

LEMMA 3.1. *The matrix $DT_B(0,0)$ in* (3.10) *is non-singular provided that (A_2) holds.*

Proof. Suppose that $DT_B(0,0)$ is singular. It means $[R_2(U_B) + \delta K_2(\Lambda_B^2)]_1 = 0$ and so

$$\lambda_2(U_B)[R_2(U_B) + \delta K_2(\Lambda_B^2)]_1 = 0. \tag{3.11}$$

Moreover, since $R_2 + \delta K_2$ is a nonzero right eigenvector of $D_u f^\delta$, we have $[R_2(U_B) + \delta K_2(\Lambda_B^2)]_2 \neq 0$ and

$$D_u f^\delta(b_B, U_B) \cdot (R_2(U_B) + \delta K_2(\Lambda_B^2)) = \lambda_2(U_B)(R_2(U_B) + \delta K_2(\Lambda_B^2)). \tag{3.12}$$

It follows by (3.11) and (3.12) that

$$\frac{\partial}{\partial u_2}(f_1 + \delta b_B f_{1a})(U_B) = \frac{\partial}{\partial u_2} f_1^\delta(b_B, U_B) = \frac{\lambda_2(U_B)[R_2(U_B) + \delta K_2(\Lambda_B^2)]_1}{[R_2(U_B) + \delta K_2(\Lambda_B^2)]_2} = 0,$$

which violates the condition (A_2) for sufficiently small δ. The proof is complete. □

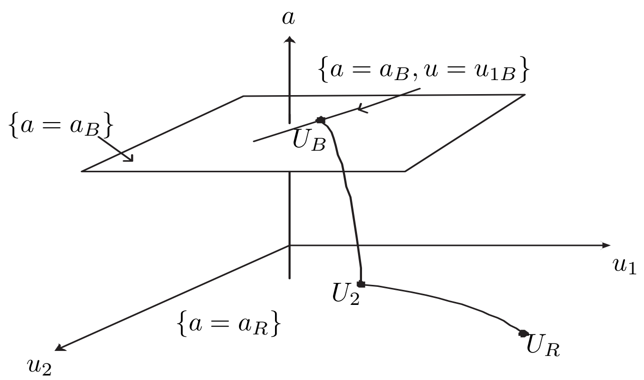

Fig. 1. The approximate solution to the PBRP in the phase plane.

We are now ready to prove the existence of approximate solutions to (3.2). First, by Lemma 3.1 and the inverse function theorem, there exist two neighborhoods N_0 of $(0, 0)$ and N_B of $[U_B]_{1,2}$ such that T_B is a diffeomorphism from N_0 to N_B. It means that we can solve T_B for σ_0 and σ_2 in terms of U_R and U_B when $|(a_B, u_{1B}) - (a_R, u_{1R})|$ is sufficiently small. Next, we determine u_{2B} when $[U_R]_{1,2} = (a_R, u_{1R}) \in N_B$ is given. In view of (3.7), we have

$$[\widetilde{T}_B(\sigma_0, \sigma_2; U_B)]_3 = u_{2B} + \sum_{j=0,2} \sigma_j [H_j^*(U_B, \Lambda_B^j)]_3 + O(|\sigma_0| + |\sigma_2|)^2. \tag{3.13}$$

By (3.13), we see that

$$\frac{d[\widetilde{T}_B(\sigma_0, \sigma_2; U_B)]_3}{du_{2B}} = 1 + \sum_{j=0,2} \sigma_j \frac{d[H_j^*(U_B, \Lambda_B^j)]_3}{du_{2B}} + O(|\sigma_0| + |\sigma_2|)^2. \tag{3.14}$$

In particular, we obtain $d[\widetilde{T}_B(0, 0; U_B)]_3/du_{2B} = 1$. Therefore, by the inverse function theorem, there exists a unique u_{2B} which is nearby u_{2R} and satisfies (3.7) for sufficiently small σ_0 and σ_2. By the analysis above, we have the following theorem regarding to the existence and uniqueness of solutions to (P_ε^B).

THEOREM 3.2. *Consider* (P_ε^B) *given in* (3.2) *with* $u_R \in \Omega$ *and sufficiently small* $|(a_B, u_{1B}) - (a_R, u_{1R})|$. *Then, under conditions* $(A_1) - (A_3)$, *there exists a unique approximate solution* $u^\varepsilon(x, t)$ *to* (P_ε^B) *obtained by the modified Lax method. The approximate solution* $u^\varepsilon(x, t)$ *consists of at most three constant states separated by a 2-wave issued from* $(\varepsilon, 0)$, *and a smooth standing wave in* $0 \le x \le \varepsilon$. *Moreover, there exists a unique* u_{2B} *such that* $u^\varepsilon(0, t) = (u_{1B}, u_{2B})$ *for* $t_0 < t < t_0 + \delta$, *see Fig. 1.*

The structure of $u^\varepsilon(x,t)$ depends on the choice of $\bar{a}^\varepsilon$. However, the wave curves for $(\bar{a}^\varepsilon, u^\varepsilon)$ are determined uniquely on the phase plane. It means that the states in $u^\varepsilon(x,t)$ are independent of $\bar{a}^\varepsilon$ and ε. By letting $\varepsilon \to 0$ in $u^\varepsilon(x,t)$, we generate a generalized boundary Riemann solver for (P_0^B).

COROLLARY 3.3. *Consider the limiting case of perturbed boundary Riemann problem (P_0^B). Then, under the same hypothesis as in Theorem 3.2, (P_0^B) admits a unique boundary Riemann solver $u(x,t) := \lim_{\varepsilon\to 0} u^\varepsilon(x,t)$ consisting of at most three constant states separated by the 2-wave and the standing wave discontinuity. Furthermore, there exists a unique constant u_{2B} such that $u(0,t) = (u_{1B}, u_{2B})$ for $t_0 < t < t_0 + \delta$.*

THEOREM 3.4. *Let $\phi \in C_c^1(E)$ and $D_{\kappa\delta}^B := [0,\kappa] \times [t_0, t_0+\delta]$. Let u^ε be the approximate solution to (P_ε^B) constructed by the modified Lax method. Then*

$$
\begin{aligned}
R_\phi(\bar{a}^\varepsilon, u^\varepsilon, D_{\kappa\delta}^B) = & \int_0^\kappa u^\varepsilon(x,t_0+\delta)\phi(x,t_0+\delta)dx - \int_0^\kappa u^\varepsilon(x,t_0)\phi(x,t_0)dx \\
& + \int_{t_0}^{t_0+\delta} \big(f + \delta b_R f_a\big)(a_R, u^\varepsilon(\kappa,t))\phi(\kappa,t)dt \\
& - \int_{t_0}^{t_0+\delta} \big(f + \delta b_B f_a\big)(a_B, u^\varepsilon(0,t))\phi(0,t)dt \\
& + O(\kappa^3) + O(\kappa^2)\big(osc.\{\eta^\varepsilon \text{ in } D_{\kappa\delta}^B\} + osc.\{a_t(\cdot,t_0) \text{ in } D_{\kappa\delta}^B\}\big) \\
& + O(\kappa)\varepsilon\|\phi\|_\infty \cdot osc.\{u^\varepsilon \text{ in } D_{\kappa\delta}^B\}.
\end{aligned} \tag{3.15}
$$

4. Generalized Glimm scheme and its stability. In this section we introduce the generalized Glimm scheme (GGS) for (1.1), (1.2) and establish its stability. Compared with the initial-boundary value problem, the analysis for the initial value problem is simpler. Therefore, we focus primarily on the case of (1.2).

To describe the scheme for (1.2), let $x_k := k\Delta x$, $t_i := i\Delta t$, $k, i = 0, 1, 2, \ldots$, and divide the first quadrant plane into the time strips of length $\Delta t = O(\Delta x)$:

$$
T_i := \{(x,t) |\ 0 \le x < \infty,\ t_i \le t < t_{i+1}\}, \quad i = 0, 1, 2, \ldots.
$$

Here Δx and Δt satisfy the generalized Courant-Friedrichs-Levy condition:

$$
\frac{\Delta x}{\Delta t} > (1-\varepsilon)^{-1} \sup\{|\lambda_i(a,u)| :\ i = 1,2\}, \quad 0 < \varepsilon \ll 1. \tag{4.1}
$$

We also define

$$
\widetilde{\Delta x} := (1+\varepsilon)^{-1}\Delta x. \tag{4.2}
$$

When introducing the GGS, we impose (4.1) to ensure that waves emanating simultaneously from points $\{(\varepsilon\widetilde{\Delta x}, t_i), (x_k \pm \varepsilon\widetilde{\Delta x}, t_i)\}$ do not interact on a time interval of length Δt. Here we use the non-staggered grid points for the computational domain. In each perturbed Riemann or boundary Riemann problem, we augment the equation $\bar{a}_t^\varepsilon = 0$ to the system. Hence, we need to provide the initial and boundary data for a and u in each time step. Let $U := (a,u)$. Following the arguments in Sections 2 and 3, we approximate initial data $U_0(x)$, boundary data $(a(0,t), u_{1B}(t))$ and $a_t(x,0)$ as follows.

(I)$_0$ The initial data $U_0^\varepsilon(x)$ on $[0,\infty)$ and the boundary data (a_B^0, u_{1B}^0) on $[0,t_1)$ are decided by

$$U_0^\varepsilon(x) = (a_0^\varepsilon(x), u_0^\varepsilon(x)) = \begin{cases} \Psi_B^0(x), & 0 \le x \le \varepsilon\widetilde{\Delta x}, \\ U_1^0, & \varepsilon\widetilde{\Delta x} < x \le x_1, \\ U_{k-1}^0, & x_{k-1} \le x < x_k - \varepsilon\widetilde{\Delta x}, \\ \Psi_k^0(x), & |x - x_k| \le \varepsilon\widetilde{\Delta x}, \\ U_{k+1}^0, & x_k + \varepsilon\widetilde{\Delta x} < x \le x_{k+1}, \end{cases} \qquad k = 2,4,6,\ldots,$$
$$(a_B^0(t), u_{1B}^0(t)) = (a(0, \Delta t/2), u_{1B}(\Delta t/2)), \quad 0 \le t < t_1,$$

where $U_{k-1}^0 = (a_{k-1}^0, u_{k-1}^0) := U_0(x_{k-1}) = (a(x_{k-1},0), u_0(x_{k-1},0))$, $\Psi_k^0(x) := (\eta_\varepsilon^0(x), \psi_k^0(x))$ is a smooth monotone function which connects U_{k-1}^0 at $x = x_k - \varepsilon\widetilde{\Delta x}$ and U_{k+1}^0 at $x = x_k + \varepsilon\widetilde{\Delta x}$, respectively, and $\Psi_B^0(x) := (\eta_B^0(x), \psi_B^0(x))$ is a smooth monotone function connecting $(a_B^0, u_{1B}^0, u_{2B}^0)$ at $x = 0$ and U_1^0 at $x = \varepsilon\widetilde{\Delta x}$, respectively. Here, u_{2B}^0 is decided by the modified Lax method. Moreover, the approximation $b_0^\varepsilon(x)$ to $a_t(x,0)$ is decided by

$$b_0^\varepsilon(x) = \begin{cases} a_{t,1}^0, & \varepsilon\widetilde{\Delta x} < x \le x_1, \\ a_{t,k-1}^0, & x_{k-1} \le x < x_k - \varepsilon\widetilde{\Delta x}, \\ a_t(x,0), & 0 \le x \le \varepsilon\widetilde{\Delta x},\ |x - x_k| \le \varepsilon\widetilde{\Delta x}, \\ a_{t,k+1}^0, & x_k + \varepsilon\widetilde{\Delta x} < x \le x_{k+1}, \end{cases} \qquad k = 2,4,6,\ldots,$$

where $a_{t,k-1}^0 := a_t(x_{k-1}, 0)$ for $k = 2,4,6,\ldots$. We then solve the PBRP and the PRPs centered at $(x_k, 0)$, $k = 2,4,6,\ldots$, by the modified Lax method. One generates an approximate solution in the zero time strip T_0.

Let $u^{\varepsilon,i-1}(x,t)$, $0 < \varepsilon \ll 1$, denote the approximate solution in the time strip T_{i-1}. For the ith time step, we choose the initial data $u_i^\varepsilon(x)$ by random choice along with the boundary data (a_B^i, u_{1B}^i), $a_i^\varepsilon(x)$ and $b_i^\varepsilon(x)$ as follows.

(I)$_i$ The initial data $U_i^\varepsilon(x)$ on $[0,\infty)$ and the boundary data (a_B^i, u_{1B}^i) on $[t_i, t_{i+1})$ are decided by

$$U_i^\varepsilon(x) = (a_i^\varepsilon, u_i^\varepsilon)(x) = \begin{cases} \Psi_B^i(x), & 0 \le x \le \varepsilon\widetilde{\Delta x}, \\ U_1^i, & \varepsilon\widetilde{\Delta x} < x \le x_1, \\ U_{k-1}^i, & x_{k-1} \le x < x_k - \varepsilon\widetilde{\Delta x}, \\ \Psi_k^i(x), & |x - x_k| \le \varepsilon\widetilde{\Delta x}, \\ U_{k+1}^i, & x_k + \varepsilon\widetilde{\Delta x} < x \le x_{k+1}, \end{cases} \qquad k = 2,4,6,\ldots,$$
$$(a_B^i(t), u_{1B}^i(t)) = (a(0, t_i + \Delta t/2), u_{1B}(t_i + \Delta t/2)), \quad t_i \le t < t_{i+1},$$

where $U_{k-1}^i = (a_{k-1}^i, u_{k-1}^i) := (a(x_{k-1}, t_i), u^{\varepsilon,i-1}(x_{k-1} + \theta_i\widetilde{\Delta x}, t_i^-))$, $\theta_i \in (-1,1)$ is a random number, $\Psi_k^i(x) := (\eta_\varepsilon^i(x), \psi_k^i(x))$ is a smooth monotone function connecting U_{k-1}^i at $x = x_k - \varepsilon\widetilde{\Delta x}$ and U_{k+1}^i at $x = x_k + \varepsilon\widetilde{\Delta x}$, respectively, and $\Psi_B^i(x) = (\eta_B^i(x), \psi_B^i(x))$ is a smooth monotone function connecting $(a_B^i, u_{1B}^i, u_{2B}^i)$ at $x = 0$ and U_1^i at $x = \varepsilon\widetilde{\Delta x}$, respectively, where u_{2B}^i is decided by the modified Lax method.

In addition, $b_i^\varepsilon(x)$ is decided by

$$b_i^\varepsilon(x) = \begin{cases} a_{t,1}^i, & \varepsilon\widetilde{\Delta x} < x \le x_1, \\ a_{t,k-1}^i, & x_{k-1} \le x < x_k - \varepsilon\widetilde{\Delta x}, \\ a_t(x,t_i), & 0 \le x \le \varepsilon\widetilde{\Delta x},\ |x-x_k| \le \varepsilon\widetilde{\Delta x}, \\ a_{t,k+1}^i, & x_k + \varepsilon\widetilde{\Delta x} < x \le x_{k+1}, \end{cases} \qquad k = 2,4,6,\ldots,$$

where $a_{t,k-1}^i := a_t(x_{k-1},t_i)$ for $k = 2,4,6,\ldots$. Again, by the results in Sections 2 and 3, we obtain an approximate solution in T_i. Applying this process to each time step, there exists an approximate solution $u_{\theta,\Delta x}^\varepsilon$ constructed by the GGS with the random sequence $\theta := (\theta_1,\theta_2,\ldots)$ in $(-1,1)$ and $0 < \varepsilon \ll 1$. By taking the $\varepsilon \downarrow 0$ limit of $\{u_{\theta,\Delta x}^\varepsilon\}$, we generate the generalized Riemann solution, denoted by $u_{\theta,\Delta x}$, of (1.2).

Here we emphasize that, in every ith time step, the choice of a_i^ε, also denoted by $a_{\Delta x}^\varepsilon$, only depend on the values of a at (x_{k-1},t_i), $k = 2,4,6,\ldots$, but not on the ones in the previous time step. Moreover, the random points $\{(x_{k-1} + \theta_i\widetilde{\Delta x}, t_i^-) : k = 2,4,6,\ldots, i \in \mathbb{N}\}$ are chosen outside the domains of standing waves to preserve the total variation of $u_{\theta,\Delta x}^\varepsilon$. This will not affect the equi-distributed property of random sequences as ε approaches 0.

To obtain the compactness of approximate solutions, the stability of the GGS is required. Due to the effect of $a(x,t)$, the interaction estimates in [9] are needed to be generalized.

We begin with the interaction of waves away from the boundary. Here we adopt the notations in [31]. Given $(x_k,t_i) \in \mathbb{R}^+ \times [0,\infty)$, $U := (a,u)$ and $V := (\bar{a},v)$. Let

$$(U,V) := [(U_0,U_1,U_2,U_3)/(\sigma_0,\sigma_1,\sigma_2)] \tag{4.3}$$

denote the approximate solution to (2.9). That is, (U,V) consists of four constant states, $U = U_0$, U_1, U_2, $U_3 = V$, separated by the j-wave curve $\widetilde{T}_j^\delta(\sigma_j;U_j) = U_{j+1}$ for $j = 0,1,2$. On account of (2.30) and (2.36), $[H_0^*,H_1^*,H_2^*]$ is nonsingular and its inverse satisfies

(4.4)
$$[H_0^*,H_1^*,H_2^*]^{-1}(U,\Lambda;x_k,t_i) = \big(\mathrm{R}^*\big)^{-1}(U) - (\Delta t)\big(\mathrm{R}^*\big)^{-1}(U)\mathrm{K}^*(\Lambda)\big(\mathrm{R}^*\big)^{-1}(U) + O(\Delta x)^2,$$

where R^* and K^* are given in (2.35). Therefore, the function $V = T^\delta(\sigma;U,x_k,t_i)$ can be inverted on a small neighborhood of $(0,0,0)$ to give

$$\sigma = (\sigma_0,\sigma_1,\sigma_2) = \Theta(V;U,x_k,t_i), \tag{4.5}$$

where Θ is smooth in $(V;U,x_k,t_i)$, $\Theta(U;U,x_k,t_i) = (0,0,0)$ and

$$D_V\Theta(U;U,x_k,t_i) = \big(\mathrm{R}^*\big)^{-1}(U) - (\Delta t)\big(\mathrm{R}^*\big)^{-1}(U)\mathrm{K}^*(\Lambda)\big(\mathrm{R}^*\big)^{-1}(U) + O(\Delta x)^2. \tag{4.6}$$

The function Θ solves the PRP (2.9) centered at (x_k,t_i), that is, for fixed $u \in \Omega$, $a \in \mathbb{R}$ and for $v \in \Omega$, $\bar{a} \in \mathbb{R}$ with $|v-u|$ small, the jump discontinuity $\{u,v\}$ is resolved into $U = U_0$, U_1, U_2 and $U_3 = V$ such that U_{j+1} is connected to U_j, on the right, by a j-wave of strength $\sigma_j = \Theta_j(V;U,x_k,t_i)$. Similarly, chose V', $U' \in \Omega$ so that (U',V') can be expressed as

$$(U',V') := [(U_0',U_1',U_2',U_3')/(\sigma_0',\sigma_1',\sigma_2')], \tag{4.7}$$

where $\sigma' := (\sigma'_0, \sigma'_1, \sigma'_2) = \Theta(V'; U', x'_k, t'_i)$. Then, by (4.6) and the fact that $|V-U| = O(1)|\Theta(V; U, x_k, t_i)| = O(1)|\sigma|$, we obtain

$$
\begin{aligned}
&\sigma' - \sigma \\
&= \int_0^1 \frac{d}{d\xi}\Theta(U' + \xi(V'-U'); U', x'_k, t'_i)d\xi - \int_0^1 \frac{d}{d\xi}\Theta(U + \xi(V-U); U, x_k, t_i)d\xi \\
&= \int_0^1 \big(D_V\Theta(U' + \xi(V'-U'); U', x'_k, t'_i) - D_V\Theta(U + \xi(V-U); U, x_k, t_i)\big)(V-U)d\xi \\
&\quad + \int_0^1 D_V\Theta(U' + \xi(V'-U'); U', x'_k, t'_i)((V'-U') - (V-U))d\xi \\
&= O(1)|\sigma|\big(|V'-V| + |U'-U| + (\Delta t)|x'_k - x_k| + (\Delta t)|t'_i - t_i| + (\Delta x)^2\big) \\
&\quad + O(1)|(V'-U') - (V-U)|.
\end{aligned}
\tag{4.8}
$$

We say that an *i-wave* and a *j-wave* are *approaching* if either (i) the wave on the left possesses larger speed or (ii) $i = j$ and at least one of them is a shock. In any case, two zero-waves are never approaching each other. Let $\theta = (\theta_0, \theta_1, \theta_2, \ldots)$, $\theta_0 = 0$, be an equi-distributed sequence of random numbers in $(-1, 1)$. The points $P_{k,i} := (x_k + \theta_i \widetilde{\Delta x}, t_i)$, $P_{0,i} := (0, (i + \frac{1}{2})\Delta t)$, $i = 0, 1, 2, \ldots$, $k = 1, 3, 5, \ldots$, are called the mesh points. We can connect those neighboring mesh points to get a set of diamond regions. In addition, the diamond regions near the boundary $x = 0$ are triangles, see Fig. 3. An unbounded piecewise linear curve I is a *mesh curve* if I lies on the boundaries of those diamond regions. Hence if I is a mesh curve, then I divides the first quadrant plane into I^+ and I^- parts such that I^- contains $t = 0$. It reminds us that every mesh curve contains some unbounded portion of the boundary. The mesh curves $I_2 > I_1$ if every point of I_2 is either on I_1 or contained in I_1^+. If $I_2 > I_1$ and if all mesh points on I_2 except one are also on I_1, then I_2 is called an immediate successor of I_1.

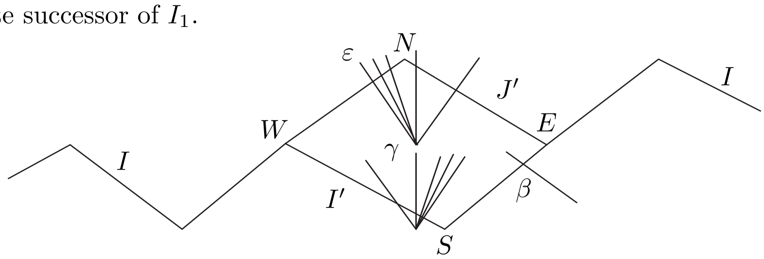

Fig. 2. A interior diamond region and in-coming and out-going waves

Let Δ be a diamond region centered at (x_m, t_{n+1}), $m \in \{2, 4, 6, \ldots\}$, and suppose that its vertices (mesh points) $N := (x_{m-1} + \theta_{n+2}\widetilde{\Delta x}, t_{n+2})$, $E := (x_{m+1} + \theta_{n+1}\widetilde{\Delta x}, t_{n+1})$, $S := (x_{m+1} + \theta_n \widetilde{\Delta x}, t_n)$ and $W := (x_{m-1} + \theta_{n+1}\widetilde{\Delta x}, t_{n+1})$, where $\{\theta_n, \theta_{n+1}, \theta_{n+2}\} \subset (-1, 1)$ are the random numbers obtained by GGS, see Fig. 2. Moreover, let $\partial\Delta^-$ stand for the lower boundary of Δ connecting W, S and E, and $\partial\Delta^+$ for the upper one connecting W, N and E. We call waves across $\partial\Delta^-$ and $\partial\Delta^+$ respectively the in-coming waves and the out-going waves of Δ. Notice that there is at most one standing wave from the in-coming waves of Δ. We have the following interaction estimate away from the boundary.

THEOREM 4.1. *(Interior wave interaction estimate) Let* $U'_L := (a'_L, u_L)$, $U'_R :=$

(a'_R, u_R), $U_L := (a_L, u_L)$, $U_R := (a_R, u_R)$, and let $U_M := (a_M, u_M)$ be a constant state near U_L and U_R. Assume that

$$\varepsilon = \Theta(U'_R; U'_L, x_m, t_{n+1})$$

are the out-going waves of Δ, and that

$$\gamma = \Theta(U_M; U_L, x_m, t_n), \quad \beta = \Theta(U_R; U_M, x_{m+2}, t_n)$$

are the in-coming waves of Δ with γ_0 across $\partial\Delta^-$, see Fig. 2. Then we have

$$\begin{aligned}\varepsilon &= \gamma + \beta + O(1)D(\gamma,\beta) + O(1)(|\gamma| + |\beta|)(|a_t| + |a_{xt}|)(\Delta t)\\ &\quad + O(1)(\Delta x)(\Delta t)|a_{xt}| + \mathcal{C}(\Delta x, |\gamma| + |\beta|)\end{aligned} \tag{4.9}$$

as $|\gamma| + |\beta| \to 0$, where a_t and a_{xt} are evaluated at (x_{m-1}, t_n), $D(\gamma,\beta) := \sum_{Approaching} |\gamma_i||\beta_j|$ and $\mathcal{C}(\Delta x, |\gamma| + |\beta|)$ denotes the cubic terms of Δx, $|\gamma|$ and $|\beta|$.

Proof. We construct $\hat\beta = \Theta(U_R; U_M, x_m, t_n)$ and $\hat\varepsilon = \Theta(U_R; U_L, x_m, t_n)$. Then, by (4.8), we have

$$\hat\beta = \beta + O(1)|\beta|(\Delta x)^2. \tag{4.10}$$

In addition, by the results of [9, 12] and (4.10), we yield

$$\hat\varepsilon = \gamma + \hat\beta + O(1)D(\gamma,\hat\beta) = \gamma + \beta + O(1)D(\gamma,\hat\beta) + \mathcal{C}(\Delta x, |\beta|) \tag{4.11}$$

as $|\gamma| + |\hat\beta| \to 0$, where $D(\gamma,\hat\beta) := \sum_{App.} |\gamma_i||\hat\beta_j|$ and $\mathcal{C}(\Delta x, |\beta|)$ denotes the cubic terms of Δx and $|\beta|$. It remains to relate ε to $\hat\varepsilon$ and $D(\gamma,\beta)$ to $D(\gamma,\hat\beta)$, respectively. By (4.10), we obtain

$$D(\gamma,\hat\beta) = D(\gamma,\beta) + O(1)|\gamma||\beta|(\Delta x)^2. \tag{4.12}$$

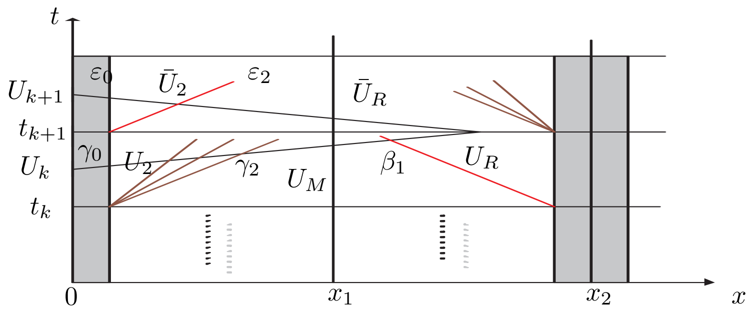

Fig. 3: Approximate solutions to the PRPs and PBRPs.

From (4.8), (4.11) and (4.12), it follows

$$\begin{aligned}\varepsilon &= \hat\varepsilon + O(1)|\hat\varepsilon|\left\{|\zeta_L| + |\zeta_R| + (\Delta x)^2\right\} + O(1)|\zeta_R - \zeta_L|\\ &= \gamma + \beta + O(1)D(\gamma,\beta) + O(1)(|\gamma| + |\beta|)(|\zeta_L| + |\zeta_R|) + O(1)|\zeta_R - \zeta_L|\\ &\quad + \mathcal{C}(\Delta x, |\gamma| + |\beta|)\end{aligned} \tag{4.13}$$

as $|\gamma| + |\beta| \to 0$, where $\zeta_R := a'_R - a_R$, $\zeta_L := a'_L - a_L$. Since

$$a'_R = a(x_{m+1}, t_{n+1}), \quad a_R = a(x_{m+1}, t_n), \quad a'_L = a(x_{m-1}, t_{n+1}), \quad a_L = a(x_{m-1}, t_n),$$

so, by the Taylor expansion, we have

$$\zeta_L = (\Delta t) a_t(x_{m-1}, t_n) + \frac{(\Delta t)^2}{2} a_{tt}(x_{m-1}, t_n) + O(\Delta x)^3, \tag{4.14}$$

$$\zeta_R = (\Delta t) a_{xt}(x_{m-1}, t_n) + 2(\Delta x)(\Delta t) a_{xt}(x_{m-1}, t_n) + O(\Delta x)^3, \tag{4.15}$$

$$\zeta_R - \zeta_L = 2(\Delta x)(\Delta t) a_{xt}(x_{m-1}, t_n) + O(\Delta x)^3. \tag{4.16}$$

It follows by (4.13)−(4.16) that (4.9) is achieved. We complete the proof. □

It remains to study the boundary wave interaction. Since a is a function of t on the boundary, the wave interaction estimate is more complicated than the one in [15]. However, we will show that the results in [15] can be extended to our problem. First, we recall the results in [15].

Lemma 4.2. *[15] (a) (Elementary wave interaction) Assume that*

$$W = \widetilde{T}_j^\delta(\beta_j; \widetilde{T}_i^\delta(\gamma_i; U)), \quad W' = \widetilde{T}_i^\delta(\gamma_i; \widetilde{T}_j^\delta(\beta_j; U)),$$

where $\widetilde{T}_i^\delta$ and $\widetilde{T}_j^\delta$ are given in (2.31). *Then there exists a continuous function η of γ_i and β_j such that*

$$W - W' = \eta(\gamma_i, \beta_j)\gamma_i\beta_j. \tag{4.17}$$

(b) (Combining waves of the same family) Assume that

$$V = \widetilde{T}_i^\delta(\gamma_i; U), \quad W = \widetilde{T}_i^\delta(\beta_i; V), \quad W' = \widetilde{T}_i^\delta(\gamma_i + \beta_i; U).$$

Then there exists a continuous function ζ of γ_i and β_i such that

$$W - W' = \begin{cases} 0, & \text{if } \gamma_i \text{ and } \beta_i \text{ are both rarefaction waves} \\ & \text{or both smooth standing waves,} \\ \zeta(\gamma_i, \beta_i)\gamma_i\beta_i, & \text{otherwise.} \end{cases} \tag{4.18}$$

By Lemma 4.2, we have the following theorem regarding to the boundary interaction estimate.

Theorem 4.3. *(Boundary wave interaction estimate) Let $U_k := (a_B^k, u_{1B}^k, u_{2B}^k)$, $U_R := (a_R^k, u_R)$, $U_{k+1} := (a_B^{k+1}, u_{1B}^{k+1}, u_{2B}^{k+1})$ and $\bar{U}_R := (a_R^{k+1}, u_R)$. Suppose that $(U_k, U_M) := [(U_k, U_2, U_M)/(\gamma_0, \gamma_2)]$ and $(U_{k+1}, \bar{U}_R) := [(U_{k+1}, \bar{U}_2, \bar{U}_R)/(\varepsilon_0, \varepsilon_2)]$ are the solutions to PBRPs on the kth and the $(k+1)$th time strips, respectively. Also, let $(U_M, U_R) := [(U_M, U_R)/(\beta_1)]$ be the 1-wave of the solution to PRP right next to (U_k, U_M) on the kth time strip, see Fig. 3. Then there exists a constant C such that for $j = 0, 2$,*

$$|\varepsilon_j - (\gamma_j + \beta_1)| \le C\Big(\sum_{App.} |\gamma_i||\beta_1| + |\beta_1| + |v_B^{k+1} - v_B^k| + |a_B^{k+1} - a_B^k| + |a_R^{k+1} - a_R^k|\Big)$$
$$+ O(\Delta x)^2. \tag{4.19}$$

Proof. Let $(\bar{U}_k, \bar{U}_R) := [(\bar{U}_k, \bar{U}_2', \bar{U}_R)/(\alpha_0, \alpha_2)]$ be the approximate solution to the PBRP $[\bar{U}_k, \bar{U}_R]$, where $\bar{U}_k := (a_B^k, v_B^k, \bar{w}_B^k)$ and $\bar{w}_B^k$ can be decided uniquely by the framework in Section 3. We emphasize that, due to the time-dependence of a, we solve the problem $[\bar{U}_k, \bar{U}_R]$ rather than $[\bar{U}_k, U_R]$ given in [15]. Then the terms of the LHS of (4.19) are bounded by

$$|\varepsilon_j - (\gamma_j + \beta_1)| \le |\varepsilon_j - \alpha_j| + |\alpha_j - (\gamma_j + \beta_1)|, \quad j = 0, 2. \tag{4.20}$$

We estimate $|\varepsilon_j - \alpha_j|$. In view of T_B in (3.9), there exists a smooth function $\widetilde{\Theta}(V; U, t) = (\widetilde{\Theta}_0(V; U, t), \widetilde{\Theta}_2(V; U, t))$ such that

$$(\varepsilon_0, \varepsilon_2) = \widetilde{\Theta}(\bar{U}_R; U_{k+1}, t_{k+1}), \quad (\alpha_0, \alpha_2) = \widetilde{\Theta}(\bar{U}_R; \bar{U}_k, t_k). \tag{4.21}$$

The function $\widetilde{\Theta}$ solves, for instance, the jump discontinuity $\{U_{k+1}, \bar{U}_R\}$ into U_{k+1}, $\bar{U}_2$ and $\bar{U}_R$ such that $\bar{U}_2$ is connected to U_{k+1} on the right by a 0-wave of strength $\varepsilon_0 = \widetilde{\Theta}_0(\bar{U}_R; U_{k+1}, t_{k+1})$, and $\bar{U}_R$ is connected to $\bar{U}_2$ on the right by a 2-wave of strength $\varepsilon_2 = \widetilde{\Theta}_2(\bar{U}_R; U_{k+1}, t_{k+1})$. Applying patterns of similar calculations in (4.8) to (4.21), we obtain

$$|\varepsilon_j - \alpha_j| \le C(|v_B^{k+1} - v_B^k| + |a_B^{k+1} - a_B^k|) + O(\Delta x)^2, \quad j = 0, 2. \tag{4.22}$$

The term $|\alpha_j - (\gamma_j + \beta_1)|$ is now estimated as follows. We notice that the map

$$U_R = \widetilde{T}_i^{\delta}(\tau_i; U_L) \text{ is smooth in } (\tau_i, U_L), \tag{4.23}$$

where U_R is the state which can be connected to U_L on the right by an i-wave of strength τ_i (Section 3). We first study the case that β_1 interacts with γ_2 and γ_0 in order, see Fig. 4. By Lemma 4.2, the states U_1', U_M' and U_R' in Fig. 4 can be completely determined by the interaction of waves β_1, γ_0 and γ_2, respectively. Furthermore, following (4.17), (4.23) and the triangle inequality, we obtain

$$|U_R' - U_R| \le C(|\gamma_0\beta_1| + |\gamma_2\beta_1|), \tag{4.24}$$

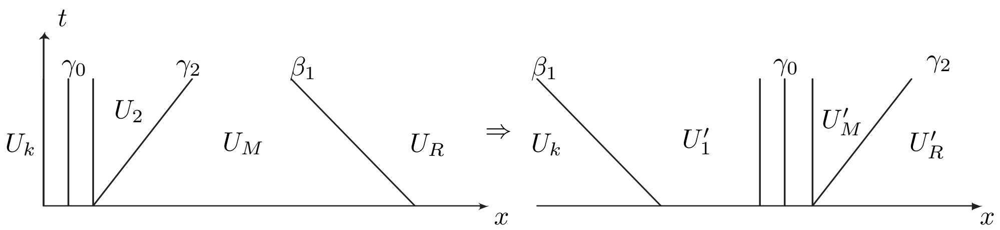

Fig. 4: Wave interaction with interchange of waves twice.

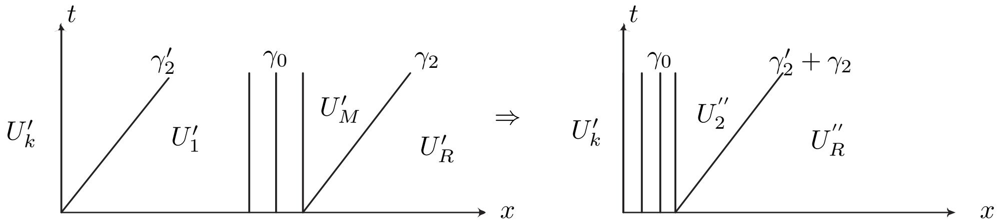

Fig. 5: Interchange and combination of waves.

where U_R' is connected to U_k on the right by waves β_1, γ_0 and γ_2. On the other hand, it is easy to see that

$$|[U_1']_{1,2} - (a_B^k, u_{1B}^k)| \le |U_1' - U_k| \le C|\beta_1|, \tag{4.25}$$

where $U_1' := (a_B^k, u_1', u_2')$ is connected to U_k on the right by β_1, and $[U_1']_{1,2}$ stands for the first two components of U_1'.

Next, we solve the PBRP $[(a_B^k, u_{1B}^k, *), U_1']$, see Fig. 5. Again, by the results in Section 3, the solution to $[(a_B^k, u_{1B}^k, *), U_1']$ exists and $*$ is decided uniquely. Let $u_{2B}^{k\prime} = *$ and $U_k' := (a_B^k, u_{1B}^k, u_{2B}^{k\prime})$, and let $(U_k', U_1') := [(U_k', U_1')/(\gamma_2')]$ be the solution to $[U_k', U_1']$. Then we estimate the case that γ_2' interchanges with γ_0 and then combines with γ_2, see Fig. 5. By the differentiability of $\widetilde{\Theta}$ and (4.25), it follows that γ_2' satisfies

$$|\gamma_2'| \le C|(a_B^k, u_{1B}^k) - [U_1']_{1,2}| \le C|\beta_1|. \tag{4.26}$$

By (4.17), (4.18), (4.23) and the triangle inequality, we obtain

$$|U_R^{''} - U_R'| \le C(|\gamma_0\gamma_2'| + |\gamma_2\gamma_2'|) \le C(|\gamma_2'|), \tag{4.27}$$

where $U_R^{''}$ is connected to U_k' on the right by waves γ_0 and $\gamma_2' + \gamma_2$. Here we used the fact that wave strengths γ_0 and γ_2 are bounded. It follows by (4.26) and (4.27) that

$$|U_R^{''} - U_R'| \le C|\beta_1|. \tag{4.28}$$

In view of $(\bar{U}_k, \bar{U}_R) = [(\bar{U}_k, \bar{U}_2', \bar{U}_R)/(\alpha_0, \alpha_2)]$, and by (4.24), (4.28) and the differentiability of $\widetilde{\Theta}$, we obtain that α_j, $j = 0, 2$, satisfy

$$\begin{aligned} |\alpha_j - (\gamma_j' + \gamma_j)| &\le C|[\bar{U}_R]_{1,2} - [U_R^{''}]_{1,2}| \le C(|\bar{U}_R - U_R| + |U_R - U_R'| + |U_R' - U_R^{''}|) \\ &\le C(|a_R^{k+1} - a_R^k| + |\gamma_0\beta_1| + |\gamma_2\beta_1| + |\beta_1|) + O(\Delta x)^2, \end{aligned} \tag{4.29}$$

where $\gamma_0' := 0$. Finally, by (4.26) and (4.29), we have

$$\begin{aligned} |\alpha_j - (\gamma_j + \beta_1)| &\le |\alpha_j - (\gamma_j' + \gamma_j)| + |\gamma_j'| + |\beta_1| \\ &\le C(|a_R^{k+1} - a_R^k| + |\gamma_0\beta_1| + |\gamma_2\beta_1| + |\beta_1|) + O(\Delta x)^2, \quad j = 0, 2. \end{aligned} \tag{4.30}$$

Therefore, by (4.20), (4.22) and (4.30), we establish (4.19). The proof is complete. ∎

Based on the estimates in Theorems 4.1 and 4.3, we begin to establish the stability of the GGS. It is sufficient to show that the total variations of approximate solutions stay uniformly bounded in time and this can be accomplished by a globally non-increasing Glimm functional introduced by Glimm [9]. Because of the presence of the boundary data, $|\beta_1|$ and $|a_R^{k+1} - a_R^k|$ in (4.19), we propose a slight modification of Glimm functional as follows.

$$F(I) := L(I) + KQ(I), \tag{4.31}$$

where I is any mesh curve, and

$$\begin{aligned} L(I) &:= \sum\{|\gamma_i| : \gamma_i \text{ crosses } I\} + K_1\Big(|\beta_1| + \sum_{k\in B(I)} l_B^k\Big), \\ Q(I) &:= \sum\{|\gamma_i||\gamma_{i'}| : \gamma_i, \gamma_{i'} \text{ cross } I \text{ and approach}\}, \\ l_B^k &:= |u_{1B}^{k+1} - u_{1B}^k| + |a_B^{k+1} - a_B^k| + (\Delta x)\big(|a_{xB}^{k+1}| + |a_{tB}^{k+1}| + |a_{xB}^k| + |a_{tB}^k|\big). \end{aligned}$$

Here, constants $K > 1$ and $K_1 > 1$ will be decided later. $B(I) := \{k : P_{0,k} = (0, t_k + \Delta t/2) \in I\}$, $u_{1B}^{k+1} := u_1(P_{0,k+1})$ and $a_{tB}^k := a_t(P_{0,k})$ etc. The term $|\beta_1|$ is involved in $L(I)$ when β_1 crosses I and locates in some boundary triangle region, see Fig. 3. We notice that, in the initial value problem (1.1), we let $K_1 = 0$ in the functional L since there is no wave-reflection on the boundary.

THEOREM 4.4. *Let* $T.V.\{U_0, u_{1B}, a_B\} := T.V.\{U_0\} + T.V.\{u_{1B}\} + T.V.\{a(0,\cdot)\}$ *and* $\omega(E) := \|a_t\|_{L^1(E)} + \|a_{xt}\|_{L^1(E)}$*, where* $E = [0,\infty) \times [0,\infty)$*. If* $T.V.\{U_0, u_{1B}, a_B\}$ *and* $\omega(E)$ *are sufficiently small, then the approximate solution* $u^{\varepsilon}_{\theta,\Delta x}$ *is well-defined for* $t > 0$ *and* $0 < \varepsilon \ll 1$.

Proof. Let I and J be two mesh curves and J an immediate successor of I. First, suppose that I and J enclose an interior diamond region centered at (x_m, t_{n+1}), see Fig. 2. Then, with the help of (4.9), we obtain

$$\begin{aligned} L(J) - L(I) &\le O(1)\big\{D(\gamma,\beta) + (|\gamma| + |\beta|)(|a_t| + |a_{xt}|)(\Delta t) + (\Delta x)(\Delta t)|a_{xt}|\big\} \\ &\quad + O(\Delta x)^3, \qquad (4.32) \\ Q(J) - Q(I) &\le O(1)L(I)\big\{D(\gamma,\beta) + (|\gamma| + |\beta|)(|a_t| + |a_{xt}|)(\Delta t) + (\Delta x)(\Delta t)|a_{xt}| \\ &\quad + O(\Delta x)^3\big\} - D(\gamma,\beta), \qquad (4.33) \end{aligned}$$

where a_t and a_{xt} are evaluated at (x_{m-1}, t_n). It follows by (4.31)–(4.33) that

$$\begin{aligned} F(J) - F(I) &\le O(1)[1 + KL(I)]\big\{D(\gamma,\beta) + (|\gamma| + |\beta|)(|a_t| + |a_{xt}|)(\Delta t) \\ &\quad + (\Delta x)(\Delta t)|a_{xt}| + O(\Delta x)^3\big\} - KD(\gamma,\beta). \qquad (4.34) \end{aligned}$$

Next, if J and I enclose a triangle region containing the mesh point $P_{0,k}$, see Fig. 3, then by (4.31) we obtain

$$\begin{aligned} F(J) - F(I) &= |\varepsilon_0| + |\varepsilon_2| - |\gamma_0| - |\gamma_2| - |\beta_1| - K_1(|\beta_1| + l^k_B) \\ &\quad + K\Big(\sum_{App.} |\alpha_i||\varepsilon_j| - \sum_{App.} |\alpha_i||\gamma_j| - \sum_{App.} |\alpha_i||\beta_1|\Big) - K(|\gamma_0\beta_1| + |\gamma_2\beta_1|). \qquad (4.35) \end{aligned}$$

Here every summation above is taken over all approaching waves $\{\alpha_i\}$ on the right of β_1, in addition, $\{\alpha_i\}$ also crosses I. It follows from (4.35) and Theorem 4.3 that

$$\begin{aligned} F(J) - F(I) &\le O(1)C(|\gamma_0\beta_1| + |\gamma_2\beta_1| + |\beta_1| + l^k_B) - K_1(|\beta_1| + l^k_B) \\ &\quad + O(1)CK\sum |\alpha_i|(|\gamma_0\beta_1| + |\gamma_2\beta_1| + |\beta_1| + l^k_B) \\ &\quad - K(|\gamma_0\beta_1| + |\gamma_2\beta_1|) + O(\Delta x)^2 \\ &\le (-K_1 + O(1)C + O(1)CK\cdot F(I))(|\beta_1| + l^k_B) \\ &\quad + (-K + O(1)C + O(1)CK\cdot F(I))(|\gamma_0\beta_1| + |\gamma_2\beta_1|) + O(\Delta x)^2 \\ &\le \nu(\Delta x)^2 \qquad (4.36) \end{aligned}$$

for some positive constant ν provided that constants K_1, $K \ge O(1)2C$, and $KF(I) \le 1$.

Now, let J_n be the mesh curve that is located on the time strip T_n for $x > 0$ and includes the half-ray $t \ge t_n + \Delta t/2$ for $x = 0$. Summing up (4.34) for even m together with (4.36), we have

$$
\begin{aligned}
F(J_{n+1}) - F(J_n) &= O(1)[1 + KL(J_n)]Q(J_n) - KQ(J_n) \\
&\quad + O(1)(\Delta t)\sum_m (|\gamma| + |\beta|)(C_m^n + \bar{C}_m^n) \\
&\quad + O(1)(\Delta x)(\Delta t)\sum_m \|a_{xt}\|_{L^\infty(D_{nm})} + O(\Delta x)^2 \\
&\le O(1)(\Delta t)L(J_n)\sup_m (C_m^n + \bar{C}_m^n) \\
&\quad + O(1)(\Delta x)(\Delta t)\sum_m \|a_{xt}\|_{L^\infty(D_{nm})} + O(\Delta x)^2, \qquad (4.37)
\end{aligned}
$$

provided that $K \geq O(1)2C$ and $KF(J_n) \leq 1$, where $D_{nm} := [x_{m-1}, x_{m+1}] \times [t_n, t_{n+1}]$, and

$$
C_m^n := \sup_{x_{m-1}\leqslant x\leqslant x_{m+1}} |a_t(x, t_n)|, \quad \bar{C}_m^n := \sup_{x_{m-1}\leqslant x\leqslant x_{m+1}} |a_{xt}(x, t_n)|.
$$

Furthermore, if $a_x(0, t)$ and $a_t(0, t)$ are bounded, then for sufficiently small $T.V.\{U_0, u_{1B}, a_B\}$ and Δx, we have

$$
\begin{aligned}
F(J_0) &\le L(J_0) + KL(J_0)^2 = (L(J_0) + KL(J_0))L(J_0) \\
&\le O(1)\bigl[1 + K_1K \cdot T.V.\{U_0, u_{1B}, a_B\}\bigr]K_1 \cdot T.V.\{U_0, u_{1B}, a_B\} \\
&\le O(1)2K_1 \cdot T.V.\{U_0, u_{1B}, a_B\}. \qquad (4.38)
\end{aligned}
$$

Therefore, by (4.37), (4.38) and by induction hypothesis, there exists a positive constant M^* such that

$$
L(J_k) \le F(J_k) \le M^* \quad \text{for } k = 0, 1, \ldots, n, \qquad (4.39)
$$

$$
\begin{aligned}
F(J_{n+1}) &\le F(J_0) + O(1)(\Delta t)M^*\sum_{k=0}^{n}\sup_m (C_m^k + \bar{C}_m^k) \\
&\quad + O(1)(\Delta x)(\Delta t)\sum_{k=0}^{n}\sum_m \|a_{xt}\|_{L^\infty(D_{km})} + O(\Delta x). \qquad (4.40)
\end{aligned}
$$

From the assumption that the constant $\omega(E)$ is finite, we see that

$$
\lim_{\Delta x\to 0}\sum_{k=0}^{\infty}\sup_m (C_m^k + \bar{C}_m^k)(\Delta t) = \omega(E), \qquad (4.41)
$$

$$
\lim_{\Delta x\to 0}\sum_{k=0}^{\infty}\sum_m \|a_{xt}\|_{L^\infty(D_{km})}2(\Delta x)(\Delta t) = \|a_{xt}\|_{L^1(E)}. \qquad (4.42)
$$

It follows by (4.39)–(4.42) that

$$
L(J_{n+1}) \le F(J_{n+1}) \le M^* + O(1)M^*\omega(E) + O(\Delta x). \qquad (4.43)
$$

According to (4.38) and (4.43), we are able to choose $T.V.\{U_0, u_{1B}, a_B\}$ and $\omega(E)$ sufficiently small such that $O(1)2K_1 \cdot T.V.\{U_0, u_{1B}, a_B\} \le M^*/3$ and $O(1)M^*\omega(E) \le M^*/3$, and this implies that

$$
L(J_{n+1}) \le F(J_{n+1}) \le M^*
$$

as Δx tends to zero. Therefore (4.39) holds for $k = n+1$. By induction on n, we show that $L(J_n)$ has a uniform bound for all $n \in \mathbb{N}$. Since the functional L is equivalent to the total variation of $U^{\varepsilon}_{\theta,\Delta x}$, the total variation of $U^{\varepsilon}_{\theta,\Delta x}$ is uniformly bounded for all $t \geq 0$ and all sufficiently small $\Delta x > 0$, so as well the L^{∞} norm of $U^{\varepsilon}_{\theta,\Delta x}$. It is worthy to indicate that $u^{\varepsilon}_{\theta,\Delta x}$ is well-defined for $t > 0$ and $\Delta x \to 0$. We complete the proof. □

It is remarked that the approximate solutions of (1.1) has the same result as that in Theorem 4.4. The following theorem is a consequence of Theorem 4.4 and the results in [31].

THEOREM 4.5. *Let $U^{\varepsilon}_{\theta,\Delta x}$ be the approximate solution to (1.2) for any $0 < \varepsilon \ll 1$. Suppose that $T.V.\{U_0, u_{1B}, a_B\}$ and $\omega(E)$ are sufficiently small. Then*

(i) $T.V.\{U^{\varepsilon}_{\theta,\Delta x}(\cdot, t)\} \leq C_1(T.V.\{U_0, u_{1B}, a_B\} + \omega)$, *where C_1 is independent of θ, Δx and ε.*

(ii) $T.V.\{U^{\varepsilon}_{\theta,\Delta x}(x, i\Delta t)\} + \sup_x [U^{\varepsilon}_{\theta,\Delta x}(x, i\Delta t)] \leq C_2(T.V.\{U_0, u_{1B}, a_B\} + \omega)$, *where C_2 is independent of θ, Δx, $i\Delta t$ and ε.*

(iii) $\displaystyle\int_0^{\infty} |U^{\varepsilon}_{\theta,\Delta x}(x, t') - U^{\varepsilon}_{\theta,\Delta x}(x, t)| dx \leq C_3(|t' - t| + \Delta t)$, *where C_3 is independent of θ, Δx and ε.*

Similarly, the above inequalities also hold for the initial value problem (1.1).

Finally, we give the compactness of subsequences of $\{U^{\varepsilon}_{\theta,\Delta x}\}$ and $\{f(U^{\varepsilon}_{\theta,\Delta x})\}$. Weak convergences of the residual and of the source will be proved in Section 5.

THEOREM 4.6. *[15] Let $\{U^{\varepsilon}_{\theta,\Delta x}\}$ be the approximate solution to (1.2) generated by the GGS. Then there exists a subsequence $\{U^{\varepsilon}_{\theta,\Delta x_i}\}$ of $\{U^{\varepsilon}_{\theta,\Delta x}\}$ such that $U^{\varepsilon}_{\theta,\Delta x_i}$ converges to some measurable function $U^{\varepsilon}(x, t)$ in $L^1_{loc}(E)$. The result also holds for sequence $\{U_{\theta,\Delta x}\}$ obtained by letting $\varepsilon \to 0$, that is, there exists a subsequence of $\{U_{\theta,\Delta x}\}$ tending to some measurable function $U(x, t)$ in $L^1_{loc}(E)$. Furthermore,*

(i) $U^{\varepsilon}(x, t) \to U(x, t)$ *in* $L^1_{loc}(E)$ *as* $\varepsilon \to 0$,

(ii) for every continuous function f, we have

$$f(U^{\varepsilon}_{\theta,\Delta x_i}) \to f(U^{\varepsilon}(x, t)) \quad \text{in } L^1_{loc}(E) \text{ as } \Delta x \to 0,$$

and

$$f(U^{\varepsilon}(x, t)) \to f(U(x, t)) \quad \text{in } L^1_{loc}(E) \text{ as } \varepsilon \to 0.$$

Similarly, properties (i), (ii) also hold for the initial value problem (1.1).

By the results in Section 3, we notice that

$$|u^{\varepsilon}_{1\theta,\Delta x}(0, t) - u_{1B}(t)| \leq |u^i_{1B} - u_{1B}(t)| \leq O(\Delta x)$$

for any $t > 0$ and $i \in \mathbb{N} \cup \{0\}$. This implies that $u^{\varepsilon}_{1\theta,\Delta x}$ will match the boundary data $u_{1B}(t)$ as Δx tends to zero.

5. Weak convergence and global existence theorem. In this section we prove the main theorem of this paper. Again, since the analysis for problems (1.1) and (1.2) are similar, we only concentrate on the problem (1.2). According to the results in Section 4, it remains to establish the consistency of the scheme, the weak convergence of the source term and the entropy inequalities.

Before we begin with the consistency of the scheme, we recall Definition 1.1 that $u(x, t)$ is a weak solution to (1.2) if and only if $R_{\phi}(a, u) = 0$ for all $\phi \in C^1_c(E)$.

Hereinafter we let $u^\varepsilon := u^\varepsilon_{\theta,\Delta x}$ be the approximate solution satisfying $u^\varepsilon_{\theta,\Delta x} \to u$ in $L^1_{loc}(E)$, and let a^ε denote $a^\varepsilon_{\Delta x}$. Since $a^\varepsilon_x g(a^\varepsilon, a^\varepsilon_x, a^\varepsilon_t, u^\varepsilon)$ may fail to converge to $a_x g(a, a_x, a_t, u)$ weakly, we use the approximation $b^\varepsilon := b^\varepsilon_{\Delta x}$ instead of a^ε_t and re-define the residual as

$$\widehat{R}_\phi(a^\varepsilon, b^\varepsilon, u^\varepsilon, E) := \iint_E u^\varepsilon \phi_t + f(a^\varepsilon, u^\varepsilon)\phi_x + a^\varepsilon_x g(a^\varepsilon, \bar{a}', b^\varepsilon, u^\varepsilon)\phi dxdt + \int_0^\infty u_0(x)\phi(x,0)dx + \int_0^\infty f(a,u)(0,t)\phi(0,t)dt, \tag{5.1}$$

where $u(0,t)$ satisfies $u(0,t) = (u_{1B}(t), u^\varepsilon_2(0,t))$ and

$$\bar{a}'(x) = \sum_{i=0}^\infty a_x(x,t_i)\chi_{\{t_i \leqslant t < t_{i+1}\}}, \quad (\chi \text{ the characteristic function}).$$

Then, to prove that the limit u is a weak solution to (1.2), it is sufficient to show for any $\phi \in C^1_c(E)$,

$$\widehat{R}_\phi(a^\varepsilon, b^\varepsilon, u^\varepsilon, E) \to 0, \quad as\ \varepsilon, \Delta x \to 0, \tag{5.2}$$

$$\iint_E \big(a^\varepsilon_x g(a^\varepsilon, \bar{a}', b^\varepsilon, u^\varepsilon) - a_x g(a, a_x, a_t, u)\big)\phi dxdt \to 0 \quad as\ \varepsilon, \Delta x \to 0. \tag{5.3}$$

First, we show (5.2). Applying the divergence theorem to $\widehat{R}_\phi(a^\varepsilon, b^\varepsilon, u^\varepsilon, E)$, together with Theorems 2.6, 3.4 and 4.5, we obtain

$$\begin{aligned}\widehat{R}_\phi(a^\varepsilon, b^\varepsilon, u^\varepsilon, E) &= -\sum_{i\geq 1} J^i_\varepsilon - \int_0^\infty \big(u^\varepsilon(x,0) - u_0(x)\big)\phi(x,0)dx \\ &\quad - \int_0^\infty \big(f(a^\varepsilon, u^\varepsilon)(0,t) - f(a,u)(0,t)\big)\phi(0,t)dt + O(\Delta x) \\ &= -J_\varepsilon + O(\Delta x) + O(\Delta x)\cdot(T.V.\{U_0, u_{1B}, a_B\} + \omega),\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}[u^\varepsilon](x, i\Delta t) &:= u^\varepsilon(x, t_i^+) - u^\varepsilon(x, t_i^-), \\ J^i_\varepsilon = J^i_\varepsilon(\theta, \Delta x, \phi) &:= \int_{-\infty}^\infty [u^\varepsilon](x, i\Delta t)\phi(x, i\Delta t)dx, \\ J_\varepsilon = J_\varepsilon(\theta, \Delta x, \phi) &:= \sum_{i\geq 1} J^i_\varepsilon.\end{aligned}$$

It remains to estimate J_ε. To this aim, we appeal to a result in [15].

THEOREM 5.1. *(See [15]) Let $\{U^\varepsilon_{\theta,\Delta x}\}$ be a family of approximate solutions constructed by the GGS in Section 4. Then for any $0 < \varepsilon \ll 1$ we can find a null set $N_\varepsilon \subset \Phi$ and a subsequence $\{\Delta x_i\} \to 0$ such that for any $\theta \in \Phi/N_\varepsilon$ and $\phi \in C^1_c(E)$, we have*

$$J_\varepsilon(\theta, \Delta x_i, \phi) = O(1)\cdot \varepsilon^{\frac{1}{2}} \quad as\ \Delta x_i \to 0.$$

Therefore, by (5.4) and Theorem 5.1, we obtain (5.2).

Next, we show the weak convergence of the source, i.e., show (5.3). Given $\delta > 0$, let $g_\delta(a, a_x, a_t, u)$ be the mollification of $g(a, a_x, a_t, u)$, that is, $g_\delta(a, a_x, a_t, u) := g(a, a_x, a_t, u) * \psi_\delta$ where ψ_δ is the standard mollifier and "$*$" denotes the convolution. Then, by the triangle inequality, we see that

$$\left|\iint_E a_x^\varepsilon g(a^\varepsilon, \bar{a}', b^\varepsilon, u^\varepsilon) - a_x g(a, a_x, a_t, u)\phi dxdt\right| \leq I_1 + I_2 + I_3 + I_4, \tag{5.5}$$

where

$$\begin{aligned}
I_1 &:= \left|\iint_E a_x^\varepsilon \big(g(a^\varepsilon, \bar{a}', b^\varepsilon, u^\varepsilon) - g_\delta(a^\varepsilon, \bar{a}', b^\varepsilon, u^\varepsilon)\big)\phi dxdt\right|,\\
I_2 &:= \left|\iint_E \big(a_x^\varepsilon - a_x\big) g_\delta(a^\varepsilon, \bar{a}', b^\varepsilon, u^\varepsilon)\phi dxdt\right|,\\
I_3 &:= \left|\iint_E a_x \big(g_\delta(a^\varepsilon, \bar{a}', b^\varepsilon, u^\varepsilon) - g(a^\varepsilon, \bar{a}', b^\varepsilon, u^\varepsilon)\big)\phi dxdt\right|,\\
I_4 &:= \left|\iint_E a_x \big(g(a^\varepsilon, \bar{a}', b^\varepsilon, u^\varepsilon) - g(a, a_x, a_t, u)\big)\phi dxdt\right|.
\end{aligned}$$

Let

$$E_{i1} := [0, x_1] \times [t_i, t_{i+1}], \quad E_{ik} := [x_{k-1}, x_{k+1}] \times [t_i, t_{i+1}], \quad i = 0, 1, 2, \ldots, k = 2, 4, \ldots.$$

To estimate I_1, we use the smoothness of standing waves, $|a_x^\varepsilon| = \frac{O(1)}{\varepsilon}|a_x| = \frac{O(1)}{\varepsilon}$ and $g_\delta \to g$ as $\delta \to 0$ at every point of continuity of g. Hence

$$\begin{aligned}
I_1 \leq& \|a_x^\varepsilon\|_{L^\infty} \|\phi\|_{L^\infty} \int_0^T \int_0^L \left|g(a^\varepsilon, \bar{a}_x, b^\varepsilon, u^\varepsilon) - g_\delta(a^\varepsilon, \bar{a}_x, b^\varepsilon, u^\varepsilon)\right| dxdt\\
\leq& \frac{C}{\varepsilon}\|\phi\|_{L^\infty}\|g - g_\delta\|_{L^1},
\end{aligned} \tag{5.6}$$

where $[0, T] \times [0, L]$ is support of ϕ and, by the property of the mollification, $I_1 \to 0$ as $\delta \to 0$. To estimate I_2, we use Taylor expansion of g_δ and $\partial^\alpha g_\delta = \partial^\alpha(g * \psi_\delta) = g * \partial^\alpha \psi_\delta$ for $\alpha = (\alpha_1, \alpha_2)$ so that I_2 can be written as

$$\begin{aligned}
I_2 =& \left|\sum_{i,k} \iint_{E_{ik}} \big(a_x^\varepsilon - a_x\big)\Big[g_\delta(x_k, t_i) + (g * \partial_x \psi_\delta)(x_k, t_i)(x - x_k)\right.\\
&\left. + (g * \partial_t \psi_\delta)(x_k, t_i)(t - t_i) + \sum_{|\alpha|=2} \frac{(x - x_k, t - t_i)^\alpha}{\alpha!}\partial^\alpha g_\delta(\bar{x}_k, \bar{t}_i)\Big]\phi dxdt\right|\\
\leq& \sum_{i,k} \big|g_\delta(x_k, t_i)\big| \cdot L_{ik}^1 + \sum_{i,k} L_{ik}^2 + O(\Delta x),
\end{aligned} \tag{5.7}$$

where $(\bar{x}_0, \bar{t}_i) \in [0, x_1] \times [t_i, t_{i+1}]$, $(\bar{x}_k, \bar{t}_i) \in [x_{k-1}, x_{k+1}] \times [t_i, t_{i+1}]$, and

$$\begin{aligned}
L_{ik}^1 &:= \left|\iint_{E_{ik}} \big(a_x^\varepsilon - a_x\big)\phi dxdt\right|,\\
L_{ik}^2 &:= \left|\iint_{E_{ik}} \big(a_x^\varepsilon - a_x\big)\big[(g * \partial_x \psi_\delta)(x_k, t_i)(x - x_k) + (g * \partial_t \psi_\delta)(x_k, t_i)(t - t_i)\big]\phi dxdt\right|.
\end{aligned}$$

Here we emphasize that it is necessary to use the mollification g_δ so that the Taylor expansion of g_δ can be apply to I_2. Now, L^1_{ik} and L^2_{ik} are estimated as follows. By the construction of a^ε along with Taylor expansion of ϕ, we have

$$\begin{aligned}
L^1_{ik} &= |\phi(x_k,t_i)|\left|\int_{t_i}^{t_{i+1}}\big[a(x_{k+1},t_i)-a(x_{k+1},t)-a(x_{k-1},t_i)+a(x_{k-1},t)\big]dt\right| \\
&\quad + O(\Delta x)^2\cdot\sup_{t_i\le t\le t_{i+1}}\big\{T.V.\{a(\cdot,t)\text{ in }[x_{k-1},x_{k+1}]\}\big\} \\
&= |\phi(x_k,t_i)|\left|\int_{t_i}^{t_{i+1}}\big\{a_t(x_{k+1},t_i)(t-t_i)-a_t(x_{k-1},t_i)(t-t_i)\big\}dt\right| + O(\Delta x)^3 \\
&\quad + O(\Delta x)^2\cdot\sup_{t_i\le t\le t_{i+1}}\big\{T.V.\{a(\cdot,t)\text{ in }[x_{k-1},x_{k+1}]\}\big\} \\
&= \left|\int_{t_i}^{t_{i+1}} a_{tx}(x_k^*,t_i)(2\Delta x)(t-t_i)dt\right| + O(\Delta x)^3 \\
&\quad + O(\Delta x)^2\cdot\sup_{t_i\le t\le t_{i+1}}\big\{T.V.\{a(\cdot,t)\text{ in }[x_{k-1},x_{k+1}]\}\big\} \\
&= O(\Delta x)^3 + O(\Delta x)^2\cdot\sup_{t_i\le t\le t_{i+1}}\big\{T.V.\{a(\cdot,t)\text{ in }[x_{k-1},x_{k+1}]\}\big\}, \qquad (5.8)
\end{aligned}$$

where $x_k^*\in(x_{k-1},x_{k+1})$ and $k=2,4,6,\ldots$. Similarly, L^1_{i1} is estimated by

$$L^1_{i1} = O(\Delta x)^3 + O(\Delta x)^2\cdot\sup_{t_i\le t\le t_{i+1}}\big\{T.V.\{a(\cdot,t)\text{ in }[0,x_1]\}\big\}. \qquad (5.9)$$

By the properties of the mollifier, we have $(g*\partial^\alpha\psi_\delta)(x,t)\le\|g\|_{L^\infty}\|\partial^\alpha\psi_\delta\|_{L^1}$ for $(x,t)\in E$, and it leads to

$$\begin{aligned}
L^2_{ik} &= O(\Delta x)\|\phi\|_\infty\|g\|_{L^\infty}\|\partial\psi_\delta\|_{L^1}\iint_{E_{ik}}\big|a^\varepsilon_x-a_x\big|dxdt \\
&= O(\Delta x)^2\cdot\sup_{t_i\le t\le t_{i+1}}\{T.V.\{a(\cdot,t)\text{ in }[x_{k-1},x_{k+1}]\}\}. \qquad (5.10)
\end{aligned}$$

Therefore, by (5.7)−(5.10), there exist positive constants C_1 and C_2 such that

$$I_2\le C_1(\Delta x)+C_2(\Delta x)\cdot\sup_{t\ge0}\{T.V.\{a(\cdot,t)\}\}. \qquad (5.11)$$

To estimate I_3, by the smoothness of a and the property of mollifiers, we obtain

$$I_3\le C_3\|\phi\|_{L^\infty}\|g-g_\delta\|_{L^1}. \qquad (5.12)$$

Finally, we estimate I_4. We notice that $|\bar a'-a_x|=O(\Delta x)$ and $|b^\varepsilon-a_t|=O(\Delta x)$ in each grid. Therefore, by the boundedness of g_a, g_{a_x}, g_{a_t}, g_u and Theorem 4.6, we obtain

$$\begin{aligned}
I_4 &\le O(1)(\big\|a^\varepsilon-a\big\|_{L^1_{loc}}+\big\|u^\varepsilon-u\big\|_{L^1_{loc}})+O(\Delta x)(\|g_{a_x}\|_\infty+\|g_{a_t}\|_\infty)\iint_E\big|a_x\big|dxdt \\
&\le O(1)(\big\|a^\varepsilon-a\big\|_{L^1_{loc}}+\big\|u^\varepsilon-u\big\|_{L^1_{loc}})+O(\Delta x)\sup_{t\ge0}\{T.V.\{a(\cdot,t)\}\}. \qquad (5.13)
\end{aligned}$$

Finally, for any $0<\varepsilon\ll1$ we choose such δ so that $\frac{1}{\varepsilon}\|g-g_\delta\|_{L^1}\to0$ as $\varepsilon\to0$. Then, by taking ε and $\Delta x\to0$ in (5.5)−(5.13), we obtain (5.3). We establish the global existence of weak solutions to the IBVP (1.2).

In the end, we show that the weak solution u is an entropy solution satisfying (1.16) for every entropy pair $(U(u), F(a,u))$. It is equivalent to show that, for any entropy pair (U, F) and positive $\phi \in C_c^1(E)$, the approximate solution u^ε satisfies the inequality

$$\sum_{i,k} \widetilde{R}_\phi(a^\varepsilon, b^\varepsilon, u^\varepsilon; E_{ik}) + \int_0^\infty U(u_0(x))\phi(x,0)dx + \int_0^\infty F(a,u)(0,t)\phi(0,t)dt$$
$$\geq O(\Delta x) + O(1)\varepsilon^{\frac{1}{2}}, \tag{5.14}$$

where

$$\widetilde{R}_\phi(a^\varepsilon, b^\varepsilon, u^\varepsilon; E_{ik}) := \iint_{E_{ik}} \big[U(u^\varepsilon)\phi_t + F(a^\varepsilon, u^\varepsilon)\phi_x + a_x^\varepsilon\big(D_u U\big(g - f_a\big) + F_a\big)\phi\big] dxdt.$$

To prove (5.14), we need to calculate $\widetilde{R}_\phi$ in each grid E_{ik}. Write $E_{ik} = D_{ik}^L \cup D_{ik}^\varepsilon \cup D_{ik}^R$, where $D_{ik}^L := \{(x,t) : x_{k-1} \leq x < x_k - \varepsilon\widetilde{\Delta x},\ t_i \leq t \leq t_{i+1}\}$, $D_{ik}^\varepsilon := \{(x,t) : x_k - \varepsilon\widetilde{\Delta x} \leq x \leq x_k + \varepsilon\widetilde{\Delta x},\ t_i \leq t \leq t_{i+1}\}$ and $D_{ik}^R := \{(x,t) : x_k + \varepsilon\widetilde{\Delta x} < x \leq x_{k+1},\ t_i \leq t \leq t_{i+1}\}$. Without loss of generality, we assume that there exist a single shock, denoted by $\mathcal{S}$, in D_{ik}^L and a rarefaction wave in D_{ik}^R. In addition, the shock $\mathcal{S}$ divides D_{ik}^L into $D_{ik}^{L^-}$ and $D_{ik}^{L^+}$, and also connects the left state u_L and the right state u_1. Also, let $a^\varepsilon = a_L$, $b^\varepsilon = b_L$ in D_{ik}^L and $a^\varepsilon = a_R$, $b^\varepsilon = b_R$ in D_{ik}^R. Then, it is easy to see

$$\begin{aligned}\widetilde{R}_\phi(a^\varepsilon, b^\varepsilon, u^\varepsilon; D_{ik}) = {}& \widetilde{R}_\phi(a_L, b_L, u^\varepsilon; D_{ik}^L) + \widetilde{R}_\phi(a^\varepsilon, b^\varepsilon, u^\varepsilon; D_{ik}^\varepsilon)\\ &+ \widetilde{R}_\phi(a_R, b_R, u^\varepsilon; D_{ik}^R).\end{aligned} \tag{5.15}$$

First, we estimate $\widetilde{R}_\phi(a_L, b_L, u^\varepsilon; D_{ik}^L)$. By $U(u^\varepsilon) = U(u_L)$ in $D_{ik}^{L^-}$ and $U(u^\varepsilon) = U(u_1)$ in $D_{ik}^{L^+}$, we have

$$\begin{aligned}\widetilde{R}_\phi(a_L, b_L, u^\varepsilon; D_{ik}^L) = {}& \iint_{D_{ik}^{L^-}} (U(u_L)\phi)_t + \big((F + (\Delta t) b_L F_a)(a_L, u_L)\phi\big)_x dxdt\\ &+ \iint_{D_{ik}^{L^+}} (U(u_1)\phi)_t + \big((F + (\Delta t) b_L F_a)(a_L, u_1)\phi\big)_x dxdt\\ &- \iint_{D_{ik}^{L^-} \cup D_{ik}^{L^+}} (\Delta t) b_L F_a(a_L, u^\varepsilon)\phi_x dxdt.\end{aligned} \tag{5.16}$$

Applying the divergence theorem to (5.16), we obtain

$$\begin{aligned}\widetilde{R}_\phi(a_L, b_L, u^\varepsilon; D_{ik}^L) = {}& Y_{ik}(U,F) + \int_{\mathcal{S}} \phi(-U(u_L) + U(u_1))dx\\ &+ \int_{\mathcal{S}} \phi\big((F + (\Delta t) b_L F_a)(a_L, u_L) - (F + (\Delta t) b_L F_a)(a_L, u_1)\big)dt\\ &- \iint_{D_{ik}^{L^-} \cup D_{ik}^{L^+}} (\Delta t) b_L F_a(a_L, u^\varepsilon)\phi_x dxdt,\end{aligned} \tag{5.17}$$

where

$$\begin{aligned}Y_{ik}(U,F):=&\int_{x_{k-1}}^{x_k-\varepsilon\widetilde{\Delta x}} U(u^\varepsilon(x,t_{i+1}))\phi(x,t_{i+1})dx - \int_{x_{k-1}}^{x_k-\varepsilon\widetilde{\Delta x}} U(u^\varepsilon(x,t_i))\phi(x,t_i)dx\\ &+\int_{t_i}^{t_{i+1}} (F+(\Delta t)b_L F_a)(a_L,u^\varepsilon(x_k-\varepsilon\widetilde{\Delta x},t))\phi(x_k-\varepsilon\widetilde{\Delta x},t)dt\\ &-\int_{t_i}^{t_{i+1}} (F+(\Delta t)b_L F_a)(a_L,u^\varepsilon(x_{k-1},t))\phi(x_{k-1},t)dt.\end{aligned}$$

The Rankine-Hugoniot condition (2.16) gives

$$s^{\Delta t}(u_L-u_1)=f^{\Delta t}(a_L,b_L,u_L)-f^{\Delta t}(a_L,b_L,u_1), \tag{5.18}$$

where $f^{\Delta t}$ is given in (2.10) with $\delta=\Delta t$ and $s^{\Delta t}$ is the speed of $\mathcal{S}$. Furthermore, by the definition of (U,F), we have $D_uF_a=(D_uU)(D_uf_a)$ and therefore

$$D_u(F+(\Delta t)b_LF_a)=(D_uU)(D_uf^{\Delta t}). \tag{5.19}$$

Then, by (5.18), (5.19) and the results in [31], we obtain

$$s^{\Delta t}[U(u_L)-U(u_1)]-[(F+(\Delta t)b_LF_a)(a_L,u_L)-(F+(\Delta t)b_LF_a)(a_L,u_1)]\le 0. \tag{5.20}$$

By (5.17) and (5.20), we obtain

$$\widetilde{R}_\phi(a_L,b_L,u^\varepsilon;D_{ik}^L)\ge Y_{ik}(U,F)+O(\Delta x)^3 \tag{5.21}$$

for any positive test function $\phi\in C_c^1(E)$.

To compute $\widetilde{R}_\phi(a_R,b_R,u^\varepsilon;D_{ik}^R)$, we notice that the rarefaction wave in D_{ik}^R is a classical solution. It follows

$$\begin{aligned}&\widetilde{R}_\phi(a_R,b_R,u^\varepsilon;D_{ik}^R)\\ &=\int_{x_k+\varepsilon\widetilde{\Delta x}}^{x_{k+1}} U(u^\varepsilon(x,t_{i+1}))\phi(x,t_{i+1})dx-\int_{x_k+\varepsilon\widetilde{\Delta x}}^{x_{k+1}} U(u^\varepsilon(x,t_i))\phi(x,t_i)dx\\ &\quad+\int_{t_i}^{t_{i+1}}(F+(\Delta t)b_RF_a)(a_R,u^\varepsilon(x_{k+1},t))\phi(x_{k+1},t)dt\\ &\quad-\int_{t_i}^{t_{i+1}}(F+(\Delta t)b_RF_a)(a_R,u^\varepsilon(x_k+\varepsilon\widetilde{\Delta x},t))\phi(x_k+\varepsilon\widetilde{\Delta x},t)dt\end{aligned} \tag{5.22}$$

for any positive test function $\phi\in C_c^1(E)$.

Finally, we calculate $\widetilde{R}_\phi(a^\varepsilon,b^\varepsilon,u^\varepsilon;D_{ik}^\varepsilon)$. Since $U(u^\varepsilon)$ is only a function of x in D_{ik}^ε, by the integration by parts, the definition of (U,F) and $D_uF_a=(D_uU)(D_uf_a)$, we obtain

$$\begin{aligned}
&\widetilde{R}_\phi(a^\varepsilon, b^\varepsilon, u^\varepsilon; D^\varepsilon_{ik})\\
&= \iint_{D^\varepsilon_{ik}} \big[U(u^\varepsilon)\phi_t + F(a^\varepsilon, u^\varepsilon)\phi_x + (\Delta t)b(x_k)F_a\phi_x - (\Delta t)b(x_k)F_a\phi_x\big]dxdt\\
&\quad + \iint_{D^\varepsilon_{ik}} a^\varepsilon_x\big(D_uU(u^\varepsilon)\big(g - f_a\big) + F_a\big)\phi dxdt\\
&= Z_{ik}(U, F) - \iint_{D^\varepsilon_{ik}} \big[(D_uF)u^\varepsilon_x - a^\varepsilon_x(D_uU)(g - f_a)\big]\phi dxdt\\
&\quad - \iint_{D^\varepsilon_{ik}} (\Delta t \cdot b(x_k)F_a)_x\phi dxdt + O(\Delta x)^3\\
&= Z_{ik}(U, F) - \iint_{D^\varepsilon_{ik}} D_uU\big[(D_uf)u^\varepsilon_x - a^\varepsilon_x(g - f_a)\big]\phi dxdt\\
&\quad - (\Delta t)b(x_k)\iint_{D^\varepsilon_{ik}} \big[F_{aa}a^\varepsilon_x + (D_uU)(D_uf_a)u^\varepsilon_x\big]\phi dxdt + O(\Delta x)^3\\
&= Z_{ik}(U, F) - \iint_{D^\varepsilon_{ik}} D_uU(u^\varepsilon)\big[\big(f + (\Delta t)b(x_k)f_a\big)_x - a^\varepsilon_x g\big]\phi dxdt\\
&\quad + (\Delta t)b(x_k)\iint_{D^\varepsilon_{ik}} a^\varepsilon_x\big[D_uU(u^\varepsilon)f_{aa} - F_{aa}\big]\phi dxdt + O(\Delta x)^3,
\end{aligned} \tag{5.23}$$

where

$$\begin{aligned}
Z_{ik}(U, F) :=& \int_{x_k-\varepsilon\widetilde{\Delta x}}^{x_k+\varepsilon\widetilde{\Delta x}} U(u^\varepsilon(x, t_{i+1}))\phi(x, t_{i+1})dx - \int_{x_k-\varepsilon\widetilde{\Delta x}}^{x_k+\varepsilon\widetilde{\Delta x}} U(u^\varepsilon(x, t_i))\phi(x, t_i)dx\\
&+ \int_{t_i}^{t_{i+1}} (F + (\Delta t)b(x_k)F_a)(a_R, u^\varepsilon(x_k + \varepsilon\widetilde{\Delta x}, t))\phi(x_k + \varepsilon\widetilde{\Delta x}, t)dt\\
&- \int_{t_i}^{t_{i+1}} (F + (\Delta t)b(x_k)F_a)(a_L, u^\varepsilon(x_k - \varepsilon\widetilde{\Delta x}, t))\phi(x_k - \varepsilon\widetilde{\Delta x}, t)dt.
\end{aligned}$$

It is easy to see

$$\iint_{D^\varepsilon_{ik}} a^\varepsilon_x\big[D_uU(u^\varepsilon)f_{aa} - F_{aa}\big]\phi dxdt = O(\Delta x)osc.\{a^\varepsilon(\cdot, t_i) \text{ in } D^\varepsilon_{ik}\}. \tag{5.24}$$

In addition, by (2.19), we have

$$\iint_{D^\varepsilon_{ik}} D_uU(u^\varepsilon)\big[\big(f + (\Delta t)b(x_k)f_a\big)_x - a^\varepsilon_x g\big]\phi dxdt = O(\Delta x)^3. \tag{5.25}$$

Then, by (5.15) and (5.21)−(5.25), we obtain

$$\begin{aligned}
\widetilde{R}_\phi(a^\varepsilon, b^\varepsilon, u^\varepsilon; E_{ik}) \geq& \int_{x_{k-1}}^{x_{k+1}} U(u^\varepsilon(x, t_{i+1}))\phi(x, t_{i+1})dx - \int_{x_{k-1}}^{x_{k+1}} U(u^\varepsilon(x, t_i))\phi(x, t_i)dx\\
&+ \int_{t_i}^{t_{i+1}} (F + (\Delta t)b_RF_a)(a_R, u^\varepsilon(x_{k+1}, t))\phi(x_{k+1}, t)dt\\
&- \int_{t_i}^{t_{i+1}} (F + (\Delta t)b_LF_a)(a_L, u^\varepsilon(x_{k-1}, t))\phi(x_{k-1}, t)dt + O(\Delta x)^3\\
&+ O(\Delta x)^2\big(osc.\{a^\varepsilon(\cdot, t_i) \text{ in } E_{ik}\} + osc.\{a_t(\cdot, t_i) \text{ in } E_{ik}\}\big)
\end{aligned} \tag{5.26}$$

for any positive test function $\phi \in C_c^1(E)$.

It follows by (5.26) and the similar results in [15] that

$$\begin{aligned}
&\sum_{i,k} \widetilde{R}_\phi(a^\varepsilon, b^\varepsilon, u^\varepsilon; E_{ik}) + \int_0^\infty U(u_0(x))\phi(x,0)dx + \int_0^\infty F(a,u)(0,t)\phi(0,t)dt,\\
&\quad \geq -\widehat{J}_\varepsilon - \int_0^\infty \big(U(u^\varepsilon(x,0)) - U(u_0(x))\big)\phi(x,0)dx\\
&\qquad - \int_0^\infty \big(F(a^\varepsilon, u^\varepsilon)(0,t) - F(a,u)(0,t)\big)\phi(0,t)dt + O(\varepsilon + \Delta x)\\
&\quad \geq -\widehat{J}_\varepsilon + O(\varepsilon + \Delta x), \qquad (5.27)
\end{aligned}$$

where

$$\begin{aligned}
\widehat{J}_\varepsilon = \widehat{J}_\varepsilon(\theta, \Delta x, \phi) &:= \sum_{i\geq 1} \int_{-\infty}^\infty [U(u^\varepsilon)](x, i\Delta t)\phi(x, i\Delta t)dx,\\
[U(u^\varepsilon)](x, i\Delta t) &:= U(u^\varepsilon)(x, t_i^+) - U(u^\varepsilon)(x, t_i^-).
\end{aligned}$$

Using (5.27) and replacing J_ε by $\widehat{J}_\varepsilon$ in Theorem 5.1, we obtain (5.14). Moreover, the weak convergence of $\{a_x^\varepsilon(D_uU\big[g - f_a\big] + F_a)(a^\varepsilon, \bar{a}', b^\varepsilon, u^\varepsilon)\}$ can be obtained by using the similar argument for $\{a_x^\varepsilon g(a^\varepsilon, \bar{a}', b^\varepsilon, u^\varepsilon)\}$. We then establish the existence of global entropy solutions to (1.2). The global existence result for (1.1) can also be obtained in the similar way. We accomplish the main theorem of this paper.

REFERENCES

[1] D. Amadori, *Initial-boundary value problems for nonlinear systems of conservation laws*, NoDEA Nonlinear Diff. Equ. Appl., 4 (1997), pp. 1–42.

[2] D. Amadori and R. M. Colombo, *Continuous dependence for* 2×2 *conservation laws with boundary*, J. Diff. Equ., 138 (1997), pp. 229–266.

[3] D. Amadori, L. Gosse and G. Guerra, *Global BV entropy solutions and uniqueness for hyperbolic systems of balance laws*, Arch. Rational Mech. Anal., 162 (2002), pp. 327–366.

[4] Y. Chang, J. M. Hong and C.-H. Hsu, *Globally Lipschitz continuous solutions to a class of quasilinear wave equations*, J. Diff. Equ., 236 (2007), pp. 504–531.

[5] C. M. Dafermos and L. Hsiao, *Hyperbolic systems of balance laws with inhomogeneity and dissipation*, Indiana U. Math. J., 31 (1982), pp. 471–491.

[6] G. Dal Maso, P. LeFloch and F. Murat, *Definition and Weak Stability of Nonconservative Products*, J. Math. Pure Appl., 74 (1995), pp. 483–548.

[7] F. Dubois and P. G. LeFloch, Boundary conditions for nonlinear hyperbolic systems of conservation laws, J. Diff. Equ., 71 (1988), pp. 93–122.

[8] H. Frid, *Initial-boundary value problems for conservation laws*, J. Diff. Eqn., 128 (1996), pp. 1–45.

[9] J. Glimm, *Solutions in the large for nonlinear hyperbolic systems of equations*, Comm. Pure Appl. Math., 18 (1965), pp. 697–715.

[10] J. B. Goodman, *Initial boundary value problem for hyperbolic systems of conservation laws*, Ph.D. dissertation, Stanford University, 1982.

[11] J. Groah, J. Smoller and B. Temple, *Shock Wave Interactions in General Relativity*, Monographs in Mathematics, Springer, Berlin, New York, 2007.

[12] J. M. Hong, *An extension of Glimm's method to inhomogeneous strictly hyperbolic systems of conservation laws by "weaker than weak" solutions of the Riemann problem*, J. Diff. Equ., 222 (2006), pp. 515–549.

[13] J. M. Hong and P. G. LeFloch, *A version of Glimm method based on generalized Riemann problems*, J. Portugal Math., 64 (2007), pp. 199-236.

[14] J. M. Hong, C.-H. Hsu and Y.-C. Su, *Global solutions for initial-boundary value problem of quasilinear wave equations*, J. Diff. Equ., 245 (2008), pp. 223–248.

[15] J. M. Hong and Y.-C. Su, *Generalized Glimm scheme to the initial boundary value problem of hyperbolic systems of balance laws*, Nonlinear Anal-Theor, 72 (2010), pp. 635–650.
[16] J. M. Hong and B. Temple, *A bound on the total variation of the conserved quantities for solutions of a general resonant nonlinear balance law*, SIAM J. Appl. Math., 64:3 (2004), pp. 819–857.
[17] E. Isaacson and B. Temple, *Nonlinear resonance in systems of conservation laws*, SIAM J. Appl. Anal., 52 (1992), pp. 1260–1278.
[18] E. Isaacson and B. Temple, *Convergence of the* 2×2 *Godunov method for a general resonant nonlinear balance law*, SIAM J. Appl. Math., 55:3 (1995), pp. 625–640.
[19] W.-L. Jin, *A kinematic wave theory of lanes-changing traffic flow*, to appear in Transportation research, Part B.
[20] B. Keyfitz and H. Kranzer, *A system of non-strictly hyperbolic conservation laws arising in elasticity theory*, Arch. Ration. Mech. Anal., 72 (1980), pp. 219–241.
[21] P. D. Lax, *Hyperbolic system of conservation laws, II*, Comm. Pure Appl. Math., 10 (1957), pp. 537–566.
[22] P. G. LeFloch, *Entropy Weak Solutions to Nonlinear Hyperbolic Systems Under Nonconservative Form*, Comm. Part. Diff. Equ., 13 (1988), pp. 669–727.
[23] P. G. LeFloch, *Shock Waves for Nonlinear Hyperbolic Systems in Nonconservative Form*, Institute for Math. and its Appl., Minneapolis, Preprint 593 (1989).
[24] P. G. LeFloch and T.-P. Liu, *Existence theory for nonlinear hyperbolic systems in nonconservative form*, Forum Math., 5 (1993), pp. 261–280.
[25] T.-P. Liu, *Initial-boundary value problems for gas dynamics*, Arch. Rat. Mech. Anal., 64 (1977), pp. 137–168.
[26] T.-P. Liu, *Quasilinear hyperbolic systems*, Comm. Math. Phys., 68 (1979), pp. 141–172.
[27] T.-P. Liu, *Approximation and qualitative behavior of admissible solutions of hyperbolic conservation laws*, Mem. Amer. Math. Soc.
[28] T. Nishida and J. Smoller, *Mixed problems for nonlinear conservation laws*, J. Diff. Equ., 23 (1977), pp. 244–269.
[29] Y.-C. Su, *Global entropy solutions to a class of quasi-linear wave equations with large time-oscillating sources* , J. Diff. Equ., 250:9, (2011), pp. 3668–3700.
[30] B. Temple, *Global solution of the Cauchy problem for a class of* 2×2 *nonstrictly hyperbolic conservation laws*, Adv. Appl. Math., 3 (1982), pp. 335–375.
[31] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, 2nd ed., Springer-Verlag, Berlin, New York, 1994.
[32] B. Whitham, *Linear and nonlinear waves*, New York, John Wiley, 1974.