METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 19, No. 4, pp. 381–402, December 2012

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TOWARDS A GENERALIZATION OF THE SEPARATION OF VARIABLES TECHNIQUE*

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Abstract. The method of separation of variables is simple, elegant and very powerful but has been applied to a limited number of differential operators both linear as well as non-linear. The underlying reason can be sought in the common belief that separation of variables for higher-order partial differential equations which include mixed derivatives is not possible. Although, the statement is valid when separation of variables is applied in its traditional form, these impediments can be bypassed introducing a generalized version of this over 250 years old technique. This will be attempted in the context of the present article. After familiarizing the reader with the concepts of the generalized form of the method of reduction, emphasis is placed on the effectiveness of the technique, providing explicit solutions to higher-order linear partial differential equations *incorporating* mixed derivatives.

Key words. Generalized separation of variables, $n-$harmonic equation, n-Helmholtz equation, $n-$metaharmonic equation.

AMS subject classifications. 31B30, 33E30.

1. Introduction.

Perhaps the technique with the longest history in constructing exact solutions for partial differential equations is the method of separation of variables. It's beginnings can be traced back to the middle of the eighteenth century where the work of Daniel Bernoulli, d' Alembert and Euler demonstrated how a linear partial differential equation can be solved by decomposing the variables (for an exciting walkthrough see [15]). This well-known technique in applied Mathematics is commonly associated with the name of Fourier who developed and utilized separation of variables to his research on conductive heat, initiating in this way the mathematical theory of approximation.

Classically, the procedure of separation of variables reduces an nth-order partial differential equation in N variables to a system of N nth-order ordinary differential equations. For *linear* partial differential equations this is accomplished by replacing the unknown function with a product of N functions which depend solely on one of the variables and introducing $N-1$ so-called separation constants. The "classical" method of variables separation will succeed if decomposition of the variables is feasible. In such manner, a set of solutions is obtained which, due to the superposition principle, can be summed up to provide a "general solution". In the case where the partial differential equation is placed in a physical setting, appropriate conditions must be applied to the solutions, restricting the summed functions to a subset, yielding the coefficients of the series. Nevertheless, the existence of mixed partial derivatives causes this approach to utterly fail.

In the early 1950's, M.H. Martin [19] noted that the identity $X''Y + XY'' = 0$, which is obtained if the method of variables separation is applied to Laplace's equation, is actually a special case of the identity

$$F_1(x)\,G_1(y) + F_2(x)\,G_2(y) = 0\,. \tag{1.0.1}$$

*Received June 19, 2012; accepted for publication February 22, 2013.
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His observation constituted a turning point for the classical method of reduction, leading to what nowadays is know as methods of generalized and functional separation of variables.

Regarding a PDE of the form

$$(\mathcal{L}u)(x,y) = 0 \tag{1.0.2}$$

the mentioned methods differ only in the way solutions are sought, viz either in the general form

$$u(x,y) = \sum_{k=1}^{m} f_k(\xi)\, g_k(\eta), \quad \xi = a_1 x + a_2 y, \quad \eta = b_1 x + b_2 y \tag{1.0.3}$$

a_1, a_2, b_1, b_2 constants, or as

$$u(x,y) = \Phi(\zeta), \quad \zeta = \sum_{k=1}^{m} f_k(x)\, g_k(y)\,, \tag{1.0.4}$$

respectively. In formula (1.0.2), $\mathcal{L}$ denotes a given linear or nonlinear partial differential operator containing quadratic and power nonlinearities, and u is the unknown.

Substituting (1.0.3) into (1.0.2) and differentiating the resulting equation, one arrives eventually at a separable two-term equation of the form (1.0.1). The ordinary differential equations obtained in this way have to be solved and their *general* solutions are then substituted into the original PDE (1.0.2). In the sequel, one has to manage all possible cases in order to evaluate the constants introduced via the general solutions in order to find $u(x,y)$. This, however, is a demanding task. The method of differentiation described briefly above, although furnishing solutions of (1.0.2), if separation of variables is permitted, introduces superfluous constants, which have to be eliminated at the final stage. Moreover, differentiating results in a two-term equation that will be of higher-order then (1.0.2). Nonetheless, these constrains can be removed by "splitting" (1.0.2) into a number of sub-problems with simpler structure, which, unfortunately, leads to loss of solutions regarding $u(x,y)$. Substituting (1.0.4) into (1.0.2) similar arguments hold. An analytic presentation of these methods is provided in [23, Supplement B].

It is remarkable that this powerful methodology, although extensively used providing exact solutions to some classes of non-linear partial differential equations (a comprehensive list is provided in [24]) as well as to integral equations [2], has not been applied to linear problems.

In the present article, exploiting concepts and ideas from generalized and functional separation of variables, a modified and simpler technique is proposed, leading to exact solutions for higher-order linear PDE's incorporating mixed derivatives. The identification of higher-order differential equations is based on their "Laplacian counterpart". For example, the equation $\Delta^n u = 0$ is recognized as the $n-$harmonic (or h_n-) equation, since $\Delta u = 0$ is labeled harmonic. This categorization will be further applied to their solutions as well.

The article is organized as follows. Section 2 provides the details of the proposed procedure which is then applied in sections 3 and 4, in order to retrieve explicit solutions for the (i) $n-$harmonic equation up to $n = 3$, (ii) the general $n-$Helmholtz equation $(\Delta^n + \kappa)u = 0$ as well as (iii) the $n-$metaharmonic equation $\sum_{j=1}^{n} \alpha_j \Delta^j u = 0$ up to $n = 3$, in the polar coordinate system. The according three-dimensional cases will be presented in a series of papers in the near future [6, 7].

Importantly, the solutions derived in the present article are "general", in the sense that we do not carry what conditions these solutions satisfy. Obviously, suitable boundary/initial conditions will restrict the generality of the solutions to be considered. However, these aspects will be neglected at present (although the main difficulty in solving boundary/initial value problems consist in satisfying the prescribed conditions). Another important feature is that, since separation of variables decomposes a partial into a set of ordinary differential equations, we can only be certain to have acquired the most general solution for these ODE's if a completeness relation exist. This issue, as well as details regarding the formulae derived in sections 3 and 4, will be addressed in subsequent papers. Concluding, henceforth we assume that the conditions for the existence and uniqueness of the solutions to the ordinary differential equations under consideration are met (for details, see, e.g. [4]).

2. A generalization of the separation of variables technique. As we strive to understand more and more complex processes encountered in the natural world, the mathematical models developed in order to describe the observed behaviour are expected to exhibit increased complexity. For example, higher-order partial differential equations occur in the theory of rotating viscous fluids [14, 17]. Such PDE's are also implemented to create, represent and manipulate surface/solid models in a Computer-aided design environment [31] and are of fundamental importance in the engineering (analysis and simulation) and medical sector (visualization of body tissues, simulation of surgical operations, etc.). An analytical treatment for these PDE's, although possible by a repeated application of variables separation is very limited.

In what follows, a step-by-step presentation of the proposed technique is provided. Consider, for simplicity, the two dimensional $n-$harmonic equation in Cartesian coordinates

$$\Delta^n u(x_1, x_2) = 0, \quad \Delta^0 = 1\,, \tag{2.0.1}$$

where Δ^n is the nth iteration of the Laplacian operator.

Introducing into Eq.(2.0.1) a multiplicatively separable solution of the form $u(x_1, x_2) = \prod_{j=1}^{2} X_j(x_j)$ where the functions $X_j(x_j)$, $j = 1, 2$ and corresponding derivatives are continuous and not identically zero, yields

$$\sum_{m=0}^{n} \binom{n}{m} X_1^{(2n-2m)} X_2^{(2m)} = 0, \quad X_j^{(0)} = X_j \tag{2.0.2}$$

where the binomial expansion has been implemented and $X_j^{(n)}(x_j)$ denotes the derivative of order n with respect to the variable x_j.

The introduced generalization builds upon the following steps:

Step 1: Isolate the highest-order derivatives and divide the resulting expression throughout the product $X_1^{(2n-2q)} X_2^{(2p)}$, *where* $p = \min m$ *and* $q = \max m$.

As a consequence, the extracted terms convert to functions depending on a single variable and Eq. (2.0.2) becomes

$$\frac{X_1^{(2n)}}{X_1} + \frac{X_2^{(2n)}}{X_2} + \sum_{m=1}^{n-1} \binom{n}{m} \frac{X_1^{(2n-2m)}}{X_1} \frac{X_2^{(2m)}}{X_2} = 0\,. \tag{2.0.3}$$

Step 2: Differentiate the outcome of step 1 with respect to both variables.

Performing step 2 eliminates the extracted single variable functions, furnishing

$$\frac{\partial^2}{\partial x_1 \partial x_2}\left[\sum_{m=1}^{n-1}\binom{n}{m}\frac{X_1^{(2n-2m)}}{X_1}\frac{X_2^{(2m)}}{X_2}\right]=0 \tag{2.0.4}$$

which holds true if the content of the bracket is either a constant or a function in one of the variables x_1 or x_2, respectively.

The latter observation needs further elaboration. In the case where the content of the bracket is represented by a constant other than zero, not all solutions are computed. On the other hand, if above expansion is set equal to a function of either variable, the resulting ordinary differential equation accepts solutions in form of certain special functions. Introducing these solutions into (2.0.3) will lead to an equation implicating both variables, therefore violating our assumption that the functions we are looking for are of one variable alone. An explicit example is given at the end of this section.

Designating the value zero to the sum in Eq.(2.0.4) and repeating the aforementioned steps k successive times the following formula is derived

$$\binom{n}{k}\left[\frac{X_1^{(2n-2k)}}{X_1^{(2k)}}+\frac{X_2^{(2n-2k)}}{X_2^{(2k)}}\right]+\sum_{m=k+1}^{n-k-1}\binom{n}{m}\frac{X_1^{(2n-2m)}}{X_1^{(2k)}}\frac{X_2^{(2m)}}{X_2^{(2k)}}=0\,. \tag{2.0.5}$$

It is readily observed that the sum present in Eq. (2.0.5) vanishes if $n \in \mathbb{Z}^+$ equals an odd number. On the other hand, if n represents even numbers only the $(k+1)$th term survives. Therefore,

$$k=\begin{cases}\frac{n-1}{2}, & \text{if } n \text{ is odd}\\ \frac{n-2}{2}, & \text{if } n \text{ is even}\end{cases}. \tag{2.0.6}$$

2.1. The case where n is an odd number. If n equals an odd number, only the term

$$\frac{X_1^{(n+1)}}{X_1^{(n-1)}}+\frac{X_2^{(n+1)}}{X_2^{(n-1)}}$$

survives from Eq. (2.0.5) providing two ODE's

$$X_1^{(n+1)}-\lambda\, X_1^{(n-1)}=0 \tag{2.1.1}$$

$$X_2^{(n+1)}+\lambda\, X_2^{(n-1)}=0 \tag{2.1.2}$$

where λ is the complex separation constant. The solutions to Eqs. (2.1.1) and (2.1.2) are

$$X_1(x_1)=\left\{1, x_1, x_1^2, \ldots, x_1^{n-2}, e^{\pm\sqrt{\lambda}\,x_1}\right\}\,, \tag{2.1.3}$$

$$X_2(x_2)=\left\{1, x_2, x_2^2, \ldots, x_2^{n-2}, e^{\pm i\sqrt{\lambda}\,x_2}\right\}\,, \tag{2.1.4}$$

representing any linear combination of its elements. Note that in the case where $n=1$ only the exponential solutions are valid.

The final steps of the procedure retrieve the solutions for the under consideration $n-$harmonic equation.

Step 3: Select the appropriate elements from (2.1.3) *or* (2.1.4) *in order for the ratio* X_j''/X_j *to be independent of the variable* x_j*.*

Obviously, step 3 diminishes the spectra of available elements acquired solving equations (2.1.1) and (2.1.2). Notwithstanding, step 3 is essential inasmuch it produces an differential equation preserving the basic assumption on which variables separation is based.

Step 4: Substitute each suitable element from (2.1.3)*,* (2.1.4) *into the initial equation* (2.0.3) *and solve the resulting ODE's of order* $2n$*.*

Replacing every element of (2.1.3) or (2.1.4) conforming step 3, into (2.0.3) furnishes an ODE of order $2n$ for the variable x_2 or x_1, respectively. Computing the $2n$ solutions of the resulting differential equations combined with the corresponding elements from which they have derived, a solution space for the unknown function $u(x_1, x_2)$ is provided.

2.2. The case where n is an even number. If n represents even numbers, only the term

$$\frac{X_1^{(n+2)}}{X_1^{(n-2)}} + \frac{X_2^{(n+2)}}{X_2^{(n-2)}}$$

together with the first term of the sum remain, and Eq. (2.0.5) simplifies as follows

$$\binom{n}{\frac{n-2}{2}} \left[\frac{X_1^{(n+2)}}{X_1^{(n-2)}} + \frac{X_2^{(n+2)}}{X_2^{(n-2)}} \right] + \binom{n}{\frac{n}{2}} \frac{X_1^{(n)}}{X_1^{(n-2)}} \frac{X_2^{(n)}}{X_2^{(n-2)}} = 0\,. \tag{2.2.1}$$

Eliminating the highest-order derivatives, yields

$$\frac{\partial}{\partial x_1} \left(\frac{X_1^{(n)}}{X_1^{(n-2)}} \right) \frac{\partial}{\partial x_2} \left(\frac{X_2^{(n)}}{X_2^{(n-2)}} \right) = 0 \tag{2.2.2}$$

which holds if

$$\frac{X_1^{(n)}}{X_1^{(n-2)}} = \lambda_1, \quad \lambda_1 \in \mathbb{C}\,, \tag{2.2.3}$$

and/or

$$\frac{X_2^{(n)}}{X_2^{(n-2)}} = \lambda_2, \quad \lambda_2 \in \mathbb{C}. \tag{2.2.4}$$

The solutions to the above equations are, respectively

$$X_1(x_1) = \left\{ 1, x_1, x_1^2, \ldots, x_1^{n-3}, e^{\pm\sqrt{\lambda_1}\, x_1} \right\}\,, \tag{2.2.5}$$

$$X_2(x_2) = \left\{ 1, x_2, x_2^2, \ldots, x_2^{n-3}, e^{\pm\sqrt{\lambda_2}\, x_2} \right\}\,. \tag{2.2.6}$$

Again, if $n = 2$ only the exponential solutions are considered.

Following previous steps, the solution to the initial equation (2.0.1) are obtained as follows: (i) Adopt the relevant elements from (2.2.5), (2.2.6) according to step 3 and retrieve the corresponding ODE of order $2n$ via equation (2.0.3). (ii) Compute the $2n$ solutions and couple with the solutions acquired through (i).

Before we proceed a few remarks are in order.

Remark 2.1. *In the case where* n *is equivalent to an odd number, the aforementioned analysis has shown that* $X_1^{(n+1)} X_2^{(n-1)} + X_1^{(n-1)} X_2^{(n+1)} = 0$ *which, rewritten reads*

$$\frac{\partial^{2n-2}}{\partial x_1^{n-1} \partial x_2^{n-1}} \Delta\, u(x_1, x_2) = 0 \tag{2.2.7}$$

and for $n = 1$ *becomes separation of variables in his traditional form.*

Remark 2.2. *If* no *assumptions are made regarding the content of the bracket of* (2.0.4)*, it is straightforward to show that* (2.0.4) *yields a more involved identity of the form* (1.0.1)*, namely*

$$\sum_{k=1}^{n-1} \binom{n}{k} F_k(x_1)\, G_k(x_2) = 0 \tag{2.2.8}$$

where

$$F_k(x_1) = \frac{\partial}{\partial x_1} \left(\frac{X_1^{(2n-2k)}}{X_1} \right) \qquad \text{and} \qquad G_k(x_2) = \frac{\partial}{\partial x_2} \left(\frac{X_2^{(2k)}}{X_2} \right). \tag{2.2.9}$$

However, following the routine so far, namely: (i) Isolate the highest-order derivatives by extracting the first and last term of the expansion (2.2.8)*; (ii) Transform the extracted terms to a function of a single variable by dividing throughout the lowest derivatives present; (iii) Eliminate these single variable functions by differentiating with respect to both variables. After several iterations we will arrive at an extremely complex two-term relation of the form* (1.0.1)*, if* n *is odd, or, an equally complex relation of the form* (2.2.2)*, if* n *is even. Nevertheless, the resulting ODE's are of higher-order than the original PDE but more importantly, there are highly non-linear! Similar conclusions are valid on the assumption that the content of the bracket equals a constant.*

Let us elucidate the particulars of the introduced technique by evaluating solutions to the 3-harmonic equation, viz

$$X_1^{(6)}\, X_2 + X_1\, X_2^{(6)} + 3X_1^{(4)}\, X_2^{(2)} + 3X_1^{(2)}\, X_2^{(4)} = 0\,. \tag{2.2.10}$$

Dividing above throughout $X_1\, X_2$ and differentiating once with respect to both variables gives

$$\frac{\partial^2}{\partial x_1\, \partial x_2} \left(\frac{X_1^{(4)}\, X_2^{(2)}}{X_1\, X_2} + \frac{X_1^{(2)}\, X_2^{(4)}}{X_1\, X_2} \right) = 0\,. \tag{2.2.11}$$

On the assumption that the content of the parenthesis vanishes leads, after simple manipulations to the separable two-term equation

$$X_1^{(4)}\, X_2^{(2)} + X_1^{(2)}\, X_2^{(4)} = 0 \tag{2.2.12}$$

and any linear combination of $x_1, x_2, e^{\pm\sqrt{\lambda}\, x_1}, e^{\pm i\sqrt{\lambda}\, x_2}$ (λ being the separation constant) is a solution to the latter. Placing, for example, $X_2 = e^{\pm i\sqrt{\lambda}\, x_2}$ into (2.2.10) yields a ODE satisfied by $e^{\pm\sqrt{\lambda}\,x_1}$, $x_1\,e^{\pm\sqrt{\lambda}\,x_1}$, $x_1^2\,e^{\pm\sqrt{\lambda}\,x_1}$. Hence, a solution to the initial PDE (2.2.10) is any linear combination of the functions $e^{\pm\sqrt{\lambda}\,x_1}\,e^{\pm i\sqrt{\lambda}\,x_2}$, $x_1\,e^{\pm\sqrt{\lambda}\,x_1}\,e^{\pm i\sqrt{\lambda}\,x_2}$ as well as $x_1^2\,e^{\pm\sqrt{\lambda}\,x_1}\,e^{\pm i\sqrt{\lambda}\,x_2}$. Further solutions are derived by substituting one-by-one the remaining solutions of the decomposed (2.2.12) into (2.2.10). Note, that Eq. (2.2.12) can be obtained at once via (2.0.5) with $n = 3$ and therefore $k = 1$. Regarding the instance wherein the constant is not zero, Eq. (2.2.12) is replaced by

$$X_1^{(4)}\,X_2^{(2)} + X_1^{(2)}\,X_2^{(4)} = \text{const.}\,X_1\,X_2 \tag{2.2.13}$$

leading to the ODE's

$$X_j^{(2)} - \lambda_j\,X_j = 0, \quad j = 1, 2\,. \tag{2.2.14}$$

Above differential equations display reduced order as the system corresponding to Eq. (2.2.12), resulting in fewer solutions.

Nevertheless, relation (2.2.11) is still satisfied when the content of the parenthesis equals a function of the form $X_1 + X_2$. Consider for economy of presentation that

$$\frac{X_1^{(4)}\,X_2^{(2)}}{X_1\,X_2} + \frac{X_1^{(2)}\,X_2^{(4)}}{X_1\,X_2} = x_1 \tag{2.2.15}$$

from which the relation

$$\frac{\partial}{\partial x_1}\left(x_1\,\frac{X_1}{X_1^{(2)}}\right)\left(\frac{\partial}{\partial x_2}\frac{X_2}{X_2^{(2)}}\right) = 0 \tag{2.2.16}$$

follows.

Equation (2.2.16) implies the system of ODE's

$$X_1^{(2)} - \xi^3\,x_1\,X_1 = 0, \quad X_2^{(2)} - \eta^2\,X_1 = 0\,. \tag{2.2.17}$$

Whereas the solutions to the second ODE are $e^{\pm\eta\,x_2}$ the first accepts solutions in form of Airy functions. Substituting either $X_1 = \mathrm{Ai}(\xi\,x_1)$ or $\mathrm{Bi}(\xi\,x_1)$ into (2.2.10) produces an equation implicating both variables violating the basic conjecture on which separation of variables is based, i.e. that the sought functions are of one variable alone.

On the other hand, dropping the assumption made equation (2.2.11) rewrites as the two term relation

$$F_1(x_1)\,G_1(x_2) + F_2(x_1)\,G_2(x_2) = 0 \tag{2.2.18}$$

where

$$\begin{aligned} F_1(x_1) &= \frac{\partial}{\partial x_1}\left(\frac{X_1^{(4)}}{X_1}\right),\; F_2(x_1) = \frac{\partial}{\partial x_1}\left(\frac{X_1^{(2)}}{X_1}\right), \\ G_1(x_2) &= \frac{\partial}{\partial x_2}\left(\frac{X_2^{(2)}}{X_2}\right),\; G_2(x_2) = \frac{\partial}{\partial x_2}\left(\frac{X_2^{(4)}}{X_2}\right). \end{aligned} \tag{2.2.19}$$

Again, equation (2.2.18) separates as $F_1 - \zeta\,F_2 = 0$ and $G_2 + \zeta\,G_1 = 0$. Both ODE's can be treated in two ways. First, by carrying out the differentiation leading to nonlinear equations. Second, by integrating introducing an additional constant. This has the disadvantage that supported boundary conditions will be harder to satisfy, if at all.

3. Linear Partial Differential Equations of order $n = 2$ and $n = 3$ of Mathematical Physics. Let us now illustrate the effectiveness of the developed framework. In what comes next, we present explicit solutions to the following PDE's, expressed in the polar coordinate system: (i) Starting with the important biharmonic equation which arises in many physical problems concerning the linear theory of elasticity, the creeping flow of a viscous incompressible fluid and other, we "move" an order higher to the (ii) triharmonic equation which appears, e.g. in the theory of rotating viscous fluids. We continue exploring the (iii) 2-Helmholtz equation originating from the study of vibrations in thin elastic plates and we complete our expedition with the (iv) 3-Helmholtz equation, which to the authors knowledge has not yet a physical interpretation.

3.1. The biharmonic equation $\Delta^2 u = 0$ in polar coordinates. In the polar coordinate system the biharmonic (2-harmonic or h_2) equation transforms, after lengthy manipulations into

$$r^4 \frac{\partial^4 u}{\partial r^4} + 2r^3 \frac{\partial^3 u}{\partial r^3} - r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + 2r^2 \frac{\partial^4 u}{\partial r^2 \partial \theta^2} - 2r \frac{\partial^3 u}{\partial r \partial \theta^2} + \frac{\partial^4 u}{\partial \theta^4} + 4 \frac{\partial^2 u}{\partial \theta^2} = 0 \tag{3.1.1}$$

given that

$$\Delta^2 = \frac{\partial^4}{\partial r^4} + \frac{2}{r} \frac{\partial^3}{\partial r^3} - \frac{1}{r^2} \frac{\partial^2}{\partial r^2} + \frac{1}{r^3} \frac{\partial}{\partial r} + \frac{2}{r^2} \frac{\partial^4}{\partial r^2 \partial \theta^2} - \frac{2}{r^3} \frac{\partial^3}{\partial r \partial \theta^2} + \frac{1}{r^4} \frac{\partial^4}{\partial \theta^4} + \frac{4}{r^4} \frac{\partial^2}{\partial \theta^2} . \tag{3.1.2}$$

The first author assessed presenting an explicit solution of (3.1.1) as

$$\begin{aligned} u(r,\theta) =& a_0 \ln r + b_0 \, r^2 + c_0 \, r^2 \ln r + d_0 \, r^2 \theta + e_0 \, \theta \\ &+ (a_1 \, r + b_1 \, r\theta + c_1 \, r^3 + d_1 \, r^{-1} + e_1 \, r \ln r) \cos\theta \\ &+ (a_2 \, r + b_2 \, r\theta + c_2 \, r^3 + d_2 \, r^{-1} + e_2 \, r \ln r) \sin\theta \\ &+ \sum_{n=2}^{\infty} (f_n \, r^n + g_n \, r^{n+2} + h_n \, r^{-n} + i_n \, r^{-n+2}) \cos n\theta \\ &+ \sum_{n=2}^{\infty} (j_n \, r^n + k_n \, r^{n+2} + l_n \, r^{-n} + m_n \, r^{-n+2}) \sin n\theta \end{aligned} \tag{3.1.3}$$

was John H. Michell [20], whence the term $r^2\theta$ was added to the solution by Timoshenko [27]. However, neither Michell nor Timoshenko specified the mathematical procedure for arriving at their results. Nonetheless, the solution (3.1.3) can be obtained via a repeated application of the variables separation technique by assuming $u(r,\theta) = rR(r)\Theta(\theta)$ [13] or $u(r,\theta) = Ke^{\lambda \ln r}\Theta(\theta)$ [8] and $\Theta(\theta)$ represented by $e^{\pm in\theta}$.

In order to address the effectiveness of the generalized variable separation technique, the investigation of the plane elasticity problem in polar coordinates leading to expression (3.1.3) is summarized in the sequel. Solving (3.1.1) by assuming $u(r,\theta) = R(r)\Theta(\theta)$ yields the following equation

$$\Theta'''' + 2R_1\Theta'' + R_2\Theta = 0 \tag{3.1.4}$$

where

$$R_1 = \frac{r^2 R'' - rR' + 2R}{R} \tag{3.1.5}$$

$$R_2 = \frac{r^4 R'''' + 2r^3 R''' - r^2 R'' + rR'}{R}\,. \tag{3.1.6}$$

Eq.(3.1.4) is derived noting that $\Delta u(r,\theta)$ furnishes two functions which share the same structure with the assumed solution. Therefore, the outcome of the application of Laplace's operator on the newly introduced functions can be calculated according to $\Delta u(r,\theta)$. The variables can be separated by eliminating the first term of the left-hand side of (3.1.4). Two cases are distinguished, either R_1' equals zero or not. In the second situation we obtain a second-order homogeneous differential equation with constant coefficients for $\Theta(\theta)$ as well as a fifth-order homogeneous differential equation of the Euler type for $R(r)$. However, one avoids the last ODE by expressing Θ'''' and Θ'' in terms of Θ so that Eq.(3.1.4) becomes a fourth-order homogeneous differential equation of the Euler type for $R(r)$. Inspecting the characteristic polynomial associated with Eq.(3.1.4) provides three sets of solutions. In the first instance, i.e. $R_1' = 0$, we arrive at two coupled equations for $R(r)$, namely

$$\left.\begin{array}{rcl} r^2 R'' - rR' + (2-c_1)R & = & 0 \\ r^4 R'''' + 2r^3 R''' - r^2 R'' + rR' - c_2 R & = & 0 \end{array}\right\}. \tag{3.1.7}$$

The real constants c_1 and c_2 must be chosen so that the foregoing two equations will have common solutions. Finally, the solution for $R(r)$ is obtained via (3.1.7) whereas the solution for $\Theta(\theta)$ from (3.1.4) by replacing the common solutions obtained through (3.1.7) using the compatibility condition for the constants c_1 and c_2. The "general" solution (3.1.3) is given by superpositioning all the particular solutions retrieved.

It is noteworthy mentioning that additional solutions of Eq.(3.1.1), which involve the terms $r^2\theta\ln r$, $\theta\ln r$, $r\theta\ln r\sin\theta$, $\sin(m\ln r)\sinh(m\theta)$, $r^2\sin(m\ln r)\sinh(m\theta)$, (including corresponding terms with cos and cosh), can be found in the book by C.B. Biezeno and R. Grammel [1]. These "complementary" solutions are acquired by a thorough examination of the characteristic polynomials linked with (3.1.4) (see for example [26]).

It is readily observed that if the origin of the coordinate system is located interior of the body, terms including $\ln r$ lead to multivalued stresses. These terms play an important role in the elastic theory of dislocations [18, 3].

In what follows, the derivation of (3.1.3) will be addressed employing the separation ansatz announced in section 3. In particular, replacing the unknown function $u(r,\theta)$ with a product of two functions for each of the variables, equation (3.1.1) reads

$$r^4\frac{R''''}{R} + 2r^3\frac{R'''}{R} - r^2\frac{R''}{R} + r\frac{R'}{R} + 2r^2\frac{R''}{R}\frac{\Theta''}{\Theta} - 2r\frac{R'}{R}\frac{\Theta''}{\Theta} + \frac{\Theta''''}{\Theta} + 4\frac{\Theta''}{\Theta} = 0\,. \tag{3.1.8}$$

Differentiating the latter with respect to r and θ we find

$$\left[\frac{\partial}{\partial r}\left(r^2\frac{R''}{R} - r\frac{R'}{R}\right)\right]\left(\frac{\partial}{\partial\theta}\frac{\Theta''}{\Theta}\right) = 0 \tag{3.1.9}$$

which implies that either

$$r^2\frac{R''}{R} - r\frac{R'}{R} = \lambda, \quad \lambda \in \mathbb{C} \tag{3.1.10}$$

and/or

$$\frac{\Theta''}{\Theta} = \tau, \quad \tau \in \mathbb{C}\,. \tag{3.1.11}$$

Introducing $\lambda = \nu(\nu-2)$ and $\tau = -\mu^2$, the solutions to Eqs. (3.1.10) and (3.1.11) are easily computed as $r^{\nu}, r^{-(\nu-2)}$ for the radial dependence and $e^{\pm i\mu\theta}$ for the angular part of the solution $u(r,\theta)$.

Remark 3.1. *The identification of the separation constants with complex variables bear a significant advantage. Allow for* $\nu = (\nu_1,\nu_2)$ *and* $\mu = (\mu_1,\mu_2)$ *so as to the radial dependence simplifies as* $r^{\nu} = r^{\nu_1}\exp(i\nu_2\ln r)$, *whereas the angular dependence gives* $\exp(\pm i\mu\theta) = \exp(\pm i\mu_1\theta)\exp(\mp\mu_2\theta)$, *thus incorporating trigonometric along with hyperbolic functions. The real numbers* ν_1 *and* ν_2 *are easily determined to be* $\nu_1 = 1\pm\sqrt{\rho}\cos\frac{\phi}{2}$ *and* $\nu_2 = \pm\sqrt{\rho}\sin\frac{\phi}{2}$, *where the parameters* ρ *and* ϕ *are related to the initial separation constant as* $\rho = \sqrt{(1+\lambda_1)^2+\lambda_2^2}$ *and* $\phi = \arctan\frac{\lambda_2}{1+\lambda_1}$, *provided that* $\lambda_1 = Re\lambda, \lambda_2 = Im\lambda$.

Substituting the radial solutions into Eq. (3.1.8) gives

$$\Theta'''' + 2[\nu(\nu-2)+2]\Theta'' + \nu^2(\nu-2)^2\Theta = 0 \tag{3.1.12}$$

which remains invariant replacing ν by $-\nu+2$.

The solutions to the latter are $\exp(\pm i\nu\theta)$ and $\exp(\pm i(\nu-2)\theta)$, providing the following sets of solutions for $u(r,\theta)$ (non-essential constants are omitted)

$$\left\{r^{\nu}, r^{-\nu+2}\right\} \cup \left\{e^{\pm i\nu\theta}, e^{\pm i(\nu-2)\theta}\right\}\,. \tag{3.1.13}$$

On the assumption that $\lambda = 0$ (equals the case $\nu = 0$ or $\nu = 2$) we find

$$\left\{1, r^2\right\} \cup \left\{1, \theta, e^{\mp i2\theta}\right\}\,. \tag{3.1.14}$$

On the other hand, substituting the angular solutions into Eq. (3.1.8) yields

$$r^4R'''' + 2r^3R''' - (2\mu^2+1)r^2R'' + (2\mu^2+1)rR' + \mu^2(\mu^2-4)R = 0 \tag{3.1.15}$$

which remains invariant replacing μ by $-\mu$.

Above equation is of Euler type and solved in a straightforward manner, contributing the subsequent eigensolutions of $u(r,\theta)$

$$\left\{e^{\pm i\mu\theta}\right\} \cup \left\{r^{\pm\mu}, r^{\pm\mu+2}\right\}\,. \tag{3.1.16}$$

In addition, when $\mu = 0$ we get

$$\{1,\theta\} \cup \left\{1, r^2, \ln r, r^2\ln r\right\}\,. \tag{3.1.17}$$

We straightforwardly notice that (3.1.17) recovers the first line of (3.1.3) as well as the terms $r^2\theta\ln r$ and $\theta\ln r$. More involved terms are obtained by properly manipulating the complex variables ν and μ. For example, in the limit $\nu \in \mathbb{Z}^+$ the series solutions (omitting non-essential constants)

$$\sum_{n=0}^{\infty}\left(r^n + r^{-n+2}\right)e^{\pm in\theta}, \quad e^{\mp i2\theta}\sum_{n=0}^{\infty}\left(r^n + r^{-n+2}\right)e^{\pm in\theta}, \quad \sum_{n=0}^{\infty}e^{\pm in\ln r}e^{\mp n\theta}\,, \tag{3.1.18}$$

are derived, the last via a 90° rotation.

3.2. The h_3−equation $\Delta^3 u = 0$ in polar coordinates.

Proposition 3.2. *The function (omitting non-essential multipliers)*

$$\begin{aligned}u(r,\theta) =& r^2(1+\theta+\ln r) + (\theta+\ln r+\theta\ln r)\sum_{k=0}^{2}(\cos k\theta+\sin k\theta)\, r^k \\ &+ (1+r^2+r^4)\sum_{n=0}^{\infty}(r^n+r^{-n})\cos n\theta + (1+r^2+r^4)\sum_{n=1}^{\infty}(r^n+r^{-n})\sin n\theta \\ &+ (1+r^2+r^4)\sum_{n=0}^{\infty}(\cos(n\ln r)+\sin(n\ln r))\cosh n\theta \\ &+ (1+r^2+r^4)\sum_{n=0}^{\infty}(\cos(n\ln r)+\sin(n\ln r))\sinh n\theta \end{aligned} \tag{3.2.1}$$

satisfies the 3-harmonic equation.

The 3-Laplacian in polar coordinates is expressed as

$$\begin{aligned}\Delta^3 =& \frac{\partial^6}{\partial r^6} + \frac{3}{r}\frac{\partial^5}{\partial r^5} - \frac{3}{r^2}\frac{\partial^4}{\partial r^4} + \frac{6}{r^3}\frac{\partial^3}{\partial r^3} - \frac{9}{r^4}\frac{\partial^2}{\partial r^2} + \frac{9}{r^5}\frac{\partial}{\partial r} \\ &+ \frac{3}{r^2}\frac{\partial^6}{\partial r^4\partial\theta^2} - \frac{6}{r^3}\frac{\partial^5}{\partial r^3\partial\theta^2} + \frac{21}{r^4}\frac{\partial^4}{\partial r^2\partial\theta^2} - \frac{45}{r^5}\frac{\partial^3}{\partial r\partial\theta^2} \\ &+ \frac{1}{r^6}\frac{\partial^6}{\partial\theta^6} + \frac{20}{r^6}\frac{\partial^4}{\partial\theta^4} + \frac{64}{r^6}\frac{\partial^2}{\partial\theta^2} + \frac{3}{r^4}\frac{\partial^6}{\partial r^2\partial\theta^4} - \frac{9}{r^5}\frac{\partial^5}{\partial r\partial\theta^4}\,. \end{aligned} \tag{3.2.2}$$

If the action of the above operator on a unknown function $u(r,\theta)$ vanishes, the h_3−equation

$$\begin{aligned}& r^6\frac{R^{(6)}}{R} + 3r^5\frac{R^{(5)}}{R} - 3r^4\frac{R^{(4)}}{R} + 6r^3\frac{R^{(3)}}{R} - 9r^2\frac{R''}{R} + 9r\frac{R'}{R} + \left(3r^4\frac{R^{(4)}}{R} - 6r^3\frac{R^{(3)}}{R}\right. \\ &\left.+ 21r^2\frac{R''}{R} - 45r\frac{R'}{R} + 64\right)\frac{\Theta''}{\Theta} + \frac{\Theta^{(6)}}{\Theta} + \left(3r^2\frac{R''}{R} - 9r\frac{R'}{R} + 20\right)\frac{\Theta^{(4)}}{\Theta} = 0 \end{aligned} \tag{3.2.3}$$

is derived, where we replaced the unknown function $u(r,\theta)$ by the product $R(r)\Theta(\theta)$ and divided throughout by the same product.

Terms which depend solely on a single variable are eliminated by differentiating Eq.(3.2.3) with respect to both variables r and θ, yielding

$$R_1\Theta'''' + R_2\Theta'' = 0 \tag{3.2.4}$$

where

$$R_1 = r^2R'' - 3rR' \tag{3.2.5}$$

and

$$R_2 = r^4R'''' - 2r^3R''' + 7r^2R'' - 15rR'\,. \tag{3.2.6}$$

Let us begin with the case where $R_1 \neq 0$. Dividing both members of (3.2.4) by $R_1\Theta''$ separates the variables as

$$r^4R'''' - 2r^3R''' - (\lambda-7)r^2R'' + 3(\lambda-5)rR' = 0 \tag{3.2.7}$$

and

$$\Theta'''' + \lambda\Theta'' = 0 \tag{3.2.8}$$

where λ is the arbitrary complex separation constant.

Introducing $\lambda = \nu(\nu - 4) + 8$ the solutions to Eq.(3.2.7) are evaluated as

$$R(r) = \{1, r^4, r^\nu, r^{-\nu+4}\} . \tag{3.2.9}$$

Placing either $R(r) = 1$ or $R(r) = r^4$ back into the triharmonic equation (3.2.3) gives

$$\Theta^{(6)} + 20\Theta^{(4)} + 64\Theta^{(2)} = 0 \tag{3.2.10}$$

delivering the unknown function $u(r, \theta)$ as

$$\left\{1, r^4\right\} \cup \left\{1, \theta, e^{\pm i2\theta}, e^{\pm i4\theta}\right\} . \tag{3.2.11}$$

On the other hand, identifying $R(r) = r^\nu$ furnishes

$$\Theta^{(6)} + \left(3(\nu-2)^2 + 8\right)\Theta^{(4)} + \left(3(\nu-2)^4 + 16\right)\Theta^{(2)} + \nu^2(\nu-2)^2(\nu-4)^2\Theta = 0 \tag{3.2.12}$$

remaining invariant if ν is replaced by $-\nu + 4$.

Solving the latter provides the following set for the function $u(r, \theta)$

$$\left\{r^\nu, r^{-\nu+4}\right\} \cup \left\{e^{\pm i\nu\theta}, e^{\pm i(\nu-2)\theta}, e^{\pm i(\nu-4)\theta}\right\} . \tag{3.2.13}$$

Reproducing above steps for Eq.(3.2.8) returns

$$\left\{1, \ln r, r^2, r^2 \ln r, r^4, r^4 \ln r\right\} \cup \{1, \theta\} \tag{3.2.14}$$

together with (modifying the separation constant to simplify calculations)

$$\left\{r^{\pm\eta}, r^{\pm\eta+2}, r^{\pm\eta+4}\right\} \cup \left\{e^{\pm i\eta\theta}\right\} \tag{3.2.15}$$

as suitable solutions $u(r, \theta)$.

Examining in the sequel the instance $\lambda = 0$ we arrive at

$$R(r) = \left\{1, r^4, r^2 e^{\pm i2\ln r}\right\}, \qquad \Theta(\theta) = \left\{1, \theta, \theta^2, \theta^3\right\} .$$

The first two solutions of each set have already been considered (see (3.2.11) and (3.2.14), respectively). If $R(r) = r^2 \exp(\pm i2 \ln r)$ we find for $u(r, \theta)$

$$\left\{r^2 e^{\pm i2\ln r}\right\} \cup \left\{e^{\pm 2\theta}, e^{\pm 2\theta} e^{\pm i2\theta}\right\} . \tag{3.2.16}$$

Postulating that $\Theta(\theta)$ equals θ^2 or θ^3 results in an ODE for R implicating the variable θ violating our opening assumption that R is a function of the variable r alone. Let us now turn to the case $R_1 = 0$, namely $r^2R'' - 3rR' = 0$ accepting 1 and r^4 as solutions, satisfying R_2 as well. It remains that $\Theta'' = 0$ providing the set (3.2.14).

Accurately managing the complex separation parameters presented, more involved terms are collected leading to relation (3.2.1).

3.3. The 2-Helmholtz equation $(\Delta^2+\kappa)u=0$ **in polar coordinates.** The study of free transverse vibrations of thin elastic plates involves the differential equation $\Delta^2 w(\mathbf{x},t)-\alpha\,\partial^2 w(\mathbf{x},t)/\partial t^2=0$ governing the transverse displacement w of the plate. The constant α incorporates the physical parameters of the problem. Separating the latter yields the modified 2-Helmholtz equation

$$\Delta^2 u(\mathbf{x})-\kappa u(\mathbf{x})=0 \tag{3.3.1}$$

where κ is the separation constant.

A solution to this equation is given by $u(\mathbf{x})=u_1(\mathbf{x})+u_2(\mathbf{x})$, where u_1 and u_2 satisfy, respectively, the differential equations

$$\Delta u_1(\mathbf{x})+\sqrt{\kappa}u_1(\mathbf{x})=0 \tag{3.3.2}$$

and

$$\Delta u_2(\mathbf{x})-\sqrt{\kappa}u_2(\mathbf{x})=0 \tag{3.3.3}$$

recognized as the Helmholtz and modified Helmholtz equations, respectively.

Equations (3.3.2) and (3.3.3) accept as solutions the Bessel and modified Bessel functions of the first and second kind J_ν, Y_ν, I_ν and K_ν, as defined and considered in [30]. Thus, a solution of the 2-Helmholtz equation (3.3.1) in polar coordinates is

$$u(r,\theta)=\Big(a\,J_\nu(\kappa^{1/4}r)+b\,Y_\nu(\kappa^{1/4}r)+c\,I_\nu(\kappa^{1/4}r)+d\,K_\nu(\kappa^{1/4}r)\Big)\,e^{\pm i\nu\theta} \tag{3.3.4}$$

where a,b,c,d are arbitrary constants.

Solutions satisfying the Bessel (or modified Bessel) differential equation are frequently called cylinder functions. For future reference we classify them as 1-cylinder or simply c_1 functions. Also, one must add to the solutions of (3.3.2) the Hankel functions of the first $H_\nu^{(1)}(\kappa^{1/4}r)$ and second $H_\nu^{(2)}(\kappa^{1/4}r)$ kind, respectively.

The equation $\Delta^2 u(r,\theta)+\kappa u(r,\theta)=0$ with the help of (3.1.2) reads as

$$\begin{aligned} &r^4\frac{\partial^4 u}{\partial r^4}+2r^3\frac{\partial^3 u}{\partial r^3}-r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+2r^2\frac{\partial^4 u}{\partial r^2\partial\theta^2}\\ &-2r\frac{\partial^3 u}{\partial r\partial\theta^2}+\frac{\partial^4 u}{\partial\theta^4}+4\frac{\partial^2 u}{\partial\theta^2}+r^4\kappa u=0\,. \end{aligned} \tag{3.3.5}$$

Adopting the proposed procedure yields equation (3.1.9) from which the differential equations (3.1.10) and (3.1.11) follow. Replacing the angular part of the solution $u(r,\theta)$, namely $e^{\pm i\mu\theta}$, into (3.3.5) provides

$$z^4y''''+2z^3y'''-(2\mu^2+1)z^2y''+(2\mu^2+1)zy'+\left[z^4+\mu^2(\mu^2-4)\right]y=0 \tag{3.3.6}$$

where the variable $z=\kappa^{1/4}r$ has been introduced.

A fourth order differential equation comparable to (3.3.6) has been announced by Everitt and Markett in 1994 [10] based on the pioneering work of H.L. Krall [16] in the late 1930's. Unlike Eq.(3.3.6), obtained by a separation of variables procedure, the Everitt-Markett equation

$$z^4y''''+2z^3y'''-\left(9+\frac{8z^2}{M}\right)z^2y''+\left(9-\frac{8z^2}{M}\right)zy'-\Lambda z^4y=0\,, \tag{3.3.7}$$

where $M \in (0, +\infty)$ and $\Lambda \in \mathbb{C}$ parameters, is derived through a limit process connected to the properties of the classical Bessel differential equation (see [5] and references therein). A linearly independent set of solutions of Eq.(3.3.7) has been presented recently [12, 11]. Evidently, these functions can be defined in terms of low order Bessel and modified functions, as

$$J_\lambda^{0,M}(z) = \left(1 + \frac{1}{4}M\lambda^2\right) J_0(\lambda z) - \frac{1}{2}M\lambda z^{-1} J_1(\lambda z),$$

$$Y_\lambda^{0,M}(z) = \left(1 + \frac{1}{4}M\lambda^2\right) Y_0(\lambda z) - \frac{1}{2}M\lambda z^{-1} Y_1(\lambda z),$$

$$I_\lambda^{0,M}(z) = -\left(1 + \frac{1}{4}M\lambda^2\right) I_0(\sqrt{8M^{-1} + \lambda^2} z) + \frac{M}{2}\sqrt{8M^{-1} + \lambda^2} z^{-1} I_1(\sqrt{8M^{-1} + \lambda^2} z),$$

$$K_\lambda^{0,M}(z) = -\left(1 + \frac{1}{4}M\lambda^2\right) K_0(\sqrt{8M^{-1} + \lambda^2} z) - \frac{M}{2}\sqrt{8M^{-1} + \lambda^2} z^{-1} K_1(\sqrt{8M^{-1} + \lambda^2} z)$$

and the parameters Λ and λ are connected via $\Lambda = \lambda^2(8M^{-1} + \lambda^2)$. Regarding the additional solutions $I_\lambda^{0,M}$ and $K_\lambda^{0,M}$, they were found employing a computer algebra program [28]. However, of particular interest is the limit case as M tends to infinity, since the resulting equation resembles (3.3.6) and the authors [12] arrive at four linearly independent solution in terms of the classical Bessel and modified Bessel functions of order 2, viz

$$J_\lambda^{0,\infty}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+2)!} \left(\frac{\lambda z}{2}\right)^{2k+2},$$

$$Y_\lambda^{0,\infty}(z) = -\frac{2}{\lambda^2 z^2} - \frac{1}{2} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+2)!} \left(\frac{\lambda z}{2}\right)^{2k+2} \left[2\log\left(\frac{\lambda z}{2}\right) - \psi(k+1) - \psi(k+3)\right],$$

$$I_\lambda^{0,\infty}(z) = \sum_{k=0}^{\infty} \frac{1}{k!(k+2)!} \left(\frac{\lambda z}{2}\right)^{2k+2},$$

$$K_\lambda^{0,\infty}(z) = \frac{2}{\lambda^2 z^2} - \frac{1}{2} - \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+2)!} \left(\frac{\lambda z}{2}\right)^{2k+2} \left[2\log\left(\frac{\lambda z}{2}\right) - \psi(k+1) - \psi(k+3)\right],$$

where ψ denotes the logarithmic derivative of the $\Gamma-$function.

Utilizing above results, the aforementioned authors provide explicit solutions for the 2-Helmholtz PDE $(\Delta^2 u)(r, \theta) = \Lambda u(r, \theta)$ in the plane, implementing a restricted form of separation [9]. As they report, application of this quasi separation has a high cost, namely leading to an angular differential equation incorporating a *fixed* separation constant, instead of an arbitrary parameter.

The generalized reduction method, in the sense that a given PDE is reduced to a system of ODE's, avoids this obstacle almost trivially, leading to Eq.(3.3.6), furnishing, after tedious but straightforward calculations, four solutions as

$$J_{\pm\mu}^{(2,0)}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!\,\Gamma(\pm\mu + 2k + 1)} \left(\frac{z}{2}\right)^{\pm\mu + 4k}, \tag{3.3.8}$$

$$J_{\pm\mu}^{(2,1)}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!\,\Gamma(\pm\mu + 2k + 2)} \left(\frac{z}{2}\right)^{\pm\mu + 4k + 2}. \tag{3.3.9}$$

The corresponding radial solutions satisfying the modified 2-Helmholtz equation (3.3.1) are obtained replacing z with $e^{i\pi/4}z$, namely

$$I_{\pm\mu}^{(2,0)}(z) = \sum_{k=0}^{\infty} \frac{1}{(2k)!\,\Gamma(\pm\mu+2k+1)} \left(\frac{z}{2}\right)^{\pm\mu+4k}, \tag{3.3.10}$$

and

$$I_{\pm\mu}^{(2,1)}(z) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!\,\Gamma(\pm\mu+2k+2)} \left(\frac{z}{2}\right)^{\pm\mu+4k+2}. \tag{3.3.11}$$

It is easily proven that the series solutions (3.3.8)-(3.3.11) converge uniformly.

Comparing the Bessel function of the first kind

$$J_{\pm\mu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\pm\mu+k+1)} \left(\frac{z}{2}\right)^{\pm\mu+2k} \tag{3.3.12}$$

with the derived formulae (3.3.8)-(3.3.11), the following relations hold

$$\begin{aligned} J_{\pm\mu}(z) &= J_{\pm\mu}^{(2,0)}(z) - J_{\pm\mu}^{(2,1)}(z) + 2\sum_{k=0}^{\infty} \frac{(-1)^k}{M_k!\,\Gamma(\pm\mu+M_k+1)} \left(\frac{z}{2}\right)^{\pm\mu+2M_k} \\ &= I_{\pm\mu}^{(2,0)}(z) - I_{\pm\mu}^{(2,1)}(z) \end{aligned}$$

where the integer sequence $M_k = 2, 3 \pmod 4$ is given as

$$M_k = \begin{cases} 2k+2, & \text{if } k \text{ is even} \\ 2k+1, & \text{if } k \text{ is odd} \end{cases}. \tag{3.3.13}$$

In section 4, an alternative representation regarding the "periodic-jump" series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{M_k!\,\Gamma(\pm\mu+M_k+1)} \left(\frac{z}{2}\right)^{\pm\mu+2M_k}$$

which resembles the Bessel function of the first kind (3.3.12), is provided.

3.4. The 3-Helmholtz equation $(\Delta^3+\kappa)u = 0$ in polar coordinates. Consider the 3-Helmholtz equation

$$(\Delta^3 + \kappa)u(\mathbf{x}) = 0 \tag{3.4.1}$$

as well as the corresponding modified equation

$$(\Delta^3 - \kappa)u(\mathbf{x}) = 0\,. \tag{3.4.2}$$

Any solution satisfying the 1-Helmholtz equation $(\Delta + \sqrt[3]{\kappa})u(\mathbf{x}) = 0$ also satisfies Eq.(3.4.1). For example, in the polar coordinate system, any linear combination of Bessel functions $J_\nu, Y_\nu, H_\nu^{(1)}, H_\nu^{(2)}$, multiplied by $\exp(\pm i\nu\theta)$, is a solution of Eq.(3.4.1). Similar, solutions satisfying the modified 1-Helmholtz equation $(\Delta - \sqrt[3]{\kappa})u(\mathbf{x}) = 0$ satisfies Eq.(3.4.2) as well. For instance, any linear combination of modified Bessel functions I_ν, K_ν multiplied by $\exp(\pm i\nu\theta)$ is a solution of Eq.(3.4.2) in polar coordinates.

However, writing out Eq.(3.4.1) with the aid of (3.2.2) and separating variables, yields

$$r^6\frac{R^{(6)}}{R}+3r^5\frac{R^{(5)}}{R}-3r^4\frac{R^{(4)}}{R}+6r^3\frac{R^{(3)}}{R}-9r^2\frac{R''}{R}+9r\frac{R'}{R}+\Bigg(3r^4\frac{R^{(4)}}{R}-6r^3\frac{R^{(3)}}{R}$$
$$+21r^2\frac{R''}{R}-45r\frac{R'}{R}+64\Bigg)\frac{\Theta''}{\Theta}+\frac{\Theta^{(6)}}{\Theta}+\Bigg(3r^2\frac{R''}{R}-9r\frac{R'}{R}+20\Bigg)\frac{\Theta^{(4)}}{\Theta}+\kappa r^6=0 \tag{3.4.3}$$

from which the differential equations (3.2.7) and (3.2.8) follow.

Replacing $\Theta(\theta)$ by $\exp(\pm i\mu\theta)$ in Eq.(3.4.3) furnishes

$$\begin{aligned}&z^6\,y^{(6)}(z)+3z^5\,y^{(5)}(z)-3(\mu^2+1)z^4\,y^{(4)}(z)+6(\mu^2+1)z^3\,y^{(3)}(z)\\&+3(\mu^4-7\mu^2-3)z^2\,y''(z)-9(\mu^4-5\mu^2-1)z\,y'(z)\\&+(z^6-\mu^2(\mu^2-4)(\mu^2-16))y(z)=0\end{aligned} \tag{3.4.4}$$

where the variable $z=\kappa^{1/6}\,r$ has been introduced.

Proposition 3.3. *Six linear independent and uniformly convergent series solutions satisfying* (3.4.4) *are*

$$J^{(3,0)}_{\pm\mu}(z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k)!\,\Gamma(\pm\mu+3k+1)}\left(\frac{z}{2}\right)^{\pm\mu+6k}, \tag{3.4.5}$$

$$J^{(3,1)}_{\pm\mu}(z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+1)!\,\Gamma(\pm\mu+3k+2)}\left(\frac{z}{2}\right)^{\pm\mu+6k+2}, \tag{3.4.6}$$

$$J^{(3,2)}_{\pm\mu}(z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+2)!\,\Gamma(\pm\mu+3k+3)}\left(\frac{z}{2}\right)^{\pm\mu+6k+4}. \tag{3.4.7}$$

Additionally, the solutions satisfying the radial part of the modified 3-Helmholtz equation (3.4.2) *are*

$$I^{(3,0)}_{\pm\mu}(z)=\sum_{k=0}^{\infty}\frac{1}{(3k)!\,\Gamma(\pm\mu+3k+1)}\left(\frac{z}{2}\right)^{\pm\mu+6k}, \tag{3.4.8}$$

$$I^{(3,1)}_{\pm\mu}(z)=\sum_{k=0}^{\infty}\frac{1}{(3k+1)!\,\Gamma(\pm\mu+3k+2)}\left(\frac{z}{2}\right)^{\pm\mu+6k+2}, \tag{3.4.9}$$

$$I^{(3,2)}_{\pm\mu}(z)=\sum_{k=0}^{\infty}\frac{1}{(3k+2)!\,\Gamma(\pm\mu+3k+3)}\left(\frac{z}{2}\right)^{\pm\mu+6k+4}. \tag{3.4.10}$$

obtained, replacing z by $e^{\frac{i\pi}{6}}\,z$.

Furthermore, the following connection formulae between c_1, c_2 and c_3 functions can be established

$$J_{\pm\mu}(z)=J^{(3,0)}_{\pm\mu}(z)-J^{(3,1)}_{\pm\mu}(z)+J^{(3,2)}_{\pm\mu}(z) \tag{3.4.11}$$

$$=I^{(3,0)}_{\pm\mu}(z)+I^{(3,1)}_{\pm\mu}(z)+I^{(3,2)}_{\pm\mu}(z)-2I^{(2,1)}_{\pm\mu}(z)\,. \tag{3.4.12}$$

4. The general n-Helmholtz equation in polar coordinates in context with time dependent phenomena.

PROPOSITION 4.1. *Assume $u \in C^{2n}$. Then*

$$u(r,\theta) = e^{\pm i\mu\theta} \sum_{m=0}^{n-1} J_{\pm\mu}^{(n,m)}(\kappa r), \tag{4.0.1}$$

where

$$J_{\pm\mu}^{(n,m)}(\kappa r) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(nk+m)!\,\Gamma(\pm\mu+nk+m+1)} \left(\frac{\kappa^{1/2n}\, r}{2}\right)^{\pm\mu+2nk+2m} \tag{4.0.2}$$

is a solution of the n-Helmholtz equation

$$(\Delta^n + \kappa)\, u(r,\theta) = 0\,. \tag{4.0.3}$$

Proof. The proof is a straightforward calculation based on a repeated application of the Laplacian. □

On the other hand, n linearly independent solutions satisfying the radial part of the modified n-Helmholtz equation $(\Delta^n - \kappa)\, u(r,\theta) = 0$ are furnished replacing r by $e^{i\pi/2n}\, r$, namely

$$I_{\pm\mu}^{(n,m)}(\kappa r) = \sum_{k=0}^{\infty} \frac{1}{(nk+m)!\,\Gamma(\pm\mu+nk+m+1)} \left(\frac{\kappa^{1/2n}\, r}{2}\right)^{\pm\mu+2nk+2m}, \quad m = 0,1,2,\ldots,n-1\,. \tag{4.0.4}$$

Additional solutions are obtained by properly factoring the $n-$Laplacian Δ^n (as seen, for example, in sections 3.3 and 3.4).

Further, taking advantage of the notation (4.0.2) and (4.0.4), the "periodic-jump" series, presented in section 3.3, can be expressed as

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{M_k!\,\Gamma(\pm\mu+M_k+1)} \left(\frac{z}{2}\right)^{\pm\mu+2M_k} = I_{\pm\mu}^{(4,2)}(z) - I_{\pm\mu}^{(4,3)}(z)\,. \tag{4.0.5}$$

It is straightforward to expand the results obtained so far to (linear) phenomena which change with time, by proposing the general operator

$$\Box_n^p(\boldsymbol{\alpha}_\ell|\,\boldsymbol{\beta}_j) \overset{def.}{=} \sum_{\ell=0}^{n} \alpha_\ell(\mathbf{x})\,\Delta^\ell + \sum_{j=0}^{p} \beta_j(\mathbf{x})\,\frac{\partial^j}{\partial t^j}\,, \tag{4.0.6}$$

where $\boldsymbol{\alpha}_\ell = (\alpha_0(\mathbf{x}), \alpha_1(\mathbf{x}), \ldots, \alpha_n(\mathbf{x}))$, and $\boldsymbol{\beta}_j = (\beta_0(\mathbf{x}), \beta_1(\mathbf{x}), \ldots, \beta_p(\mathbf{x}))$.

Conclusively, more involved partial differential equations *incorporating* mixed derivatives can be solved *explicitly* performing separation of variables as demonstrated. We illustrate this statement further by taking steps beyond section 3, providing the exact solutions of the first kind to the 3-metaharmonic equation $\sum_{\ell=0}^{3} \alpha_\ell \Delta^\ell\, u = 0$ for simple values of the coefficients α_ℓ, $\ell = 0,1,2,3$, displayed in Table 1.

Table 1: Explicit solutions of Equations $(\alpha_3\Delta^3 + \alpha_2\Delta^2 + \alpha_1\Delta + \alpha_0)\, u(r,\theta) = 0$ (omitting non-essential constants).

	α_3	α_2	α_1	α_0	$u(r,\theta)$	Details can be found where
1.	0	0	1	κ	$J^{(1,0)}_{\pm\mu}(\kappa^{1/2}r)\, e^{\pm i\mu\theta}$	[30]
2.	0	0	1	$-\kappa$	$I^{(1,0)}_{\pm\mu}(\kappa^{1/2}r)\, e^{\pm i\mu\theta}$	
3.	0	1	0	κ	$J^{(2,0)}_{\pm\mu}(\kappa^{1/4}r)\, e^{\pm i\mu\theta}$	Section 3.3
					$J^{(2,1)}_{\pm\mu}(\kappa^{1/4}r)\, e^{\pm i\mu\theta}$	
4.	0	1	0	$-\kappa$	$I^{(2,0)}_{\pm\mu}(\kappa^{1/4}r)\, e^{\pm i\mu\theta}$	
					$I^{(2,1)}_{\pm\mu}(\kappa^{1/4}r)\, e^{\pm i\mu\theta}$	
5.	1	0	0	κ	$J^{(3,0)}_{\pm\mu}(\kappa^{1/6}r)\, e^{\pm i\mu\theta}$	Section 3.4
					$J^{(3,1)}_{\pm\mu}(\kappa^{1/6}r)\, e^{\pm i\mu\theta}$	
					$J^{(3,2)}_{\pm\mu}(\kappa^{1/6}r)\, e^{\pm i\mu\theta}$	
6.	1	0	0	$-\kappa$	$I^{(3,0)}_{\pm\mu}(\kappa^{1/6}r)\, e^{\pm i\mu\theta}$	
					$I^{(3,1)}_{\pm\mu}(\kappa^{1/6}r)\, e^{\pm i\mu\theta}$	
					$I^{(3,2)}_{\pm\mu}(\kappa^{1/6}r)\, e^{\pm i\mu\theta}$	

Continued on next page

Table 1 – continued from previous page

	α_3	α_2	α_1	α_0	$u(r,\theta)$	Details can be found where
7.	0	1	1	0	$J^{(1,0)}_{\pm\mu}(r)\, e^{\pm i\mu\theta}$ $J^{(1,1)}_{\pm\mu}(r)\, e^{\pm i\mu\theta}$	Appendix A
8.	0	1	-1	0	$I^{(1,0)}_{\pm\mu}(r)\, e^{\pm i\mu\theta}$ $I^{(1,1)}_{\pm\mu}(r)\, e^{\pm i\mu\theta}$	
9.	1	0	1	0	$J^{(2,0)}_{\pm\mu}(r)\, e^{\pm i\mu\theta}$ $J^{(2,1)}_{\pm\mu}(r) e^{\pm i\mu\theta}$ $J^{(2,2)}_{\pm\mu}(r)\, e^{\pm i\mu\theta}$	
10.	1	0	-1	0	$I^{(2,0)}_{\pm\mu}(r)\, e^{\pm i\mu\theta}$ $I^{(2,1)}_{\pm\mu}(r) e^{\pm i\mu\theta}$ $I^{(2,2)}_{\pm\mu}(r)\, e^{\pm i\mu\theta}$	
11.	1	1	0	0	$J^{(1,0)}_{\pm\mu}(r)\, e^{\pm i\mu\theta}$ $J^{(1,1)}_{\pm\mu}(r) e^{\pm i\mu\theta}$ $J^{(1,2)}_{\pm\mu}(r)\, e^{\pm i\mu\theta}$	
12.	1	-1	0	0	$I^{(1,0)}_{\pm\mu}(r)\, e^{\pm i\mu\theta}$ $I^{(1,1)}_{\pm\mu}(r) e^{\pm i\mu\theta}$ $I^{(1,2)}_{\pm\mu}(r)\, e^{\pm i\mu\theta}$	
13.	1	1	1	0	$\left(I^{(3,0)}_{\pm\mu}(r) - I^{(3,1)}_{\pm\mu}(r)\right)\, e^{\pm i\mu\theta}$ $\left(I^{(3,1)}_{\pm\mu}(r) - I^{(3,2)}_{\pm\mu}(r)\right)\, e^{\pm i\mu\theta}$ $\left(I^{(3,2)}_{\pm\mu}(r) - I^{(3,3)}_{\pm\mu}(r)\right)\, e^{\pm i\mu\theta}$	
14.	1	-1	1	0	$\left(J^{(3,0)}_{\pm\mu}(r) - J^{(3,1)}_{\pm\mu}(r)\right)\, e^{\pm i\mu\theta}$ $\left(J^{(3,1)}_{\pm\mu}(r) - J^{(3,2)}_{\pm\mu}(r)\right)\, e^{\pm i\mu\theta}$ $\left(J^{(3,2)}_{\pm\mu}(r) - J^{(3,3)}_{\pm\mu}(r)\right)\, e^{\pm i\mu\theta}$	

5. Conclusions. The method of separation of variables (in the sense of determining a set of separable solutions) is one of the earliest approaches in the history of science to solve linear second- as well as higher-order partial differential equations as long as mixed derivatives are not present. If such derivatives exist, the technique of variables separation fails in its "traditional" form. Nonetheless, the present article, by reassessing established concepts provides a "modus operandi" which allows to split a general linear partial differential equation with constant coefficients into a, dimension dependent, number of ordinary differential equations. Although, nowadays computational methods are promoted in order to solve higher-order differential equations, deriving exact solutions bears two significant advantages: (i) They explicitly illustrate how the parameters of the differential equation (and therefore of the problem under consideration) influence on the solution and (ii) they display structure.

Indeed, this is shown in sections 3.3 and 3.4 where series solutions for higher-order Laplacians' are, for the first time, evaluated. This has been achieved, not by the usual factorization of the operator, thus yielding eigensolutions satisfying the corresponding Laplacian as well (see introductory remarks of section 3.3), but implementing a separation ansatz on the initial partial differential equation furnishing "genuine" eigensolutions which *do not* satisfy the corresponding factorized differential equations. More importantly, these special functions share similarities providing the means to create n linearly independent solutions for the *general* $n-$Helmholtz equation $(\Delta^n + \kappa)u = 0$ in polar coordinates.

Separation of variables "offers" not only explicit solutions, but more importantly, it render guidance to the most essential class of special functions, namely the hypergeometric functions ${}_pF_q$. From the eleven orthogonal coordinate systems in which both the Laplace and Helmholtz equation (as well as their higher-order counterparts) separate[1], the ordinary differential equations arising for nine of them are of Fuchsian type with three regular singularities and can be expressed in terms of hypergeometric functions. These ordinary differential equations, if transformed to the corresponding hypergeometric equation, provide the $p + q$ parameters which allow the solution to be expressed in terms of ${}_pF_q$. Admittedly, this is not always an easy task, still, the significance is obvious. Identities, special values and transformation formulas are well recorded [25, 22] and new properties are continuously added.

Appendix A. The equations $(\alpha_3\Delta^3 + \alpha_2\Delta^2 + \alpha_1\Delta + \alpha_0)\,u(r,\theta) = 0$**.** The equation $(\Delta^2 + \Delta)\,u(r,\theta) = 0$ is a special case of the equation $\Box_n^p(\boldsymbol{\alpha}_\ell|\,\boldsymbol{\beta}_j)u = 0$ in the case where $p = 0, \beta_0 = 0$ and $n = 2, \alpha_0 = 0, \alpha_1 = \alpha_2 = 1$. Separating variables yields the differential equations

$$
\begin{aligned}
&r^4R'''' + 2r^3R''' + \left[\pm r^2 - (2\mu^2+1)\right] r^2R'' + \left[\pm r^2 + (2\mu^2+1)\right] rR' \\
&\quad - \left[\pm\mu^2r^2 - \mu^2(\mu^2-4)\right] R = 0 \qquad \text{(A.0.1)}
\end{aligned}
$$

corresponding to $(\Delta^2 \pm \Delta)\,u(r,\theta) = 0$, respectively. However, utilizing Frobenius' method [29], four linear independent and convergent series solutions can be retrieved (see cases 7 and 8 of Table 1).

An analogous development for the equations $(\Delta^3 \pm \Delta)\,u(r,\theta) = 0$ and $(\Delta^3 \pm \Delta^2)\,u(r,\theta) = 0$ reveals that the equivalent radial components are

$$
\begin{aligned}
&r^6R^{(6)} + 3r^5R^{(5)} - 3(\mu^2+1)r^4R'''' + 6(\mu^2+1)r^3R''' + \left[\pm r^4 + 3(\mu^4 - 7\mu^2 - 3)\right] r^2R'' \\
&+ \left[\pm r^4 - 9(\mu^4 - 5\mu^2 - 1)\right] rR' + \left[\mp\mu^2r^4 - \mu^2(\mu^2-4)(\mu^2-16)\right] R = 0 \qquad \text{(A.0.2)}
\end{aligned}
$$

[1] These coordinate systems, according to [21], are: (1) Cartesian, (2) Cylindrical, (3) Elliptic cylinder, (4) Parabolic cylinder, (5) Spherical, (6) Prolate spheroidal, (7) Oblate spheroidal, (8) Parabolic, (9) Paraboloidal, (10) Conical and (11) Ellipsoidal.

as well as

$$
\begin{aligned}
&r^6 R^{(6)} + 3r^5 R^{(5)} + \left[\pm r^2 - 3(\mu^2+1)\right] r^4 R'''' + 2\left[\pm r^2 + 3(\mu^2+1)\right] r^3 R''' \\
&+ \left[\mp(2\mu^2+1)r^2 + 3(\mu^4 - 7\mu^2 - 3)\right] r^2 R'' + \left[\pm(2\mu^2+1)r^2 - 9(\mu^4 - 5\mu^2 - 1)\right] rR' \\
&+ \left[\mp\mu^2(\mu^2-4)r^2 - \mu^2(\mu^4 - 20\mu^2 + 64)\right] R = 0\,, \qquad \text{(A.0.3)}
\end{aligned}
$$

respectively. Again, a straightforward application of Frobenius' method provides the desired solutions (see cases 9-12 of Table 1), augmented by the following sets of solutions

$$
\left\{J_0^{(2,0)}(r), J_0^{(2,1)}(r), J_0^{(2,2)}(r)\right\} \cup \{1,\theta\}, \quad \left\{I_0^{(2,0)}(r), I_0^{(2,1)}(r), I_0^{(2,2)}(r)\right\} \cup \{1,\theta\} \qquad \text{(A.0.4)}
$$

for equations $(\Delta^3 \pm \Delta)\, u(r,\theta) = 0$, respectively, and

$$
\left\{J_0^{(1,0)}(r), J_0^{(1,1)}(r), J_0^{(1,2)}(r)\right\} \cup \{1,\theta\}, \quad \left\{I_0^{(1,0)}(r), I_0^{(1,1)}(r), I_0^{(1,2)}(r)\right\} \cup \{1,\theta\} \qquad \text{(A.0.5)}
$$

for equations $(\Delta^3 \pm \Delta^2)\, u(r,\theta) = 0$, respectively.

Last, but not least, separation for the equations $(\Delta^3 \pm \Delta^2 + \Delta)\, u(r,\theta) = 0$ leads to the subsequent differential equations

$$
\begin{aligned}
&r^6 R^{(6)} + 3r^5 R^{(5)} + \left[\pm r^2 - 3(\mu^2+1)\right] r^4 R'''' + 2\left[\pm r^2 + 3(\mu^2+1)\right] r^3 R''' \\
&+ \left[r^4 \mp (2\mu^2+1)r^2 + 3(\mu^4 - 7\mu^2 - 3)\right] r^2 R'' \\
&+ \left[r^4 \pm (2\mu^2+1)r^2 - 9(\mu^4 - 5\mu^2 - 1)\right] rR' \\
&+ \left[-\mu^2 r^4 \pm \mu^2(\mu^2-4)r^2 - \mu^2(\mu^4 - 20\mu^2 + 64)\right] R = 0\,, \qquad \text{(A.0.6)}
\end{aligned}
$$

which accept "periodic-jump" power series with basis the number $M_k = 0, 1 (\mathrm{mod}\ 3)$, as solutions. Nonetheless, rearranging terms and employing the notation introduced in section 4, a linear combination of $J_{\pm\mu}^{(3,m)}$, $I_{\pm\mu}^{(3,m)}$, $m = 0,1,2,3$ is obtained (see cases 13-14 of Table 1). These solutions have to be further enlarged by a set similar to (A.0.5).

REFERENCES

[1] C. B. Biezeno and R. Grammel, *Theory of Elasticity: Analytical and Experimental Methods*, volume 1 of *Engineering Dynamics*, Blackie and Son, London, 1955.

[2] V. Biletskyy and S. Yaroshko, *A Method of Generalized Separation of Variables for Solving Three-Dimensional Integral Equations*, in "Direct and Inverse Problems of Electromagnetic and Acoustic Wave Theory, Proceedings of XIth International Seminar/Workshop on", pp. 164–168, 2006.

[3] J. L. Bogdanoff, *On the Theory of Dislocations*, J. Appl. Phys., 21 (1950), pp. 1258–1263.

[4] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[5] J. Das, W. N. Everitt, D. B. Hinton, L. L. Littlejohn, and C. Markett, *The fourth-order besseltype differential equation*, Applicable Anal., 83 (2004), pp. 325–362.

[6] M. Doschoris, *Biharmonic functions. Eigensolutions and beyond. Part I: Cylindrical coordinates*, in preparation.

[7] M. Doschoris, *Biharmonic functions. Eigensolutions and beyond. Part II: Spherical coordinates*, in preparation.

[8] A. J. Durelli, E. A. Phillips, and C. H. Tsao, *Introduction to the theoretical and experimental analysis of stress and strain*, McGraw-Hill, New York, 1958.

[9] W. N. Everitt, L. L. Johansson, B. T. Littlejohn, and C. Markett, *Quasi-separation of the biharmonic partial differential equation*, IMA J. Appl. Math, 74 (2009), pp. 685–709.
[10] W. N. Everitt and C. Markett, *On a generalization of bessel functions satisfying higher-order differential equations*, J. Comput. Appl. Math., 54 (1994), pp. 325–349.
[11] W. N. Everitt and C. Markett, *Solution bases for the fourth-order bessel-type and laguerre-type differential equations*, Applicable Anal., 90 (2011), pp. 515–531.
[12] W. N. Everitt, C. Markett, and L. L. Littlejohn, *Properties of the solutions of the fourth-order bessel-type differential equation*, J. Math. Anal. Appl., 359 (2009), pp. 252–264.
[13] M. Filonenko-Borodich, *Theory of Elasticity*, Peace Publishers, Moscow, 1963.
[14] H. P. Greenspan, *The Theory of Rotating Fluids*, Cambridge University Press, 1968.
[15] M. Kline, *Mathematical Thought from Ancient to Modern Times*, volume 2, Oxford University Press, New York, 1990.
[16] H. L. Krall, *On orthogonal polynomials satisfying a certain fourth order differential equation*, The Pennsylvania State College studies, no. 6, State College, Pa., 1940.
[17] B. Lonsdale, M. I. G. Bloor, and M. A. Kelmanson, *An iterative integral-equation method for 6th-order inhomogeneous partial differential equations*, in C.A. Brebbia, R.C. Ertekin, M. Tanaka, and R. Shaw, editors, "Boundary Element Technology XI. Transaction on Modelling and Simulation", volume 15, pp. 369–378. WIT Press, 1996.
[18] E. H. Mann, *An Elastic Theory of Dislocations*, Proc. R. Soc. London Ser. A, 199 (1949), pp. 376–394.
[19] M. H. Martin, *A generalization of the method of separation of variables*, Indiana Univ Math J., 2 (1953), pp. 315–327.
[20] J. H. Michell, *On the direct determination of stress in an elastic solid with application to the theory of plates*, Proc. Lond. Math. Soc., s1-31 (1899), pp. 100–124.
[21] P. Moon and D. E. Spencer, *Field Theory Handbook*, Springer Verlag, Berlin, 2nd edition, 1988.
[22] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors, *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010.
[23] A. D. Polyanin, *Handbook of Linear Partial Differential Equations for Engineers and Scientists*, Chapman and Hall/CRC, New York, 1st edition, 2001.
[24] A. D. Polyanin, *Handbook of Nonlinear Partial Differential Equations*, Chapman and Hall/CRC, New York, 1st edition, 2003.
[25] L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, New York, 1966.
[26] I. H. Stampouloglou and E. E. Theotokoglou, *Additional Separated-Variable Solutions of the Biharmonic Equation in Polar Coordinates*, J. Appl. Mech., 77 (2010), 021003 (8 pages).
[27] S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity*, McGraw-Hill, New York, 3rd edition, 1970.
[28] M. van Hoeij, *Formal solutions and factorization of differential operators with power series coefficients*, J. Symbolic Comput., 24 (1997), pp. 1–30.
[29] Z. X. Wang and D. R. Guo, *Special Functions*, World Scientific, 1989.
[30] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, New York, 2nd edition, 1995.
[31] L. H. You, J. Changa, X. S. Yanga, and J. J. Zhanga, *Solid modelling based on sixth order partial differential equations*, Computer-Aided Design, 43 (2011), pp. 720–729.