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FLOWS AND A TANGENCY CONDITION FOR EMBEDDABLE CR STRUCTURES IN DIMENSION 3*

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Dedicated to Professor Stephen Yau on his 60th birthday

Abstract. We study the fillability (or embeddability) of 3-dimensional CR structures under the geometric flows. Suppose we can solve a certain second order equation for the geometric quantity associated to the flow. Then we prove that if the initial CR structure is fillable, then it keeps having the same property as long as the flow has a solution. We discuss the situation for the torsion flow and the Cartan flow. In the second part, we show that the above mentioned second order operator is used to express a tangency condition for the space of all fillable or embeddable CR structures at one embedded in $\mathbb{C}^2$.

Key words. CR structure, fillable, embeddable, pseudohermitian structure, torsion, Tanaka-Webster curvature, Cartan flow, Torsion flow.

AMS subject classifications. Primary 32G07, 32V30; Secondary 32V20, 32V05.

1. Introduction.

A closed CR manifold M is fillable if M bounds a complex manifold in the smooth (C^∞) sense (i.e. there exists a complex manifold with smooth boundary M, and the complex structure restricts to the CR structure on M). The notion of fillability is weaker than that of embeddability. Recall that a CR manifold is embeddable if it can be embedded in C^N for large N with the CR structure being the one induced from the complex structure of C^N. The embeddability is a special property for 3-dimensional CR manifolds since any closed CR manifold of dimension ≥ 5 is embeddable by a result of Boutet de Monvel ([2]). It is easy to see that a closed embeddable (strongly pseudoconvex) CR 3-manifold is fillable by some well-known results (see the argument on page 543 in [19]).

Conversely, if there exists a smooth strictly plurisubharmonic function defined in a neighborhood of a fillable M, then M is embeddable (see Theorem 5.3 in [19]; in fact, any compact complex surface with nonempty strongly pseudoconvex boundary can be made Stein by deforming it and blowing down any exceptional curves according to [4]).

In this paper, we first investigate the fillability of 3-dimensional CR structures under the (curvature) flows. Let (M, ξ) denote a closed (compact with no boundary) contact 3-manifold with contact bundle (structure) ξ coorientable meaning that both ξ and TM/ξ, the normal bundle of ξ in TM, are orientable. Let $J_{(t)}$ be a family of CR structures compatible with ξ, i.e., $J_{(t)} \in End(\xi)$ satisfying $J_{(t)}^2 = -I$. Set

$$\frac{\partial J_{(t)}}{\partial t} = 2E_{J_{(t)}} \tag{1.1}$$

where $E_{J_{(t)}}$ satisfies

$$E_{J_{(t)}} \circ J_{(t)} + J_{(t)} \circ E_{J_{(t)}} = 0. \tag{1.2}$$

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The most interesting case is to take E_J to be the Cartan (curvature) tensor Q_J (see below for more details). Another intriguing situation is to take E_J to be the torsion tensor.

Choose a (global) contact form θ (exists since TM/ξ is orientable). We will always assume the CR structure J is nondegenerate and (J, θ) is strictly pseudoconvex (see Section 2 for definitions). Associated to (J, θ) we have a unitary coframe $\{\theta^1,\ \theta^{\bar{1}}\}$ and its dual $\{Z_1,\ Z_{\bar{1}}\}$. We then have the torsion (tensor) A_{11} and the Tanaka-Webster curvature W, etc. (see Section 2 for details). We define a second order linear differential operator D_J from real functions to endomorphism fields by

$$D_J f = (f_{,11} + iA_{11} f)\theta^1 \otimes Z_{\bar{1}} + (f_{,\bar{1}\bar{1}} - iA_{\bar{1}\bar{1}} f)\theta^{\bar{1}} \otimes Z_1. \tag{1.3}$$

We have lowered all the upper indices. Let T denote the Reeb vector field associated to θ, i.e., the unique vector field satisfying $\theta(T) = 1$ and $L_T\theta = 0$. We say a closed CR manifold (M, J) is fillable in the convex side with respect to θ if (J, θ) is strictly pseudoconvex and there exists a complex manifold N with boundary $\partial N = M$ such that $\hat{J}$, the (almost) complex structure, restricts to J on M and $\hat{J}(T)$ points inwards to N at the boundary M. In Section 4 we construct an almost complex structure $\check{J}$ on $M \times [0, \tau)$ (see (4.1)).

Theorem A. *Suppose equation (1.1) has a solution for* $0 \le t < \tau$ *with* $(J_{(t)}, \theta)$ *being strictly pseudoconvex. Suppose*

$$J_{(t)} \circ D_{J_{(t)}} f + D_{J_{(t)}} g = E_{J_{(t)}} \tag{1.4}$$

has a solution of real functions (f, g) *defined on* $M \times [0, \tau)$ *with* $f > 0$. *Assume* $J_{(0)}$ *is fillable in the convex side with respect to* θ *and* $\hat{J} = \check{J}$ *at* $M \times \{0\}$. *Then* $J_{(t)}$ *is fillable in the convex side with respect to* θ *for* $0 \le t < \tau$.

Recall that $D_{J_{(t)}} f = \frac{1}{2} L_{X_f} J_{(t)}$ ([11]) in which $X_f = -fT + i(Z_{1(t)}f)Z_{\bar{1}(t)}$ $-i(Z_{\bar{1}(t)}f)Z_{1(t)}$ is the infinitesimal contact diffeomorphism induced by f. So the image of $D_{J_{(t)}}$ describes the tangent space of the orbit of the symmetry group acting on $J_{(t)}$ by the pullback (in this case, the contact diffeomorphisms are our symmetries). Now condition (1.4) means that $E_{J_{(t)}}$ sits in the "complexification" of the infinitesimal orbit of contact diffeomorphism group for all $t \in [0, \tau)$.

Write $u = f + ig$ and $E_{J_{(t)}} = E_{11(t)}\theta^1_{(t)} \otimes Z_{\bar{1}(t)} + E_{\bar{1}\bar{1}(t)}\theta^{\bar{1}}_{(t)} \otimes Z_{1(t)}$. We can then reformulate (1.4) as an equation for u:

$$u_{,11} + iuA_{11(t)} = iE_{11(t)} \tag{1.5}$$

(cf. (4.8), (4.9) in Section 4). Note that equation (1.5) has a solution $u = 1$ for $E_{11(t)} = A_{11(t)}$ and $\alpha = \beta = 0$ by (4.6). Let $A_{J_{(t)}}$ or $A_{J_{(t)},\theta}$ denote the torsion tensor $A_{11(t)}\theta^1_{(t)} \otimes Z_{\bar{1}(t)} + A_{\bar{1}\bar{1}(t)}\theta^{\bar{1}}_{(t)} \otimes Z_{1(t)}$. We have the following corollary.

Corollary B. *Suppose the torsion flow*

$$\frac{\partial J_{(t)}}{\partial t} = 2A_{J_{(t)}} \tag{1.6}$$

has a solution for $0 \le t < \tau$. *Assume* $J_{(0)}$ *is fillable in the convex side (with respect to* θ*) and* $\hat{J}(T) = -\frac{\partial}{\partial t}$ *at* $M \times \{0\}$. *Then* $J_{(t)}$ *is fillable in the convex side (with respect to* θ*) for* $0 \le t < \tau$.

Note that (1.6) may not have a short-time solution for a general smooth initial value. For the case $E_{J_{(t)}} = 0$, $J_{(t)} = J_{(0)}$ for all t and $f = 1$, $g = 0$ ($u = 1$, resp.) is a solution to (1.4) ((1.5), resp.) provided $A_{J_{(0)}} = 0$ (and hence $A_{J_{(t)}} = 0$ for all t). In this case Theorem A is obvious (note that $A_{J_{(0)}} = 0$ implies that $J_{(0)}$ is embeddable, and hence fillable. See the remark after Corollary C). For the case $E_{J_{(t)}} = Q_{J_{(t)}}$, we do have a short-time solution for (1.1) (see [11]). In [11], we study an evolution equation for CR structures $J_{(t)}$ on (M, ξ) according to their Cartan (curvature) tensor $Q_{J_{(t)}}$(see also Section 2):

$$\frac{\partial J_{(t)}}{\partial t} = 2Q_{J_{(t)}}. \tag{1.7}$$

We will often call this evolution equation (1.7) Cartan flow. Since (1.7) is invariant under a big symmetry group, namely, the contact diffeomorphisms, we add a gauge-fixing term on the right-hand side to break the symmetry. The gauge-fixed (called "regularized" in [11]) Cartan flow reads as follows:

$$\frac{\partial J_{(t)}}{\partial t} = 2Q_{J_{(t)}} - \frac{1}{6}D_{J_{(t)}}F_{J_{(t)}}K \tag{1.8}$$

(see [11] or Section 2 for the meaning of notations). Now it is natural to ask the following question:

QUESTION 1.1. *Is the fillability (in the convex side) or embeddability preserved under the (gauge-fixed) Cartan flow (1.7) (or (1.8))?*

An affirmative answer to the above question has an application in determining the topology of the space of all fillable (embeddable, resp.) CR structures. For instance, one can apply such a result plus the convergence of the long time solution to (1.8) (expected for S^3) to prove that the space of all fillable (embeddable, resp.) CR structures on S^3 is contractible (cf. Remark 4.3 in [15]). For other topological applications of solving (1.8), we refer the reader to [10].

In view of Theorem A we make the following conjecture.

CONJECTURE 1.2. *We can find real functions $f \neq 0$ and g (u with $\operatorname{Re} u \neq 0$, resp.) such that $J \circ D_J f + D_J g = Q_J$ ($u_{,11} + iuA_{11} = -Q_{11}$, resp.) for J fillable or embeddable.*

On the other hand, we examine a family of CR structures embedded in $\mathbb{C}^2$ and find its tangent to be of the form $J \circ D_J f + D_J g$ (see Section 6). Let $W_{J,\theta}$ denote the Tanaka-Webster curvature of a pseudohermitian structure (J, θ) (see Section 2 for the definition).

COROLLARY C. *Suppose $J_{(0)}$ is fillable in the convex side with respect to θ with $A_{J_{(0)},\theta} = 0$ and $W_{J_{(0)},\theta} < 0$. Assume $\hat{J} = \check{J}$ at $M \times \{0\}$. Then the solution $J_{(t)}$ to (1.8) with $K = J_{(0)}$ stays fillable in the convex side with respect to θ for a short time.*

The idea of the proof of Corollary C is to show $A_{J_{(t)},\theta} = 0$ for a short time (see Lemma 3.1) and then solve (1.4) for $E_{J_{(t)}} = Q_{J_{(t)}} - \frac{1}{12}D_{J_{(t)}}F_{J_{(t)}}K$ in Theorem A (see Section 3).

The proof of Theorem A is a direct construction of an almost complex structure $\check{J}$ on $M \times [0, \tau)$ (integrable on $M \times (0, \tau)$) so that $\check{J}|_\xi = J_{(t)}$ at $M \times \{t\}$ (see Section 4 for details). Then we glue this complex structure $\check{J}$ with $\hat{J}$, the one induced by the complex surface that $(M, J_{(0)})$ bounds along $M \times \{0\}$ (identified with M). After we obtained the above result, László Lempert pointed out to the author that the existence of a CR vector field T is sufficient to imply the embeddability of the CR structure (see [22]). So by Lemma 3.1 we can remove the condition in Corollary C on the Tanaka-Webster curvature according to [22]. We speculate that the embeddability (or fillability) is preserved under the (gauge-fixed) Cartan flow without any conditions (Question 1.1).

Our method can also be applied to the problem of local embeddability. An obvious case is that $(M, J_{(0)})$ with $A_{J_{(0)},\theta} = 0$ is locally embeddable. The reason is that the above $\check{J}$ is integrable on $M \times (-\tau, \tau)$ since the torsion flow (1.6) has an obvious solution $J_{(t)} = J_{(0)}$ for $t \in (-\tau, \tau)$ (with $A_{J_{(t)}} = A_{J_{(t)},\theta} = 0$). We single it out as a corollary.

Corollary D. *$(M, J_{(0)})$ with $A_{J_{(0)},\theta} = 0$ is locally embeddable.*

Another remark is that we may couple equation (1.1) with an evolution equation in contact form θ :

$$\frac{\partial \theta_{(t)}}{\partial t} = 2h\theta_{(t)}. \tag{1.9}$$

This does not affect the condition (1.4) for integrability of $\check{J}$. For instance, we may take $h = W_{J_{(t)},\theta_{(t)}}$ to couple with the (pure) torsion flow (1.6). (this coupled torsion flow is the negative gradient flow of the pseudohermitian Einstein-Hilbert action : $-\int W_{J,\theta}\theta \wedge d\theta$). See a recent paper [7] for more information about this flow.

One may speculate that $J \circ D_J f + D_J g$ is a tangent at J of the space of all embeddable or fillable (compatible) CR structures on a fixed contact 3-manifold. Starting from Section 5, we will justify this statement for J associated to a real hypersurface embedded in $\mathbb{C}^2$.

As we mentioned in the beginning, the (global) embeddability of a compact (strongly pseudoconvex) CR manifold (of hypersurface type) is of special interest in dimension 3 since it is always embeddable for dimension ≥ 5 and not always so for dimension $= 3$. The analytic reason is that the operator $\bar{\partial}_b$ associated to the concerned CR structure is solvable for type (0,1)-form when dimension ≥ 5 and the space is compact ([16], [2]). In dimension 3, the analysis of $\bar{\partial}_b$ is delicate. Lewy's example ([23]) tells that there are no solutions at all for the equation $\bar{\partial}_b u = \psi$ even with certain C^∞ functions ψ (or, say, (0,1)-form). Using ψ, Nirenberg ([24]) was able to find an example which is not embeddable locally. On the other hand, an example of real-analytic perturbation of S^3 was constructed ([26], [1]), which (surely is locally embeddable) is not globally embeddable. It is now understood that CR structures with "exotic" underlying contact structures are generally non-embeddable ([14]). In [3], Burns and Epstein made a detailed study on perturbations of the standard CR structure on the three sphere. They gave a "pointwise" criterion for embeddability in terms of the spectrum of $\Box_b$. Also they showed that structures which are infinitesimally obstructed can not be embedded as "small perturbations" of the standard sphere. In fact, an explicit decomposition of the tangent space T_0 to the perturbations of S^3 is given in their paper:

$$T_0 = N \oplus E \oplus O$$

where O is tangent to the orbit of symmetries (in this case, they are contact diffeomorphisms of S^3), E is tangent to a family of perturbations all of which are embedded in $\mathbb{C}^2$ as small perturbations of S^3, and N is tangent to a family of structures which generically embed in no $\mathbb{C}^n$, and none of which can embed in $\mathbb{C}^2$ as small perturbations of S^3. Later J. Bland, T. Duchamp and L. Lempert studied the case of CR structures induced by strictly linearly convex domains extending the above case.

In general, we want to give a qualitative description of a11 embeddable CR structures near an embedded one in smooth tame or Banach category. Let (M, ξ) denote a closed (compact with no boundary) contact 3-manifold.with contact structure ξ coorientable. The set of all CR structures compatible with ξ is denoted by $\mathfrak{J}_\xi$. It is known [11] that $\mathfrak{J}_\xi$ can be parametrized by sections of a certain real 2-dimensional subbundle of $End(\xi)$. Therefore $\mathfrak{J}_\xi$ is a tame Fréchet or Banach manifold. Our goal is to solve the following problem:

Conjecture 1.3. *Given a CR structure (M, ξ, J) embedded in $\mathbb{C}^2$, there is a submanifold $\mathfrak{J}_c \subset \mathfrak{J}_\xi$ passing through J and a neighborhood $\mathfrak{U}$ of J in $\mathfrak{J}_\xi$ such that $\tilde{J} \in \mathfrak{U}$ is embeddable in $\mathbb{C}^2$ and realized by a nearby embedding if and only if $\tilde{J} \in \mathfrak{J}_c$.*

In this paper, we look at the above problem infinitesimally. We will describe the tangent space $T_J\mathfrak{J}_c$, of $\mathfrak{J}_c$, at J (meaning the space of all $\frac{d}{dt}|_{t=0}\ J_{(t)}$ for $J_{(t)} \in \mathfrak{J}_c$ with $J_{(0)} = J$) as $J \circ D_J f + D_J g$ for all real functions f and g. The statment "and realized by a nearby embedding" in Conjecture 1.3 may be removed due to a result of Lempert [22] for the stability of embeddings if (M, ξ, J) is the boundary of a strictly linearly convex domain in $\mathbb{C}^2$. But in general unstable CR embeddings do exist ([6]).

Let $M \subset \mathbb{C}^2$ be a closed strongly pseudoconvex real hypersurface with induced CR structure (ξ, J). Let $\mathfrak{E}_c$ denote the set of all contact embeddings: $(M, \xi) \to \mathbb{C}^2$ near the inclusion map (see Section 5). Two contact embeddings φ, ψ are equivalent (in notation, $\varphi \sim \psi$) if $\varphi^* J_{\mathbb{C}^2} = \psi^* J_{\mathbb{C}^2}$. Define (with respect to unitary (co)frame)

$$\mathfrak{D}_J h = (h_{,\bar{1}\bar{1}} - iA_{\bar{1}\bar{1}} h)\theta^{\bar{1}} \otimes Z_1$$

for $h \in C^\infty(M, \mathbb{C})$, say (cf. (6.9)). Note that

$$2\,\mathrm{Re}\,\mathfrak{D}_J h = J \circ D_J f + D_J g$$

for $h = g + if$. Define a type (1,0) vector field Y_h by

$$Y_h := i(h\zeta + h^{,1} Z_1)$$

where $i\zeta$, Z_1 are type (1,0) (local) vector fields dual to $\theta = -i\partial\gamma$, θ^1 near $M := \{\gamma = 0\}$ in C^2 (see Section 6).

Theorem E. *In the situation described above, we have the following commutative diagram:*

(1.10)

$$\begin{array}{ccccc}
(\text{complex version})T_{[i_M]}(\mathfrak{E}_c/_\sim) & \overset{4Im\circ\bar{\partial}_b}{\longrightarrow} & T_J\mathfrak{J}_c & \subset & T_J\mathfrak{J}_\xi \\
[Y_\cdot]\uparrow & & 4\mathbb{R}e\uparrow & & 4\mathbb{R}e\uparrow \\
C^\infty(M,\mathbb{C})/Ker\mathfrak{D}_J & \overset{\mathfrak{D}_J}{\longrightarrow} & \text{Range } \mathfrak{D}_J & \subset & \Gamma(T^*_{0,1}(M) \otimes T_{1,0}(M))
\end{array}$$

where the maps indicated by "$\to$" or "$\uparrow$" are all one-one correspondences.

The results in this paper were essentially obtained in nineties. The author would like to thank László Lempert, Jack Lee, and I-Hsun Tsai for discussions during those

years. In recent years, embeddability has played an important role in the study of CR positive mass theorem of $3D$ and positivity of CR Paneitz operator ([13], [8], [9]). In this respect, the author would like to thank Paul Yang, Andrea Malchiodi, Hung-Lin Chiu, and Sagun Chanillo for many useful discussions. Theorem E is related to such a study.

2. Review in CR and pseudohermitian geometries. For most of basic material we refer the reader to [28], [27] or [20]. Throughout the paper, our base space M is a closed (compact with no boundary) contact 3-manifold with a cooriented contact structure ξ meaning that ξ and TM/ξ are orientable. A CR structure J (compatible with ξ) is an endomorphism on ξ with $J^2 = -identity$.

By choosing a (global) contact form θ (exists since the normal bundle TM/ξ of ξ in TM is orientable), we can talk about pseudohermitian geometry. The Reeb vector field T is uniquely determined by $\theta(T) = 1$ and $T\rfloor d\theta = 0$. We choose a (local) complex vector field Z_1, an eigenvector of J with eigenvalue i, and a (local) complex 1-form θ^1 such that $\{\theta, \theta^1, \theta^{\bar{1}}\}$ is dual to $\{T, Z_1, Z_{\bar{1}}\}$ (here $\theta^{\bar{1}}$ and $Z_{\bar{1}}$ mean the complex conjugates of θ^1 and Z_1 resp.). It follows that $d\theta = ih_{1\bar{1}}\theta^1\wedge\theta^{\bar{1}}$ for some real function $h_{1\bar{1}}$. We will always assume J is nondegenerate, i.e., $h_{1\bar{1}}$ never vanishes. (may assume $h_{1\bar{1}} > 0$; otherwise, replace θ by $-\theta$). Call (J, θ) strictly pseudoconvex if $h_{1\bar{1}} > 0$ (or equivalently, the Levi form defined by $d\theta(\cdot, J\cdot)$ on ξ is positive definite). We can then choose a Z_1 (hence θ^1) such that $h_{1\bar{1}} = 1$. That is to say

$$d\theta = i\theta^1\wedge\theta^{\bar{1}}. \tag{2.1}$$

We will always assume our pseudohermitian structure (J, θ) satisfies (2.1), i.e., $h_{1\bar{1}} = 1$ throughout the paper. The pseudohermitian connection of (J, θ) is the connection $\nabla^{\psi.h.}$ on $TM\otimes C$ (and extended to tensors) given by

$$\nabla^{\psi.h.}Z_1 = \omega_1{}^1\otimes Z_1, \nabla^{\psi.h.}Z_{\bar{1}} = \omega_{\bar{1}}{}^{\bar{1}}\otimes Z_{\bar{1}}, \nabla^{\psi.h.}T = 0$$

in which the connection 1-form $\omega_1{}^1$ is uniquely determined by the following equation and associated normalization condition:

$$\begin{aligned} d\theta^1 &= \theta^1\wedge\omega_1{}^1 + A^1{}_{\bar{1}}\theta\wedge\theta^{\bar{1}}, \\ 0 &= \omega_1{}^1 + \omega_{\bar{1}}{}^{\bar{1}}. \end{aligned} \tag{2.2}$$

The coefficient $A^1{}_{\bar{1}}$ in (2.2) and its complex conjugate $A^{\bar{1}}{}_1$ are components of the torsion (tensor) $A_{J,\theta} = A^{\bar{1}}{}_1\theta^1\otimes Z_{\bar{1}} + A^1{}_{\bar{1}}\theta^{\bar{1}}\otimes Z_1$. Since $h_{1\bar{1}} = 1$, $A_{\bar{1}\bar{1}} = h_{1\bar{1}}A^1{}_{\bar{1}} = A^1{}_{\bar{1}}$. Further A_{11} is just the complex conjugate of $A_{\bar{1}\bar{1}}$. Write $J = i\ \theta^1\otimes Z_1 - i\theta^{\bar{1}}\otimes Z_{\bar{1}}$. It is not hard to see from (2.1) and (2.2) that

$$L_T J = 2J \circ A_{J,\theta} \tag{2.3}$$

where L_T denotes the Lie differentiation in the direction T (this is the case when $f = -1$ in Lemma 3.5 of [11]). So the vanishing torsion is equivalent to T being an infinitesimal CR diffeomorphism. We can define the covariant differentiations with respect to the pseudohermitian connection. For instance, $f_{,1} = Z_1 f$, $f_{,1\bar{1}} = Z_{\bar{1}}Z_1 f - \omega_1{}^1(Z_{\bar{1}})Z_1 f$ for a (smooth) function f (see, e.g., Section 4 in [20]). Now differentiating $\omega_1{}^1$ gives

$$d\omega_1{}^1 = W\theta^1\wedge\theta^{\bar{1}} + 2iIm(A_{11,\bar{1}}\theta^1\wedge\theta) \tag{2.4}$$

where W or $W_{J,\theta}$ (to emphasize the dependence of the pseudohermitian structure) is called the (scalar) Tanaka-Webster curvature.

There are distinguished CR structures J, called spherical, if (M, ξ, J) is locally CR equivalent to the standard 3-sphere $(S^3, \hat{\xi}, \hat{J})$, or equivalently if there are contact coordinate maps into open sets of $(S^3, \hat{\xi})$ so that the transition contact maps can be extended to holomorphic transformations of open sets in $\mathbb{C}^2$. In 1930's, Elie Cartan ([5]; see also [11]) obtained a geometric quantity, denoted as Q_J, by solving the local equivalence problem for 3-dimensional CR structures so that the vanishing of Q_J characterizes J to be spherical. We will call Q_J the Cartan (curvature) tensor. Note that Q_J depends on a choice of contact form θ. It is CR-covariant in the sense that if $\tilde{\theta} = e^{2f}\theta$ is another contact form and $\tilde{Q}_J$ is the corresponding Cartan tensor, then $\tilde{Q}_J = e^{-4f}Q_J$. We can express Q_J in terms of pseudohermitian invariants. Write $Q_J = iQ_{11}\theta^1 \otimes Z_{\bar{1}} - iQ_{\bar{1}\bar{1}}\theta^{\bar{1}} \otimes Z_1$ (note that $Q_1{}^{\bar{1}} = Q_{11}$ and $Q_{\bar{1}}{}^1 = Q_{\bar{1}\bar{1}}$ since we always assume $h_{1\bar{1}} = 1$). We have the following formula (Lemma 2.2 in [11]):

$$Q_{11} = \frac{1}{6}W_{,11} + \frac{i}{2}WA_{11} - A_{11,0} - \frac{2i}{3}A_{11,\bar{1}1.} \tag{2.5}$$

In terms of local unitary coframe fields we can express the Cartan flow (1.7) as follows:

$$\dot{\theta}^1 = -Q_{\bar{1}\bar{1}}\theta^{\bar{1}} \tag{2.6}$$

(cf. (2.16) in [11] with $E_{\bar{1}}{}^1$ replaced by $-iQ_{\bar{1}\bar{1}}$). The torsion evolves under the Cartan flow as shown in the follow formula:

$$\dot{A}_{11} = -Q_{11,0} \tag{2.7}$$

(this is the complex conjugate of (2.18) in [11] with $E_{\bar{1}}{}^1$ replaced by $-iQ_{\bar{1}\bar{1}}$). Since the Cartan flow is invariant under the pullback action of contact diffeomorphisms (cf. the argument in the proof of Proposition 3.6 in [11]), we need to add a gauge-fixing term to the right-hand side of (1.7) to get the subellipticity of its linearized operator. Let us recall what this term is. First we define a quadratic differential operator F_J from endomorphism fields to functions by ([11], p.236 and note that $h_{1\bar{1}} = 1$ here)

$$F_J E = (iE_{1\bar{1}}E_{11,\bar{1}\bar{1}} + iE_{\bar{1}\bar{1}}E_{11,\bar{1}1}) + conjugate. \tag{2.8}$$

Also we define a linear differential operator D_J from functions to endomorphism fields and its formal adjoint D_J^* by

$$\begin{aligned} D_J f &= (f_{,11} + iA_{11}f)\theta^1 \otimes Z_{\bar{1}} + (f_{,\bar{1}\bar{1}} - iA_{\bar{1}\bar{1}}f)\theta^{\bar{1}} \otimes Z_1, \\ D_J^* E &= E_{11,\bar{1}\bar{1}} + E_{\bar{1}\bar{1},11} - iA_{\bar{1}\bar{1}}E_{11} + iA_{11}E_{\bar{1}\bar{1}} \end{aligned} \tag{2.9}$$

(note that we have used the notations D_J, D_J^* instead of $B_J^{'}, B_J$ in [11], resp.). Now let K be a fixed CR structure. The Cartan flow with a gauge-fixing term reads as follows: (this is (1.8))

$$\frac{\partial J_{(t)}}{\partial t} = 2Q_{J_{(t)}} - \frac{1}{6}D_{J_{(t)}}F_{J_{(t)}}K.$$

We also need the following commutation relations often:

$$\begin{aligned} C_{I,01} - C_{I,10} &= C_{I,\bar{1}}A_{11} - kC_I A_{11,\bar{1}} \\ C_{I,0\bar{1}} - C_{I,\bar{1}0} &= C_{I,1}A_{\bar{1}\bar{1}} + kC_I A_{\bar{1}\bar{1},1} \\ C_{I,1\bar{1}} - C_{I,\bar{1}1} &= iC_{I,0} + kC_I W. \end{aligned} \tag{2.10}$$

Here C_I denotes a coefficient of a tensor with multi-index I consisting of 1 and $\bar{1}$, and k is the number of 1 in I minus the number of $\bar{1}$ in I (an extension of formulas in [21]).

3. Proof of Corollary C.

LEMMA 3.1. *Suppose there is a contact form θ such that the torsion $A_{J_{(0)},\theta}$ vanishes. Then under the gauge-fixed Cartan flow (1.8) (assuming smooth solution) with $K = J_{(0)}$, $A_{J_{(t)},\theta}$ stays vanishing.*

Let T denote the Reeb vector field associated with the contact form θ. The vanishing of the torsion is equivalent to saying that T is an infinitesimal CR diffeomorphism (see (2.3)). We say a CR manifold has transverse symmetry if the infinitesimal generator of a one-parameter group of CR diffeomorphisms is everywhere transverse to ξ. Such an infinitesimal generator can be realized as the Reeb vector field for a certain contact form $\hat{\theta}$ ([21]). From Lemma 3.1 we have

COROLLARY 3.2. *The CR structures $J_{(t)}$ stay having the same transverse symmetry as $J_{(0)}$ does under the gauge-fixed Cartan flow (1.8) with $K = J_{(0)}$ and $\theta = \hat{\theta}$.*

Proof of Lemma 3.1. We will compute the evolution of the torsion under the flow (1.8) (with K being the initial CR structure $J_{(0)}$). First, instead of (2.7), we have

$$\dot{A}_{11} = -Q_{11,0} - \frac{i}{12}(D_J F_J K)_{11,0}. \tag{3.1}$$

From the formula (2.5) for Q_{11}, we compute $Q_{11,0}$. Using the commutation relations (2.10) and the Bianchi identity: $W_{,0} = A_{11,\bar{1}\bar{1}} + A_{\bar{1}\bar{1},11}$ ([21]), we can express $Q_{11,0}$ only in terms of $A_{11}, A_{\bar{1}\bar{1}}$ and their covariant derivatives as follows:

$$Q_{11,0} = \frac{1}{6}(A_{11,\bar{1}\bar{1}11} + A_{\bar{1}\bar{1},1111}) - A_{11,00} - \frac{2i}{3}A_{11,\bar{1}10} + l.w.t. \tag{3.2}$$

where $l.w.t.$ means a lower weight term in A_{11} and $A_{\bar{1}\bar{1}}$.We count covariant derivatives in 1 or $\bar{1}$ direction (0 direction, resp.) as weight 1 (weight 2, resp.) and we call a term of weight m if its total weight of covariant derivatives is m. For instance, $A_{11,\bar{1}\bar{1}11}$, $A_{11,00}$ and $A_{11,\bar{1}10}$ are all of weight 4. So more precisely each single term in $l.w.t.$ must contain terms of weight ≤ 3 in A_{11} or $A_{\bar{1}\bar{1}}$.In particular, if $A_{11} = 0$, then $l.w.t. = 0$. Note that $A_{\bar{1}\bar{1},1111}$ is a "bad"term in the sense that we need a gauge-fixing term to cancel it and obtain a fourth order subelliptic operator in A_{11}. Now by (2.9) the gauge-fixing term in (3.1) (up to a multiple) reads as

$$\begin{aligned}(D_J F_J K)_{11,0} &= (F_J K)_{,110} + i[A_{11}(F_J K)]_{,0} \\ &= (F_J K)_{,011} + l.w.t.(\text{in } A_{11})\end{aligned} \tag{3.3}$$

(we have used the commutation relations (2.10) for the last equality). Write $K = K_{11}\theta^1 \otimes Z_{\bar{1}} + K_{1\bar{1}}\ \theta^1 \otimes Z_1 + K_{\bar{1}\bar{1}}\ \theta^{\bar{1}} \otimes Z_1 + K_{\bar{1}1}\theta^{\bar{1}} \otimes Z_{\bar{1}}$ where $K_{\bar{1}\bar{1}}, K_{\bar{1}1}$ are the complex conjugates of $K_{11}, K_{1\bar{1}}$, respectively. We compute

$$\begin{aligned}K_{11,0} &= T[\theta^{\bar{1}}(KZ_1)] - 2\omega_1{}^1(T)K_{11} \\ &= (L_T\theta^{\bar{1}})(KZ_1) + \theta^{\bar{1}}[(L_T K)Z_1] + \theta^{\bar{1}}[K(L_T Z_1)] - 2\omega_1{}^1(T)K_{11}.\end{aligned} \tag{3.4}$$

It is easy to compute the first term using the (complex conjugate of) structure equation (2.2) and the third term using the formula $[T, Z_1] = -A_{11}Z_{\bar{1}} + \omega_1{}^1(T)Z_1$ ([20]). For the second term, if we take K to be the initial CR structure $J_{(0)}$,then

$$L_T K = L_T J_{(0)} = 2A_{J_{(0)},\theta} = 0 \tag{3.5}$$

by (2.3) and the assumption. So altogether we obtain

$$K_{11,0} = A_{11}(K_{1\bar{1}} - K_{\bar{1}1}) = 2A_{11}K_{1\bar{1}}. \tag{3.6}$$

Note that $K^2 = -I$ implies that $K_{1\bar{1}} = \pm i(1 + |K_{11}|^2)^{\frac{1}{2}}$ and $K_{\bar{1}1} = -K_{1\bar{1}.}$ It follows that

$$K_{1\bar{1},0} = -A_{11}K_{\bar{1}\bar{1}} + A_{\bar{1}\bar{1}}K_{11}. \tag{3.7}$$

Here the point is that both $K_{11,0}$ and $K_{1\bar{1},0}$ are linear in A_{11} and $A_{\bar{1}\bar{1}}$ with coefficients being "0th-order" in a (co)frame. Using (3.6), (3.7), we can express $(F_J K)_{,011}$ as follows:

$$\begin{aligned}(F_J K)_{,011} &= iK_{1\bar{1}}K_{11,\bar{1}\bar{1}011} + iK_{\bar{1}\bar{1}}K_{11,\bar{1}1011} \\ &\qquad -iK_{\bar{1}1}K_{\bar{1}\bar{1},11011} - iK_{11}K_{\bar{1}\bar{1},1\bar{1}011} + l.w.t.\end{aligned} \tag{3.8}$$

Here and hereafter $l.w.t.$ will mean a lower weight term in $A_{11}, A_{\bar{1}\bar{1}}$ up to weight 3 with coefficients in $K_{1\bar{1}}$, $K_{\bar{1}\bar{1}}$, $K_{\bar{1}1}$, K_{11} and their covariant derivatives up to weight 5. Note that $A_{11}, A_{\bar{1}\bar{1}}$ are of weight 2 in $K_{1\bar{1}}, K_{\bar{1}\bar{1}}$, $K_{\bar{1}1}$, K_{11}. The first four terms on the right-hand side of (3.8) contain the hightest weight terms of weight 4 in $A_{11}, A_{\bar{1}\bar{1}}$ in view of the commutation relations (2.10) and (3.6), (3.7) as will be shown below. Using (2.10) repeatedly and (3.6), we compute

$$\begin{aligned}K_{11,\bar{1}\bar{1}011} &= K_{11,\bar{1}0\bar{1}11} + l.w.t. = K_{11,0\bar{1}\bar{1}11} - 2K_{11}A_{\bar{1}\bar{1},1\bar{1}11} + l.w.t. \\ &= 2K_{1\bar{1}}A_{11,\bar{1}\bar{1}11} - 2K_{11}A_{\bar{1}\bar{1},1\bar{1}11} + l.w.t.\end{aligned} \tag{3.9}$$

Similarly we obtain

$$\begin{aligned}K_{11,\bar{1}1011} &= 2K_{1\bar{1}}A_{11,\bar{1}111} - 2K_{11}A_{\bar{1}\bar{1},1111} + l.w.t. \\ K_{\bar{1}\bar{1},11011} &= 2K_{\bar{1}1}A_{\bar{1}\bar{1},1111} - 2K_{\bar{1}\bar{1}}A_{11,\bar{1}111} + l.w.t. \\ K_{\bar{1}\bar{1},1\bar{1}011} &= 2K_{\bar{1}1}A_{\bar{1}\bar{1},1\bar{1}11} - 2K_{\bar{1}\bar{1}}A_{11,\bar{1}\bar{1}11} + l.w.t.\end{aligned} \tag{3.10}$$

Substituting (3.9), (3.10) in (3.8), we get, in view of (3.3),

$$(D_J F_J K)_{11,0} = -2iA_{11,\bar{1}\bar{1}11} + 2iA_{\bar{1}\bar{1},1111} + l.w.t. \tag{3.11}$$

Now substituting (3.2) and (3.11) in (3.1) gives

$$\dot{A}_{11} = -\frac{1}{3}A_{11,\bar{1}\bar{1}11} + A_{11,00} + \frac{2i}{3}A_{11,\bar{1}10} + l.w.t. \tag{3.12}$$

(note that the "bad"terms cancel). Define $L_\alpha A_{11} = -A_{11,1\bar{1}} - A_{11,\bar{1}1} + i\alpha A_{11,0}$ for a complex number α. Let L_α^* be the formal adjoint of L_α. It is a direct computation (cf. p.1257 in [12]) that

$$\begin{aligned}L_\alpha^* L_\alpha A_{11} &= 2(A_{11,11\bar{1}\bar{1}} + A_{11,\bar{1}\bar{1}11}) - i(3 + \alpha + \bar{\alpha} - |\alpha|^2)A_{11,1\bar{1}0} \\ &\qquad + i(3 - \alpha - \bar{\alpha} - |\alpha|^2)A_{11,\bar{1}10} + l.w.t.\end{aligned} \tag{3.13}$$

Using the commutation relations (2.10), we can easily obtain

$$A_{11,11\bar{1}\bar{1}} = A_{11,\bar{1}\bar{1}11} + 2iA_{11,\bar{1}10} + 2iA_{11,1\bar{1}0} + l.w.t. \qquad (3.14)$$
$$A_{11,00} = -iA_{11,1\bar{1}0} + iA_{11,\bar{1}10} + l.w.t.$$

In view of (3.14) and (3.13), we can rewrite (3.12) as follows:

$$\dot{A}_{11} = -\frac{1}{12}L_\alpha^* L_\alpha A_{11} + l.w.t. \qquad (3.15)$$

for $\alpha = 4 + i\sqrt{3}$. Since α is not an odd integer, L_α and hence $L_\alpha^* L_\alpha$ (note $L_\alpha^* = L_{\bar{\alpha}}$) are subelliptic (e.g. [11]). Taking the complex conjugate of (3.15) gives a similar equation for $A_{\bar{1}\bar{1}}$ only with α replaced by $-\bar{\alpha}$. On the other hand, we observe that $A_{11} = 0, A_{\bar{1}\bar{1}} = 0$ for all (valid) time is a solution to (3.15) and its conjugate equation (note that $l.w.t.$ vanishes if A_{11} and $A_{\bar{1}\bar{1}}$ vanish as remarked previously). Therefore by the uniqueness of the solution to a (or system of) subparabolic equation(s), we conclude that A_{11} stays vanishing under the flow (1.8). □

Proof of Corollary C. By the existence of a short-time solution to (1.8) (which is C^k smooth for any given large k, see [11]) and Lemma 3.1, we can find $\tau_1 > 0$ such that $A_{11(t)} = 0$ for $0 \le t < \tau_1$. It follows from (2.5) that

$$Q_{11(t)} = \frac{1}{6}W_{,11(t)}. \qquad (3.16)$$

Here $W_{,11(t)} = (Z_{1(t)})^2 W_{(t)} - \omega_1{}^1_{(t)}(Z_{1(t)})Z_{1(t)}W_{(t)}$ and $W_{(t)}$ is the Tanaka-Webster curvature with respect to $J_{(t)}$ (and fixed θ). Therefore $u = -\frac{1}{6}W_{(t)} + \frac{i}{12}F_{J_{(t)}}J_{(0)}$ is a solution to (1.5) by (3.16) for $0 \le t < \tau_2 \le \tau_1$ with τ_2 so small that $W_{(t)} < 0$ (hence $\mathrm{Re}\, u = -\frac{1}{6}W_{(t)} > 0$). Since (1.5) is equivalent to (1.4), we conclude the result by Theorem A (which still holds true in C^k category for large k). □

4. Proof of Theorem A. Let $J_{(t)}$ be a solution to (1.1) for $0 \le t < \tau$ with given initial $J_{(0)}$ being fillable. We are going to construct an almost complex structure $\check{J}$ on $M \times [0, \tau)$, integrable on $M \times (0, \tau)$.

There is a canonical choice of the (unitary) frame $Z_{1(t)}$ with respect to $J_{(t)}$ ([11]). Write $Z_{1(t)} = \frac{1}{2}(e_{1(t)} - ie_{2(t)})$ where $e_{1(t)}, e_{2(t)} \in \xi$ and $J_{(t)}e_{1(t)} = e_{2(t)}$. Let $\{\theta, e^1_{(t)}, e^2_{(t)}\}$ be a coframe dual to $\{T, e_{1(t)}, e_{2(t)}\}$ on M. We will identify $M \times \{t\}$ with M (hence $T(M \times \{t\})$ with TM). Now we define an almost complex structure $\check{J}$ at each point in $M \times \{t\}$ as follows:

$$\check{J}\mid_\xi = J_{(t)},\ \check{J}T = -a\frac{\partial}{\partial t} + bT + a(\alpha e_{1(t)} + \beta e_{2(t)}). \qquad (4.1)$$

Here a, b, α, β are some real (smooth) functions of space variable and t, and $a > 0$ (so $\check{J}\,\frac{\partial}{\partial t}$ is completely determined from the above formulas, $\check{J}^2 = -identity$, and $\check{J}T$ points in the direction of $-\frac{\partial}{\partial t}$ modulo the tangent directions of $M \times \{t\}$). Strictly speaking, α, β depend on the choice of frame while a, b are global). It is easy to see that the coframe dual to $\{e_{1(t)},\ e_{2(t)},\ \frac{\partial}{\partial t} - (b/a)T - \alpha e_{1(t)} - \beta e_{2(t)},\ (1/a)T\}$ is $\{e^1_{(t)} + \alpha dt,\ e^2_{(t)} + \beta dt,\ dt,\ a\theta + bdt\}$. So the following complex 1-forms:

$$\Theta^1 = (e^1_{(t)} + \alpha dt) + i(e^2_{(t)} + \beta dt) = \theta^1_{(t)} + \gamma^1 dt, \qquad (4.2)$$

$$\eta = (a\theta + bdt) - idt = a\theta + (b-i)dt \tag{4.3}$$

are type (1,0) forms with respect to $\check{J}$. Here $\gamma^1 = \alpha + i\beta$ is really the $Z_{1(t)}$ coefficient of the vector field $\alpha e_{1(t)} + \beta e_{2(t)}$. Let $\Lambda^{p,q}$ denote the space of type (p,q) forms. The integrability of $\check{J}$ is equivalent to $d\Lambda^{1,0} \subset \Lambda^{2,0} + \Lambda^{1,1}$ or $\Lambda^{2,0} \wedge d\Lambda^{1,0} = 0$. In terms of Θ^1, η, the integrability conditions read as follows:

$$\eta \wedge \ \Theta^1 \wedge d\eta = 0, \tag{4.4}$$

$$\eta \wedge \ \Theta^1 \wedge d\Theta^1 = 0. \tag{4.5}$$

Substituting (4.2), (4.3) into (4.4) and making use of $d\theta = d_M\theta = i\theta^1_{(t)} \wedge \theta^{\bar{1}}_{(t)}$(here d_M denotes the exterior differentiation on M and $d = d_M + dt\frac{\partial}{\partial t}$ on $M \times (0,\tau)$), we obtain

$$0 = \eta \wedge \ \Theta^1 \wedge d\eta = [ab_{,\bar{1}} - (b-i)a_{,\bar{1}} + ia^2\gamma^1]\theta \wedge \theta^1_{(t)} \wedge \theta^{\bar{1}}_{(t)} \wedge dt.$$

Here $b_{,\bar{1}} = Z_{\bar{1}(t)}b$, $a_{,\bar{1}} = Z_{\bar{1}(t)}a$. Therefore (4.4) is equivalent to the relation between a, b and γ^1 as shown below:

$$\gamma^1 = ia^{-1}b_{,\bar{1}} - ia^{-2}(b-i)a_{,\bar{1}}. \tag{4.6}$$

Next note that $d\theta^1_{(t)} = d_M\theta^1_{(t)} + dt \wedge \dot{\theta}^1_{(t)}$ and

$$\dot{\theta}^1_{(t)} = -iE_{\bar{1}\bar{1}(t)}\theta^{\bar{1}}_{(t)}$$

($E_{\bar{1}\bar{1}(t)}$ is the $\bar{1}\bar{1}-$component of $E_{J_{(t)}}$with respect to $J_{(t)}$). So substituting (4.2), (4.3) into (4.5) and making use of (2.2) for $\theta^1_{(t)}$, we obtain

$$\begin{aligned} 0 &= \eta \wedge \Theta^1 \wedge d\Theta^1 \\ &= (aiE_{\bar{1}\bar{1}(t)} + (b-i)A_{\bar{1}\bar{1}(t)} + a\gamma^1_{,\bar{1}})\theta \wedge \theta^1_{(t)} \wedge \theta^{\bar{1}}_{(t)} \wedge dt. \end{aligned}$$

Here $A_{\bar{1}\bar{1}(t)}$ is the $\bar{1}\bar{1}-$component of the torsion tensor with respect to $J_{(t)}$ and $\gamma^1_{,\bar{1}}$ $:= Z_{\bar{1}(t)}\gamma^1 + \omega_1{}^1_{(t)}(Z_{\bar{1}(t)})\gamma^1$ where $\omega_1{}^1_{(t)}$ is the pseudohermitian connection form with respect to $\theta^1_{(t)}$. Therefore (4.5) is equivalent to the following relation between a, b and $\gamma^1_{,\bar{1}}$:

$$\gamma^1_{,\bar{1}} = -iE_{\bar{1}\bar{1}(t)} - a^{-1}(b-i)A_{\bar{1}\bar{1}(t)}. \tag{4.7}$$

Substituting (4.6) into (4.7) and letting $f = a^{-1} \neq 0, g = -ba^{-1}, u = f + ig$, we obtain an equation for a complex valued function u :

$$u_{,11} + iuA_{11(t)} = iE_{11(t)}. \tag{4.8}$$

In view of (2.9), we can express (4.8) in an intrinsic form:

$$J_{(t)} \circ D_{J_{(t)}}f + D_{J_{(t)}}g = E_{J_{(t)}}. \tag{4.9}$$

Recall that $D_{J_{(t)}} f = \frac{1}{2} L_{X_f} J_{(t)}$ ([11]) in which $X_f = -fT + i(Z_{1(t)}f)Z_{\bar{1}(t)} -i(Z_{\bar{1}(t)}f)Z_{1(t)}$ is the infinitesimal contact diffeomorphism induced by f. So the image of $D_{J_{(t)}}$ describes the tangent space of the orbit of the symmetry group acting on $J_{(t)}$ by the pullback (in this case, the contact diffeomorphisms are our symmetries). Now by assuming (1.4) (which is the same as (4.9)) in Theorem A, we obtain that $\check{J}$ is integrable on $M \times (0, \tau)$.

On the other hand, $(M, J_{(0)})$ bounds a complex surface N by our assumption that $J_{(0)}$ is fillable. So we have another almost complex structure $\hat{J}$ on $M \times (-\delta, 0]$ induced from N, integrable on $M \times (-\delta, 0)$, and restricting to $J_{(0)}$ on (M, ξ). By assumption we have that $\check{J}$ and $\hat{J}$ coincide at $M \times \{0\}$ where they may not coincide up to C^k for $k \geq 1$, however. We want to find a local orientation-preserving diffeomorphism Φ from a neighborhood U of a point in M times $(-\delta_1, 0]$ to a similar set so that Φ is an identity on $U \times \{0\}$, and $\Phi^* \hat{J}$ coincides with $\check{J}$ up to C^k for some large integer k at $U \times \{0\}$. Let $x^i, 0 \leq i \leq 3$ denote the coordinates of $U \times (-\delta_1, 0]$ with x^0 being the time variable for $(-\delta_1, 0]$. Let $y^i, 0 \leq i \leq 3$ denote the corresponding coordinates of the image of Φ with y^0 being the time variable. If we express $\hat{J}_1 = \Phi^* \hat{J} = \Phi_*^{-1}(\hat{J} \circ \Phi)\Phi_*$ in coordinates, we usually write

$$(\hat{J}_1)^l_m = \hat{J}^j_i \frac{\partial y^i}{\partial x^m} \frac{\partial x^l}{\partial y^j}$$

for $\hat{J}_1 = (\hat{J}_1)^l_m dx^m \otimes \frac{\partial}{\partial x^l}$ and $\hat{J} = \hat{J}^j_i dy^i \otimes \frac{\partial}{\partial y^j}$. Let $\eta = \Phi_*^{-1}$. Then η^{-1} has the expression $(\frac{\partial y^i}{\partial x^m})$, the Jacobian matrix of Φ, in coordinates. We require $\eta^{-1} = \mathit{identity}$ at each point with $x^0 = 0$ where $\hat{J}$ coincides with $\check{J}$. Differentiating $\hat{J}_1 = \Phi^* \hat{J} = \Phi_*^{-1}(\hat{J} \circ \Phi)\Phi_* = \eta(\hat{J} \circ \Phi)\eta^{-1}$(considered as a matrix equation with respect to the above-mentioned bases) in x^0 at $x^0 = 0$, we obtain

$$\hat{J}_1' - \hat{J}' = \eta' \hat{J} - \hat{J} \eta'. \tag{4.10}$$

Here the prime of $\hat{J}'$ means the y^0-derivative at $y^0 = 0$ while the prime of $\hat{J}_1'$ and η'means the x^0-derivative at $x^0 = 0$. Finding Φ such that $\hat{J}_1 = \Phi^* \hat{J}$ coincides with $\check{J}$ up to C^1 at $U \times \{0\}$ is reduced to solving the above equation (4.10) for η' with $\hat{J}_1' = \check{J}'$. Here the prime of $\check{J}'$ means the t-derivative at $t = 0$. And this can be done by simple linear algebra as follows. First note that $C = \check{J}' - \hat{J}'$ satisfies $\hat{J}C + C\hat{J} = 0$ since $\check{J} = \hat{J}$ at $U \times \{0\}$ and both $\check{J}'$ and $\hat{J}'$ satisfies the same relation as C does. With respect to a suitable basis, $\hat{J}$ has a canonical matrix representation:

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Then C has the matrix form $\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ where each C_{ij} is a 2×2 matrix $\begin{pmatrix} a_{ij} & b_{ij} \\ b_{ij} & -a_{ij} \end{pmatrix}$. Now the solution η' to (4.10) has the matrix form $\begin{pmatrix} \eta'_{11} & \eta'_{12} \\ \eta'_{21} & \eta'_{22} \end{pmatrix}$ where each η'_{ij} is a 2×2 matrix $\begin{pmatrix} u_{ij} & v_{ij} \\ w_{ij} & s_{ij} \end{pmatrix}$ satisfying the relations: $v_{ij} + w_{ij} = -a_{ij}$, $u_{ij} - s_{ij} = b_{ij}$. Once η' is determined by equation (4.10), it is easy to construct the "local" diffeomorphism Φ_1 such that the inverse Jacobian and its x^0-derivative at $x^0 = 0$

of Φ_1 is η =the identity and η', resp. (we may need to shrink the time interval $(-\delta_1, 0]$). So if we start with $\hat{J}_1 = \Phi_1^* \hat{J}$ instead of $\hat{J}$ and repeat the above procedure looking for Φ_2 so that $\hat{J}_2 = \Phi_2^* \hat{J}_1$ coincides with $\check{J}$ at $U \times \{0\}$ up to C^2, we differentiate $\hat{J}_2 = \eta_1 \hat{J}_1 \eta_1^{-1}$ twice with respect to x^0 at $x^0 = 0$. Here η_1 denotes the inverse Jacobian matrix of Φ_2 (to be determined). Requiring $\hat{J}_2' = \hat{J}_1'$ and $\eta_1 = identity$ (at $x^0 = 0$) implies $\eta_1' = 0$. It then follows that η_1'',the second derivative of η_1 in x^0 at $x^0 = 0$, satisfies a similar equation as in (4.10):

$$\eta_1'' \hat{J}_1 - \hat{J}_1 \eta_1'' = \hat{J}_2'' - \hat{J}_1''. \tag{4.11}$$

We can verify that the right-hand side anti-commutes with $\hat{J}_1$ as follows: $(\hat{J}_2'' - \hat{J}_1'')\hat{J}_1 + \hat{J}_1(\hat{J}_2'' - \hat{J}_1'') = (\hat{J}_2''\hat{J}_1 + \hat{J}_1\hat{J}_2'') - (\hat{J}_1''\hat{J}_1 + \hat{J}_1\hat{J}_1'') = -2(\hat{J}_2')^2 + 2(\hat{J}_1')^2 = 0$ (here we have used $\hat{J}_2 = \hat{J}_1$, $\hat{J}_2' = \hat{J}_1'$ and $J''J + 2(J')^2 + JJ'' = 0$ for any almost complex structure J by differentiating $J^2 = -I$ twice. So we can solve (4.11) for η_1'' with $\hat{J}_2'' = \check{J}''$ and hence find a Φ_2 with the required properties as before. In general, suppose we have found Φ_{n-1}such that $\hat{J}_{n-1} = \Phi_{n-1}^* \hat{J}_{n-2} = \eta_{n-2} \hat{J}_{n-2} \eta_{n-2}^{-1}$ coincides with $\check{J}$ up to C^{n-1} at $x^0 = 0$. Then by the similar procedure we can find Φ_n such that $\hat{J}_n = \Phi_n^* \hat{J}_{n-1} = \eta_{n-1} \hat{J}_{n-1} \eta_{n-1}^{-1}$ coincides with $\check{J}$ up to C^n at $x^0 = 0$, and the x^0-derivatives of η_{n-1} vanish up to the order $n-1$. Furthermore the n-th x^0-derivative $\eta_{n-1}^{(n)}$ satisfies a similar equation as in (4.10) or (4.11):

$$\eta_{n-1}^{(n)} \hat{J}_{n-1} - \hat{J}_{n-1} \eta_{n-1}^{(n)} = \check{J}^{(n)} - \hat{J}_{n-1}^{(n)}. \tag{4.12}$$

Here $\check{J}^{(n)}$ denotes the n-th t-derivative of $\check{J}$ at $t = 0$ while $\hat{J}_{n-1}^{(n)}$ means the n-th x^0-derivative of $\hat{J}_{n-1}$ at $x^0 = 0$.

Now $\hat{J}_n$ defined on $U \times (-\delta_n, 0]$ and $\check{J}$ defined on $U \times [0, \delta_n)$ for a small $\delta_n > 0$ together form a C^n integrable almost complex structure on $U \times (-\delta_n, \delta_n)$. Therefore $U \times (-\delta_n, \delta_n)$ is a complex manifold for $n \geq 4$ by a theorem of Newlander-Nirenberg ([25]). Since M is compact, we can cover it by a finite number of $U's$ and have corresponding $\delta_n's$. By the method of cutting off we can glue associated finitely many $\Phi_j's$ to get an orientation-preserving diffeomorphism $\tilde{\Phi}_j$ on $M \times (-\tilde{\delta}_j, 0]$ for small $\tilde{\delta}_j$, $1 \leq j \leq n$, such that $\tilde{\Phi}_n^* \circ \tilde{\Phi}_{n-1}^* \circ \cdots \circ \tilde{\Phi}_1^* \hat{J}$ coincides with $\check{J}$ up to C^n at M. Denote $\tilde{\Phi}_1 \circ \tilde{\Phi}_2 \circ \cdots \circ \tilde{\Phi}_n$ by Ψ_n. Again using the cutoff construction we get an orientation-preserving diffeomorphism $\tilde{\Psi}_n$ on the complex surface N that $(M, J_{(0)})$ bounds in the convex side, satisfying the property that $\tilde{\Psi}_n = \Psi_n$ on $M \times (-\bar{\delta}, 0]$ for $0 < \bar{\delta} < \min\{\tilde{\delta}_1$, ..., $\tilde{\delta}_n\}$ and $\tilde{\Psi}_n$ = identity for the part of N far away from the boundary M. We have shown that for $0 < t < \tau$, $J_{(t)}$ bounds the complex surface, obtained by gluing (N, $\tilde{\Psi}_n^* \hat{J}$) with $(M \times [0, t), \check{J})$, in the convex side.

5. Describing embeddable CR structures. Let $M \subset \mathbb{C}^2$ be a closed strongly pseudoconvex real hypersurface with the inclusion map $i_M : M \to \mathbb{C}^2$. Let $\mathfrak{E}_m$ denote the (tame Fréchet) manifold of C^∞ smooth embeddings from M into $\mathbb{C}^2$, which are isotopic to i_M (see, e.g., [18]). Define an equivalent relation "$\sim$", in $\mathfrak{E}_m$ by

$$\varphi, \psi \in \mathfrak{E}_m, \varphi \sim \psi \Longleftrightarrow \exists \text{ a } CR \text{ diffeomorphism } \rho : \varphi(M) \to \psi(M)$$
$$\text{such that } \rho \circ \varphi = \psi.$$

Often we use the notation $\varphi^* J_{\mathbb{C}_2}$ to mean $\varphi_*^{-1} \circ J_{\mathbb{C}_2} \circ \varphi_*$ restricted to $\varphi_*^{-1} \xi_\varphi$, where $J_{\mathbb{C}^2}$ is the complex structure of $\mathbb{C}^2$ and $\xi_\varphi := T(\varphi(M)) \cap J_{\mathbb{C}^2}(T(\varphi(M)))$. So

$\varphi^* J_{\mathbb{C}^2}$ is an induced CR structure compatible with the contact bundle $\varphi_*^{-1}\xi_\varphi$ on M. In this notation, for $\varphi, \psi \in \mathfrak{E}_m$

$$\varphi \sim \psi \Longleftrightarrow \varphi^* J_{\mathbb{C}^2} = \psi^* J_{\mathbb{C}^2}.$$

Now let $\mathfrak{J}_e$ denote the set of all CR structures on M, induced from $\mathfrak{E}_m$ (or say, embeddable and realized by a nearby embedding). That is to say,

$$\mathfrak{J}_e = \{\varphi^* J_{\mathbb{C}^2} : \varphi \in \mathfrak{E}_m\}.$$

(Or more precisely, $\mathfrak{J}_e = \{(M, \varphi_*^{-1}\xi_\varphi, \varphi^* J_{\mathbb{C}^2}) : \varphi \in \mathfrak{E}_m\}$ to specify the possibly different contact structures) Obviously $[\varphi] \in \mathfrak{E}_m/_\sim \mapsto \varphi^* J_{\mathbb{C}^2} \in \mathfrak{J}_e$ is a one-one correspondence. Next let $\xi = \xi_{i_M}$ denote the standard contact structure on M induced from the inclusion map i_M. Let $\mathfrak{E}_c \subset \mathfrak{E}_m$ denote the set of all contact embeddings φ: $M \to \mathbb{C}^2$, i.e., φ is an embedding such that $\varphi_*\xi = \xi_\varphi$. Also let $Diff(M)$ and $Cont(M)$ denote the groups of diffeomorphisms and contact (w.r.t. ξ) diffeomorphisms, resp.. Then by a theorem of Gray ([17] or [18]): two close enough contact structures are isotopically equivalent, the map: $Cont(M)\backslash\mathfrak{E}_c \to Diff(M)\backslash\mathfrak{E}_m$ induced from the identity is a one-one correspondence. Denote by $\mathfrak{J}_c$, the set of all CR structures on M, induced from $\mathfrak{E}_c$. That is to say,

$$\mathfrak{J}_c = \{\varphi^* J_{\mathbb{C}^2} : \varphi \in \mathfrak{E}_c\}.$$

Note that elements in $\mathfrak{J}_c$, are all compatible with the contact bundle ξ. Easy to see that the map:

$$[\varphi] \in \mathfrak{E}_c/_\sim \to \varphi^* J_{\mathbb{C}^2} \in \mathfrak{J}_c \tag{5.1}$$

is a one-one correspondence too. Let $\mathfrak{J} \supset \mathfrak{J}_e$ denote the set of all CR structures on M, whose associated contact stuctures are isotropic to ξ. Let $\mathfrak{J}_\xi \subset \mathfrak{J}$ denote those which are compatible with ξ. It follows that the maps: $Cont(M)\backslash\mathfrak{J}_c \to Diff(M)\backslash\mathfrak{J}_e$ and $Cont(M)\backslash\mathfrak{J}_\xi \to Diff(M)\backslash\mathfrak{J}$ induced by the identity are also one-one correspondences. Thus in summary we have the following commutative diagram:

$$\begin{array}{ccccc} Diff(M)\backslash\mathfrak{E}_m/_\sim & \twoheadrightarrow & Diff(M)\backslash\mathfrak{J}_e & \subset & Diff(M)\backslash\mathfrak{J} \\ \Uparrow & & \Uparrow & & \Uparrow \\ Cont(M)\backslash\mathfrak{E}_c/_\sim & \twoheadrightarrow & Cont(M)\backslash\mathfrak{J}_c & \subset & Cont(M)\backslash\mathfrak{J}_\xi \end{array} \tag{5.2}$$

where the maps indicated by $\twoheadrightarrow$ or $\Uparrow$ are all one-one correspondences. We can put, say, C^∞ topology on the above spaces in (5.2) so that all the maps are homeomorphisms.

Remark 5.1. After we mod out the symmetry group, we get the genuine spaces $Diff(M)\backslash\mathfrak{J}$ ($Diff(M)\backslash\mathfrak{J}_e$, resp.) of all CR structures (all CR structures on M, embeddable in $\mathbb{C}^2$ and realized by an embedding isotopic to i_M, resp.) on M. According to (5.2), we can reduce our spaces to those in the contact category. So forgetting about symmetries, we should try to parametrize the space $\mathfrak{E}_c/_\sim$ in such a way that $\mathfrak{J}_c$, becomes a submanifold of $\mathfrak{J}_\xi$. We will look at this problem infinitesimally in the next section.

Remark 5.2. Every object in this section can be similarly defined for general n provided M is a closed strongly pseudoconvex hypersurface in $\mathbb{C}^{n+1}$. The diagram (5.2) still holds for general dimensions.

6. Parametrizing $\mathfrak{J}_c$ infinitesimally. In this section, we will observe what the tangent space $T_J\mathfrak{J}_c$ of $\mathfrak{J}_c$, at J would be if we know that $\mathfrak{J}_c$ has a manifold structure. In view of (5.1), we look at $\mathfrak{C}_c$ first. Suppose $\mathfrak{C}_c$ is a good space (manifold, say). Can we describe the tangent space $T_{i_M}\mathfrak{C}_c$ of $\mathfrak{C}_c$, at the inclusion map i_M quantitatively? Since the results below hold for general n, we will discuss the general case hereafter.

Let $M \supset \mathbb{C}^{n+1}$ be the boundary of a strongly pseudoconvex bounded domain, defined by $\gamma = 0$ with $d\gamma \neq 0$ at M. Let $\theta = -i\partial\gamma$ (type (1,0) in $\mathbb{C}^{n+1}$; when restricted on M, θ is a contact form). Choose type (1, 0)-form θ^α, $\alpha = 1, \dots, n$, near M (locally) such that θ, θ^α, $\alpha = 1, \dots, n$, are independent and

$$d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}} + \rho\theta \wedge \bar{\theta} \tag{6.1}$$

(summation convention hereafter) at (points of) M (see Section 7: Appendix for a proof). Let $i\zeta$, Z_α, $\alpha = 1, \dots, n$ be type (1,0) vector fields dual to θ, θ^α, $\alpha = 1, \dots, n$. With this θ on M (note that $d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}}$ on M since $\theta = \bar{\theta}$ on TM), we have the pseudohermitian geometry and can talk about the torsion and covariant derivatives, etc. ([20]). Recall that $C^\infty(M, \mathbb{C})$ denote the set of C^∞ smooth complex-valued functions on M. Let $T_{i_M}\mathfrak{C}_c$ denote the space of all $\partial F_t/\partial t|_{t=0}$ where $F_t \in \mathfrak{C}_c : M \subset \mathbb{C}^{n+1} \to M_t \subset \mathbb{C}^{n+1}$ is a family of contact diffeomorphisms (w.r.t. contact structures induced from $\mathbb{C}^{n+1}$) with $F_t = i_M$ at $t = 0$.

LEMMA 6.1. *$T_{i_M}\mathfrak{C}_c$ (complex version) $= \{Y_f = i(f\zeta + f^{,\alpha}Z_\alpha) : f \in C^\infty(M, \mathbb{C})\}$, i.e., every "external" contact vector field on M with value in $T_{1,0}\mathbb{C}^{n+1}$ is of the form Y_f and vice versa.*

Proof. Take $Y \in T_{i_M}\mathfrak{C}_c$. That is to say, Y is an "external" contact vector field with value in $T_{1,0}\mathbb{C}^{n+1}$, i.e. (real version) $2\mathbb{R}eY = \partial F_t/\partial t|_{t=0}$ where $F_t \in \mathfrak{C}_c : M \subset \mathbb{C}^{n+1} \to M_t \subset \mathbb{C}^{n+1}$ is a family of contact diffeomorphisms with $F_t = i_M$ at $t = 0$. Let $\tilde{F}_t$ be a (C^∞) smooth extension of F_t to $\mathbb{C}^{n+1}$ so that $\gamma_t = \gamma \circ \tilde{F}_t^{-1}$ is a defining function of M_t. Let $\theta_t = -i\partial\gamma_t$ be a contact form on M_t. Since F_t is contact, we have

$$\tilde{F}_t^*\theta_t = \lambda_t\theta \tag{6.2}$$

on M. Differentiating (6.2) in t at $t = 0$ gives

$$L_{2\mathbb{R}e\tilde{Y}}\theta + \frac{\partial\theta_t}{\partial t}\Big|_{t=0} = \lambda\theta \tag{6.3}$$

where $\widetilde{Y}$ is an extension of Y near M and $\lambda = \partial\lambda_t/\partial t|_{t=0}$. Now write $Y = if\zeta + g^\alpha Z_\alpha$. It follows from the basic formula: $L_x = d \circ i_X + i_X \circ d$ where i_X is the interior product in the direction X that

$$L_{2\mathbb{R}e\tilde{Y}}\theta = df + ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}}(2\mathbb{R}eY, \cdot) \tag{6.4}$$

on $T_{1,0}(M) \oplus T_{0,1}(M)$ by (6.1), where $T_{1,0}(M)$ ($T_{0,1}(M)$, resp.) denotes the space of type (1,0) ((0,1), resp.) tangent vectors. On the other hand, compute

$$\begin{aligned} \frac{\partial\theta_t}{\partial t}\Big|_{t=0} &= -i\partial\Big(\frac{\partial\gamma_t}{\partial t}\Big|_{t=0}\Big) \\ &= -\partial(d\gamma(-2\mathbb{R}eY))(\text{since } \frac{\partial\tilde{F}_t^{-1}}{\partial t}\Big|_{t=0} = -2\mathbb{R}eY) \\ &= -\partial(f - \bar{f}) \quad (\text{by } d\gamma = \partial\gamma + \bar{\partial}\gamma). \end{aligned} \tag{6.5}$$

Applying (6.3) to $Z_{\bar{\alpha}}$ and using (6.4) and (6.5), we obtain

$$f_{,\bar{\alpha}} + ih_{\beta\bar{\alpha}}g^{\beta} = 0.$$

Raising the indices gives $g^{\alpha} = if^{,\alpha}$, as expected. Conversely, given Y_f, we need to construct a one-parameter family of contact embeddings $\Sigma_t \in \mathfrak{E}_c$, such that $\dot{\Sigma}_t|_{t=0} := \frac{d}{dt}|_{t=0}\Sigma_t = 2\mathbb{R}eY_f$. First we can find embeddings $\varphi_t \in \mathfrak{E}_m$ for t small with

$$\dot{\varphi}_t|_{t=0} = 2\mathbb{R}eY_{iImf} \quad \text{and } \varphi_0 = i_M. \tag{6.6}$$

Second, by Gray's theorem, there exists a one-parameter family of diffeomorphisms ψ_t on M with $\psi_0 = Id$ such that $\varphi_t \circ \psi_t \in \mathfrak{E}_c$ for all t. Since $\psi_t = (\varphi_t \circ \psi_t) - \varphi_t$ at $t = 0$ and $(\varphi_t \circ \psi_t)' = 2\mathbb{R}eY_g$ for some g from the previous argument, hence

$$\dot{\psi}_t|_{t=0} = 2\mathbb{R}eY_h \tag{6.7}$$

for some h by the linearity of Y_f in f. Moreover, note that $\dot{\psi}_t|_{t=0}$ is a tangent vector field on M and $\mathrm{Im}\,\zeta$ is tangent to M. Therefore h must be a real function. By a result in [12] (for the smooth tame category), there exists a one-parameter family of contact diffeomorphisms ρ_t on M with $\rho_0 = Id$ such that

$$\dot{\rho}_t|_{t=0} = 2\mathbb{R}eY_{\mathbb{R}ef-h}. \tag{6.8}$$

Now set $\Sigma_t = \varphi_t \circ \psi_t \circ \rho_t \in \mathfrak{E}_c$. Easy to see that $\dot{\Sigma}_t|_{t=0} = 2\mathbb{R}e(Y_{iImf} + Y_h + Y_{\mathbb{R}ef-h}) = 2\mathbb{R}eY_f$ by (6.6), (6.7), and (6.8). □

We remark that Y_f can be characterized by conditions: $Y_f \lrcorner\, \theta = f$, $Y_f \lrcorner\, d\theta = -\bar{\partial}_b f \pmod{\bar{\theta}}$. Define a second order operator $\mathfrak{D}_J$ as follows:

$$\mathfrak{D}_J f = (f_{,\bar{\alpha}}{}^{\beta} - iA_{\bar{\alpha}}{}^{\beta} f)\theta^{\bar{\alpha}} \otimes Z_{\beta} \tag{6.9}$$

for $f \in C^{\infty}(M, \mathbb{C})$, say, where $A_{\bar{\alpha}}{}^{\beta}$ is the torsion ([20]). Let T' denote $T_{1,0}\mathbb{C}^{n+1}$ restricted on M. We often do not distinguish between bundles and their sections. Define $\bar{\partial}_b : T' \to T^*_{0,1}(M) \otimes T'$ by

$$\bar{\partial}_b(f^{\ell}\frac{\partial}{\partial z^{\ell}}) = (\bar{\partial} f^{\ell}) \otimes \frac{\partial}{\partial z^{\ell}}$$

for $f^{\ell} \in C^{\infty}(M, \mathbb{C})$, $\ell = 1, \ldots, n+1$. Easy to check that the above definition is independent of choices of local coordinates. For $\bar{Z} \in T_{0,1}(M)$, $Y \in T'$, we have the formula

$$(\bar{\partial}_b Y)(\bar{Z}) = \pi_{1,0}([\tilde{(\bar{Z})}, \tilde{Y}]) \text{ restricted to } M \tag{6.10}$$

where $\tilde{(\bar{Z})}, \tilde{Y}$ denote local (C^{∞} smooth) extensions of $\bar{Z}$, Y to $\mathbb{C}^{n+1}$ near M, resp. and $\pi_{1,0}$ denotes the projection to $T_{1,0}\mathbb{C}^{n+1}$.

LEMMA 6.2. $\bar{\partial}_b Y_f = i\mathfrak{D}_J f$.

Proof. Let $\tilde{f}$ be a local extension of f to $\mathbb{C}^{n+1}$ near M. From (6.10) we compute

$$\begin{aligned}(\bar{\partial}_b Y_f)(Z_{\bar{\alpha}}) &= \pi_{1,0}([Z_{\bar{\alpha}}, Y_{\tilde{f}}] \text{ restricted to } M \\ &= \theta([Z_{\bar{\alpha}}, Y_{\tilde{f}}])i\xi + \theta^{\beta}([Z_{\bar{\alpha}}, Y_{\tilde{f}}])Z_{\beta}.\end{aligned} \tag{6.11}$$

On the other hand,

$$\theta([Z_{\bar\alpha},Y_{\tilde f}]) = Z_{\bar\alpha}(\theta(Y_{\tilde f})) - Y_{\tilde f}(\theta(Z_{\bar\alpha})) - d\theta(Z_{\bar\alpha},Y_{\tilde f}) \\ = \tilde f_{,\bar\alpha} - 0 - \tilde f_{,\bar\alpha} = 0 \tag{6.12}$$

by (6.1) (or (7.1) in Section 7: Appendix). We also have

$$\theta^\beta([Z_{\bar\alpha},Y_{\tilde f}]) = Z_{\bar\alpha}(\theta^\beta(Y_{\tilde f})) - Y_{\tilde f}(\theta^\beta(Z_{\bar\alpha})) - d\theta^\beta(Z_{\bar\alpha},Y_{\tilde f}) \\ = if_{,\bar\alpha}{}^\beta - 0 + A_{\bar\alpha}{}^\beta f \tag{6.13}$$

on M by (7.3) in Section 7: Appendix. Substituting (6.12) and (6.13) into (6.11), we obtain the desired formula. □

Define a map $\chi_f : M \to \mathbb{C}^{n+1}$ by

$$\chi_f(p) = p + Y_f(p) \tag{6.14}$$

for $p \in M$ considered as a position vector in $\mathbb{C}^{n+1}$. For f small, χ_f is an embedding.

LEMMA 6.3. *The map : $f \in Ker\mathfrak{D}_J \to \chi_f \in \{CR \text{ maps} : (M,\xi,J) \to \mathbb{C}^{n+1}\}$ is a one-one correspondence. Note that for f small, the set of CR maps can be replaced by the set of CR embeddings near the inclusion map i_M.*

Proof. It is clear that χ_f is CR for $f \in Ker\mathfrak{D}_J$ by Lemma 6.2. Also obviously the map is injective since $Y_f = Y_g$ implies $f = g$. For surjectivity, we observe that $Y(p) = \varphi(p) - p$ for a given CR map $\varphi : M \to \mathbb{C}^{n+1}$ is an "external" CR (hence contact) vector field on M. By Lemma 6.1, $Y = Y_f$ for some $f \in C^\infty(M,\mathbb{C})$. Moreover, we have $i\mathfrak{D}_J f = \bar\partial_b Y_f = \bar\partial_b Y = 0$. Hence $f \in Ker\mathfrak{D}_j$ and $\varphi = \chi_f$. □

Proof of Theorem E. Note that the equivalence class $\{\varphi \in \mathfrak{E}_c : \varphi \sim i_M\}$ of i_M is equal to the set of CR embeddings near (isotopic to) i_M. Hence the tangent space $T_{[i_M]}(\mathfrak{E}_c/_\sim)$ of $\mathfrak{E}_c/_\sim$ at $[i_M]$ is expected to identify with $\{Y_f\}/\{Y_f : CR\}$ which is in one-one correspondence to $C^\infty(M,\mathbb{C})/Ker\mathfrak{D}_J$ according to Lemma 6.1 and Lemma 6.2.

The map $\varphi_t \in \mathfrak{E}_c \to \varphi_t^* J_{\mathbb{C}^2} \in \mathfrak{J}_c$ induces a tangential map:

$$2\mathbb{R}eY_f \text{ (real version)} = \varphi_t|_{t=0} \in T_I\mathfrak{E}_c \to \frac{d}{dt}\Big|_{t=0}\varphi_t^* J_{\mathbb{C}^2} = L_{2\mathbb{R}eY_f} J_{\mathbb{C}^2} \in T_J\mathfrak{J}_c.$$

A similar argument as in the proof of Lemma 3.5 in [11] shows that

$$L_{Y_f} J_{\mathbb{C}^2} = 2\mathfrak{D}_J f = -2i\bar\partial_b Y_f \tag{6.15}$$

(the last equality is due to Lemma 6.2.). Hence

$$L_{2\mathbb{R}eY_f} J_{\mathbb{C}^2} = 4\mathbb{R}e(\mathfrak{D}_J f) = 4Im(\bar\partial_b Y_f)$$

by (6.15) (note that the operator $\mathfrak{D}_J$ and the real operator D_J (=B'_J in [11]) is related as below:

$$2\mathbb{R}e\mathfrak{D}_J f = D_J f_r + JD_J f_c$$

for $f = f_r + if_c$). In summary, we have shown the commutative diagram (1.10). □

7. Appendix. Let M be a strongly pseudoconvex real hypersurface in $\mathbb{C}^{n+1}$. Let γ be a defining function of M, i.e., $\gamma = 0$, $d\gamma \neq 0$ at M. Set $\theta = -i\partial\gamma$. Then

LEMMA A.1. *There exist θ^α, $\alpha = 1, \dots, n$ of type $(1,0)$-forms in $\mathbb{C}^{n+1}$ locally near M such that θ and θ^α, $\alpha = 1, \dots, n$ are independent and*

$$d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}} + \rho\theta \wedge \bar{\theta} \tag{7.1}$$

at (points of) M with some function ρ and $(h_{\alpha\bar{\beta}})$ being positive hermitian. Furthermore, $h_{\alpha\bar{\beta}}$ can be chosen to be $c_\alpha\delta_{\alpha\beta}$ with c_α positive constant on M.

Proof. Since $d\theta = -i\bar{\partial}\partial\gamma$ is of type (1,1), we have

$$d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}} + g_\alpha\theta \wedge \theta^{\bar{\alpha}} + h_\alpha\theta^\alpha \wedge \bar{\theta} + \rho\theta \wedge \bar{\theta}$$

for any choice of θ^α, $\alpha = 1, \dots, n$ (such that θ and θ^α's are independent). On TM, $\theta = \bar{\theta}$. Hence $(h_{\alpha\bar{\beta}})$ is hermitian and positive due to strongly pseudoconveacity of M and $g_\alpha = -\bar{h}_\alpha$ on M. Choose U^α_β, v^α in $\mathbb{C}^{n+1}$ near M such that $\tilde{h}_{\alpha\bar{\beta}}U^\alpha_m U^{\bar{\beta}}_{\bar{\ell}} = h_{m\bar{\ell}}$ with $\tilde{h}_{\alpha\bar{\beta}}$ (positive hermitian) prescribed, and $i\tilde{h}_{\alpha\bar{\beta}}v^\alpha U^{\bar{\beta}}_{\bar{\ell}} = g_\ell$. Let $\tilde{\theta}^\alpha = U^\alpha_\beta\theta^\beta + v^\alpha\theta$. Then it follows that

$$\begin{aligned} d\theta &= i\tilde{h}_{\alpha\bar{\beta}}\tilde{\theta}^\alpha \wedge \tilde{\theta}^{\bar{\beta}} + (h_\alpha + \bar{g}_\alpha)\theta^\alpha \wedge \bar{\theta} + \rho\theta \wedge \bar{\theta} \text{ near } M \\ &= i\tilde{h}_{\alpha\bar{\beta}}\tilde{\theta}^\alpha \wedge \tilde{\theta}^{\bar{\beta}} + \rho\theta \wedge \bar{\theta} \text{ at } M. \end{aligned}$$

∎

From the above proof we also see that there exist θ^α's such that

$$d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}} + f_\alpha\theta^\alpha \wedge \bar{\theta} + \rho\theta \wedge \bar{\theta} \tag{7.2}$$

near M with $h_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$ and $f_\alpha = 0$ on M.

LEMMA A.2. *With θ^α, $\alpha = 1, \dots, n$ satisfying (7.2), there exist connection forms ${\omega_\beta}^\alpha$, torsion ${A^\alpha}_{\bar{\beta}}$ such that*

$$d\theta^\alpha = \theta^\beta \wedge {\omega_\beta}^\alpha + {A^\alpha}_{\bar{\beta}}\theta \wedge \theta^{\bar{\beta}} + \lambda\theta \wedge \bar{\theta} \tag{7.3}$$

with ${A^\alpha}_{\bar{\beta}} = {A_{\bar{\alpha}}}^\beta$ near M and ${\omega_\beta}^\alpha + {\omega_{\bar{\alpha}}}^{\bar{\beta}} = 0$ at M.

Note that last terms in (7.1) and (7.3) disappear if restricted to TM (where $\theta = \bar{\theta}$).

Proof. It is clear that near M,

$$d\theta^\alpha = \theta^\beta \wedge {\omega_\beta}^\alpha + \theta \wedge \tau^\alpha + \lambda\theta \wedge \bar{\theta} \tag{7.4}$$

for certain 1-forms ${\omega_\beta}^\alpha$ and τ^α being of type $(0,1)$ since $\{\theta, \theta^\alpha = 1, \dots, n\}$ spans $T_{1,0}\mathbb{C}^{n+1}$. Let $\omega_{\gamma\bar{\beta}} = {\omega_\gamma}^\beta$, $\omega_{\bar{\beta}\gamma} = \overline{({\omega_\beta}^\gamma)}$, $\tau_\alpha = \overline{(\tau^\alpha)}$ and $\tau_{\bar{\alpha}} = \tau^\alpha$ (note that $h_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$). Write $df_\alpha - f_\beta\omega^\beta_\alpha = f_{\alpha,\beta}\theta^\beta + f_{\alpha,\bar{\beta}}\theta^{\bar{\beta}} (\text{mod } \theta, \bar{\theta})$. Now we differentiate (7.2) using (7.4) to get

$$\begin{aligned} 0 = {} & i(-\omega_{\alpha\bar{\beta}} - \omega_{\bar{\beta}\alpha} + E_{\alpha\bar{\beta}}) \wedge \theta^\alpha \wedge \theta^{\bar{\beta}} + i\theta \wedge (\tau_{\bar{\beta}} \wedge \theta^{\bar{\beta}}) \\ & -i\bar{\theta} \wedge (\tau_\alpha \wedge \theta^\alpha + if_{\alpha\beta}\theta^\beta \wedge \theta^\alpha) (\text{mod } \theta \wedge \bar{\theta}) \end{aligned} \tag{7.5}$$

where the error term $E_{\alpha\bar{\beta}} = -\delta_{\alpha\beta} f_\ell \theta^\ell + (if_\alpha \bar{f}_\beta - \rho\delta_{\alpha\beta})\theta + (if_{\alpha\bar{\beta}} + \rho\delta_{\alpha\beta})\bar{\theta}$ satisfies the condition:

$$E_{\alpha\bar{\beta}} = 0 \text{ at } M. \tag{7.6}$$

From (7.5) and writing $\tau^\alpha = A^\alpha{}_{\bar{\beta}}\theta^{\bar{\beta}}$, we can express

$$-\omega_{\alpha\bar{\beta}} - \omega_{\bar{\beta}\alpha} + E_{\alpha\bar{\beta}} = A_{\alpha\bar{\beta}\ell}\theta^\ell + B_{\alpha\bar{\beta}\bar{\ell}}\theta^{\bar{\ell}} \tag{7.7}$$

with $A_{\alpha\bar{\beta}\ell} = A_{\ell\bar{\beta}\alpha}$, $B_{\alpha\bar{\beta}\bar{\ell}} = B_{\alpha\bar{\ell}\bar{\beta}}$ and clearly $A^\alpha{}_{\bar{\beta}} = A_{\bar{\alpha}}{}^\beta$. We adjust $\omega_{\alpha\bar{\beta}}$ by $\tilde{\omega}_{\alpha\bar{\beta}} = \omega_{\alpha\bar{\beta}} + A_{\alpha\bar{\beta}\ell}\theta^\ell$. Easy to verify that (7.4) holds with $\omega_\beta{}^\alpha$ replaced by $\tilde{\omega}_\beta{}^\alpha$. Compute

$$\tilde{\omega}_{\alpha\bar{\beta}} + \tilde{\omega}_{\bar{\beta}\alpha} = (A_{\bar{\beta}\alpha\bar{\ell}} - B_{\alpha\bar{\beta}\bar{\ell}})\theta^{\bar{\ell}} + E_{\alpha\bar{\beta}} \tag{7.8}$$

by (7.7), where $A_{\bar{\beta}\alpha\bar{\ell}} = \overline{(A_{\beta\bar{\alpha}\ell})}$. Note that from (7.7), $A_{\bar{\beta}\alpha\bar{\ell}} = B_{\alpha\bar{\beta}\bar{\ell}}$ at M by hermitian symmetry of $\omega_{\alpha\bar{\beta}}$ and (7.6). Therefore it follows from (7.8) that at M

$$\tilde{\omega}_{\alpha\bar{\beta}} + \tilde{\omega}_{\bar{\beta}\alpha} = 0.$$

□

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