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SELF-ADJOINT STURM-LIOUVILLE PROBLEMS WITH DISCONTINUOUS BOUNDARY CONDITIONS*

AIPING WANG† AND ANTON ZETTL‡

Abstract. We present a general framework for the study of self-adjoint Sturm-Liouville problems with discontinuous boundary conditions specified at interior points of the underlying interval. Some regular such conditions have been studied and are known by various names including transmission conditions, interface conditions, multi-point conditions, point interactions (in the Physics literature) etc. Using this framework we generate additional regular conditions and singular analogues of all these regular conditions. In the singular case the solutions and their (quasi) derivatives may have finite as well as infinite jump discontinuities.

Key words. Self-adjoint domains, transmission and interface conditions, point interactions.

AMS subject classifications. Primary 34B20, 34B24; Secondary 47B25.

1. Introduction. Recently there has been a lot of interest in the literature in self-adjoint Sturm-Liouville (SL) problems with discontinuous boundary conditions specified at regular interior points of the underlying interval. Such conditions are known by various names including: transmission conditions [8], [9], [21], [24], [25], [36], interface conditions [17], [20], [33], [41], discontinuous conditions [14], [29], [22], multi-point conditions [16], [20], [12], [40], point interactions (in the Physics literature) [13], [2], [3], [5] conditions on trees graphs or networks [26], [24], [25], etc. For an informative survey of such problems arising in applications including an extensive bibliography and historical notes, see Pokornyi-Borovskikh [24] and Prokornyi-Pryadiev [25].

Such problems are not covered by the classical SL theory since, in this theory, solutions and their (quasi) derivatives are continuous at all interior points. In particular this applies to all eigenfunctions.

Here we present a framework for studying self-adjoint SL problems with interior discontinuities. This framework provides a unified theory for the study of the problems referenced above and enlarges this class of problems. Furthermore, it generates natural singular analogues of these regular problems. At singular interior points the jump discontinuities are infinite, in general.

In this paper by a self-adjoint Sturm-Liouville problem we mean a problem which generates a self-adjoint operator in some Hilbert space. Thus its spectrum is real and the well developed theory of self-adjoint operators in Hilbert space, such as the spectral theorem for unbounded self-adjoint operators, can be applied to study its spectrum. We do not consider problems in Krein or Symplectic Geometry spaces here.

The celebrated classical special functions of Mathematics associated with the names of Bessel, Fourier, Heun, Ince, Jacobi, Jörgens, Latzko, Legendre, Littlewood-McLeod, Mathieu, Meissner, Morse etc. are eigenfunctions of self-adjoint SL problems in Hilbert space. These problems also play an important role in the study of the

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†Mathematics Department, Harbin Institute of Technology, Harbin, 150001, China (wapxf@163.com).

‡Mathematics Department, Northern Illinois University, DeKalb, Il. 60115, USA (zettl@math.niu.edu).

harmonic oscillator and hydrogen atom equations, the one-dimensional Schrödinger equation, the Laplace tidal wave equation, etc. See the book [42] for these and other examples and for basic facts and definitions about SL problems.

Spectral properties of the classic SL problems are studied by considering the symmetric SL equation

$$My = -(py')' + qy = \lambda wy \quad \text{on} \quad J = (a,b),\ \lambda \in \mathbb{C},\ -\infty \le a < b \le \infty, \tag{1.1}$$

with coefficients p, q and weight function w satisfying

$$p^{-1}, q, w \in L_{loc}(J, \mathbb{R}), \quad w > 0 \ \ a.e.\ on\ J, \tag{1.2}$$

where $L_{loc}(J, \mathbb{R})$ denotes the real valued functions which are Lebesgue integrable on all compact subintervals of J. For later reference we note that there is no sign restriction on q *or* p in (1.2).

For convenience we let $y^{[1]} = (py')$, this is called the quasi-derivative of y. Under conditions (1.2) every solution y of (1.1) and its quasi-derivative $y^{[1]}$ is defined and continuous on J (but $y'(t)$ may not exist for some t in J).

Equation (1.1) generates minimal and maximal operators $S_{\min}$ and $S_{\max}$ in the Hilbert space $L^2(J, w)$ and self-adjoint operators S in this space. Each of the above mentioned classical problems has such an operator realization S. These operators S satisfy

$$S_{\min} \subset S = S^* \subset S_{\max}. \tag{1.3}$$

From (1.3) it is clear that these operators S are distinguished from each other only by their domains. These domains can be determined by boundary conditions specified only at the endpoints a, b of the interval J. (See Section 2 below.)

The organization of this paper is as follows. In Section 2 we review the boundary conditions characterizing the operators S satisfying (1.3). Here we call this the 1-interval theory. From (1.2) it follows that all solutions y of (1.1) together with their quasi-derivatives $y^{[1]}$ are continuous on the open interval J. In particular, the eigenfunctions and their derivatives $y^{[1]}$ of any self-adjoint operator S satisfying (1.3) are continuous on J. The 2-interval theory developed by Everitt and Zettl in [4] is presented in Section 3. In Section 4 we show that this 2-interval theory generates self-adjoint boundary conditions with discontinuities at interior points and that this includes all regular self-adjoint discontinuous conditions in the literature that we are aware of and some additional ones including coupled condition determined by any real nonsingular coupling matrix K and *nonreal* self-adjoint discontinuous conditions. In the 1-interval theory K must satisfy $\det(K) = 1$. Mukhtarov and Yakubov [21] showed that in the 2-interval theory $\det(K) > 0$ is sufficient and here we extend this further to K nonsingular. Furthermore, this theory produces natural singular analogues of all these regular conditions. A number of illustrative examples are given.

Section 5 is devoted to the Legendre equation on the interval $(-1, \infty)$. Here we give explicit examples of singular self-adjoint transmission and interface conditions at the interior singular point 1. We have chosen the Legendre equation to illustrate some of the singular self-adjoint conditions primarily for two reasons: (i) it is one of the celebrated SL equations and (ii) the singular self-adjoint conditions can be given explicitly.

Remarks on the title of this paper are given at the end of Sections 2 and 4.

2. The One Interval Theory. Let $\mathbb{R}$ denote the reals, $\mathbb{C}$ the complex numbers, and $M_2(F)$ the 2×2 matrices over F for $F = \mathbb{R}$ or $F = \mathbb{C}$.

For definitions of technical terms used here such as the operators $S_{\min}$ and $S_{\max}$ and their domains $D_{\max}$ and $D_{\min}$ the reader is referred to the book [42]. Also proofs or references to proofs not given here can be found in this book. Below we use the notation $D_{\max}(a, c)$ etc. to indicate the dependence on the interval (a, c). Note that (1.2) holds when (a, b) is replaced by (a, c) or (c, b) for any $c \in (a, b)$.

To characterize the domains of the operators S satisfying (1.3) we start with some definitions.

Definition 1. The endpoint a is regular if $p^{-1}, q, w \in L(a, c)$ for some (and hence any) $c \in (a, b)$. Similarly b is regular if $p^{-1}, q, w \in L(c, b)$ for $a < c < b$. If an endpoint is not regular it is called singular. If a is singular it is said to be in the limit-circle (LC) case if all solutions of (1.1) are in the Hilbert space $H = L^2((a, c), w)$. This is known to hold for some $\lambda \in \mathbb{C}$ if and only if it holds for all $\lambda \in \mathbb{C}$. If a is singular and not LC it is said to be in the limit-point (LP) case. Similarly for the endpoint b. We say that an operator S in the Hilbert space $H = L^2(J, w)$ is a self-adjoint realization of equation (1.1) if and only if (1.3) holds.

Definition 2. The Lagrange form $[\cdot, \cdot]$ is defined, for all $y, z \in D_{\max}$ by

$$[y, z] = y\,(p\overline{z}') - \overline{z}(py'). \tag{2.1}$$

Definition 3 (Boundary Condition Basis). Assume the endpoint a is either regular or LC. A real valued function pair $(u, v) \in D_{\max}(a, c)$ is said to be a boundary condition basis at a if there exists a point $c \in (a, b)$ such that each of u, v is linearly independent modulo $D_{\min}(a, c)$ and normalized to satisfy $[u, v](a) = 1$. A similar definition is made for the endpoint b. A simple way to get such (u, v) at a is to take linearly independent real-valued solutions for any real λ in some interval (a, c) and normalize them as indicated. Similarly for b.

The number of boundary conditions needed to characterize the operators S satisfying (1.3) depends on the deficiency index d of $S_{\min}$ which depends on the classification of the endpoints a, b as regular, LC or LP. This classification depends on the coefficients p, q, w and this dependence is implicit and complicated. There is a vast literature on this dependence and much is known but there still exist equations (1.1) for which the LC/LP classification is not known, see [15]. The number d is given in terms of the endpoint classifications by the next Proposition.

Proposition 1. *The deficiency index d of $S_{\min}$ in $L^2(J, w)$ satisfies $0 \leq d \leq 2$ and all three values are realized. Furthermore*

1. *If $d = 0$ then $S_{\min}$ is self-adjoint and has no proper self-adjoint extension.*
2. *$d = 1$ if and only if one endpoint is LP and the other regular or LC.*
3. *$d = 2$ if and only if each endpoint is either regular or LC.*

Proof. See [42], [23], [38] for proofs. □

We can now state the characterization of all operators S which satisfy (1.3).

Theorem 1. *Let (1.1) and (1.2) hold. Then*

1. *If both endpoints are LP, then $S_{\min}$ is self-adjoint with no proper self-adjoint extension.*

2. *Suppose that a is LP and b is LC. Assume that (u,v) is a boundary condition basis at b. If $c,d\in\mathbb{R}$, $(c,d)\neq(0,0)$ and*

$$D(S)=\{y\in D_{\max}: c\,[y,u](b)+d\,[y,v](b)=0\} \tag{2.2}$$

then the operator S with domain $D(S)$ satisfies (1.3).
If a is LC, b is LP and (u,v) is a boundary condition basis at a, then replace b by a in (2.2).
If a is LP and b is regular, then (2.2) reduces to (but not necessarily with the same c,d.)

$$D(S)=\{y\in D_{\max}: c\,y(b)+d\,(py')(b)=0\}. \tag{2.3}$$

If a is regular and b is LP, then (2.2) reduces to

$$D(S)=\{y\in D_{\max}: c\,y(a)+d\,(py')(a)=0\}. \tag{2.4}$$

3. *Assume each of a and b are, independently, regular or LC and let (u_a,v_a), (u_b,v_b) be boundary condition bases at a and b, respectively. Suppose $A,B\in M_2(\mathbb{C})$ satisfy*

$$rank(A:B)=2 \text{ and } AEA^*=BEB^*,\quad E=\begin{pmatrix}0&-1\\1&0\end{pmatrix}. \tag{2.5}$$

If $D(S)=\{y\in D_{\max}:$

$$A\begin{bmatrix}[y,u_a](a)\\ [y,v_a](a)\end{bmatrix}+B\begin{bmatrix}[y,u_b](b)\\ [y,v_b](b)\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\}, \tag{2.6}$$

then $D(S)$ is the domain of a self-adjoint extension S satisfying (1.3). Moreover, for fixed (u_a,v_a), (u_b,v_b) all operators S satisfying (1.3) are generated this way.
Furthermore, if a is regular, then the term in (2.6) multiplied by A can be replaced by

$$\begin{bmatrix}y(a)\\ (py')(a)\end{bmatrix}. \tag{2.7}$$

Similarly, if b is regular, then the term in (2.6) multiplied by B can be replaced by

$$\begin{bmatrix}y(b)\\ (py')(b)\end{bmatrix}. \tag{2.8}$$

Thus if both a and b are regular, then (2.6) can be reduced to the more familiar regular self-adjoint boundary conditions

$$A\begin{bmatrix}y(a)\\ (py')(a)\end{bmatrix}+B\begin{bmatrix}y(b)\\ (py')(b)\end{bmatrix}=0. \tag{2.9}$$

It is well known that the boundary conditions (2.9) can be categorized into two mutually exclusive classes: separated and coupled. The separated conditions have the form (2.3) when b is regular and the same form with b replaced by a, *but not necessarily with the same* c, d*, when* a *is regular and these separated conditions have the familar canonical form*

$$\cos\beta\, y(b) - \sin\beta\, (py')(b) = 0,\ 0 < \beta \leq \pi \tag{2.10}$$

when b *is regular; and the canonical form*

$$\cos\alpha\, y(a) - \sin\alpha\, (py')(a) = 0,\ 0 \leq \alpha < \pi \tag{2.11}$$

when a *is regular. (The different parameterizations for* α *and* β *are customary and used for convenience in using the Prüfer transformation and stating results but these different parameterizations play no role in this paper.) The coupled singular conditions (2.6) have the canonical form*

$$Y(b) = e^{i\gamma} K\, Y(a),\ -\pi < \gamma \leq \pi,\ i = \sqrt{-1} \tag{2.12}$$

where $K \in M_2(\mathbb{R})$ *satisfies* $\det(K) = 1$ *and*

$$Y(a) = \begin{bmatrix} [y, u_a](a) \\ [y, v_a](a) \end{bmatrix},\ Y(b) = \begin{bmatrix} [y, u_b](b) \\ [y, v_b](b) \end{bmatrix}. \tag{2.13}$$

When b *is regular* $Y(b)$ *can be replaced by (2.8) and* $Y(a)$ *can be replaced by (2.7) when* a *is regular. So when both* a *and* b *are regular (2.12) can be reduced to*

$$Y(b) = \begin{bmatrix} y(b) \\ (py')(b) \end{bmatrix} = e^{i\gamma} K \begin{bmatrix} y(a) \\ (py')(a) \end{bmatrix},\ -\pi < \gamma \leq \pi,\ i = \sqrt{-1} \tag{2.14}$$

with $K \in M_2(\mathbb{R})$, $\det(K) = 1$.

Proof. See [42]. □

REMARK 1. *See Chapter 14 in [42] for the explicit boundary conditions which determine the problems generating the special functions associated with the names of Bessel, Chebychev, Fourier, Jacobi, Legendre, Morse, etc.*

REMARK 2. *For reference below, we note that the characterization of the self-adjoint operators* S *satisfying (1.3) given by Theorem 1 is unchanged if the usual inner product*

$$(f, g) = \int_J f\,\overline{g}\, w \tag{2.15}$$

in $H = L^2(J, w)$ *is replaced by*

$$(f, g) = h \int_J f\,\overline{g}\, w \tag{2.16}$$

for any $h > 0$.

REMARK 3. From another perspective, we can say that the characterization of the self-adjoint operators S satisfying (1.3) given by Theorem 1 is unchanged if the weight function w is replaced by $h\,w$ where h is a positive constant. The positivity of

h is important for (2.16) to be an inner product and for the weight function to satisfy (1.2). However we make the following interesting observation: The characterization given by Theorem 1 remains valid and unchanged if the weight function w is replaced by $h\,w$ where h is any positive or **negative** constant. In the negative case write the right hand side of equation (1.1) as $(-\lambda)(-h\,w)y$ and use the previous observation for the positive weight function $(-h\,w)$. This leaves the characterization of the self-adjoint operators given by Theorem 1 unchanged. However, the spectrum of the operator changes, in particular if λ is changed to $-\lambda$ then the spectrum is 'flipped' accordingly.

Remark 4. The simple observations of Remarks 2 and 3 take on added significance in the 2-interval theory as we will see below. In particular, they are used to extend the self-adjointness condition $\det(K) = 1$ to $\det(K) > 0$ using Remark 2 as observed by Mukhtarov and Yakubov [21]. Using Remark 3 we obtain the further extension to K nonsingular. This seems to be new.

Remark 5. From the perspective of the modern classical 1-interval theory, we can say that Theorem 1 characterizes all self-adjoint SL operators in the Hilbert space $L^2(J, w)$. It follows from (1.2) that all solutions of equation (1.1) and their quasi-derivatives are continuous on the interval $J = (a, b)$. In particular, the eigenfunctions of every self-adjoint operator S satisfying (1.3) are continuous on J.

Remark 6. However, we will see in Section 4 that the 2-interval theory, when specialised to adjacent intervals, produces more self-adjoint operators in $L^2(J, w)$.

3. The Two-Interval Theory. Motivated by applications, in particular the paper of Boyd [10] and its references, in 1986 Everitt and Zettl [4] introduced a framework for the rigorous study of Sturm-Liouville problems which have a singularity in the interior of the domain interval since the existing theory did not cover such cases. The Boyd paper, which was based on several previous papers by Atmospheric Scientists, studies eddies in the atmosphere using a mathematical model based on the SL problem

$$-y'' + \frac{1}{x}\,y = \lambda y,\ y(-1) = 0 = y(1),\ -1 < x < 1. \tag{3.1}$$

Note that 0 is a singular point in the interior of the underlying interval $(-1, 1)$ and condition (1.2) of the 1-interval theory does not hold.

The framework introduced in [4] is the direct sum of Hilbert spaces, one for each interval $(-1, 0)$ and $(0, 1)$. The primary goal of this study is the characterization of all self-adjoint realizations from the two intervals. A simple way of getting self-adjoint operators in the direct sum space is to take the direct sum of operators from the separate spaces. However, there are many self-adjoint operators in the direct sum space which are not obtained this way. These 'new' operators involve interactions between the two intervals.

Mukhtarov and Yakubov [21] observed that the set of self-adjoint operator realizations developed in [4] can be further enlarged by using different multiples of the usual inner products associated with each of the intervals. In [37], [34], Sun, Wang and Zettl use the Mukhtarov and Yakubov modification of the Everitt-Zettl theory in [4] to obtain very general self-adjoint two interval boundary conditions. In particular it is shown in [37], [34] that for coupled boundary conditions determined by a coupling matrix K the condition that $\det(K) = 1$ which is required in the 1-interval case can be replaced by $\det(K) > 1$. Here we extend this condition further to $\det(K) \neq 0$. This seems to be new.

Let

$$J_1 = (a,b),\ -\infty \leq a < b \leq \infty,\ J_2 = (c,d),\ -\infty \leq c < d \leq \infty,$$

and assume the coefficients and weight functions satisfy

$$p_r^{-1}, q_r, w_r \in L_{loc}(J_r, \mathbb{R}), \quad w_r > 0 \ a.e. \, on \ J_r, \quad r = 1, 2. \tag{3.2}$$

Note that the intervals J_1 and J_2 are independent; they may be disjoint, abut (have a common endpoint), overlap, or be identical.

Define differential expressions M_r by

$$M_r y = -(p_r \, y')' + q_r \, y \ on \ J_r, \quad r = 1, 2, \tag{3.3}$$

and consider the equations

$$M_r y = \lambda w_r \, y \ on \ J_r, \quad r = 1, 2. \tag{3.4}$$

Let

$$H_r = L^2(J_r, w_r),\ r = 1, 2.$$

A simple way of getting self-adjoint operators S in the direct sum space

$$H_u = H_1 + H_2,\ where\ H_r = L^2(J_r, w_r),\ r = 1, 2$$

is to take the direct sum of self-adjoint operators from H_1 and H_2. If these were all the self-adjoint operator realizations from the two intervals there would be no need for a "2-interval" theory. As noted in [4] there are many self-adjoint operators which are not merely the sum of self-adjoint operators from each of the separate intervals. These "new" self-adjoint operators involve interactions between the two intervals. Characterizing these interactions as explicitly as possible is our primary goal in this section.

Below we use the notation with a subscipt r to denote the $r - th$ interval. The subscript r is sometimes omitted when it is clear from the context. For basic facts, notation and terminology see the book [42].

Elements of $H_u = H_1 + H_2$ will be denoted in bold face type: $\mathbf{f} = \{f_1, f_2\}$ with $f_1 \in H_1,\ f_2 \in H_2$. The usual inner product in H_u is given by

$$(\mathbf{f}, \mathbf{g}) = (f_1, g_1)_1 + (f_2, g_2)_2, \tag{3.5}$$

where $(\cdot, \cdot)_r$ is the usual inner product in H_r :

$$(f_r, g_r)_r = \int_{J_r} f_r \, \overline{g_r} \, w_r,\ r = 1, 2. \tag{3.6}$$

Following Mukhtarov and Yakubov [21], we replace the direct sum inner product (3.5) by

$$\langle \mathbf{f}, \mathbf{g} \rangle = h \ (f_1, g_1)_1 + k \ (f_2, g_2)_2,\ h > 0,\ k > 0, \tag{3.7}$$

and study operator theory in the direct sum space

$$H = (L^2(J_1, w_1) \dotplus L^2(J_2, w_2),\ \langle\cdot,\cdot\rangle). \tag{3.8}$$

REMARK 7. Note that (3.7) is an inner product in H for any positive numbers h and k. The elements of the Hilbert space H defined by (3.7) are the same as those of the usual direct sum Hilbert space H_u, thus these spaces are differentiated from each other only by their inner products. As we will see below the parameters h, k influence the boundary conditions which yield self-adjoint realizations of the Sturm-Liouville equations in the 2-interval case. From another perspective, the Hilbert space (3.8) can be viewd as a 'usual' direct sum space H_u with summands $H_r = L^2(J_r, w_r)$ but with w_1 replaced by $h\,w_1$ and w_2 replaced by $k\,w_2$.

REMARK 8. Note that $w > 0$ ensures that $L^2(J, w)$ is a Hilbert space. However, if $w < 0$ on J we can multiply equation (1.1) by -1 to obtain

$$-(-py')' + (-q)y = \lambda(-w)y \text{ on } J, \tag{3.9}$$

and observe that the 1-interval theory applies to (3.9) since there is no sign restriction in (1.2) on either p or q and $-w > 0$. Also the boundary conditions are homogeneous and thus invariant with respect to multiplication by -1. Below we will apply these observations to one or both equations (3.3) to extend the restriction $\det(K) > 0$ to $\det(K) \neq 0$. The assumption that $p > 0$ in (1.2) is commonly used in the literature and in books but it is not needed for the characterization of the self-adjoint operators characterized by the equation. It is this fact which allows us to extend the Mukhtarov-Yakubov restriction $h > 0$, $k > 0$ to any $h, k \in \mathbb{R}$, $h \neq 0 \neq k$. (However we note that, in general, the spectral properties, the oscillatory behavior of eigenfunctions, etc. of a given self-adjoint operator S are different when p is not positive.)

As in the 1-interval case the Lagrange sesquilinear form $[\cdot,\cdot]$ is fundamental to the study of boundary value problems. It is defined, for $\mathbf{y} = \{y_1, y_2\}$, $\mathbf{z} = \{z_1, z_2\}$, $y_1, z_1 \in D_{\max}(J_1)$, $y_2, z_2 \in D_{\max}(J_2)$, by

$$[\mathbf{y}, \mathbf{z}] = h\,[y_1, z_1]_1(b) - h\,[y_1, z_1]_1(a) + k\,[y_2, z_2]_2(d) - k\,[y_2, z_2]_2(c), \tag{3.10}$$

where

$$[y_r, z_r]_r = y_r(p_r\overline{z_r'}) - \overline{z_r}(p_r y_r') = Z_r^* E Y_r \tag{3.11}$$

and

$$Y_r = \begin{pmatrix} y_r \\ y_r^{[1]} \end{pmatrix},\quad Z_r = \begin{pmatrix} z_r \\ z_r^{[1]} \end{pmatrix},\quad y_r^{[1]} = (p_r y_r'),\quad r = 1, 2;\quad E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.12}$$

Note that the 2-interval Lagrange form $[\mathbf{y}, \mathbf{z}]$ 'connects' all four endpoints with each other and depends on h and k.

The 2-interval maximal and minimal domains and operators are simply the direct sums of the corresponding 1-interval domains and operators:

$$D_{\max}(J_1, J_2) = D_{\max}(J_1) \dotplus D_{\max}(J_2);\quad D_{\min}(J_1, J_2) = D_{\min}(J_1) \dotplus D_{\min}(J_2). \tag{3.13}$$

$$S_{\max}(J_1, J_2) = S_{\max}(J_1) + S_{\max}(J_2);\quad S_{\min}(J_1, J_2) = S_{\min}(J_1) + S_{\min}(J_2). \tag{3.14}$$

Note that the maximal and minimal domains and operators do not depend on h and k.

DEFINITION 4. Let the hypotheses and notation above hold. By a self-adjoint realization of the two equations

$$-(p_r y')' + q_r y = \lambda w_r y \quad \text{on} \quad J_r, \quad r = 1, 2, \tag{3.15}$$

in the space $H = (L^2(J_1, w_1) \dot{+} L^2(J_2, w_2), \langle \cdot, \cdot \rangle)$ we mean an operator S from H into H which satisfies

$$S_{\min}(J_1, J_2) \subset S = S^* \subset S_{\max}(J_1, J_2). \tag{3.16}$$

From (3.16) it is clear that the 2-interval self-adjoint realizations are distinguished from each other only by their domains. It is the characterization of *these* domains in terms of boundary conditions which is a major goal of the 2-interval theory. Each operator S satisfying (3.16) can be considered an extension of the minimal operator $S_{\min}(J_1, J_2)$ or, equivalently, a restriction of the maximal operator $S_{\max}(J_1, J_2)$. Let d_1 be the deficiency index on J_1 and d_2 the deficiency index on J_2. For definitions of these and other technical terms used here see [42].

Our starting point for the 2-interval theory is

LEMMA 1. *We have*

1. $S^*_{\min}(J_1.J_2) = S^*_{\min}(J_1) + S^*_{\min}(J_2) = S_{\max}(J_1) + S_{\max}(J_2) = S_{\max}(J_1, J_2)$; $S^*_{\max}(J_1, J_2) = S^*_{\max}(J_1) + S^*_{\max}(J_2) = S_{\min}(J_1) + S_{\min}(J_2) = S_{\min}(J_1, J_2)$. *In particular,* $D_{\max}(J_1, J_2) = D(S_{\max}(J_1, J_2)) = D(S_{\max}(J_1)) + D(S_{\max}(J_2))$; $D_{\min}(J_1, J_2) = D(S_{\min}(J_1, J_2)) = D_{\min}(J_1) + D_{\min}(J_2)$.
2. *The minimal operator* $S_{\min}(J_1, J_2)$ *is a closed, symmetric, densely defined operator in the Hilbert space* H *with deficiency index* $d = d_1 + d_2$.

Proof. See Lemma 13.3.1 in [42]. Since the coefficients and weight functions are all real the upper and lower deficiency indices are equal and the common value is denoted by d in the 2-interval case and by d_1, d_2 for intervals 1 and 2, respectively. □

Using the 2-interval extension of the GKN Theorem developed by Everitt and Zettl in [4] together with the Mukhtarov and Yakubov modification [21] as further developed in [37], [34] together with our own modification indicated in Remark 8, we can now give our characterization of the self-adjoint operators of the 2-interval theory. We first prove the theorem for both of h, k positive and then use the observation of Remark 8 to extend the result to the case when one or both of h, k is negative.

We state the results for endpoints which are either LP or LC but indicate at the end of the theorem how the characterizations can be simplified at each regular endpoint.

THEOREM 2. *Let the 2-interval minimal and maximal domains* $D_{\min} = D_{\min}(J_1, J_2)$, $D_{\max} = D_{\max}(J_1, J_2)$, *and operators* $S_{\min} = S_{\min}(J_1, J_2)$, $S_{\max} = S_{\max}(J_1, J_2)$ *be defined as above. Let* d *denote the deficiency index of* $S_{\min}$ *in* H. *Then* $0 \leq d \leq 4$ *and all values in this range are realized. Let the Lagrange form* $[\cdot, \cdot]$ *be given by (3.10). Then all self-adjoint operators* S *satisfying (3.16) can be characterized as follows:*

Case 1. $d = 0$. *This case occurs if and only if all four endpoints are LP. In this case* $S_{min} = S_{max}$ *and* S_{min} *is a self-adjoint operator in* H *with no proper self-adjoint extension. Thus there are no boundary conditions in this case required or allowed.*

Also note that for all $\mathbf{f} = \{f_1, f_2\}$, $\mathbf{g} = \{g_1, g_2\} \in D_{max}$ *we have* $[f_1, g_1]_1(b) = 0$, $[f_1, g_1]_1(a) = 0$, $[f_2, g_2]_2(d) = 0$, $[f_2, g_2]_2(c) = 0$ *and therefore*

$$[\mathbf{f}, \mathbf{g}] = h\,[f_1, g_1]_1(b) - h\,[f_1, g_1]_1(a) + k\,[f_2, g_2]_2(d) - k\,[f_2, g_2]_2(c) = 0. \tag{3.17}$$

Case 2. $d = 1$. *This case occurs if and only if exactly three endpoints are LP; the other, say* $s \in \{a, b, c, d\}$, *is regular or LC. Let* (u, v) *be a boundary condition basis at* s. *Then*

$$\begin{aligned} D(S) = \{\mathbf{y} = \{\mathbf{y}_1, y_2\} \in D_{max} : c_{11}[y, u](s) + c_{12}[y, v](s) = 0, \\ c_{11}, c_{12} \in \mathbb{R}, \quad (c_{11}, c_{12}) \neq (0, 0)\} \end{aligned} \tag{3.18}$$

is a self-adjoint domain. Conversely, if $D(S)$ *is a self-adjoint domain, then there exist* $c_{11}, c_{12} \in \mathbb{R}$ *with* $(c_{11}, c_{12}) \neq (0, 0)$, *such that (3.18) holds.*

To summarize this case we say that all self-adjoint extensions of the minimal operator are determined by a separated boundary conditions of the form (3.18) at the non LP endpoint s.

Case 3. $d = 2$. *This case occurs if and only if exactly two of the four endpoints are LP. There are two subcases.*

(i) Say a, b *are the two non LP endpoints.*

In this case all the self-adjoint extensions S *in* H *are given by* $S = S(J_1) \dotplus S_{\min}(J_2)$ *where* $S(J_1)$ *is an arbitrary self-adjoint extension in* H_1 *obtained from the 1-interval theory on* J_1. *Note that* $S_{\min}(J_2)$ *is self-adjoint by the 1-interval theory since both endpoints* c, d *are LP and all* $S(J_1)$ *are obtained from the 1-interval theory discussed in Section 2. There is a similar result when* a, b *are both LP and* c, d *are non LP.*

To summarize this case we can say that all self-adjoint operators in H *are obtained simply as direct sums of the minimal operator from the interval with the two LP endpoints together with all the self-adjoint operators from the other interval and these are charaterized by the 1-interval theory of Section 2.*

(ii) The two non-LP endpoints are from different intervals. In this subcase there are nontrivial interactions between the two intervals which are not directly obtainable from the 1-interval theory. These depend on h *and* k *when the boundary conditions are coupled.*

Assume that b *and* c *are the two non LP endpoints. Let* (u_1, v_1) *be a boundary basis at* b *and* (u_2, v_2) *a boundary basis at* c.*If the matrices* $B, C \in M_2(\mathbb{C})$ *satisfy the two conditions:*

1. *The matrix* $(B : C)$ *has full rank.*
2. *For some* $h, k \in \mathbb{R}$, $h \neq 0 \neq k$,

$$k\,BEB^* - h\,CEC^* = 0, \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{3.19}$$

then $D(S) = \{\mathbf{y} = \{y_1, y_2\} \in D_{\max}$ *such that*

$$B\,\mathbf{Y}_1(b) + C\,\mathbf{Y}_2(c) = 0\}, \tag{3.20}$$

where

$$\mathbf{Y}_1(b) = \begin{bmatrix} [y_1, u_1]_1(b) \\ [y_1, v_1]_1(b) \end{bmatrix}, \ \mathbf{Y}_2(c) = \begin{bmatrix} [y_2, u_2]_2(c) \\ [y_2, v_2]_2(c) \end{bmatrix}, \tag{3.21}$$

is the domain of a self-adjoint operator S in H which satisfies (3.16) and every operator S in H satisfying (3.16) is obtained this way.

Assume that a and d are the two non LP endpoints. Let (u_1, v_1) be a boundary basis at a and (u_2, v_2) a boundary basis at d.If $A, D \in M_2(\mathbb{C})$ satisfy the two conditions:

1. *The matrix $(A : D)$ has full rank.*
2. *For some $h, k \in \mathbb{R}$, $h \neq 0 \neq k$,*

$$k\, AEA^* - h\, DED^* = 0, \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{3.22}$$

then $D(S) = \{\mathbf{y} = (y_1, y_2) \in D_{\max}$ such that

$$A\mathbf{Y}_1(a) + D\mathbf{Y}_2(d) = 0\}, \tag{3.23}$$

where

$$\mathbf{Y}_1(a) = \begin{bmatrix} [y_1, u_1]_1(a) \\ [y_1, v_1]_1(a) \end{bmatrix}, \ \mathbf{Y}_2(d) = \begin{bmatrix} [y_2, u_2]_2(d) \\ [y_2, v_2]_2(d) \end{bmatrix}, \tag{3.24}$$

is the domain of a self-adjoint operator S in H satisfying (3.16) and every operator S in H satisfying (3.16) is obtained this way.

Assume that a and c are the two non LP endpoints. Let (u_1, v_1) is a boundary basis at a and (u_2, v_2) be a boundary basis at c. If $A, C \in M_2(\mathbb{C})$ satisfy the two conditions:

1. *The matrix $(A : C)$ has full rank.*
2. *For some $h, k \in \mathbb{R}$, $h \neq 0 \neq k$,*

$$k\, AEA^* + h\, CEC^* = 0, \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{3.25}$$

then $D(S) = \{\mathbf{y} = (y_1, y_2) \in D_{\max}$:

$$A\mathbf{Y}_1(a) + C\mathbf{Y}_2(c) = 0\}, \tag{3.26}$$

where

$$\mathbf{Y}_1(a) = \begin{bmatrix} [y_1, u_1]_1(a) \\ [y_1, v_1]_1(a) \end{bmatrix}, \ \mathbf{Y}_2(c) = \begin{bmatrix} [y_2, u_2]_2(c) \\ [y_2, v_2]_2(c) \end{bmatrix}, \tag{3.27}$$

is the domain of a self-adjoint operator S in H satisfying (3.16) and every operator S in H satisfying (3.16) is obtained this way.

Assume that b and d are the two non LP endpoints. Let (u_1, v_1) be a boundary basis at b and (u_2, v_2) a boundary basis at d. If $B, D \in M_2(\mathbb{C})$ satisfy the two conditions:

1. *The matrix $(B : D)$ has full rank.*
2. *For some $h, k \in \mathbb{R}$, $h \neq 0 \neq k$,*

$$k\, BEB^* + h\, DED^* = 0, \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{3.28}$$

then $D(S) = \{\mathbf{y} = (y_1, y_2) \in D_{\max}$ such that

$$B\,\mathbf{Y}_1(b) + D\,\mathbf{Y}_2(d) = 0\}, \tag{3.29}$$

where

$$\mathbf{Y}_1(b) = \begin{bmatrix} [y_1, u_1]_1(b) \\ [y_1, v_1]_1(b) \end{bmatrix}, \ \mathbf{Y}_2(d) = \begin{bmatrix} [y_2, u_2]_2(d) \\ [y_2, v_2]_2(d) \end{bmatrix}, \tag{3.30}$$

is the domain of a self-adjoint operator S in H satisfying (3.16) and every operator S in H satisfying (3.16) is obtained this way.

Case 4. $d = 3$. *In this case there is exactly one LP endpoint. Assume that a is LP. Let (u_1, v_1) be a boundary basis at b, (u_2, v_2) a boundary basis at c and (u_3, v_3) a boundary basis at d. If $B = (b_{ij})$, $C = (c_{ij})$, $D = (d_{ij})$ are 3×2 matrices with complex entries satisfying the two conditions:*

1. *The matrix (B, C, D) has full rank,*
2. *For some $h, k \in \mathbb{R}$, $h \neq 0 \neq k$,*

$$k\, BEB^* - h\, CEC^* + h\, DED^* = 0, \tag{3.31}$$

then $D(S) = \{\mathbf{y} = (y_1, y_2) \in D_{\max}$ such that

$$B\,\mathbf{Y}_1(b) + C\mathbf{Y}_2(c) + D\,\mathbf{Y}_3(d) = 0\}, \tag{3.32}$$

where
(3.33)
$$\mathbf{Y}_1(b) = \begin{bmatrix} [y_1, u_1]_1(b) \\ [y_1, v_1]_1(b) \end{bmatrix}, \mathbf{Y}_2(c) = \begin{bmatrix} [y_2, u_2]_2(c) \\ [y_2, v_2]_2(c) \end{bmatrix}, \ \mathbf{Y}_3(d) = \begin{bmatrix} [y_2, u_3]_2(d) \\ [y_2, v_3]_2(d) \end{bmatrix},$$

is the domain of a self-adjoint operator S in H satisfying (3.16) and every operator S in H satisfying (3.16) is obtained this way.

The cases when exactly one of b, c, d is LP are similar.

Case 5. $d = 4$. *This is the case when there is no LP endpoint, i.e. each endpoint is either regular or LC. Let (u_1, v_1) be a boundary basis at a, (u_2, v_2) a boundary basis at b, (u_3, v_3) a boundary basis at c and (u_4, v_4) a boundary basis at d. A linear submanifold $D(S)$ of D_{max} is the domain of a self-adjoint extension S of S_{min} satisfying (3.16) if there exist 4 by 2 matrices $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, $D = (d_{ij})$ with complex entries such that the 4 by 8 matrix (A, B, C, D) whose first two columns are those of A, the second two columns are those of B, etc. satisfies the followind two conditions:*

1. *The matrix (A, B, C, D) has full rank.*
2. *For some $h, k \in \mathbb{R}$, $h \neq 0 \neq k$,*

$$k\, AEA^* - k\, BEB^* + h\, CEC^* - h\, DED^* = 0, \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{3.34}$$

and $D(S)$ is the set of $\mathbf{y} = \{y_1, y_2\} \in D_{\max}$ satisfying
(3.35)
$$A\mathbf{Y}_1(a) + B\,\mathbf{Y}_1(b) + C\,\mathbf{Y}_2(c) + D\,\mathbf{Y}_2(d) = 0, \ \mathbf{Y}_i = \begin{bmatrix} [y_i, u_i]_i \\ [y_i, v_i]_i \end{bmatrix}, \ \ i = 1, 2.$$

Furthermore every operator S satisfying (3.16) is obtained this way.

In each of the above cases if $t \in \{a, b, c, d\}$ is a regular endpoint, then $Y_r(t)$ can be replaced by

$$\begin{bmatrix} y_r(t) \\ y_r^{[1]}(t) \end{bmatrix}, \ r = 1, 2. \tag{3.36}$$

Proof. See [34]. □

Next we give some illustrative examples for both regular, singular, and mixed problems.

3.1. Regular endpoints. Although, as stated in Theorem 2, the conditions at a regular endpoint can be obtained from the LC conditions at that point, we give some examples here to illustrate this in view of the wide interest in regular problems.

EXAMPLE 1. Separated boundary conditions at all four regular endpoints:

$$\begin{aligned} &A_1y(a)+A_2y^{[1]}(a)=0,\ A_1,A_2\in\mathbb{R},\ (\ A_1,A_2)\neq(0,0);\\ &B_1y(b)+B_2y^{[1]}(b)=0,\ B_1,B_2\in\mathbb{R},\ (\ B_1,B_2)\neq(0,0);\\ &C_1y(c)+C_2y^{[1]}(c)=0,\ C_1,C_2\in\mathbb{R},\ (\ C_1,C_2)\neq(0,0);\\ &D_1y(d)+D_2y^{[1]}(d)=0,\ D_1,D_2\in\mathbb{R},\ (\ D_1,D_2)\neq(0,0).\end{aligned} \tag{3.37}$$

Let

$$A=\begin{bmatrix} A_1 & A_2\\ 0 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix},\ B=\begin{bmatrix} 0 & 0\\ B_1 & B_2\\ 0 & 0\\ 0 & 0\end{bmatrix},\ C=\begin{bmatrix} 0 & 0\\ 0 & 0\\ C_1 & C_2\\ 0 & 0\end{bmatrix},\ D=\begin{bmatrix} 0 & 0\\ 0 & 0\\ 0 & 0\\ D_1 & D_2\end{bmatrix}.$$

In this case the 4×8 matrix (A,B,C,D) has full rank and

$$0=AEA^*=BEB^*=CEC^*=DED^*. \tag{3.38}$$

Note that this case is independent of h,k.

EXAMPLE 2. Separated boundary conditions at a and at d and coupled conditions at b,c.

$$\begin{aligned} &A_1\,y(a)+A_2(py')(a)=0,\ A_1,A_2\in\mathbb{R},\ (\ A_1,A_2)\neq(0,0);\\ &D_1\,y(d)+D_2(py')(d)=0,\ D_1,D_2\in\mathbb{R},\ (\ D_1,D_2)\neq(0,0).\end{aligned}$$

and

$$\begin{aligned} &Y(c)=e^{i\gamma}KY(b),\ Y=\begin{bmatrix} y\\ y^{[1]}\end{bmatrix},\\ &K=(k_{ij}),\ k_{ij}\in\mathbb{R},\ 1\leq i,j\leq 2,\ \det K\neq 0,-\pi<\gamma\leq\pi.\end{aligned} \tag{3.39}$$

Let A,D be as in Example 1, then $rank(A,D)=2$ and $k\,AEA^*-h\,DED^*=0$ for any h,k since $0=AEA^*=DED^*$. Let

$$C=\begin{bmatrix} 0 & 0\\ -1 & 0\\ 0 & -1\\ 0 & 0\end{bmatrix},\ B=e^{i\gamma}\begin{bmatrix} 0 & 0\\ k_{11} & k_{12}\\ k_{21} & k_{22}\\ 0 & 0\end{bmatrix},\ -\pi<\gamma\leq\pi. \tag{3.40}$$

Then a straightforward computation shows that

$$h\,CEC^*=k\,BEB^*$$

is equivalent with

$$h\,E = k\,(\det K)\,E$$

which is equivalent with

$$h = k\det K. \tag{3.41}$$

Since (3.41) holds for any $h, k \in \mathbb{R}$, $h \neq 0 \neq k$, it follows from Theorem 2 that the boundary conditions of this example are self-adjoint for any $K \in M_2(\mathbb{R})$ with $det(K) \neq 0$.

Remark 9. In the 1-interval theory of Section 2 $\det K = 1$ is required for self-adjointness. We find it remarkable that the 1-interval condition $\det K = 1$ of Section 2 extends to $det(K) \neq 0$ in the 2-interval theory and that this generalization follows from two simple observations: (i) The Mukhtarov-Yakubov [21] observation that for $h > 0$ and $k > 0$ using inner product muliples produces an interaction between the two intervals yielding $det(K) > 0$, and (ii) our observation that the boundary value problem is invariant under muliplication by -1 yields the further extension $det(K) \neq 0$. This is optimal in the sense that when K is singular the boundary condition is separated not coupled. In (3.41) assume $h < 0$ and $k > 0$. Now apply Theorem 2 to the two equations

$$M_1 y = -(-p_1\, y')' + (-q_1)\, y = \lambda(-h\, w_1) y \ on\ J_1$$

and

$$M_2 y = -(p_2 y')' + q_2 y = \lambda(k w_2)\, y \ on\ J_2.$$

To obtain (3.41) for any $h, k \in \mathbb{R}$, $h \neq 0 \neq k$. Note that for both equation $r = 1$ and its boundary conditions are invariant under multiplication by -1. If $h > 0$ and $k < 0$ the proof is the same with the roles of equations $r = 1$ and $r = 2$ interchanged.

If the boundary conditions are coupled for the endpoint pair a, d as well as the pair b, c then the parameters h, k play a role in both sets of coupled boundary conditions. The next example illustrates this point.

Example 3. Two pairs of coupled conditions, with $-\pi < \gamma_1, \gamma_2 \leq \pi$,

$$\begin{aligned} &Y(d) = e^{i\gamma_1} G Y(a),\ G = (g_{ij}),\ g_{ij} \in \mathbb{R},\ i, j = 1, 2,\ \det G \neq 0, \\ &Y(c) = e^{i\gamma_2} K Y(b),\ K = (k_{ij}),\ k_{ij} \in \mathbb{R},\ i, j = 1, 2,\ \det K \neq 0,\ Y = \begin{bmatrix} y \\ y^{[1]} \end{bmatrix}. \end{aligned} \tag{3.42}$$

Proceeding as in the previous example we obtain the equivalence of the conditions for self-adjointness:

$$k\, GEG^* = h\, E \ \ and\ \ k\, KEK^* = hE;$$

$$k \det G = h \ \ and\ \ k \det K = h;$$

i.e.,

$$\det G = \det K = \frac{h}{k}.$$

This shows that (3.42) are self-adjoint boundary conditions for any h, k positive or negative.

See Section 5 below for more examples with discontinuous boundary conditions.

3.2. Singular endpoints. Here we illustrate the self-adjoint boundary conditions given by Theorem 2 when at least one endpoint is singular. The conditions when $d = 0$ or 1 are the same as in the one interval case and are independent of h and k. In these cases the self-adjoint extensions in the Hilbert space H defined by (3.7) are the same as those of the usual direct sum Hilbert space H_u. So we give examples here only for $d = 2$, $d = 3$ and $d = 4$.

NOTATION 1. *In the examples below in this section and the next* (u_1, v_1) *denotes a boundary condition basis at* a, (u_2, v_2) *a boundary condition basis at* $b, (u_3, v_3)$ *a boundary condition basis at* c, *and* (u_4, v_4) *a boundary condition basis at* d. *Also we use* $[y, u_r]$ *as an abbreviation for* $[y_r, u_r]$ *and* $[y, v_r]$ *as an abbreviation for* $[y_r, v_r]$, $r = 1, 2, 3, 4$.

EXAMPLE 4. Assume $d = 2$. Let a and d be the two non LP endpoints. Suppose that the boundary conditions at a and d are coupled:

$$\begin{bmatrix} [y, u_1](a) \\ [y, v_1](a) \end{bmatrix} = K \begin{bmatrix} [y, u_3](d) \\ [y, v_3](d) \end{bmatrix}, \quad K = (k_{ij}),\ k_{ij} \in \mathbb{R},\ i, j = 1, 2,\ \det K \neq 0. \tag{3.43}$$

Let

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad D = K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}.$$

Then $rank(A, D) = 2$. From a straightforward computation, it follows that

$$k\, AEA^* = h\ DED^*$$

is equivalent with

$$k = h \det K.$$

By Theorem 2, we have that if $h = 1$, $k > 0$ satisfies $\det K = k$ then the boundary conditions (3.43) are self-adjoint.

Using Remark 9 this result extends to any h, k positive or negative as in the previous examples.

EXAMPLE 5. Assume $d = 2$. Let a and c be the two non LP endpoints. Let the boundary conditions at a, c be given by:

$$\begin{bmatrix} [y, u_3](c) \\ [y, v_3](c) \end{bmatrix} = K \begin{bmatrix} [y, u_1](a) \\ [y, v_1](a) \end{bmatrix}, \quad K = (k_{ij}),\ k_{ij} \in \mathbb{R},\ i, j = 1, 2,\ \det K \neq 0. \tag{3.44}$$

Let

$$A = K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}, \qquad C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then $rank(A, C) = 2$. By a straightforward computation we see that

$$k\, AEA^* + h\, CEC^* = 0$$

is equivalent with

$$k \det K = -h.$$

Therefore if $k = 1$, $h > 0$ and $\det K = -h$ then the boundary conditions (3.44) are self-adjoint. This extends to $\det K = +h$ as in the above examples.

REMARK 10. By changing the weight function w_1 to $h\, w_1$ we can generate self-adjoint operators for any real coupling matrix K satisfying $\det K \neq 0$.

EXAMPLE 6. Assume $d = 3$. Let b, c, d be regular or LC endpoints. Separated boundary conditions at d and coupled conditions at b, c.

$$\begin{gathered} A_1[y, u_4](d) + A_2[y_2, v_4](d) = 0, \quad A_1, A_2 \in \mathbb{R}, \quad (A_1, A_2) \neq (0,0); \\ \begin{bmatrix} [y_2, u_3](c) \\ [y_2, v_3](c) \end{bmatrix} = K \begin{bmatrix} [y_1, u_2](b) \\ [y_1, v_2](b) \end{bmatrix}, \\ K = (k_{ij}),\ k_{ij} \in \mathbb{R},\ i, j = 1, 2,\ \det K \neq 0. \end{gathered} \tag{3.45}$$

Let

$$B = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ A_1 & A_2 \end{bmatrix}.$$

In this case $rank(B, C, D) = 3$ and $DED^* = 0$. Then, in terms of Theorem 2, we obtain the equivalence of the conditions for self-adjointness:

$$h = k \det K.$$

Thus, if $k = 1$, and $h > 0$ satisfies $\det K = h$, then the boundary conditions (3.45) are self-adjoint and this extends to $\det K = -h$ as above.

In the following example, we still let three endpoints b, c and d be regular or LC, but let boundary the conditions at c be separated and the boundary conditions at b, d be coupled.

EXAMPLE 7. Assume $d = 3$. Let

$$\begin{gathered} C_1[y, u_3](c) + C_2[y, v_3](c) = 0, \quad C_1, C_2 \in \mathbb{R}, \quad (C_1, C_2) \neq (0,0); \\ \begin{bmatrix} [y, u_4](d) \\ [y, v_4](d) \end{bmatrix} = K \begin{bmatrix} [y, u_2](b) \\ [y, v_2](b) \end{bmatrix}, \\ K = (k_{ij}),\ k_{ij} \in \mathbb{R},\ i, j = 1, 2,\ \det K \neq 0. \end{gathered} \tag{3.46}$$

Let

$$B = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ C_1 & C_2 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

Then $rank(B, C, D) = 3$ and $h\, CEC^* = 0$ for any h since $CEC^* = 0$. Proceeding as in the previous example we obtain the equivalence of the conditions for self-adjointness:

$$k \det K + h = 0.$$

This shows that (3.46) are self-adjoint boundary conditions when $k = 1$, $h > 0$ and $\det K = -h$ and extends to $\det K = +h$ as above.

EXAMPLE 8. Assume $d = 3$. Separated boundary conditions at b and coupled conditions at c, d:

$$\begin{aligned} &B_1[y,u_2](b) + B_2[y,v_2](b) = 0, \quad B_1, B_2 \in \mathbb{R}, \quad (B_1, B_2) \neq (0,0); \\ &C \begin{bmatrix} [y,u_3](c) \\ [y,v_3](c) \end{bmatrix} + D \begin{bmatrix} [y,u_4](d) \\ [y,v_4](d) \end{bmatrix} = 0. \end{aligned} \tag{3.47}$$

Then $k\,BEB^* = 0$ for any k since $BEB^* = 0$. In terms of Theorem 2, the boundary conditions (3.47) are self-adjoint if and only if $rank(C, D) = 2$ and

$$CEC^* - DED^* = 0. \tag{3.48}$$

Note that these conditions are independent of h and k and are simply the one-interval self-adjointness conditions for each of the two intervals separately. Thus Example 8 just gives the two-interval self-adjointness conditions which are generated by the direct sum of self-adjoint operators from each of the two intervals separately.

EXAMPLE 9. Assume $d = 4$, i.e. each endpoint is either regular or LC. Separated boundary conditions at b and at d and coupled conditions at a, c:

$$\begin{aligned} &B_1[y,u_2](b) + B_2[y,v_2](b) = 0,\ B_1, B_2 \in \mathbb{R},\ (\ B_1, B_2) \neq (0,0); \\ &D_1[y,u_4](d) + D_2[y,v_4](d) = 0,\ D_1, D_2 \in \mathbb{R},\ (\ D_1, D_2) \neq (0,0). \end{aligned} \tag{3.49}$$

$$\begin{aligned} &\begin{bmatrix} [y,u_3](c) \\ [y,v_3](c) \end{bmatrix} = e^{i\gamma} K \begin{bmatrix} [y,u_1](a) \\ [y,v_1](a) \end{bmatrix}, \\ &K = (k_{ij}),\ k_{ij} \in \mathbb{R},\ i,j = 1,2,\ \det K \neq 0. \end{aligned} \tag{3.50}$$

Let

$$A = \begin{bmatrix} 0 & 0 \\ k_{11} & k_{12} \\ k_{21} & k_{22} \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix},\ D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ D_1 & D_2 \end{bmatrix}.$$

In this case $rank(A, B, C, D) = 4$, and $k\,BEB^* + h\,DED^* = 0$ for any h, k since $0 = BEB^* = DED^*$. Therefore the boundary conditions (3.49) and (3.50) are self-adjoint if

$$k \det K + h = 0.$$

If we choose $k = 1$, $h > 0$ such that $\det K = -h < 0$, then the boundary conditions are self-adjoint. As above this extends to $\det K \neq 0$.

When $d = 4$, for more examples see [41, 42, 34, 37] and the next section.

4. Transmission and interface conditions. In this section we show that the special case of the 2-interval theory:

$$(4.1) \qquad -\infty \leq a < b = c < d \leq +\infty, \ J_1 = (a,b), \ J_2 = (c,d), \ J = (a,d),$$

produces the regular transmission and interface conditions used in the cited references as well as more general regular conditions and singular analogues of all these regular conditions.

NOTATION 2. *Below when comparing the results with those of the 1-interval theory from Section 2 it is important to note that the interval (a,d) in (4.1) plays the role of the interval (a,b) of Section 2.*

We start with two simple but important observations which help to illustrate how the special case (4.1) of the 2-interval theory produces regular and singular transmission and interface conditions.

REMARK 11. To connect the 2-interval theory discussed in Section 3 above to the transmission and interface conditions used in the referenced papers, a key observation is that the direct sum Hilbert space $L^2(J_1, w_1) \dot{+} L^2(J_2, w_2)$ can be identified with the space $L^2(J, w)$ where $w = w_1$ on J_1 and $w = w_2$ on J_2. Note that even though $b = c$ there are still four endpoint classifications since the endpoint c may have different LC and LP classifications on (a,c) than on (c,d). To emphasize this point as well as to relate to the notation commonly used for regular transmission and interface conditions we use the notation c^+ when c is a right endpoint i.e. for the interval (a,c) and c^- for c as an endpoint of the interval (c,d).

REMARK 12. In this section we will show that the self-adjointness conditions for the 'interval' (c^+, c^-) which produce the transmission and interface conditions are, surprisingly, more general than the corresponding 1-interval conditions of Section 2. This is due to the influence of the parameters h, k and the observation that the 1-interval problems are invariant with respect to muliplication by -1.

Throughout this section we assume that (3.2) holds and note that (3.2) implies that (1.2) holds when c is a regular endpoint for both intervals (a,c) and (c,b). In this case the 1-interval theory of Section 1 can be applied to the interval $J = (a,d)$.

REMARK 13. Note that c is the right endpoint of the interval J_1 and the left endpoint of the interval J_2. When c is a regular endpoint for both intervals (a,c) and (c,b) the 1-interval theory of Section 1 can be applied to the interval $J = (a,d)$ but this theory does not produce any self-adjoint operator in $L^2(J,w)$ with a condition that requires a jump discontinuity at c since condition (1.2) implies that all functions in the maximal domain $D_{\max}(a,d)$ and thus all solutions of (1.1) as well as their quasi-derivatives are continous at c. But, as we will see below, the 2-interval theory generates self-adjoint operators in $L^2(J,w)$ with boundary conditions that specify jump discontinuities at regular interior points and self-adjoint boundary conditions at interior singular points which, in general, have infinite jumps. Such boundary conditions may be separated or coupled; in the separated case they are generally called 'transmission' conditions in the literature, in the coupled case they are often referred to as 'interface' conditions. But both the 'transmission' and the 'interface' conditions transmit conditions from one interval to the other.

We start with the case when both outer endpoints are LP and the interior point c is regular from both sides because this case highlights the jump discontinuities at c.

Corollary 1. *Let (4.1) hold and assume that c is a regular endpoint for both intervals (a,c) and (c,d) and that both outer endpoints a and d are LP. Let $h,k\in\mathbb{R}$, $h\neq 0\neq k$. Suppose the matrices C, $D\in M_2(\mathbb{C})$ satisfy*

$$rank(C,D)=2,\quad k\,CEC^{*}=h\,DED^{*},\qquad E=\begin{bmatrix}0&-1\\1&0\end{bmatrix}. \tag{4.2}$$

Let

$$D(S)=\{y\in D_{\max}(J_1,J_2):CY(c^{+})+DY(c^{-})=0,\ Y=\begin{bmatrix}y\\(py')\end{bmatrix}\}. \tag{4.3}$$

Define an operator S in $L^2(J,w)$ by $S(y)=S_{\max}(J_1,J_2)\,y$ for $y\in D(S)$. Then S is self-adjoint in $L^2(J,w)$.

Here

$$\begin{aligned}&y(c^{+})=\lim_{t\to c^{+}}y(t),\ y(c^{-})=\lim_{t\to c^{-}}y(t),\\&(py')(c^{+})=\lim_{t\to c^{+}}(py')(t),\ (py')(c^{-})=\lim_{t\to c^{-}}(py')(t);\end{aligned} \tag{4.4}$$

note that these limits exist and are finite.

As in Theorem 1 the boundary conditions (4.3) can be categorized into two mutually exclusive classes: separated and coupled. The separated conditions have the general form

$$h_1\,y(c^{-})+k_1\,(py')(c^{-})=0,\ h_1,k_1\in\mathbb{R},\ (h_1,k_1)\neq(0,0) \tag{4.5}$$

$$h_2\,y(c^{+})+k_2\ (py')(c^{+})=0,\ h_2,k_2\in\mathbb{R},\ (h_2,k_2)\neq(0,0) \tag{4.6}$$

and these have the canonical form

$$\cos\alpha\ y(c^{-})-\sin\alpha\,(py')(c^{-})=0,\ \alpha\in[0,\pi), \tag{4.7}$$

$$\cos\beta\ y(c^{+})-\sin\beta\,(py')(c^{+})=0,\ \beta\in(0,\pi]. \tag{4.8}$$

Note that these separated conditions do not depend on h and k. (We follow the customary parameterizations for α and β even though these play no role in this paper.)

The coupled conditions have the canonical form

$$\begin{bmatrix}y(c^{+})\\(py')(c^{+})\end{bmatrix}=e^{i\gamma}K\begin{bmatrix}y(c^{-})\\(py')(c^{-})\end{bmatrix},\ -\pi<\gamma\leq\pi,\ i=\sqrt{-1} \tag{4.9}$$

with $K\in M_2(\mathbb{R})$ and $\det(K)\neq 0$.

Proof. Note that in this case we can identify the direct sum space $L^2(J_1,w_1)\dotplus L^2(J_2,w_2)$ with the space $L^2(J,w)$ where $w=w_1$ on J_1 and $w=w_2$ on J_2. For $y=(y_1,y_2)\in D_{\max}(J_1,J_2)$ let $y=y_1$ on J_1 and y_2 on J_2, then define $y(c)$ using (4.4) and note that $y\in D_{\max}(J)$. The conclusion then follows from case 3 part (ii) of Theorem 2 and Remark 8. □

Remark 14. We comment on Collorary 1. Note the similarity between the conditions of Corollary 1 and the regular 1-interval self-adjoint boundary conditions of

Theorem 1. These are similar to the conditions (2.10), (2.11), and (2.12). Comparing Collorary 1 with the regular 1-interval theory we see that the regular separated self-adjointness conditions on the nondegenerate interval (a, d) are the same as the separated jump conditions on the 'interval' (c^+, c^-). However, remarkably, the coupled condition on the 'interval' (c^+, c^-) only require $\det K \neq 0$ in contrast with the requirement that $\det K = 1$ in the 1-interval theory of Section 2. This is due to the influence of the inner product parameters h, k and the observation that on each interval the boundary value problem is invariant under multiplication by -1. Athough we have conditions on the narrow 'interval' (c^+, c^-) rather than the nondegenerate interval (a, d) of Section 2 the influence of k is felt by c^- and the influence of h by c^+. Note that when $K = I$, the identity matrix, and $\gamma = 0$ condition (4.9) is just the continuity condition for y and (py') at c and therefore this case generates the 1-interval minimal operator $S_{\min}(a, d)$ and this is the only self-adjoint operator in this case since a, d are both LP.

REMARK 15. In much of the literature the separated jump conditions (4.5), (4.6) and (4.7), (4.8) are called 'transmission' conditions; while special cases of the coupled conditions (4.9) are called 'interface' conditions. The separated conditions are real but note that the coupled jump conditions are nonreal when $\gamma \neq 0$ and $\gamma \neq \pi$. We are not aware of any nonreal self-adjoint jump conditions having been explicitly discussed in the literature.

Next we give the analogue of Collorary 1 when c is LC.

COROLLARY 2. *Let (4.1) hold. Assume that c is LC for both intervals (a, c) and (c, d) and that both outer endpoints a and d are LP. Let (u_1, v_1) be a boundary condition bases at c for the interval (a, c) and (u_2, v_2) a boundary condition bases at c for the interval (c, d).*

Suppose the matrices C, $D \in M_2(\mathbb{C})$ satisfy (4.2).

Let

$$D(S) = \{\mathbf{y} \in D_{\max}(J_1, J_2) : C \begin{bmatrix} [y, u_1](c^+) \\ [y, v_1](c^+) \end{bmatrix} + D \begin{bmatrix} [y, u_2](c^-) \\ [y, v_2](c^-) \end{bmatrix} = 0.\} \tag{4.10}$$

Define an operator S in $L^2(J, w)$ by $S(\mathbf{y}) = M_{\max}(J_1, J_2)\,\mathbf{y}$ for $\mathbf{y} \in D(S)$. Then S is self-adjoint in $L^2(J, w)$.

Here

$$[y, u_r](c^+),\ [y, v_r](c^+),\ [y, u_r](c^-),\ [y, v_r](c^-),\ r = 1, 2 \tag{4.11}$$

exist as finite limits.

As in Theorem 1 the boundary conditions (4.3) to (4.6) can be categorized into two mutually exclusive classes: separated and coupled. The separated conditions have the general form

$$h_1\,[y, u_1](c^+) + k_1[y, v_1](c^+) = 0,\ h_1, k_1 \in \mathbb{R},\ (h_1, k_1) \neq (0, 0) \tag{4.12}$$

$$h_2\,[y, u_2](c^-) + k_2\ [y, v_2](c^-) = 0,\ h_2, k_2 \in \mathbb{R},\ (h_2, k_2) \neq (0, 0) \tag{4.13}$$

and these have the canonical form

$$\cos\alpha\ \ [y, u_1](c^+) - \sin\alpha\ [y, v_1](c^+) = 0,\ \alpha \in [0, \pi), \tag{4.14}$$

(4.15) $$\cos\beta\ [y,u_2](c^-) - \sin\beta\ [y,v_2](c^-) = 0,\ \beta \in (0,\pi].$$

The coupled conditions have the canonical form

(4.16) $$\begin{bmatrix} [y,u_2](c^-) \\ [y,v_2](c^+) \end{bmatrix} = e^{i\gamma} K \begin{bmatrix} [y,u_1](c^+) \\ [y,v_1](c^+) \end{bmatrix},\ -\pi < \gamma \le \pi,\ i = \sqrt{-1}$$

with $K \in M_2(\mathbb{R})$ *and* $\det(K) \neq 0$.

Proof. As in Corollary 1 we note that we can identify the direct sum space $L^2(J_1, w_1) \dot{+} L^2(J_2, w_2)$ with the space $L^2(J, w)$ where $w = w_1$ on J_1 and $w = w_2$ on J_2 and then use Theorem 2. □

REMARK 16. Note that Corollary 2 parallels Corollary 1 with the jump conditions on the Lagrange forms rather than on y and $y^{[1]}$. Such a parallel result holds generally when the assumption that an endpoint is regular is replaced by the assumption that this endpoint is LC. At an LP endpoint there is no boundary condition. The conditions (4.12), (4.13) and (4.14), (4.15) are the singular analogues of the regular separated jump conditions (4.5), (4.6) and (4.7), (4.8); condition (4.16) is the singular analogue of the regular jump condition (4.9). Thus (4.5), (4.6) and (4.7), (4.8) could be called singular transmission conditions and (4.16) singular coupled interface conditions but we have not seen any of these singular jump conditions studied in the literature. Note that while the Lagrange brackets $[y,u_1]$, $[y,v_1]$, $[y,u_2](c^+)$, $[y,v_2](c^+)$ exist and are finite at c^+ and c^- the solutions y and their quasi-derivatives (py'), in general are not continuous at c; they may blow up i.e. be infinite at c^+ or c^- or they may oscillate wildly at c^+ or c^-.

The next corollary shows how case 5 of Theorem 2 can be used to get self-adjoint jump conditions at an interior point c when c is LC for both intervals (a,c) and (c,d) and each of a, d is LC. The cases when one or more of these four endpoints is regular then follows as above in Sections 1 and 2 and in Corollary 1.

COROLLARY 3. *Let (4.1) hold. Assume that c is LC for both intervals (a,c) and (c,d) and that both outer endpoints a and d are LC. Let (u_1,v_1) be a boundary condition basis at a, (u_2,v_2) a boundary condition basis at c^-, (u_3,v_3) a boundary condition bases at c^+, (u_4,v_4) a boundary condition basis at d, respectively.*

Suppose that for some $h,k \in \mathbb{R}$, $h \neq 0 \neq k$, the matrices $A, B, C, D \in M_2(\mathbb{C})$ satisfy

(4.17) $$rank(A,D) = 2,\ \ k\,AEA^* = h\,DED^*,\ rank(B,C) = 2,\ \ h\,BEB^* = k\,CEC^*$$

where E is given in (2.5).

With $Y_r = \begin{bmatrix} [y,u_r] \\ [y,v_r] \end{bmatrix}$, $r = 1,2,3,4$ *define $D(S)$ to be the set of all $\mathbf{y} \in D_{\max}(J_1,J_2)$ satisfying*

(4.18) $$AY_1(a) + DY_4(d) = 0,\ CY_3(c^+) + BY_2(c^-) = 0,$$

and define the operator S in $L^2(J,w)$ by $S(\mathbf{y}) = M_{\max}(J_1,J_2)\,\mathbf{y}$ for $\mathbf{y} \in D(S)$, $\mathbf{y} = \{y_1,y_2\}$. Then S is self-adjoint in $L^2(J,w)$.

Here

(4.19) $$[y,u_r](t),\ [y,v_r](t),\ for\ t = a, c^-, c^+, d;\ r = 1,2,3,4$$

exist as finite limits.

As in Theorem 1 and Corollary 2 each of the two boundary conditions (4.18) can be categorized into two mutually exclusive classes: separated and coupled and these have the canonical forms given there.

Proof. As above we indentify the direct sum space $L^2(J_1, w_1) \dotplus L^2(J_2, w_2)$ with the space $L^2(J, w)$ where $w = w_1$ on J_1 and $w = w_2$ on J_2. For $y = (y_1, y_2) \in D_{\max}(J_1, J_2)$ let $y = y_1$ on J_1 and y_2 on J_2, then define $y(c)$ using (4.4) and note that $y \in D_{\max}(J)$. Now we apply case 5 of Theorem 2. Let

$$A_0 = \begin{bmatrix} A \\ O \end{bmatrix}, B_0 = \begin{bmatrix} O \\ B \end{bmatrix}, C_0 = \begin{bmatrix} O \\ C \end{bmatrix}, D_0 = \begin{bmatrix} D \\ O \end{bmatrix} \tag{4.20}$$

where O denotes the 2×2 zero matrix. Now note that these 4×2 matrices satisfy the conditions of case 5 of Theorem 2; the matrix (A_0, B_0, C_0, D_0) has full rank, conditions (3.34) holds for the matrices A_0, B_0, C_0, D_0 and boundary condition (3.35) reduces to (4.18). □

As above at each regular endpoint t, $Y_r(t) = \begin{bmatrix} [y, u_r](t) \\ [y, v_r](t) \end{bmatrix}$ in (4.18) can be replaced by $\begin{bmatrix} y(t) \\ (py')(t) \end{bmatrix}$ and the boundary conditions can be given in canonical form. We do this in the next Corollary of Corollary 3 for the case when c is regular for both intervals (a, c) and (c, d). But we emphasize that this can be done for each combination of endpoints i.e. for each Y_r independent of the other three Y_r.

Corollary 4. *Let (4.1) hold. Assume that c is regular for both intervals (a, c) and (c, d) and that both outer endpoints a and d are LC. Let (u_1, v_1) be a boundary condition bases at a and (u_4, v_4) a boundary condition bases at d. Suppose the matrices $A, B, C, D \in M_2(\mathbb{C})$ satisfy (4.17).*

Define $D(S)$ to be the set of all $y \in D_{\max}(J_1, J_2)$ satisfying

$$AY_1(a) + DY_4(d) = 0,\ CY_3(c^+) + BY_2(c^-) = 0, \tag{4.21}$$

where

$$\begin{aligned} Y_1(a) &= \begin{bmatrix} [y, u_1](a) \\ [y, v_1](a) \end{bmatrix}, Y_4(d) = \begin{bmatrix} [y, u_4](d) \\ [y, v_4](d) \end{bmatrix}, \\ Y_2(c^-) &= \begin{bmatrix} y(c^-) \\ (py')(c^-) \end{bmatrix}, Y_3(c^+) = \begin{bmatrix} y(c^+) \\ (py')(c^+) \end{bmatrix} \end{aligned} \tag{4.22}$$

and define the operator S in $L^2(J, w)$ by $S(y) = S_{\max}(J_1, J_2)\, y$ for $y \in D(S)$. Then S is self-adjoint in $L^2(J, w)$.

Recall that each of the two boundary conditions in (4.21) consists of separated and coupled conditions and these have the canonical forms:

$$\begin{aligned} &\cos\alpha\, [y, u_1](a) - \sin\alpha [y, v_1](a) = 0,\ 0 \le \alpha < \pi, \\ &\cos\beta\, [y, u_4](d) - \sin\beta\, [y, v_4](d) = 0,\ 0 < \beta \le \pi; \\ &Y(d) = e^{i\gamma} K\, Y(a),\ -\pi < \gamma \le \pi, \end{aligned} \tag{4.23}$$

where $Y(a)$, $Y(d)$ are given in (4.22); and

$$\begin{aligned} &\cos\alpha\ y(c^-) - \sin\alpha\, (py')(c^-) = 0,\ \alpha \in [0, \pi), \\ &\cos\beta\ y(c^+) - \sin\beta\, (py')(c^+) = 0,\ \beta \in (0, \pi]; \\ &\begin{bmatrix} y(c^+) \\ (py')(c^+) \end{bmatrix} = e^{i\gamma} K \begin{bmatrix} y(c^-) \\ (py')(c^-) \end{bmatrix},\ -\pi < \gamma \le \pi, \end{aligned} \tag{4.24}$$

with $K \in M_2(\mathbb{R})$ *and* $\det(K) \neq 0$.

Proof. This follows from Theorem 2 and Corollaries 1, 2, and 3. □

EXAMPLE 10. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ m & -1 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{4.25}$$

It is easy to check that if $h = k > 0$, then the self-adjointness conditions of Theorem 2 are satisfied for any $m \in \mathbb{R}$. These four matrices yield the boundary conditions:

$$y(a) = 0 = y(d), \quad y(b) = y(c), \quad (py')(b) - (py')(c) = -m\, y(c). \tag{4.26}$$

Thus, if $b = c$, conditions (4.25) require y to be continuous at $b = c$ but allow the quasi-derivative to have a jump discontinuity at c. If this jump is proportional to the value of y at c with a real proportionality constant $-m$ ($m = 0$ is allowed and reduces to the continuous case) then the jump is self-adjoint. Note that the conditions at a, d are independent of those at c, b and the conditions at a, d can be replaced by any self-adjoint conditions at these two endpoints i.e. by

$$A_1 E A_1^* = D_1 E D_1^*, \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad rank(A_1, D_1) = 2,$$

where A_1, D_1 are 2 by 2 matrices and A, D are the 4 by 2 matrices respectively obtained by inserting two rows of zeros between the two rows of A_1 and between the two rows of D_1.

EXAMPLE 11. Replacing the matrix C in the previous Example by

$$C = \begin{bmatrix} 0 & 0 \\ -1 & m \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

we get a self-adjoint problem for any real m by choosing $h = k > 0$. When $b = c$ the quasi-derivatives are continuous at b but the solutions are discontinuous when $m \neq 0$. In this case the self-adjoint boundary conditions are:

$$y(a) = 0 = y(d), \quad (py')(c^+) = (py')(c^-), \quad y(c^+) - y(c^-) = -m\,(py')(c).$$

Examples 10 and 11 can be found in [41] where they were established by a completely different method using Green's functions.

REMARK 17. We remark that Corollaries 1, 2, 3, 4 are only a few of the special cases of case 5 of Theorem 2 when the endpoints satisfy (4.1). There are many more. Others can be obtained with other choices of the matrices A_0, B_0, C_0, D_0 and with other endpoint classifications. Also corollaries of cases 3 and 4 of Theorem 2 can be obtained similarly.

Remark 18. From the perspective of the 2-interval theory and the corollaries discussed in this section, there are many self-adjoint S-L operators in $L^2(J,w)$ Hilbert spaces and direct sums of such spaces which are generated by discontinuous regular and singular, separated and coupled, boundary conditions.

Remark 19. The extension of the 2-interval theory discussed in Section 3 above to any finite number of intervals is routine, see the paper of Everitt and Zettl [12] where this is done for S-L and higher order problems. An infinite interval theory is also developed in this paper but the extension to an infinite number of intervals in [12] is not routine and requires additional technical considerations.

Remark 20. See the paper by Everitt, Shubin, Stolz and Zettl [5] for a discussion of self-adjoint Sturm-Liouville problems with an infinite number of interior singularities including an extension of the Titchmarsh-Weyl dichotomy for square-integrable solutions and the corresponding m-coefficient. Applications to the one-dimensional Schrödinger equation extend the earlier work of Gesztesy and Kirsch [13].

Remark 21. From the perspective of the finite and infinite interval theories we see that there are many self-adjoint S-L operators in the $L^2(J,w)$ spaces and their direct sums which are not covered by the classical modern 1-interval theory. Which of these will become celebrated classical operators in Applied Mathematics corresponding to the Bessel, Legendre, Laguerre, Jacobi etc. operators mentioned in Section 2?

5. Singular transmission and interface conditions for the Legendre equation. Consider the Legendre equation

$$-(py')' = \lambda y,\ p(t) = 1 - t^2. \tag{5.1}$$

In this section we illustate Theorem 2 by describing the 2-interval self-adjoint realizations of (5.1) in the Hilbert space $L^2(-1,\infty)$ with singular transmission and interface conditions at the singular interior point 1 and give some explicit examples for this equation. We have chosen this equation because it is one of the celebrated equations of Applied Mathematics and because the singular transmission and interface conditions can be given explicitly.

Let

$$J_1 = (-1,1),\ J_2 = (1,\infty),\ J = (-1,\infty). \tag{5.2}$$

Note that the direct sum Hilbert space $L^2(J_1,1) \dotplus L^2(J_2,1)$ can be identified with $H = L^2(-1,\infty)$ so in this section we will describe 2-interval self-adjoint realizations of the Legendre equation in this Hilbert space H. See [19] and [7] for additional information on the equation and its operators.

Below we use the notation -1^- and 1^- for the left endpoints of the intervals J_1 and J_2, respectively, and 1^+ for the right endpoint of J_1.

For clarity of exposition we state case 4 of Theorem 2 for the case when the endpoint d is LP and the other three endpoints are singular LC as Corollary 5. Then in Corollary 6 we apply Corollary 5 to the 2-interval Legendre problem on the intervals (5.2). In this case $a = -1^-$, $b = 1^+$, $c = 1^-$ and $d = \infty$. Recall that d is LP and the other three endpoints are singular LC; from this it follows that the deficiency index is 3 [42].

Corollary 5. *Consider the 2-interval problem consisting of the Legendre equation (5.1) on the intervals (5.2) with endpoints $a = -1^-$, $b = 1^+$, $c = 1^-$ and $d = \infty$. Let (u_1, v_1) be a boundary basis at a, (u_2, v_2) a boundary basis at b and (u_3, v_3) a boundary basis at c. If $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ are 3×2 matrices with complex entries satisfying the two conditions:*

1. *The matrix (A, B, C) has full rank,*
2. *For some $h, k \in \mathbb{R}$, $h \neq 0 \neq k$,*

$$k\,AEA^* - k\,BEB^* + h\,CEC^* = 0. \tag{5.3}$$

Then $D(S) = \{\mathbf{y} = (y_1, y_2) \in D_{\max}$ such that

$$A\,\mathbf{Y}_1(a) + B\,\mathbf{Y}_2(b) + C\,\mathbf{Y}_3(c) = 0\}, \tag{5.4}$$

where

$$\mathbf{Y}_1(a) = \begin{bmatrix} [y, u_1]_1(a) \\ [y, v_1]_1(a) \end{bmatrix}, \mathbf{Y}_2(b) = \begin{bmatrix} [y, u_2]_2(b) \\ [y, v_2]_2(b) \end{bmatrix}, \ \mathbf{Y}_3(c) = \begin{bmatrix} [y, u_3]_3(c) \\ [y, v_3]_3(c) \end{bmatrix}, \tag{5.5}$$

is the domain of a self-adjoint operator S in $H = L^2(-1, \infty)$ satisfying (3.16) and every operator S in H satisfying (3.16) is obtained this way.

Proof. This is case 4 of Theorem 2 when the endpoint d is LP and each of a, b, c is singular LC. In this case the deficiency index is 3. □

Next we make some observations about the Legendre equation; these will be used below to give explicit singular self-adjoint transmission and interface conditions at the interior singular point 1.

For $\lambda = 0$ two linearly independent solutions of (5.1) are given by

$$u(t) = 1, \ v(t) = \frac{-1}{2}\ln(|\frac{1-t}{t+1}|). \tag{5.6}$$

Since these two functions u, v play an important role below we list some of their properties.

Observe that for all $t \in \mathbb{R}$, $t \neq \pm 1$, we have

$$v^{[1]}(t) = (pv')(t) = +1. \tag{5.7}$$

Thus the quasi derivative (pv') can be continuously extended so that it is well defined and continuous on the whole real line $\mathbb{R}$ including the two singular points -1 and $+1$. It is interesting to observe that u, (pu') and the extended (pv') can be defined to be continuous on $\mathbb{R}$ and only v blows up logarithmically at the singular points -1 and $+1$.

Note that

$$[u, v](t) = u(t)(pv')(t) - v(t)(pu')(t) = 1, \ -\infty < t < \infty \tag{5.8}$$

where we have taken appropriate one sided limits at ± 1.

For all $y = (y_1, y_2) \in D_{\max}(J_1, J_2)$ we have

$$[y, u] = -(py'), \ [y, v] = y - v\,(py') \tag{5.9}$$

and again by taking appropriate one sided limits, if necessary, we see that $[y,u](t)$ is defined and finite for all $t \in \mathbb{R}$. Thus the vector

$$Y(t) = \begin{pmatrix} [y,u](t) \\ [y,v](t) \end{pmatrix} = \begin{pmatrix} -(py')(t) \\ (y - v\,(py'))(t) \end{pmatrix} \tag{5.10}$$

is well defined for all $t \in \mathbb{R}$. In particular,

$$Y(-1^-),\ Y(1^-),\ Y(1^+) \tag{5.11}$$

are well defined and finite. Note also that $Y(\infty)$ is well defined and

$$Y(\infty) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{5.12}$$

since ∞ is LP.

Using these observations we now apply Corollary 5 to the Legendre equation and obtain:

Corollary 6. *Consider the 2-interval problem consisting of the Legendre equation (5.1) on the intervals (5.2) with endpoints $a = -1^-$, $b = 1^+$, $c = 1^-$ and $d = \infty$ and note that the deficiency index is* 3. *If $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ are 3×2 matrices with complex entries satisfying the two conditions:*

1. *The matrix (A, B, C) has full rank,*
2. *For some $h, k \in \mathbb{R}$, $h \neq 0 \neq k$,*

$$k\,AEA^* - k\,BEB^* + h\,CEC^* = 0. \tag{5.13}$$

Then $D(S) = \{\mathbf{y} = (y_1, y_2) \in D_{\max}(J_1, J_2)$ such that

$$A\,\mathbf{Y}_1(-1^-) + B\,\mathbf{Y}_1(1^+) + C\,\mathbf{Y}_2(1^-) = 0\}, \tag{5.14}$$

where

$$\begin{aligned} &\mathbf{Y}_1(-1^-) = \begin{bmatrix} -(py')(-1^-) \\ (y - v\,(py'))(-1^-) \end{bmatrix}, \mathbf{Y}_1(1^+) = \begin{bmatrix} -(py')(1^+) \\ (y - v\,(py'))(1^+) \end{bmatrix}, \\ &\mathbf{Y}_2(1^-) = \begin{bmatrix} -(py')(1^-) \\ (y - v\,(py'))(1^-) \end{bmatrix}, \end{aligned} \tag{5.15}$$

is the domain of a self-adjoint operator S in $H = L^2(-1,\infty)$ satisfying (3.16) and every operator S in H satisfying (3.16) is obtained this way.

Proof. Note that (u_1, v_1) where $u_1 = u$ on J_1 and $v_1 = v$ on J_1 is a boundary condition basis at both endpoints -1^- and 1^+. Also $u_3 = u$ and $v_3 = v$ is a boundary condition basis for 1^-. The explicit form of the singular boundary conditions (5.14), (5.15) then follows from the observations (5.6) to (5.10). □

Remark 22. We comment on Corollary 6. Let $\mathbf{y} = \{y_1, y_2\} \in D_{\max}(J_1, J_2)$. In (5.9) since $(py_r')(t)$ and $y_r - v\,(py_r')$ are finite for all $t \in \mathbb{R}$ and v blows up at $a = -1^-$, $b = 1^+$, $c = 1^-$ it follows that y_r also blows up at these points, $r = 1, 2$. In particular this holds for any solution of equation (5.1) on J_1 and on J_2 for any $\lambda \in \mathbb{R}$.

Example 12. Let $a_{1j}, b_{1j}, c_{1j} \in \mathbb{R}$, $j = 1, 2$ with at least one member of each pair not 0.

Next we give some examples.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ b_{11} & b_{12} \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ c_{11} & c_{12} \end{bmatrix}. \tag{5.16}$$

$$\begin{aligned} 0 &= -a_{11}(py')(-1^-) + a_{12}(y - v\,(py'))(-1^-) \\ &= -b_{11}(py')(1^+) + b_{12}(y - v\,(py'))(1^+) \\ &= -c_{11}(py')(1^-) + c_{12}(y - v\,(py'))(1^-). \end{aligned} \tag{5.17}$$

EXAMPLE 13. Note that these singular transmission conditions are independent of the parameters h, k. We mention a couple of special cases:

- $$0 = (py')(-1^-) = (py')(1^+) = (py')(1^-). \tag{5.18}$$

  It is interesting to note that each of these conditions looks like a regular Neumann conditions but is actually a singular analogue of the regular Dirichlet condition [42]. The singular analogues of the regular Neumann conditions are given by

- $$0 = (y - v\,(py'))(-1^-) = (y - v\,(py'))(1^+). \tag{5.19}$$

  These depend on the function v.

The next example illustrates singular self-adjoint interface conditions. These are 'jump' conditions involving a solution y which blows up i.e. has an infinite jump at the singular interior point 1 where the condition is specified.

EXAMPLE 14. In this example we have a separated condition at (-1^-) and a coupled condition 'coupling' the endpoints (1^+) and (1^-). For $a_{11}, a_{12} \in \mathbb{R}$, $(a_{11}, a_{12}) \neq (0, 0)$.

$$0 = -a_{11}(py')(-1^-) + a_{12}(y - v\,(py'))(-1^-); \tag{5.20}$$

and

$$Y_2(1^-) = e^{i\gamma} K\, Y_1(1^+),\ -\pi < \gamma \leq \pi,\ i = \sqrt{-1} \tag{5.21}$$

where $Y_1(1^+)$ and $Y_2(1^-)$ are given by (5.15) and K is a real 2×2 nonsingular matrix.

REMARK 23. The special case $K = I$ and $\gamma = 0$ is a singular analogue of the regular periodic boundary condition. Similarly, the case when $K = -I$ and $\gamma = 0$ is a singular analogue of the regular semi-periodic (anti-periodic) boundary condition. However, in both cases, these conditions depend on the function v.

REMARK 24. Consider $\gamma = 0$ and

$$K = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix},\ r \in \mathbb{R}.$$

Then (5.21) yields:

$$(py')(1^-) = (py')(1^+) \tag{5.22}$$

i.e. the quasi-derivative (py') is continuous at the singular interior point 1.

And

$$(y - v\,(py'))(1^-) - (y - v\,(py'))(1^+) = r\,\{-(py')(1)\}. \tag{5.23}$$

Note that the right hand side of (5.23) is finite by (5.22). On the left hand side of (5.23), as remarked above, y must blow up asymptotically like v so that each of $(y - v\,(py'))(1^-)$ and $(y - v\,(py'))(1^+)$ is finite and the self-adjointness condition is that the difference between these two finite numbers is equal to the right hand side of (5.23).

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