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GLOBAL EXISTENCE AND DECAY PROPERTY FOR SOLUTIONS IN NONLINEAR ELASTIC SOLIDS WITH VOIDS*

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Abstract. In this paper, we consider a nonlinear Cauchy problem for a system of elastic solids with voids. First, we prove that the damping in the porous equation alone is weak and the solutions of the corresponding system are of regularity-loss type. In addition, we show a global existence result for solutions in $H^s(\mathbb{R})$ for large s. Second, we prove that by considering an additional viscoelastic damping, then the solutions can gain some regularity and all solutions in H^s with $s \geq 4$ are global in time.

Key words. Decay rate, stability, regularity-loss, regularity gain, energy method.

AMS subject classifications. 35B35, 35L55, 74D05, 93D15, 93D20.

1. Introduction. The theory of elastic solids with voids has attracted a great deal of attention since Goodman and Cowin [5] first introduced the concept of a continuum theory granular materials with intersticial voids. It is considered to be a simple extension of the classical elasticity theory to porous media, where, in addition to the elastic effects, these materials (with voids) possess a microstructure with the property that the mass at each point is obtained as the product of the mass density of the material matrix by the volume fraction. This latter idea came from Nuziato and Cowin [14] in 1979 to develop a nonlinear theory of elastic materials with voids. This type of problems have been considered by many authors and several results concerning existence, and asymptotic behavior have been established.

Quintanilla [16] considered

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x \\ \rho\kappa\varphi_{tt} = \alpha\varphi_{xx} - bu_x - \tau\varphi_t - a\varphi \end{cases} \tag{1.1}$$

where u is the longitudinal displacement, φ is the volume fraction, $\rho > 0$ is the mass density, $\kappa > 0$ is the equilibrated inertia, and μ, α, τ, a are the constitutive constants which are positive and satisfy

$$\mu a > b^2. \tag{1.2}$$

He showed that the damping $(-\tau\varphi_t)$ in the porous equation is not strong enough to obtain an exponential decay. Only the slow decay has been proved.

Subsequently, many contributions have been made where the decay of solutions to the problems in elasticity with voids have been treated. See for instance [1, 2, 11, 12, 15] and the references therein. In [13] the authors investigated the problem (1.1) with $\tau = 0$ and a viscoelastic damping term of the form γu_{txx} acting on the right hand side of the first equation. They proved that the decay rate of the solution is polynomial and cannot be exponential. We point out that the assumption (1.2) played a decisive role in their proof.

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In this paper, we consider the one-dimensional nonlinear porous-elastic model:

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\varphi_x = 0, \\ J\varphi_{tt} - \sigma(\varphi_x)_x + bu_x + a\varphi + \tau\varphi_t = 0, \end{cases} \quad x \in \mathbb{R},\ t > 0, \tag{1.3}$$

where u is the longitudinal displacement, φ is the volume fraction, $\rho > 0$ is the mass density and a, b, μ, τ are the constitutive constants which are positive and satisfy (1.2). Here σ is a smooth function such that $\sigma'(\eta) > 0$, for any $\eta > 0$, with

$$\sigma'(0) = \delta,$$

where $\delta > 0$. We consider the following initial conditions

$$(u, u_t, \varphi, \varphi_t)(x, 0) = (u_0, u_1, \varphi_0, \varphi_1). \tag{1.4}$$

The corresponding linearized system

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\varphi_x = 0, \\ J\varphi_{tt} - \delta\varphi_{xx} + bu_x + a\varphi + \tau\varphi_t = 0, \end{cases} \quad x \in \mathbb{R},\ t > 0, \tag{1.5}$$

has been considered in [3] and under the assumption (1.2), the authors proved the following decay estimates:

$$\left\|\partial_x^k U(t)\right\|_{L^2} \le C(1+t)^{-1/4-k/2} \left\|U_0\right\|_{L^1} + C(1+t)^{-l/6} \left\|\partial^{k+l} U_0\right\|_{L^2}, \tag{1.6}$$

where $U(x,t) = (u_x, u_t, \varphi_x, \varphi_t, \varphi)(x,t)$, $U_0 = U(x, 0)$ and C is a positive constant. The decay estimate in (1.6) is of a regularity-loss type and is based on the following pointwise estimate of the Fourier image of the solution:

$$\left|\hat{U}(\xi, t)\right|^2 \le Ce^{-c\rho(\xi)t} \left|\hat{U}(\xi, 0)\right|^2, \tag{1.7}$$

where

$$\rho(\xi) = \frac{\xi^2}{(1+\xi^4)(1+\xi^2+\xi^4)}. \tag{1.8}$$

It is obvious that for high frequencies ($|\xi| \ge 1$), the function $\rho(\xi)$ behaves like ξ^{-6}, this means that the damping effect is very weak in the high frequency part and lead to a regularity-loss. On the other hand, over the low frequency domain ($|\xi| \le 1$), the solution decays polynomially with the rate of the heat kernel. The loss in the regularity yields technical difficulty when dealing with the nonlinear problem (1.3). In fact we will show (see Theorem 3.3 below) only the solutions with high regularity (i.e. solutions in $H^s(\mathbb{R})$ with $s \ge 24$) can be continued globally in time. The regularity-loss properties has been also shown for other models, such as Timoshenko systems [7, 8, 19, 18], hyperbolic-elliptic problems [6, 10] and the Maxwell systems [4, 21].

It is worth noting that the regularity assumption $s \ge 24$ comes from the estimate (1.7) obtained in [3], which is not optimal. Consequently, any improvement in the estimate (1.7) will lead to a low regularity assumption on s. This is an interesting open question.

This paper is organized as follows: In Section 2, we introduce some notations and some inequalities that will be used in the paper. The global existence and decay estimates in case of regularity loss are stated in Section 3 and the proof is given in Section 4. Section 5 is devoted to the regularity gain case.

2. Preliminaries. In this section, we introduce some notations and a technical lemma to be used throughout this paper. Throughout this paper, $\|.\|_q$ (or $\|.\|_{L^q}$) stand for the $L^q(\mathbb{R})$-norm ($1 \leq q \leq \infty$) and $\|.\|_{H^l}$ denotes the $H^l(\mathbb{R})$-norm. Let us also denote by $\hat{f} = \mathcal{F}(f)$ the Fourier transform of f with inverse $\mathcal{F}^{-1}$:

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x} dx,$$

Furthermore, the next lemma has been proved for instance in [6, Lemma 4.1].

LEMMA 2.1. *Let* $1 \leq p, q, r \leq \infty$ *and* $1/p = 1/q + 1/r$. *Then, we have*

$$\|\partial_x^k(uv)\|_p \leq C(\|u\|_q\|\partial_x^k v\|_r + \|v\|_q\|\partial_x^k u\|_r), \quad k \geq 0, \tag{2.1}$$

and

$$\|[\partial_x^k, u]v_x\|_p \leq C(\|u_x\|_q\|\partial_x^k v\|_r + \|v_x\|_q\|\partial_x^k u\|_r), \quad k \geq 1. \tag{2.2}$$

3. Statement of the problem. In order to state and prove our results, and to make our computations easy to handle, let us first transform our problem to a first order system. Indeed, we introduce the new variables:

$$v = u_x, \qquad h = u_t, \qquad z = \varphi_x, \qquad y = \varphi_t, \tag{3.1}$$

then, the resulting system takes the form

$$\begin{cases} v_t - h_x = 0, \\ \rho h_t - \mu v_x - bz = 0, \\ z_t - y_x = 0, \\ J y_t - \sigma(z)_x + bv + a\varphi + \tau y = 0, \\ \varphi_t - y = 0, \end{cases} \qquad x \in \mathbb{R},\ t > 0, \tag{3.2}$$

with the initial data

$$(v, h, z, y, \varphi)(x, 0) = (v_0, h_0, z_0, y_0, \varphi_0)(x). \tag{3.3}$$

The system (3.2)-(3.3) can be also rewritten as:

$$\begin{cases} U_t + F(U)_x + LU = 0, \\ U(x, 0) = U_0. \end{cases} \tag{3.4}$$

where $U = (v, h, z, y, \varphi)^T$, $F(U) := -\left(h, \frac{\mu}{\rho}v, y, \frac{1}{J}\sigma(z), 0\right)^T$ and L is defined as

$$L := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -b/\rho & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ b/J & 0 & 0 & \tau/J & a/J \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

and $U_0 := (v_0, h_0, z_0, y_0, \varphi_0)^T$. It is clear that the matrix L is not real symmetric.

REMARK 3.1. It is well known (see for instance [17, Theorem 5. 1] and [9]) that for $U_0 \in H^s(\mathbb{R}), s \in \mathbb{N}$ and $s \geq 2$, then problem (3.4) has a unique solution U such that

$$U \in C^0([0,\infty), H^s(\mathbb{R})) \cap C^1([0,\infty), H^{s-1}(\mathbb{R})).$$

The *linearized* problem of (3.4) can be obtained by taking the Jacobian of F at $U = 0$. Thus, we get the problem

$$\begin{cases} U_t + AU_x + LU = 0, \\ U(x,0) = U_0. \end{cases} \tag{3.5}$$

where the matrix A is defined as

$$A := D_U F(0) = -\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \mu/\rho & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of the matrix A are: $\lambda_0 = 0, \quad \lambda_{1|2} = \pm\sqrt{\delta}, \quad \lambda_{3|4} = \pm\sqrt{\frac{\mu}{\rho}}$.

Taking the Fourier transform of (3.5), then we get

$$\begin{cases} \hat{U}_t(\xi,t) = \Lambda(\xi)\hat{U}(\xi,t), \\ \hat{U}_t(\xi,0) = \hat{U}_0(\xi), \end{cases} \quad \xi \in \mathbb{R},\, t > 0, \tag{3.6}$$

where $\Lambda(\xi) = -L - i\xi A$. Consequently, solving the above first order ordinary differential equation, we obtain

$$\hat{U}(\xi,t) = e^{\Lambda(\xi)t}\hat{U}_0(\xi). \tag{3.7}$$

To compute the term $e^{\Lambda(\xi)t}$ is a challenging problem and in many situations, this cannot be done. Consequently, in order to show the asymptotic behavior of the solution, it suffices to find a positive function $\rho(\xi)$ such that

$$e^{\Lambda(\xi)t} \leq Ce^{-c\rho(\xi)t}, \tag{3.8}$$

for two positive constants C and c. Thus, the behavior of the solution depends on a critical way on the behavior of the function $\rho(\xi)$. If A and L are symmetric matrices and under the so-called Shizuta−Kawashima algebraic condition (SK) (see [20]):

(SK) $\qquad \forall\xi \in \mathbb{R}-\{0\}, \quad Ker(L) \cap \{\text{eigenvectors of } (\xi A)\} = \{0\},$

the function $\rho(\xi)$ has the form

$$\rho(\xi) = \frac{\xi^2}{1+\xi^2}. \tag{3.9}$$

Concerning our system (3.6), the function $\rho(\xi)$ in (3.8) has been found in [3] and has the form (1.8). This leads to the following decay estimate of the solution of (3.5):

THEOREM 3.2. *([3]) Let s be a nonnegative integer and assume $U_0 = (v_0, h_0, z_0, y_0, \varphi_0)^T \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$. Assume that (1.2) holds. Then the solution $U = (v, h, z, y, \varphi)^T$ of the system (3.5) satisfies the following decay estimates:*

$$\left\|\partial_x^k U(t)\right\|_{L^2} \leq C(1+t)^{-1/4-k/2} \|U_0\|_{L^1} + C(1+t)^{-l/6} \left\|\partial^{k+l} U_0\right\|_{L^2}, \tag{3.10}$$

for $k + l \leq s$. Here C and c are two positive constants.

Now, we present the results on the global existence and on the asymptotic stability of the *nonlinear* problem (3.4). In order to state our main result, and led by [8], we introduce the time weighed energy norm $E(t)$ and the corresponding dissipation norm $D(t)$ as follows:

$$E^2(t) \equiv \sum_{j=0}^{[s/4]} \sup_{0\leq r\leq t} (1+r)^{j-\frac{1}{2}} \left\|\partial_x^j U(r)\right\|_{H^{s-4j}}^2 \tag{3.11}$$

and

$$\begin{aligned} D^2(t) \equiv & \sum_{j=0}^{[s/4]} \int_0^t (1+r)^{j-\frac{3}{2}} \left\|\partial_x^j U(r)\right\|_{H^{s-4j}}^2 dr \\ & + \sum_{j=0}^{[s/4]-1} \int_0^t (1+r)^{j-\frac{1}{2}} \left\|\partial_x^j z(r)\right\|_{H^{s-1-4j}}^2 dr \\ & + \sum_{j=0}^{[s/4]} \int_0^t (1+r)^{j-\frac{1}{2}} \left\|\partial_x^j y(r)\right\|_{H^{s-4j}}^2 dr. \end{aligned} \tag{3.12}$$

Our main result reads as follows:

THEOREM 3.3. *Assume that $\sigma'(\eta) > 0$. Let $U_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ with $s \geq 24$ and put $E_0 := \|U_0\|_{H^s} + \|U_0\|_{L^1}$. Then, there exists a positive constant $\delta_0 > 0$ such that if $E_0 \leq \delta_0$, then problem (3.4) has a unique global solution U satisfying*

$$U \in C([0,\infty); H^s(\mathbb{R})) \cap C^1([0,\infty); H^{s-1}(\mathbb{R})). \tag{3.13}$$

Moreover, the solution satisfies the weighted energy estimate

$$E^2(t) + D^2(t) \leq C E_0^2, \tag{3.14}$$

and the decay estimate

$$\left\|\partial_x^k U(t)\right\|_2 \leq C E_0 (1+t)^{-1/4-k/2}, \tag{3.15}$$

where C is a positive constant and $0 \leq k \leq [s/4] - 4$.

REMARK 3.4. It is obvious that from the estimate (3.15), we deduce that the optimal decay rate is given for the solution U and its lower order derivatives only, but not for all derivatives up to the s-order. This is a result of the regularity-loss type of the solution.

4. Proof of the main results. In this section, we prove Theorem 3.3. The proof is a direct consequence of Lemma 4.1. Due to the regularity-loss property in Theorem 3.2, the proof is based on the energy method with negative weight. This method has been used before in [6] and [8].

Let us first define the following quantities

$$M_0(t) := \sup_{0\le r\le t} (1+r)^{\frac{1}{2}} \|U(r)\|_{L^\infty},$$
$$M_1(t) := \sup_{0\le r\le t} (1+r) \|\partial_x U(r)\|_{L^\infty},$$

and inspired by the estimates (3.10), we define

$$M(t) := \sum_{j=0}^{[s/4]-4} \sup_{0\le r\le t} (1+r)^{1/4+j/2} \left\|\partial_x^j U(r)\right\|_2.$$

We have the following lemma.

LEMMA 4.1. *Assume that the assumptions of Theorem 3.3 hold. Let $T>0$ and $s\ge 2$ and let U be the solution to the problem (3.4) satisfying*

$$U \in C([0,T]; H^s(\mathbb{R})) \cap C^1(0,T]; H^{s-1}(\mathbb{R})).$$

Then we have the a priori estimates

$$E^2(T) + D^2(T) \le CE_0^2, \tag{4.1}$$
$$M(T) \le CE_0, \tag{4.2}$$

where E_0 is given in Theorem 3.3 and C is a positive constant independent of T.

We shall derive the energy estimates under the a priori assumption

$$\sup_{0\le t\le T} \|U(t)\|_{L^\infty} \le \bar{\alpha} \tag{4.3}$$

where $\bar{\alpha}$ is a fixed small number, independent of T.

PROPOSITION 4.2. *Suppose that the assumptions in Theorem 3.3 hold. Let $T>0$ and $s\ge 2$, and let U be the solution of problem (3.2)-(3.3) satisfying (3.13) and (4.3). Then, the estimate*

$$E(t)^2 + D(t)^2 \le C\|U_0\|_{H^s}^2 + C(M_0(t) + M_1(t)) D^2(t), \tag{4.4}$$

holds true for all $t\in[0,T]$, where C is a positive constant which is independent of T.

Proof. We proceed with the basic energy estimate by multiplying the first equation in (3.2) by μv, the second equation by h, the third by $(\sigma(z)-\sigma(0))$, the fourth by y, the fifth by $a\varphi$, respectively, adding the resulting equations, and integrating with respect to x over $\mathbb{R}$, we obtain

$$\frac{d}{dt}E^{(0)}(t) + \tau\|y\|_2^2 = 0, \tag{4.5}$$

where

$$E^{(0)}(t) := \frac{1}{2}\left\{\mu\|v\|_2^2 + \rho\|h\|_2^2 + J\|y\|_2^2 + a\|\varphi\|_2^2 + 2b(v\varphi)\right\} + \int_{\mathbb{R}} F^{(0)}(z)\,dx \tag{4.6}$$

and

$$F^{(0)}(z) := \int_0^z (\sigma(s) - \sigma(0))\, ds.$$

In (3.2), we have used (3.1) and the fact that

$$\begin{aligned} bvy - bzh &= bu_x\varphi_t - b\varphi_x u_t \\ &= \frac{d}{dt}(bu_x\varphi) - \frac{d}{dx}(bu_t\varphi). \end{aligned}$$

To obtain the energy estimates on higher-order terms, applying, for $k \geq 1$, ∂_x^k to (3.2), we get

$$\begin{cases} \partial_x^k v_t - \partial_x^{k+1} h = 0, \\ \rho\partial_x^k h_t - \mu\partial_x^{k+1} v - b\partial_x^k z = 0, \\ \partial_x^k z_t - \partial_x^{k+1} y = 0, \\ J\partial_x^k y_t - \sigma'(z)\partial_x^{k+1} z + b\partial_x^k v + a\partial_x^k\varphi + \tau\partial_x^k y = [\partial_x^k, \sigma'(z)]z_x, \\ \partial_x^k\varphi_t - \partial_x^k y = 0, \end{cases} \quad x \in \mathbb{R},\ t > 0, \tag{4.7}$$

where we have used the notation $[\partial_x^k, A]B := \partial_x^k(AB) - A\partial_x^k B$.

Now, we define the energy associated to system (4.7) as

$$\begin{aligned} E^{(k)}(t) := &\frac{1}{2}\left\{\mu\left\|\partial_x^k v\right\|_2^2 + \rho\|\partial_x^k h\|_2^2 + J\left\|\partial_x^k y\right\|_2^2 + a\left\|\partial_x^k\varphi\right\|_2^2 + 2b(\partial_x^k v\partial_x^k\varphi)\right\} \\ &+ \int_{\mathbb{R}} F^{(k)}(z)\, dx, \end{aligned} \tag{4.8}$$

where

$$F^{(k)}(z) := \frac{1}{2}\sigma'(z)(\partial_x^k z)^2.$$

Next, multiplying the first equation in (4.7) by $\mu\partial_x^k v$, the second equation by $\partial_x^k h$, the third equation by $(\sigma'(z)\partial_x^k z)$, the fourth equation by $\partial_x^k y$ and the fifth equation by $a\partial_x^k\varphi$, respectively, adding the resulting equations, using the identity

$$b\partial_x^k v\partial_x^k y - b\partial_x^k z\partial_x^k h = \frac{d}{dt}\left(\partial_x^{k+1} u\partial_x^k\varphi\right) - \frac{d}{dx}\left(\partial_x^k u_t\partial_x^k\varphi\right)$$

and integrating with respect to x, we obtain

$$\frac{d}{dt}E^{(k)}(t) + \tau\left\|\partial_x^k y\right\|_2^2 = R_0^{(k)}, \tag{4.9}$$

where

$$R_0^{(k)} := \int_{\mathbb{R}}\left\{\frac{1}{2}\sigma'(z)_t\left(\partial_x^k z\right)^2 - \sigma'(z)_x\left(\partial_x^k z\right)\partial_x^k y + \partial_x^k y\left[\partial_x^k, \sigma'(z)\right](z)_x\right\}dx. \tag{4.10}$$

Recalling the fact that $z_t = y_x$, using the assumption (4.3) and as in [8], we get

$$\left|R_0^{(k)}\right| \leq C\int_{\mathbb{R}} |y_x|\left|\partial_x^k z\right|^2 + |z_x|\left|\partial_x^k z\right|\left|\partial_x^k y\right| + \left|\partial_x^k y\right|\left|\left[\partial_x^k, \sigma'(z)\right]\right||z_x|,$$

where $C = C(\bar{\alpha})$ and $\bar{\alpha}$ is defined in (4.3). This implies that, by using Lemma 2.1 (see [8] for details),

$$\left|R_0^{(k)}\right| \leq C \left\|\partial_x U\right\|_{L^\infty} \left\|\partial_x^k U\right\|_{L^2}^2 . \tag{4.11}$$

Now, for $k \geq 0$, we have, by applying Young's inequality

$$2b(\partial_x^k v \partial_x^k \varphi) \leq 2b\lambda_1 \left\|\partial_x^k v\right\|_2^2 + 2b\frac{1}{4\lambda_1}\|\partial_x^k \varphi\|_2^2, \qquad \lambda_1 > 0.$$

The above formula together with (4.6), (4.8) and the assumption (1.2) lead to

$$c_1 \hat{E}^{(k)}(t) \leq E^{(k)}(t) \leq c_2 \hat{E}^{(k)}(t), \qquad \forall t \geq 0, \tag{4.12}$$

for c_1, $c_2 > 0$, where $k \geq 0$, and $\hat{E}^{(k)}(t)$ is defined as

$$\hat{E}^{(k)} := \left\{\left\|\partial_x^k v\right\|_2^2 + \|\partial_x^k h\|_2^2 + \left\|\partial_x^k y\right\|_2^2 + \left\|\partial_x^k \varphi\right\|_2^2\right\} + \int_{\mathbb{R}} F^{(k)}(z)\, dx. \tag{4.13}$$

On the other hand, recalling (4.3), we deduce that there exist two positive constants c_3 and c_4 depending on $\bar{\alpha}$, such that

$$c_3 \left\|\partial_x^k U\right\|_{L^2}^2 \leq \hat{E}^{(k)}(t) \leq c_4 \left\|\partial_x^k U\right\|_{L^2}^2, \qquad k \geq 0. \tag{4.14}$$

Combining (4.12) and (4.14), we deduce that there exist two positive constants β_1 and β_2 such that

$$\beta_1 \left\|\partial_x^k U\right\|_{L^2}^2 \leq E^{(k)}(t) \leq \beta_2 \left\|\partial_x^k U\right\|_{L^2}^2, \qquad k \geq 0. \tag{4.15}$$

Consequently, multiplying (4.5) by $(1+t)^\alpha$, with $\alpha \in \mathbb{R}$ and integrating with respect to t and using (4.15), we get

$$\begin{aligned} &(1+t)^\alpha \|U(t)\|_{L^2}^2 + \frac{\tau}{\beta_1}\int_0^t (1+r)^\alpha \|y(s)\|_{L^2}^2\, dr \\ \leq\; &\frac{\beta_1}{\beta_2}\|U_0\|_{L^2}^2 + \frac{\alpha}{\beta_1}\int_0^t (1+r)^{\alpha-1}\|U(s)\|_{L^2}^2\, dr. \end{aligned} \tag{4.16}$$

Similarly, for $k \geq 1$, the estimates (4.11), (4.15) together with (4.9) yield, after a multiplication by $(1+t)^\alpha$ and integration with respect to t over $(0,t)$

$$\begin{aligned} &(1+t)^\alpha \left\|\partial_x^k U(t)\right\|_{L^2}^2 + \frac{\tau}{\beta_1}\int_0^t (1+r)^\alpha \|\partial_x^k y(r)\|_{L^2}^2 dr \\ \leq\; &\frac{\beta_1}{\beta_2}\left\|\partial_x^k U_0\right\|_{L^2}^2 + \frac{\alpha}{\beta_1}\int_0^t (1+r)^{\alpha-1}\|\partial_x^k U(r)\|_{L^2}^2 dr \\ &+C\int_0^t (1+r)^\alpha \|\partial_x U(r)\|_{L^\infty}\|\partial_x^k U(r)\|_{L^2}^2 dr, \end{aligned} \tag{4.17}$$

where C here and in the sequel is a generic positive constant that may take different values in different places.

Adding the estimate (4.16) to (4.17) and taking the summation for $1 \le k \le s$, we get the main estimate

$$\begin{aligned} &(1+t)^{\alpha}\left\|U(t)\right\|_{H^s}^2+\frac{\tau}{\beta_1}\int_0^t(1+r)^{\alpha}\left\|y(r)\right\|_{H^s}^2dr \\ &\le C\left\|U_0\right\|_{H^s}^2+\frac{\alpha}{\beta_1}\int_0^t(1+r)^{\alpha-1}\left\|U(r)\right\|_{H^s}^2dr \\ &+C\int_0^t(1+r)^{\alpha}\left\|\partial_xU(r)\right\|_{L^\infty}\left\|\partial_xU(r)\right\|_{H^{s-1}}^2dr. \end{aligned} \tag{4.18}$$

On the other hand, applying ∂_x^k to the system (3.2) and put $\partial_x^k(v,h,z,y,\varphi) = (\tilde v,\tilde h,\tilde z,\tilde y,\tilde\varphi)$, then we get

$$\begin{cases} \tilde v_t-\tilde h_x=0, \\ \rho\tilde h_t-\mu\tilde v_x-b\tilde z=0, \\ \tilde z_t-\tilde y_x=0, & x\in\mathbb{R},\ t>0, \\ J\tilde y_t-\delta\tilde z_x+b\tilde v+a\tilde\varphi+\tau\tilde y=\partial_x^k g(z)_x\,, \\ \tilde\varphi_t-\tilde y=0, \end{cases} \tag{4.19}$$

where $g(z) := \sigma(z)-\sigma(0)-\sigma'(0)z = O(z^2)$ near $z=0$.

Multiplying the second equation in (4.19) by $-\tilde z$ and the third equation by $-\rho\tilde h$ and adding the resulting equations, we obtain

$$\frac{d}{dt}\left(-\rho\tilde h\tilde z\right)+\rho\tilde y_x\tilde h+\mu\tilde v_x\tilde z+b\tilde z^2=0. \tag{4.20}$$

Next, multiplying the fourth equation in (4.19) by $-\tilde z_x$ and the third equation by $J\tilde y_x$ and summing up the results, we find

$$\frac{d}{dx}(J\tilde y\tilde z_t)-\frac{d}{dt}(J\tilde y\tilde z_x)+\delta\tilde z_x^2-b\tilde v\tilde z_x-a\tilde\varphi\tilde z_x-\tau\tilde y\tilde z_x-J\tilde y_x^2=-\partial_x^k g(z)_x\,\tilde z_x, \tag{4.21}$$

which can be rewritten as

$$\frac{d}{dx}(J\tilde y\tilde z_t-a\tilde{\varphi\tilde z})-\frac{d}{dt}(J\tilde y\tilde z_x)+\delta\tilde z_x^2+a\tilde z^2-b\tilde v\tilde z_x-\tau\tilde y\tilde z_x-J\tilde y_x^2=-\partial_x^k g(z)_x\,\tilde z_x, \tag{4.22}$$

where we have used the fact that $\tilde\varphi_x=\tilde z$. Adding $-\frac{b}{\mu}$(4.20) to (4.22), we find

$$\begin{aligned} &\frac{d}{dx}\left(J\tilde y\tilde z_t-a\tilde{\varphi\tilde z}-b\tilde v\tilde z-\frac{\rho b}{\mu}\tilde y\tilde h\right)-\frac{d}{dt}\left(J\tilde y\tilde z_x+\frac{\rho b}{\mu}\tilde h\tilde z\right)+\delta\tilde z_x^2+\left(a-\frac{b^2}{\mu}\right)\tilde z^2 \\ &+\frac{\rho b}{\mu}\tilde y\tilde h_x-\tau\tilde y\tilde z_x-J\tilde y_x^2=-\partial_x^k g(z)_x\,\tilde z_x. \end{aligned} \tag{4.23}$$

The above identity can be rewritten as

$$\begin{aligned} &\frac{d}{dt}F^{(k)}(t)+\delta\left(\partial_x^k z_x\right)^2+\left(a-\frac{b^2}{\mu}\right)\left(\partial_x^k z\right)^2-J\left(\partial_x^k y_x\right)^2 \\ &=\frac{d}{dx}A^{(k)}-\frac{\rho b}{\mu}\partial_x^k y\partial_x^k h_x+\tau\partial_x^k y\partial_x^k z_x-\partial_x^k g(z)_x\,\partial_x^k z_x, \end{aligned} \tag{4.24}$$

where

$$F^{(k)}(t) := -\left(J\partial_x^k y \partial_x^k z_x + \frac{\rho b}{\mu}\partial_x^k h \partial_x^k z\right), \quad A^{(k)} := J\partial_x^k \partial_x^k z_t - a\partial_x^k \varphi \partial_x^k z - b\partial_x^k v \partial_x^k z - \frac{\rho b}{\mu}\partial_x^k \partial_x^k h$$

Now, for all $k \geq 0$, we have

$$\begin{aligned}
&\frac{d}{dt}\left(F^{(k)}(t) + F^{(k+2)}(t)\right) + \delta\left\{\left(\partial_x^k z_x\right)^2 + \left(\partial_x^{k+2} z_x\right)^2\right\} \\
&+\left(a - \frac{b^2}{\mu}\right)\left\{\left(\partial_x^k z\right)^2 + \left(\partial_x^{k+2} z\right)^2\right\} \\
&= J\left\{\left(\partial_x^k y_x\right)^2 + \left(\partial_x^{k+2} y_x\right)^2\right\} + \frac{d}{dx}\left(A^{(k)} + A^{(k+2)}\right) \\
&-\frac{\rho b}{\mu}\left\{\partial_x^k y \partial_x^k h_x + \partial_x^{k+2} y \partial_x^{k+2} h_x\right\} + \tau\left\{\partial_x^k y \partial_x^k z_x + \partial_x^{k+2} y \partial_x^{k+2} z_x\right\} \\
&-\left\{\partial_x^k g(z)_x \partial_x^k z_x + \partial_x^{k+2} g(z)_x \partial_x^{k+2} z_x\right\}.
\end{aligned} \tag{4.25}$$

On the other hand, we have

$$-\partial_x^{k+2} y \partial_x^{k+2} h_x = \frac{d}{dx}\left(\partial_x^{k+2} y \partial_x^{k+1} h_x - \partial_x^{k+3} y \partial_x^k h_x\right) + \partial_x^{k+4} y \partial_x^k h_x. \tag{4.26}$$

Plugging (4.26) into (4.26), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(F^{(k)}(t) + F^{(k+2)}(t)\right) + \delta\left\{\left(\partial_x^k z_x\right)^2 + \left(\partial_x^{k+2} z_x\right)^2\right\} \\
&+\left(a - \frac{b^2}{\mu}\right)\left\{\left(\partial_x^k z\right)^2 + \left(\partial_x^{k+2} z\right)^2\right\} \\
&= J\left\{\left(\partial_x^k y_x\right)^2 + \left(\partial_x^{k+2} y_x\right)^2\right\} - \left\{\partial_x^k g(z)_x \partial_x^k z_x + \partial_x^{k+2} g(z)_x \partial_x^{k+2} z_x\right\}. \\
&-\frac{\rho b}{\mu}\left\{\partial_x^k y \partial_x^k h_x - \partial_x^{k+4} y \partial_x^k h_x\right\} + \tau\left\{\partial_x^k y \partial_x^k z_x + \partial_x^{k+2} y \partial_x^{k+2} z_x\right\} \\
&+\frac{d}{dx}\left(A^{(k)} + A^{(k+2)} + \frac{\rho b}{\mu}\partial_x^{k+2} y \partial_x^{k+1} h_x - \frac{\rho b}{\mu}\partial_x^{k+3} y \partial_x^k h_x\right).
\end{aligned} \tag{4.27}$$

Applying Young's inequality, we obtain for any $\epsilon_1 > 0$,

$$-\frac{\rho b}{\mu}\left\{\partial_x^k y \partial_x^k h_x - \partial_x^{k+4} y \partial_x^k h_x\right\} \leq \epsilon_1\left(\partial_x^k h_x\right)^2 + C(\epsilon_1)\left\{\left(\partial_x^k y\right)^2 + \left(\partial_x^{k+4} y\right)^2\right\} \tag{4.28}$$

and

$$\begin{aligned}
\tau\left\{\partial_x^k y \partial_x^k z_x + \partial_x^{k+2} y \partial_x^{k+2} z_x\right\} \leq\ & \epsilon_1\left\{\left(\partial_x^k z_x\right)^2 + \left(\partial_x^{k+2} z_x\right)^2\right\} \\
&+ C(\epsilon_1)\left\{\left(\partial_x^k y\right)^2 + \left(\partial_x^{k+2} y\right)^2\right\}.
\end{aligned} \tag{4.29}$$

Plugging (4.28) and (4.29) into (4.28) and integrating with respect to x, we obtain for all $0 \leq k \leq s-4$

$$\begin{aligned}
&\frac{d\mathscr{F}^{(k)}(t)}{dt} + (\delta - \epsilon_1)\left\{\left\|\partial_x^k z_x\right\|_2^2 + \left\|\partial_x^{k+2} z_x\right\|_2^2\right\} \\
&+\left(a - \frac{b^2}{\mu}\right)\left\{\left\|\partial_x^k z\right\|_2^2 + \left\|\partial_x^{k+2} z\right\|_2^2\right\} \\
&\leq \epsilon_1\left\|\partial_x^k h_x\right\|_2^2 + C(\epsilon_1)\left\{\left\|\partial_x^k y\right\|_2^2 + \left\|\partial_x^k y_x\right\|_2^2 + \left\|\partial_x^{k+2} y\right\|_2^2\right. \\
&\qquad\left. + \left\|\partial_x^{k+2} y_x\right\|_2^2 + \left\|\partial_x^{k+4} y\right\|_2^2\right\} + R_1^{(k)},
\end{aligned} \tag{4.30}$$

where

$$\mathscr{F}^{(k)}(t) := -\int_{\mathbb{R}} \left(J\partial_x^k y \partial_x^k z_x + \frac{\rho b}{\mu}\partial_x^k h \partial_x^k z + J\partial_x^{k+2} y \partial_x^{k+2} z_x + \frac{\rho b}{\mu}\partial_x^{k+2} h \partial_x^{k+2} z \right) dx \tag{4.31}$$

and

$$R_1^{(k)} := \int_{\mathbb{R}} \left(|\partial_x^k z_x| \left|\partial_x^k g(z)_x\right| + |\partial_x^{k+2} z_x| \left|\partial_x^{k+2} g(z)_x\right| \right) dx. \tag{4.32}$$

On the other hand, multiplying the fourth equation in (4.19) by $\tilde{v}$ and the first equation by $J\tilde{y}$ and adding the results, we obtain

$$\frac{d}{dt}(J\tilde{y}\tilde{v}) - \frac{d}{dx}(J\tilde{h}\tilde{y}) + b\tilde{v}^2 - \delta\tilde{z}_x\tilde{v} + a\tilde{\varphi}\tilde{v} + \tau\tilde{y}\tilde{v} + J\tilde{h}\tilde{y}_x = \partial_x^k g(z)_x \tilde{v}. \tag{4.33}$$

Using Young's inequality and integrating (4.33) with respect to x, we get for any $\epsilon_2 > 0$,

$$\begin{aligned} &\frac{d\mathscr{P}^{(k)}(t)}{dt} + (b-\epsilon_2)\|\partial_x^k v\|_{L^2}^2 \\ &\le \epsilon_2 \|\partial_x^k h\|_{L^2}^2 + C(\epsilon_2)\left(\|\partial_x^k z_x\|_{L^2}^2 + \|\partial_x^k \varphi\|_{L^2}^2 + \|\partial_x^k y\|_{H^1}^2\right) + R_2^{(k)}, \end{aligned} \tag{4.34}$$

where

$$\mathscr{P}^{(k)}(t) := \int_{\mathbb{R}} J\partial_x^k y \partial_x^k v dx, \qquad R_2^{(k)} := \int_{\mathbb{R}} \partial_x^k g(z)_x \partial_x^k v. \tag{4.35}$$

Next, multiplying the second equation in (4.19) by $\tilde{v}_x$ and the first equation by $-\rho\tilde{h}_x$, we get

$$\frac{d}{dt}\left(\rho\tilde{h}\tilde{v}_x\right) - \frac{d}{dx}\left(\rho\tilde{h}\tilde{v}_t\right) - \mu\tilde{v}_x^2 + \rho\tilde{h}_x^2 - b\tilde{z}\tilde{v}_x = 0. \tag{4.36}$$

Using Young's inequality and integrating the result with respect to x, we get for each $\epsilon_2 > 0$

$$\frac{d\mathscr{V}^{(k)}(t)}{dt} + \rho\|\partial_x^k h_x\|_{L^2}^2 - (\mu+\epsilon_2)\|\partial_x^k v_x\|_{L^2}^2 \le C(\epsilon_2)\|\partial_x^k z\|_{L^2}^2, \tag{4.37}$$

where

$$\mathscr{V}^{(k)}(t) := \int_{\mathbb{R}} \rho\partial_x^k h \partial_x^k v_x. \tag{4.38}$$

Now, using (4.34) and (4.37), we get for all $0 \le k \le s-2$

$$\begin{aligned} &\frac{d}{dt}\left(\frac{a\mu}{b}\mathscr{P}^{(k+1)}(t) + \frac{b^2}{\mu}\mathscr{V}^{(k)}(t)\right) + \left\{(a\mu - b^2) - \left(\frac{a\mu}{b} + \frac{b^2}{\mu}\right)\epsilon_2\right\}\|\partial_x^k v_x\|_{L^2}^2 \\ &+ \left(\frac{b^2\rho}{\mu} - \epsilon_2\right)\|\partial_x^k h_x\|_{L^2}^2 \\ &\le C(\epsilon_2)\left\{\|\partial_x^k z\|_{L^2}^2 + \|\partial_x^{k+1} z_x\|_{L^2}^2 + \|\partial_x^{k+1}\varphi\|_{L^2}^2 + \|\partial_x^{k+1} y\|_{H^1}^2\right\} + \frac{a\mu}{b}R_2^{(k+1)}. \end{aligned} \tag{4.39}$$

Let us now define the functional $\Psi^{(k)}(t)$ as follows:

$$\Psi^{(k)}(t) := \alpha_1\mathscr{F}^{(k)}(t) + \frac{a\mu}{b}\mathscr{P}^{(k+1)}(t) + \frac{b^2}{\mu}\mathscr{V}^{(k)}(t), \tag{4.40}$$

where α_1 is a positive constants that will be chosen later. Consequently, the estimates (4.30) and (4.40) lead to, for all $0 \leq k \leq s-4$,

$$\begin{aligned}
&\frac{d}{dt}\Psi^{(k)}(t) + \alpha_1(\delta-\epsilon_1)\left\|\partial_x^k z_x\right\|_2^2 + \alpha_1(\delta-\epsilon_1)\left\|\partial_x^{k+2} z_x\right\|_{L^2}^2 \\
&+\left\{\alpha_1\left(a-\frac{b^2}{\mu}\right)-2C(\epsilon_2)\right\}\left\|\partial_x^k z\right\|_{L^2}^2 + \left\{\alpha_1\left(a-\frac{b^2}{\mu}\right)-C(\epsilon_2)\right\}\left\|\partial_x^{k+2} z\right\|_{L^2}^2 \\
&+\left\{(a\mu-b^2)-\left(\frac{a\mu}{b}+\frac{b^2}{\mu}\right)\epsilon_2\right\}\|\partial_x^k v_x\|_{L^2}^2 \\
&+\left\{\left(\frac{b^2\rho}{\mu}-\epsilon_2\right)-\alpha_1\epsilon_1\right\}\|\partial_x^k h_x\|_{L^2}^2 \\
&\leq C(\epsilon_1,\epsilon_2,\alpha_1)\left\|\partial_x^k y\right\|_{H^4}^2 + \alpha_1 R_1^{(k)} + \frac{a\mu}{b}R_2^{(k+1)},
\end{aligned} \tag{4.41}$$

where we have used the fact that $\|\partial_x^{k+1}\varphi\|_{L^2}^2 = \|\partial_x^k z\|_{L^2}^2$. Now, we are going to choose our constants in (4.42) as follows: first, recalling (1.2) and choosing ϵ_2 small enough such that

$$\epsilon_2 < \min\left\{\frac{b^2\rho}{\mu}, (a\mu-b^2)/\left(\frac{a\mu}{b}+\frac{b^2}{\mu}\right)\right\}.$$

After that, we select α_1 large enough so that

$$\alpha_1\left(a-\frac{b^2}{\mu}\right)-2C(\epsilon_2) > 0.$$

Once α_1 and ϵ_2 are fixed, then we take ϵ_1 small enough such that

$$\epsilon_1 < \min\left\{\delta, \left(\frac{b^2\rho}{\mu}-\epsilon_2\right)/\alpha_1\right\}.$$

Consequently, there exists a positive constant $\tilde{c}$ such that (4.42) becomes

$$\begin{aligned}
&\frac{d}{dt}\Psi^{(k)}(t) + \tilde{c}\left\{\left\|\partial_x^k z\right\|_{H^3}^2 + \|\partial_x^k v_x\|_{L^2}^2 + \|\partial_x^k h_x\|_{L^2}^2\right\} \\
&\leq C(\epsilon_1,\epsilon_2,\alpha_1)\left\|\partial_x^k y\right\|_{H^4}^2 + \alpha_1 R_1^{(k)} + \frac{a\mu}{b}R_2^{(k+1)}.
\end{aligned} \tag{4.42}$$

This last estimate can be rewritten as

$$\begin{aligned}
&\frac{d}{dt}\Psi^{(k)}(t) + \tilde{c}_1\left\{\left\|\partial_x^k z\right\|_{H^3}^2 + \left\|\partial_x^{k+1} U\right\|_{L^2}^2\right\} \\
&\leq C_1(\epsilon_1,\epsilon_2,\alpha_1)\left\|\partial_x^k y\right\|_{H^4}^2 + \alpha_1 R_1^{(k)} + \frac{a\mu}{b}R_2^{(k+1)}.
\end{aligned} \tag{4.43}$$

for some $\tilde{c}_1$, $C_1(\epsilon_1,\epsilon_2,\alpha_1) > 0$.

On the other had, from (4.40), (4.31), (4.35) and (4.38) it is obvious that there exists $\tilde{c_2} > 0$ such that

$$\left|\Psi^{(k)}(t)\right| \leq \tilde{c_2}\left\|\partial_x^k U(t)\right\|_{H^3}^2, \qquad \forall t \geq 0. \tag{4.44}$$

Now, following [8], we get

$$R_1^{(k)} \leq C\|z\|_{L^\infty}\left(\left\|\partial_x^{k+1} z\right\|_{L^2}^2 + \left\|\partial_x^{k+3} z\right\|_{L^2}^2\right) \tag{4.45}$$

and

$$
\begin{aligned}
R_2^{(k+1)} &\leq \int_{\mathbb{R}} \left|\partial_x^{k+1} g(z)_x\right| \left|\partial_x^{k+1} v\right| \\
&\leq C \|z\|_{L^\infty} \left\|\partial_x^{k+1} v\right\|_{L^2} \left\|\partial_x^{k+2} z\right\|_{L^2}.
\end{aligned} \tag{4.46}
$$

Now, multiplying (4.43) by $(1+t)^\alpha$, integrating with respect to t and exploiting (4.44)-(4.46), we arrive at

$$
\begin{aligned}
&\int_0^t (1+r)^\alpha \left(\left\|\partial_x^k z(r)\right\|_{H^3}^2 + \left\|\partial_x^{k+1} U(r)\right\|_{L^2}^2\right) dr \\
&\leq C \left\|\partial_x^k U_0\right\|_{H^3}^2 + C(1+t)^\alpha \left\|\partial_x^k U(t)\right\|_{H^3}^2 \\
&\quad + C\alpha \int_0^t (1+r)^{\alpha-1} \left\|\partial_x^k U(r)\right\|_{H^3}^2 dr + C \int_0^t (1+r)^\alpha \left\|\partial_x^k y\right\|_{H^4}^2 dr \\
&\quad + C \int_0^t (1+r)^\alpha \|z\|_{L^\infty} \left(\left\|\partial_x^{k+1} z\right\|_{L^2}^2 + \left\|\partial_x^{k+3} z\right\|_{L^2}^2\right. \\
&\qquad\qquad \left. + \left\|\partial_x^{k+1} v\right\|_{L^2} \left\|\partial_x^{k+2} z\right\|_{L^2}\right) dr,
\end{aligned} \tag{4.47}
$$

for all $t \geq 0$ and $0 \leq k \leq s-4$. Taking the summation in (4.47) over k with $0 \leq k \leq s-4$, we obtain

$$
\begin{aligned}
&\int_0^t (1+r)^\alpha \left(\|z(r)\|_{H^{s-1}}^2 + \|\partial_x U(r)\|_{H^{s-4}}^2\right) dr \\
&\leq C \|U_0\|_{H^{s-1}}^2 + C(1+t)^\alpha \|U(t)\|_{H^{s-1}}^2 \\
&\quad + \alpha C \int_0^t (1+r)^{\alpha-1} \|U(r)\|_{H^{s-1}}^2 dr + C \int_0^t (1+r)^\alpha \|y\|_{H^s}^2 dr \\
&\quad + C \int_0^t (1+r)^\alpha \|z\|_{L^\infty} \left(\|z\|_{H^{s-1}}^2 + \|\partial_x v\|_{H^{s-4}} \|\partial_x z\|_{H^{s-3}}\right) dr.
\end{aligned} \tag{4.48}
$$

Now, let $\lambda > 0$ be a small positive constant, then multiplying (4.48) by λ, adding the result to (4.18) and choosing λ small enough, we get

$$
\begin{aligned}
&(1+t)^\alpha \|U(t)\|_{H^s}^2 + \int_0^t (1+r)^\alpha \|y(r)\|_{H^s}^2 dr \\
&+\lambda \int_0^t (1+r)^\alpha \left(\|z(r)\|_{H^{s-1}}^2 + \|\partial_x U(r)\|_{H^{s-4}}^2\right) dr \\
&\leq C \|U_0\|_{H^s}^2 + \alpha C \int_0^t (1+r)^{\alpha-1} \|U(r)\|_{H^s}^2 dr \\
&+C \int_0^t (1+r)^\alpha \|\partial_x U(r)\|_{L^\infty} \|\partial_x U(r)\|_{H^{s-1}}^2 dr \\
&+C\lambda \int_0^t (1+r)^\alpha \|z\|_{L^\infty} \left(\|z\|_{H^{s-1}}^2 + \|\partial_x v\|_{H^{s-4}} \|\partial_x z\|_{H^{s-3}}\right) dr,
\end{aligned} \tag{4.49}
$$

where C is a generic positive constant depending on λ. It is quite obvious from (4.49) that the term $\int_0^t (1+r)^\alpha \|\partial_x U(r)\|_{L^\infty} \|\partial_x U(r)\|_{H^{s-1}}^2 dr$ can not be controlled by the term $\int_0^t (1+r)^\alpha \|\partial_x U(r)\|_{H^{s-4}}^2 dr$ due to the regularity-loss property. In order to solve this technical difficulty, and following [8], we have to choose $\alpha < 0$ in (4.49) in order to gain the dissipative term $\alpha C \int_0^t (1+r)^{\alpha-1} \|U(r)\|_{H^s}^2 \, dr$ which is strong enough to restore the regularity in (4.49). Consequently, choosing $\alpha = -1/2$, then the estimate (4.49) can be rewritten as

$$
\begin{aligned}
&(1+t)^{-1/2} \|U(t)\|_{H^s}^2 + \int_0^t (1+r)^{-1/2} \|y(r)\|_{H^s}^2 \, dr \\
&+\lambda \int_0^t (1+r)^{-1/2} \left(\|z(r)\|_{H^{s-1}}^2 + \|\partial_x U(r)\|_{H^{s-4}}^2 \right) dr \\
&+ \int_0^t (1+r)^{-3/2} \|U(r)\|_{H^s}^2 \, dr \\
\leq\ & C \|U_0\|_{H^s}^2 + C \int_0^t (1+r)^{-1/2} \|\partial_x U(r)\|_{L^\infty} \|\partial_x U(r)\|_{H^{s-1}}^2 \, dr \\
&+C\lambda \int_0^t (1+r)^{-1/2} \|z\|_{L^\infty} \left(\|z\|_{H^{s-1}}^2 + \|\partial_x v\|_{H^{s-4}} \|\partial_x z\|_{H^{s-3}} \right) dr.
\end{aligned} \tag{4.50}
$$

The last two term in the right-hand side of (4.51) can be estimated as

$$
\begin{aligned}
&\int_0^t (1+r)^{-1/2} \|\partial_x U(r)\|_{L^\infty} \|\partial_x U(r)\|_{H^{s-1}}^2 \, dr \\
\leq\ & M_1(t) \int_0^t (1+r)^{-3/2} \|\partial_x U(r)\|_{H^{s-1}}^2 \, dr \\
\leq\ & M_1(t) D^2(t).
\end{aligned} \tag{4.51}
$$

On the other hand, using the estimate $\|z\|_{L^\infty} \leq (1+t)^{-1/2} M_0(t)$, we have

$$
\begin{aligned}
&\int_0^t (1+r)^{-1/2} \|z\|_{L^\infty} \left(\|z\|_{H^{s-1}}^2 + \|\partial_x v\|_{H^{s-4}} \|\partial_x z\|_{H^{s-3}} \right) dr \\
=\ & \int_0^t (1+r)^{-1/2} \|z\|_{L^\infty} \|z\|_{H^{s-1}}^2 \, dr \\
&+ \int_0^t (1+r)^{-1/2} \|z\|_{L^\infty} \|\partial_x v\|_{H^{s-4}} \|\partial_x z\|_{H^{s-3}} \, dr \\
\leq\ & M_0(t) \int_0^t (1+r)^{-1} \|z\|_{H^{s-1}}^2 \, dr + M_0(t) \int_0^t (1+r)^{-1} \|\partial_x v\|_{H^{s-4}} \|\partial_x z\|_{H^{s-3}} \, dr \\
\leq\ & C M_0(t) D^2(t) \\
&+M_0(t) \left(\int_0^t (1+r)^{-1/2} \|\partial_x v\|_{H^{s-4}}^2 \, dr \right)^{1/2} \left(\int_0^t (1+r)^{-3/2} \|\partial_x z\|_{H^{s-3}}^2 \, dr \right)^{1/2} \\
\leq\ & C M_0(t) D^2(t) \\
&+M_0(t) \left(\int_0^t (1+r)^{-3/2} \|v\|_{H^s}^2 \, dr \right)^{1/2} \left(\int_0^t (1+r)^{-1/2} \|z\|_{H^{s-2}}^2 \, dr \right)^{1/2} \\
\leq\ & C M_0(t) D^2(t).
\end{aligned} \tag{4.52}
$$

Consequently, plugging the estimates (4.52) and (4.52) into (4.51), then (4.4) holds true for $j = 0$. It is sufficient to use induction on j to show that (4.4) is fulfilled. Assume that (4.4) is satisfied for $j-1$; we will show that (4.4) is also valid for j.

Indeed, assume that (4.4) is satisfied for $j-1$. We have to prove that (4.4) remains also valid for j. To this end, take $\alpha = j - 1/2$ in (4.17) and taking the summation over k with $j \leq k \leq s-3j$ with $0 < j \leq [s/4]$, we get

$$\begin{aligned}
&(1+t)^{j-1/2}\left\|\partial_x^j U(t)\right\|^2_{H^{s-4j}} + \frac{\tau}{\beta_1}\int_0^t (1+r)^{j-1/2}\left\|\partial_x^j y(r)\right\|^2_{H^{s-4j}} dr \\
&\leq C\left\|\partial_x^j U_0\right\|^2_{H^{s-4j}} + C\int_0^t (1+r)^{j-3/2}\left\|\partial_x^j U(r)\right\|^2_{H^{s-4j}} dr \qquad (4.53)\\
&+C\int_0^t (1+r)^{j-1/2}\left\|\partial_x U(r)\right\|_{L^\infty}\left\|\partial_x^j U(r)\right\|^2_{H^{s-4j}} ds.
\end{aligned}$$

Similarly, taking $\alpha = j - 1/2$ in (4.47) and taking the summation over k with $j \leq k \leq s - 3j - 4$ with $0 < j \leq [s/4] - 1$, we find

$$\begin{aligned}
&\int_0^t (1+r)^{j-1/2}\left(\left\|\partial_x^j z(r)\right\|^2_{H^{s-4j-1}} + \left\|\partial_x^{j+1} U(r)\right\|^2_{H^{s-4j-4}}\right) dr \\
&\leq C\left\|\partial_x^j U_0\right\|^2_{H^{s-4j-1}} + C(1+t)^{j-1/2}\left\|\partial_x^j U(t)\right\|^2_{H^{s-4j-1}} \\
&+C\int_0^t (1+r)^{j-3/2}\left\|\partial_x^j U(r)\right\|^2_{H^{s-4j-1}} dr + C\int_0^t (1+r)^{j-1/2}\left\|\partial_x^j y\right\|^2_{H^{s-4j}} dr \qquad (4.54)\\
&+C\int_0^t (1+r)^{j-1/2}\left\|z\right\|_{L^\infty}\left(\left\|\partial_x^{j+1} z\right\|^2_{H^{s-4j-4}} + \left\|\partial_x^{j+3} z\right\|^2_{H^{s-4j-4}}\right.\\
&\qquad\left. + \left\|\partial_x^{j+1} v\right\|_{H^{s-4j-4}}\left\|\partial_x^{j+2} z\right\|_{H^{s-4j-4}}\right) dr.
\end{aligned}$$

As above, for $1 \leq j \leq [s/4] - 1$, then (4.54) + $\tilde{\lambda}$(4.54) gives for $\tilde{\lambda}$ sufficiently small

$$\begin{aligned}
&(1+t)^{j-1/2}\left\|\partial_x^j U(t)\right\|^2_{H^{s-4j}} + C\int_0^t (1+r)^{j-1/2}\left\|\partial_x^j y(r)\right\|^2_{H^{s-4j}} dr \\
&+\int_0^t (1+r)^{j-1/2}\left(\left\|\partial_x^j z(r)\right\|^2_{H^{s-4j-1}} + \left\|\partial_x^{j+1} U(r)\right\|^2_{H^{s-4j-4}}\right) dr \\
&\leq C\left\|\partial_x^j U_0\right\|^2_{H^{s-4j}} + C\int_0^t (1+r)^{j-3/2}\left\|\partial_x^j U(r)\right\|^2_{H^{s-4j}} dr \\
&+C\int_0^t (1+r)^{j-1/2}\left\|\partial_x^j y\right\|^2_{H^{s-4j}} dr \qquad (4.55)\\
&+C\int_0^t (1+r)^{j-1/2}\left\|\partial_x U(r)\right\|_{L^\infty}\left\|\partial_x^j U(r)\right\|^2_{H^{s-4j}} dr \\
&+C\int_0^t (1+r)^{j-1/2}\left\|z\right\|_{L^\infty}\left(\left\|\partial_x^{j+1} z\right\|^2_{H^{s-4j-4}} + \left\|\partial_x^{j+3} z\right\|^2_{H^{s-4j-4}}\right.\\
&\qquad\left. + \left\|\partial_x^{j+1} v\right\|_{H^{s-4j-4}}\left\|\partial_x^{j+2} z\right\|_{H^{s-4j-4}}\right) dr,
\end{aligned}$$

where C is a positive constant depending on $\tilde{\lambda}$. The main step here is to estimate the second term on the right-hand side of (4.56). This can be done by induction as follows: since we assumed that (4.4) holds for $j-1$ (hence the above estimate (4.56) is fulfilled for $j-1$ instead of j), then we have

$$\int_0^t (1+r)^{j-3/2}\left\|\partial_x^j U(r)\right\|^2_{H^{s-4j}} dr \leq C\left\|U_0\right\|^2_{H^s} + C\left(M_0(t) + M_1(t)\right) D^2(t).$$

Since the estimate (4.54) holds also for $[s/4]$, then from (4.51), (4.54), (4.56) and in order to prove (4.4), it is enough to prove the following two estimates:

$$\int_0^t (1+r)^{j-1/2} \|\partial_x U(r)\|_{L^\infty} \|\partial_x^j U(r)\|^2_{H^{s-4j}} ds \leq CM_1(t) D^2(t), \quad \text{for } 1 \leq j \leq [s/4] \tag{4.56}$$

and

$$\begin{aligned}&\int_0^t (1+r)^{j-1/2} \|z\|_{L^\infty} \Big(\big\|\partial_x^{j+1} z\big\|^2_{H^{s-4j-4}} + \big\|\partial_x^{j+3} z\big\|^2_{H^{s-4j-4}} \\ &\qquad + \big\|\partial_x^{j+1} v\big\|_{H^{s-4j-4}} \big\|\partial_x^{j+2} z\big\|_{H^{s-4j-4}} \Big) dr \\ &\leq CM_0(t) D^2(t), \quad \text{for } 1 \leq j \leq [s/4]-1.\end{aligned} \tag{4.57}$$

The estimate (4.56) is straightforward. Let us now prove (4.58). Indeed, we have

$$\begin{aligned}&\int_0^t (1+r)^{j-1/2} \|z\|_{L^\infty} \big\|\partial_x^{j+1} z\big\|^2_{H^{s-4j-4}} dr \\ &\leq CM_0(t) \int_0^t (1+r)^{j-1} \Big(\big\|\partial_x^{j+1} z(r)\big\|^2_{H^{s-4j-4}} + \big\|\partial_x^{j+3} z\big\|^2_{H^{s-4j-4}} \Big) dr \\ &\leq CM_0(t) \int_0^t (1+r)^{j-1} \big\|\partial_x^{j} z(r)\big\|^2_{H^{s-4j-1}} dr \\ &\leq CM_0(t) D^2(t).\end{aligned}$$

On the other hand, we have for $1 \leq j \leq [s/4]-1$,

$$\begin{aligned}&\int_0^t (1+r)^{j-1/2} \|z\|_{L^\infty} \Big\|\partial_x^{j+1} v(r)\Big\|_{H^{s-4j-4}} \Big\|\partial_x^{j+2} z(r)\Big\|_{H^{s-4j-4}} dr \\ &\leq M_0(t) \int_0^t (1+r)^{j-1} \Big\|\partial_x^{j+1} v(r)\Big\|_{H^{s-4j-4}} \Big\|\partial_x^{j} z(r)\Big\|_{H^{s-4j-1}} dr \\ &\leq M_0(t) \left(\int_0^t (1+r)^{j-3/2} \Big\|\partial_x^{j} v(r)\Big\|^2_{H^{s-4j}} dr \right)^{1/2} \left(\int_0^t (1+r)^{j-1/2} \Big\|\partial_x^{j} z(r)\Big\|^2_{H^{s-4j-1}} dr \right)^{1/2} \\ &\leq CM_0(t) D^2(t).\end{aligned}$$

Consequently, the proof of Proposition 4.2 is finished. □

Lemma 4.3. *Under the same assumptions as in Proposition 4.2, and supposing that $U_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ with $s \geq 2$, we have*

$$M(t) \leq CE_0 + CM(t)^2 + CM_0(t) E(t) \tag{4.58}$$

for all $t \in [0,T]$, where C is a positive constant which is independent of T, and E_0 is given in Theorem 3.3.

Proof. In order to prove (4.58), it is suffices to show the estimate

$$\Big\|\partial_x^k U(s)\Big\|_{L^2} \leq CE_0 (1+t)^{-1/4-k/2} + C\left(M(t)^2 + M_0(t) E(t)\right)(1+t)^{-1/4-k/2}, \tag{4.59}$$

for $0 \leq k \leq [s/4]-4$. To show the above estimate, we follow some steps in [8].

Using the Duhamel principle, the solution of problem (3.4) can be written as an integral equation of the form

$$U(t) = e^{t\Phi}U_0 + \int_0^t e^{(t-\tau)\Phi}G(U)_x(\tau)\,d\tau, \tag{4.60}$$

where

$$\left(e^{t\Phi}\omega\right)(x) := \mathcal{F}^{-1}\left[e^{t\hat{\Phi}(i\xi)}\hat{\omega}(\xi)\right](x)$$

with $\hat{\Phi}(i\xi) := -(i\xi A + L)$ and $G(U) := (0,0,0,g(z),0)$.

The arguments used to prove the estimate (4.59) are very similar to ones employed by Ide and Kawashima [8], so many of the details will therefore be omitted.

Let, $0 \le k \le [s/4]-1$, then applying ∂_x^k to (4.60) and taking the L^2 norm, we conclude

$$\begin{aligned}\left\|\partial_x^k U(t)\right\|_2 &\le \left\|\partial_x^k e^{t\Phi}U_0\right\|_2 + \int_0^t \left\|\partial_x^{k+1}e^{(t-r)\Phi}G(U)\right\|_2 dr\\ &= I_1 + I_2.\end{aligned} \tag{4.61}$$

Since $e^{t\Phi}U_0$ is the solution of the linear problem, then from (3.10), we get for $l = 3k+2$

$$\begin{aligned}I_1 &\le C(1+t)^{-1/4-k/2}\left\|U_0\right\|_{L^1} + C(1+t)^{-k/2-1/3}\left\|\partial^{4k+2}U_0\right\|_{L^2}\\ &\le CE_0(1+t)^{-1/4-k/2},\end{aligned} \tag{4.62}$$

where we have used the fact that $4k+2 \le s$ for $k \le [s/4]-1$.

The estimate of I_2 is standard. Let us split it into two parts:

$$\begin{aligned}I_2 &= \int_0^{t/2}\left\|\partial_x^{k+1}e^{(t-\tau)\Phi}G(U(r))\right\|_2 dr + \int_{t/2}^t \left\|\partial_x^{k+1}e^{(t-r)\Phi}G(U(r))\right\|_2 dr\\ &= J_1 + J_2,\end{aligned}$$

and applying (3.10), with $l = 3k+2$, we infer that

$$\begin{aligned}J_1 &\le C\int_0^{t/2}(1+t-r)^{-3/4-k/2}\left\|G(U(r))\right\|_{L^1} dr\\ &\quad + C\int_0^{t/2}(1+t-r)^{-1/3-k/2}\left\|\partial_x^{4k+3}G(U(r))\right\|_{L^2} dr\\ &\le C\int_0^{t/2}(1+t-r)^{-3/4-k/2}\left\|U(r)\right\|_{L^2}^2 dr\\ &\quad + C\int_0^{t/2}(1+t-r)^{-1/3-k/2}\left\|\partial_x^{4k+3}G(U(r))\right\|_{L^2} dr\\ &\le CM(t)^2\int_0^{t/2}(1+t-r)^{-3/4-k/2}(1+r)^{-1/2} dr\\ &\quad + C\int_0^{t/2}(1+t-r)^{-1/2-k/2}\left\|\partial_x^{4k+3}G(U(r))\right\|_{L^2} dr,\end{aligned} \tag{4.63}$$

where we have used the fact that $\|U(t)\|_{L^2} \le M(t)(1+t)^{-1/4}$. The first term in the right-hand side of (4.63) can be estimated as

$$CM(t)^2\int_0^{t/2}(1+t-r)^{-3/4-k/2}(1+r)^{-1/2}d\tau \le CM(t)^2(1+t)^{-1/4-k/2}.$$

Also, following [8], we have $\left\|\partial_x^{4k+3}G\left(U\left(r\right)\right)\right\|_{L^2} \leq C\left\|U\right\|_{L^\infty}\left\|\partial_x^{4k+3}U\right\|_{L^2}$. Since $\left\|U\left(t\right)\right\|_{L^\infty} \leq M_0\left(t\right)\left(1+t\right)^{-1/2}$ and

$$\left\|\partial_x^{4k+3}U\right\|_{L^2} \leq \left\|\partial_x U\right\|_{H^{s-4}} \leq E\left(t\right)\left(1+t\right)^{-1/4}, \tag{4.64}$$

for $4k+2 \leq s-4$ with $0 \leq k \leq [s/4]-1$, consequently, we have

$$\begin{aligned}
&\int_0^{t/2}\left(1+t-r\right)^{-1/3-k/2}\left\|\partial_x^{4k+3}G\left(U\left(r\right)\right)\right\|_{L^2}dr\\
&\leq M_0\left(t\right)E\left(t\right)\int_0^{t/2}\left(1+t-r\right)^{-1/3-k/2}\left(1+r\right)^{-3/4}dr\\
&\leq CM_0\left(t\right)E\left(t\right)\left(1+t\right)^{-1/4-k/2}.
\end{aligned}$$

Thus, we get

$$J_1 \leq CM\left(t\right)^2\left(1+t\right)^{-1/4-k/2} + CM_0\left(t\right)E\left(t\right)\left(1+t\right)^{-1/4-k/2}.$$

On the other hand, applying the estimate (3.10) with $k=1$, $l=3$, we obtain

$$\begin{aligned}
J_2 &= \int_{t/2}^t\left\|\partial_x e^{(t-\tau)\Phi}\partial_x^k G\left(U\left(r\right)\right)\right\|_2 dr\\
&\leq C\int_{t/2}^t\left(1+t-\tau\right)^{-3/4}\left\|\partial_x^k G\left(U\left(r\right)\right)\right\|_{L^1}dr\\
&\quad+C\int_{t/2}^t\left(1+t-r\right)^{-1/2}\left\|\partial_x^{k+4}G\left(U\left(r\right)\right)\right\|_{L^2}dr.
\end{aligned}$$

This implies that for $k \leq [s/4]-2$ we have (cp. [8]) $\left\|\partial_x^k G\left(U\left(r\right)\right)\right\|_{L^1} \leq C\left\|U\right\|_{L^2}\left\|\partial_x^k U\right\|_{L^2}$ and $\left\|\partial_x^k U\left(t\right)\right\|_{L^2} \leq M\left(t\right)\left(1+t\right)^{-1/4-k/2}$. Consequently, we get

$$\begin{aligned}
\int_{t/2}^t\left(1+t-r\right)^{-3/4}\left\|\partial_x^k G\left(U\left(r\right)\right)\right\|_{L^1}dr &\leq M\left(t\right)^2\int_{t/2}^t\left(1+t-r\right)^{-3/4}\left(1+\tau\right)^{-1/2-k/2}\\
&\leq CM\left(t\right)^2\left(1+t\right)^{-1/4-k/2}.
\end{aligned}$$

Also, we have (cp. [8]) $\left\|\partial_x^{k+4}G\left(U\left(r\right)\right)\right\|_{L^2} \leq C\left\|U\right\|_{L^\infty}\left\|\partial_x^{k+4}U\right\|_{L^2}$ and for $k \leq [s/4]-4$, then we deduce that

$$\left\|\partial_x^{k+4}U\right\|_{L^2} \leq \left\|\partial_x^{k+4}U\right\|_{H^{s-4k-16}} \leq E\left(t\right)\left(1+t\right)^{-k/2-7/4}.$$

This last estimate leads to

$$\begin{aligned}
&\int_{t/2}^t\left(1+t-r\right)^{-1/2}\left\|\partial_x^{k+4}G\left(U\left(r\right)\right)\right\|_{L^2}dr\\
&\leq C\int_{t/2}^t\left(1+t-r\right)^{-1/2}\left\|U\right\|_{L^\infty}\left\|\partial_x^{k+4}U\left(r\right)\right\|_{L^2}dr\\
&\leq CM_0\left(t\right)E\left(t\right)\int_{t/2}^t\left(1+t-r\right)^{-1/2}\left(1+r\right)^{-k/2-9/4}dr\\
&\leq CM_0\left(t\right)E\left(t\right)\left(1+t\right)^{-1/4-k/2}.
\end{aligned}$$

Plugging all the above estimates into (4.61), we get

$$\left\|\partial_x^k U(t)\right\|_2 \le CE_0(1+t)^{-1/4-k/2} + \left(CM(t)^2 + CM_0(t)E(t)\right)(1+t)^{-1/4-k/2},$$

for $0 \le k \le [s/4] - 4$. Thus the estimate (4.58) holds which completes the proof of Lemma 4.3. □

The proof of Lemma 4.1 can be done exactly as in [8, 18]. We omit the details.

5. Regularity gain. In this section, we show that an additional damping of a viscoelastic type of the form u_{txx} acting on the first equation of the system (1.3) gives a regularity gain to the solution and we show that the global existence of the solution holds for $s \ge 4$ rather than $s \ge 24$. Thus we are interested in the following system:

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\varphi_x - u_{txx} = 0, \\ J\varphi_{tt} - \sigma(\varphi_x)_x + bu_x + a\varphi + \tau\varphi_t = 0, \end{cases} \quad x \in \mathbb{R},\ t > 0. \tag{5.1}$$

Taking the same change of variables (3.1), then the above system can be rewritten as

$$\begin{cases} v_t - h_x = 0, \\ \rho h_t - \mu v_x - bz - h_{xx} = 0, \\ z_t - y_x = 0, \\ Jy_t - \sigma(z)_x + bv + a\varphi + \tau y = 0, \\ \varphi_t - y = 0, \end{cases} \quad x \in \mathbb{R},\ t > 0. \tag{5.2}$$

The system (5.2), (3.3) can be rewritten as

$$\begin{cases} U_t + F(U)_x + LU = BU_{xx}, \\ U(x,0) = U_0, \end{cases} \tag{5.3}$$

with U, F, L are as before and B is the matrix

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The linearized system associated to (5.2) has the following decay rate:

THEOREM 5.1. *([3]) Let s be a nonnegative integer and let $U_0 = (v_0, h_0, z_0, y_0, \varphi_0)^T \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$. Assume that (1.2) holds. Then the solution $U = (v, h, z, y, \varphi)^T$ of the linearized system associate to (5.3) satisfies the following decay estimates:*

$$\left\|\partial_x^k U(t)\right\|_{L^2} \le C(1+t)^{-1/4-k/2}\|U_0\|_{L^1} + Ce^{-ct}\left\|\partial_x^k U_0\right\|_{L^2}, \tag{5.4}$$

for any $t \ge 0$ and $k \le s$, where C and c are two positive constants independent of t and U_0.

As in the previous section, define

$$\tilde{E}^2(t) \equiv \sum_{j=0}^{s-1} \sup_{0\leq r\leq t} (1+r)^j \left\|\partial_x^j U(r)\right\|_{H^{s-j-1}}^2 \tag{5.5}$$

and

$$\begin{aligned}\tilde{D}^2(t) \equiv & \sum_{j=0}^{s-2} \int_0^t (1+r)^j \left\|\partial_x^{j+1} U(r)\right\|_{H^{s-j-2}}^2 dr \\ & + \sum_{j=0}^{s-1} \int_0^t (1+r)^j \left\|\partial_x^j y(r)\right\|_{H^{s-j-1}}^2 dr \\ & + \sum_{j=0}^{s-1} \int_0^t (1+r)^j \left\|\partial_x^{j+1} h(r)\right\|_{H^{s-j-1}}^2 dr.\end{aligned} \tag{5.6}$$

Our main result in this section reads as follows:

THEOREM 5.2. *Assume that $\sigma'(\eta) > 0$. Let $U_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ with $s \geq 4$ and put $E_0 := \|U_0\|_{H^s} + \|U_0\|_{L^1}$. Then, there exists a positive constant $\delta_1 > 0$ such that if $E_0 \leq \delta_1$, then problem (5.3) has a unique global solution U satisfying*

$$U \in C([0,\infty); H^s(\mathbb{R})) \cap C^1([0,\infty); H^{s-1}(\mathbb{R})). \tag{5.7}$$

Moreover, the solution satisfies the weighted energy estimate

$$\tilde{E}^2(t) + \tilde{D}^2(t) \leq C E_0^2, \tag{5.8}$$

and the decay estimate

$$\left\|\partial_x^k U(t)\right\|_2 \leq C E_0 (1+t)^{-1/4-k/2}, \tag{5.9}$$

where C is a positive constant and $0 \leq k \leq s-2$.

Now define

$$\tilde{M}(t) := \sum_{j=0}^{s-2} \sup_{0\leq r\leq t} (1+r)^{1/4+j/2} \left\|\partial_x^j U(r)\right\|_2.$$

Then, we have the following proposition:

PROPOSITION 5.3. *Suppose that the assumptions in Theorem 5.2 hold. Let $T > 0$ and $s \geq 2$, and let U be the solution of the problem (5.3) satisfying (5.7) and (4.3). Then, the estimate*

$$\tilde{E}(t)^2 + \tilde{D}(t)^2 \leq C \|U_0\|_{H^{s-1}}^2 + C(M_0(t) + M_1(t)) \tilde{D}^2(t), \tag{5.10}$$

holds true for all $t \in [0,T]$, where C is a positive constant which is independent of T.

Proof. The proof of Proposition 5.3 will be similar (in some parts) to the proof of Proposition 4.2. In order to show (5.10) it is suffices to show that for any $t \in [0,T]$

and for any $0 \leq j \leq s-2$

$$(5.11)\qquad \begin{aligned} &(1+t)^j \left\|\partial_x^j U(t)\right\|_{H^{s-j-1}}^2 + \int_0^t (1+r)^j \left\|\partial_x^{j+1} U(r)\right\|_{H^{s-j-2}}^2 dr \\ &+ \int_0^t (1+r)^j \left\|\partial_x^j y(r)\right\|_{H^{s-j-1}}^2 dr + \int_0^t (1+r)^j \left\|\partial_x^{j+1} h(r)\right\|_{H^{s-j-1}}^2 dr \\ \leq{}& C \left\|U_0\right\|_{H^{s-1}}^2 + C\left(M_0(t) + M_1(t)\right) \tilde{D}^2(t) \end{aligned}$$

and

$$(5.12)\qquad \begin{aligned} &(1+t)^{s-1} \left\|\partial_x^{s-1} U(t)\right\|_{L^2}^2 + \int_0^t (1+r)^{s-1} \left\|\partial_x^{s-1} y(r)\right\|_{L^2}^2 dr \\ &+ \int_0^t (1+r)^{s-1} \left\|\partial_x^s h(r)\right\|_{H^{s-j-1}}^2 dr \\ \leq{}& C \left\|U_0\right\|_{H^{s-1}}^2 + C\left(M_0(t) + M_1(t)\right) \tilde{D}^2(t) \end{aligned}$$

hold true.

First, the identity (4.5) takes the form

$$(5.13)\qquad \frac{d}{dt} E^{(0)}(t) + \tau \left\|y\right\|_2^2 + \left\|h_x\right\|_2^2 = 0,$$

where $E^{(0)}$ is defined in (4.6). Applying, for $k \geq 1$, ∂_x^k to (5.2), we get

$$(5.14)\qquad \begin{cases} \partial_x^k v_t - \partial_x^{k+1} h = 0, \\ \rho \partial_x^k h_t - \mu \partial_x^{k+1} v - b \partial_x^k z - \partial_x^{k+2} h = 0, \\ \partial_x^k z_t - \partial_x^{k+1} y = 0, & x \in \mathbb{R},\ t > 0, \\ J \partial_x^k y_t - \sigma'(z) \partial_x^{k+1} z + b \partial_x^k v + a \partial_x^k \varphi + \tau \partial_x^k y = [\partial_x^k, \sigma'(z)] z_x, \\ \partial_x^k \varphi_t - \partial_x^k y = 0, \end{cases}$$

Applying the same techniques as in Section 4, then the identity (4.9) becomes

$$(5.15)\qquad \frac{d}{dt} E^{(k)}(t) + \tau \left\|\partial_x^k y\right\|_2^2 + \left\|\partial_x^{k+1} h\right\|_2^2 = R_0^{(k)},$$

where $E^{(k)}(t)$ and $R_0^{(k)}$ are given in (4.8) and (4.10) respectively. Consequently, as in Section 4, the estimate (4.18) becomes

$$(5.16)\qquad \begin{aligned} &(1+t)^\alpha \left\|\partial_x^k U(t)\right\|_{L^2}^2 + \frac{\tau}{\beta_1} \int_0^t (1+r)^\alpha \left\|\partial_x^k y(r)\right\|_{L^2}^2 dr \\ &+ \frac{1}{\beta_1} \int_0^t (1+r)^\alpha \left\|\partial_x^{k+1} h(r)\right\|_{L^2}^2 dr \\ \leq{}& C \left\|\partial_x^k U_0\right\|_{L^2}^2 + \frac{\alpha}{\beta_1} \int_0^t (1+r)^{\alpha-1} \left\|\partial_x^k U(r)\right\|_{L^2}^2 dr \\ &+ C \int_0^t (1+r)^\alpha \left\|\partial_x U(r)\right\|_{L^\infty} \left\|\partial_x^k U(r)\right\|_{L^2}^2 dr. \end{aligned}$$

We take the summation over $0 \leq k \leq s-1$, we get

$$
\begin{aligned}
&(1+t)^{\alpha} \|U(t)\|_{H^{s-1}}^2 + \frac{\tau}{\beta_1} \int_0^t (1+r)^{\alpha} \|y(r)\|_{H^{s-1}}^2 \, dr \\
&+ \frac{1}{\beta_1} \int_0^t (1+r)^{\alpha} \|\partial_x h(r)\|_{H^{s-1}}^2 \, dr \\
\leq\, & C \|U_0\|_{H^{s-1}}^2 + \frac{\alpha}{\beta_1} \int_0^t (1+r)^{\alpha-1} \|U(r)\|_{H^{s-1}}^2 \, dr \\
&+ C \int_0^t (1+r)^{\alpha} \|\partial_x U(r)\|_{L^\infty} \|\partial_x U(r)\|_{H^{s-2}}^2 \, dr.
\end{aligned} \tag{5.17}
$$

Now, following the same steps as in the proof of Proposition 4.2, we rewrite system (5.14) as

$$
\begin{cases}
\tilde{v}_t - \tilde{h}_x = 0, \\
\rho \tilde{h}_t - \mu \tilde{v}_x - b\tilde{z} - \tilde{h}_{xx} = 0, \\
\tilde{z}_t - \tilde{y}_x = 0, \\
J\tilde{y}_t - \delta \tilde{z}_x + b\tilde{v} + a\tilde{\varphi} + \tau \tilde{y} = \partial_x^k g(z)_x, \\
\tilde{\varphi}_t - \tilde{y} = 0,
\end{cases} \qquad x \in \mathbb{R},\ t > 0, \tag{5.18}
$$

and (4.24) is easy reformulated as

$$
\begin{aligned}
&\frac{d}{dt} F^{(k)}(t) + \delta \left(\partial_x^k z_x\right)^2 + \left(a - \frac{b^2}{\mu}\right) \left(\partial_x^k z\right)^2 - J \left(\partial_x^k y_x\right)^2 \\
=\, & \frac{d}{dx} A^{(k)} - \frac{\rho b}{\mu} \partial_x^k y \partial_x^k h_x + \tau \partial_x^k y \partial_x^k z_x - \frac{b}{\mu} \partial_x^k h_{xx} \partial_x^k z - \partial_x^k g(z)_x \partial_x^k z_x.
\end{aligned} \tag{5.19}
$$

Applying Young's inequality, we obtain for any $\epsilon_3 > 0$,

$$
\begin{aligned}
&\frac{d}{dt} F^{(k)}(t) + (\delta - \epsilon_3) \left(\partial_x^k z_x\right)^2 + \left(a - \frac{b^2}{\mu} - \epsilon_3\right) \left(\partial_x^k z\right)^2 - J \left(\partial_x^k y_x\right)^2 \\
\leq\, & \frac{d}{dx} A^{(k)} + C(\epsilon_3) \left\{ \left(\partial_x^k y\right)^2 + \left(\partial_x^{k+1} h\right)^2 + \left(\partial_x^{k+2} h\right)^2 \right\}.
\end{aligned}
$$

Integrating the above inequality with respect to x, we obtain for all $0 \leq k \leq s-2$

$$
\begin{aligned}
&\frac{d\tilde{\mathscr{F}}^{(k)}(t)}{dt} + (\delta - \epsilon_3) \left\|\partial_x^k z_x\right\|_2^2 + \left(a - \frac{b^2}{\mu} - \epsilon_3\right) \left\|\partial_x^k z\right\|_2^2 \\
\leq\, & C(\epsilon_3) \left\{ \left\|\partial_x^k y\right\|_2^2 + \left\|\partial_x^{k+1} y\right\|_2^2 + \left\|\partial_x^{k+1} h\right\|_2^2 + \left\|\partial_x^{k+2} h\right\|_2^2 \right\} + \tilde{R}_1^{(k)},
\end{aligned} \tag{5.20}
$$

where

$$
\tilde{\mathscr{F}}^{(k)}(t) := -\int_{\mathbb{R}} \left(J \partial_x^k y \partial_x^k z_x + \frac{\rho b}{\mu} \partial_x^k h \partial_x^k z \right) dx \tag{5.21}
$$

and

$$
\tilde{R}_1^{(k)} := \int_{\mathbb{R}} \left|\partial_x^k z_x\right| \left|\partial_x^k g(z)_x\right| dx. \tag{5.22}
$$

On the other hand, multiplying the first equation in (5.18) by $\rho\tilde{h}_x$ and the second equation by $-\tilde{v}_x$ and adding the results, we get

$$\begin{aligned}&\frac{d}{dt}\left(\rho\partial_x^k v\partial_x^k h_x\right)-\frac{d}{dx}\left(\rho\partial_x^k v\partial_x^k h_t\right)+b\left(\partial_x^k v_x\right)^2-\rho\left(\partial_x^k h_x\right)^2\\&=-b\partial_x^k z\partial_x^k v_x-\partial_x^k h_{xx}\partial_x^k v_x.\end{aligned}$$

Applying Young's inequality to the terms on the right-hand side of the above estimate and integrating the result with respect to x, we obtain for any $\epsilon_3>0$,

$$\frac{d}{dt}G^{(k)}(t)+(b-\epsilon_3)\left\|\partial_x^{k+1}v\right\|_2^2\leq C(\epsilon_3)\left\{\left\|\partial_x^k z\right\|_2^2+\left\|\partial_x^{k+1}h\right\|_2^2+\left\|\partial_x^{k+2}h\right\|_2^2\right\},\tag{5.23}$$

where

$$G^{(k)}(t):=\int_{\mathbb{R}}\rho\partial_x^k v\partial_x^k h_x dx.\tag{5.24}$$

Next, we define the functional $\tilde{\mathscr{L}}(t)$ as

$$\tilde{\mathscr{L}}(t):=\alpha_2\tilde{\mathscr{F}}^{(k)}(t)+G^{(k)}(t),\tag{5.25}$$

for some large positive constant $\alpha_2>0$. Thus, we obtain from (5.20) and (5.23) for any $0\leq k\leq s-2$

$$\begin{aligned}&\frac{d}{dt}\tilde{\mathscr{L}}(t)+\alpha_2(\delta-\epsilon_3)\left\|\partial_x^k z_x\right\|_2^2\\&+\left\{\alpha_2\Big(a-\frac{b^2}{\mu}-\epsilon_3\Big)-C(\epsilon_3)\right\}\left\|\partial_x^k z\right\|_2^2+(b-\epsilon_3)\left\|\partial_x^{k+1}v\right\|_2^2\\&\leq C(\epsilon_3,\alpha_2)\left\{\left\|\partial_x^k y\right\|_2^2+\left\|\partial_x^{k+1}y\right\|_2^2+\left\|\partial_x^{k+1}h\right\|_2^2+\left\|\partial_x^{k+2}h\right\|_2^2\right\}+\alpha_2\tilde{R}_1^{(k)}.\end{aligned}$$

Keeping in mind (1.2) and choosing ϵ_3 small enough such that

$$\epsilon_3<\min\Big(\delta,b,a-\frac{b^2}{\mu}\Big).$$

Once ϵ_3 is fixed, we pick α_2 large enough such that

$$\alpha_2\Big(a-\frac{b^2}{\mu}-\epsilon_3\Big)-C(\epsilon_3)>0.$$

Thus, we deduce that there exists $\tilde{c}_3>0$ such that

$$\begin{aligned}&\frac{d}{dt}\tilde{\mathscr{L}}(t)+\tilde{c}_2\left\{\left\|\partial_x^k z\right\|_2^2+\left\|\partial_x^k z_x\right\|_2^2+\left\|\partial_x^{k+1}v\right\|_2^2\right\}\\&\leq C(\epsilon_3,\alpha_2)\left\{\left\|\partial_x^k y\right\|_2^2+\left\|\partial_x^{k+1}y\right\|_2^2+\left\|\partial_x^{k+1}h\right\|_2^2+\left\|\partial_x^{k+2}h\right\|_2^2\right\}+\alpha_2\tilde{R}_1^{(k)}.\end{aligned}\tag{5.26}$$

Also, it is not hard to see that

$$\left|\tilde{\mathscr{L}}(t)\right|\leq\tilde{c}_4\left\|\partial_x^k U(t)\right\|_{H^1}^2,\qquad\forall t\geq 0,\tag{5.27}$$

and

$$\tilde{R}_1^{(k)}\leq C\left\|z\right\|_{L^\infty}\left\|\partial_x^{k+1}z\right\|_{L^2}^2.\tag{5.28}$$

Now, plugging the estimates (5.27) and (5.28) into (5.26), multiplying the result by $(1+t)^\alpha$ and integrating with respect to t, we find

$$
\begin{aligned}
&\int_0^t (1+r)^\alpha \left\|\partial_x^{k+1} U(r)\right\|_{L^2}^2 dr \\
&\le C\left\|\partial_x^k U_0\right\|_{H^1}^2 + C(1+t)^\alpha \left\|\partial_x^k U(t)\right\|_{H^1}^2 \\
&\quad + \int_0^t (1+r)^\alpha \left(\left\|\partial_x^k y(r)\right\|_{H^1}^2 + \left\|\partial_x^{k+1} h(r)\right\|_{H^1}^2\right) dr \\
&\quad + \alpha \int_0^t (1+r)^{\alpha-1} \left\|\partial_x^k U(r)\right\|_{H^1}^2 dr \\
&\quad + \int_0^t (1+r)^\alpha C \|z\|_{L^\infty} \left\|\partial_x^{k+1} z(r)\right\|_{L^2}^2 dr,
\end{aligned} \tag{5.29}
$$

for all $t \ge 0$ and $0 \le k \le s-2$. Taking the summation in (4.47) over k with $0 \le k \le s-2$, we obtain

$$
\begin{aligned}
&\int_0^t (1+r)^\alpha \|\partial_x U(r)\|_{H^{s-2}}^2 dr \\
&\le C\|U_0\|_{H^{s-1}}^2 + C(1+t)^\alpha \|U(t)\|_{H^{s-1}}^2 \\
&\quad + \int_0^t (1+r)^\alpha \left(\|y(r)\|_{H^{s-1}}^2 + \|\partial_x h(r)\|_{H^{s-1}}^2\right) dr \\
&\quad + \alpha \int_0^t (1+r)^{\alpha-1} \|U(r)\|_{H^{s-1}}^2 dr \\
&\quad + \int_0^t (1+r)^\alpha C \|z\|_{L^\infty} \|\partial_x z(r)\|_{H^{s-2}}^2 dr.
\end{aligned} \tag{5.30}
$$

Now, multiplying (5.30) by $\tilde\lambda$, adding the result to (5.17) and choosing $\tilde\lambda$ small enough, we get

$$
\begin{aligned}
&(1+t)^\alpha \|U(t)\|_{H^{s-1}}^2 + \int_0^t (1+r)^\alpha \|\partial_x U(r)\|_{H^{s-2}}^2 dr \\
&+ \int_0^t (1+r)^\alpha \|y(r)\|_{H^{s-1}}^2 dr + \int_0^t (1+r)^\alpha \|\partial_x h(r)\|_{H^{s-1}}^2 dr \\
&\le C\|U_0\|_{H^{s-1}}^2 + \alpha C \int_0^t (1+r)^{\alpha-1} \|U(r)\|_{H^{s-1}}^2 dr \\
&+ C\int_0^t (1+r)^\alpha \|\partial_x U(r)\|_{L^\infty} \|\partial_x U(r)\|_{H^{s-2}}^2 dr \\
&+ \int_0^t (1+r)^\alpha C \|z\|_{L^\infty} \|\partial_x z(r)\|_{H^{s-2}}^2 dr,
\end{aligned} \tag{5.31}
$$

where C is a generic positive constant depending on $\tilde\lambda$. Our goal now is to prove (5.11) which will be done by induction on j. Indeed, take $\alpha = 0$, in (5.31) and plugging the estimates

$$\begin{aligned}&\int_0^t \|\partial_x U(r)\|_{L^\infty} \|\partial_x U(r)\|_{H^{s-2}}^2 \, dr \\ &\le CM_1(t) \int_0^t (1+r)^{-1} \|\partial_x U(r)\|_{H^{s-2}}^2 \, dr \\ &\le CM_1(t) \int_0^t \|\partial_x U(r)\|_{H^{s-2}}^2 \, dr \\ &\le CM_1(t) \tilde{D}(t)^2\end{aligned} \tag{5.32}$$

and

$$\begin{aligned}&\int_0^t \|z\|_{L^\infty} \|\partial_x z(r)\|_{H^{s-2}}^2 \, dr \\ &\le CM_0(t) \int_0^t (1+r)^{-1/2} \|\partial_x z(r)\|_{H^{s-2}}^2 \, dr \\ &\le CM_0(t) \int_0^t \|\partial_x z(r)\|_{H^{s-2}}^2 \, dr \\ &\le CM_0(t) \tilde{D}(t)^2\end{aligned} \tag{5.33}$$

into (5.31), then we deduce that (5.11) holds for $j = 0$.

Now, let $0 \le l \le s-2$ and suppose that (5.11) holds for $j = l$, and we will prove that (5.11) also holds for $j = l+1$. Taking $\alpha = l+1$ in the estimate (5.16) and adding over k with $l+1 \le k \le s-1$, we obtain

$$\begin{aligned}&(1+t)^{l+1} \left\|\partial_x^{l+1} U(t)\right\|_{H^{s-l-2}}^2 + \int_0^t (1+r)^{l+1} \|\partial_x^{l+1} y(r)\|_{H^{s-l-2}}^2 dr \\ &+ \int_0^t (1+r)^{l+1} \left\|\partial_x^{l+2} h(r)\right\|_{H^{s-l-2}}^2 dr \\ &\le C\|U_0\|_{H^{s-1}}^2 + C\int_0^t (1+r)^l \|\partial_x^{l+1} U(r)\|_{H^{s-l-2}}^2 dr \\ &+ C\int_0^t (1+r)^{l+1} \|\partial_x U(r)\|_{L^\infty} \|\partial_x^{l+1} U(r)\|_{H^{s-l-2}}^2 dr,\end{aligned} \tag{5.34}$$

where we have used the fact that $\left\|\partial_x^{l+1} U_0\right\|_{H^{s-l-2}}^2 \le \|U_0\|_{H^{s-1}}^2$. Similarly, taking $\alpha = l+1$ in (5.29) and adding over k with $l+1 \le k \le s-2$, we deduce

$$\begin{aligned}&\int_0^t (1+r)^{l+1} \left\|\partial_x^{l+2} U(r)\right\|_{H^{s-l-3}}^2 dr \\ &\le C\|U_0\|_{H^{s-1}}^2 + C(1+t)^{l+1} \left\|\partial_x^{l+1} U(t)\right\|_{H^{s-l-2}}^2 \\ &+ C\int_0^t (1+r)^{l+1} \left(\left\|\partial_x^{l+1} y(r)\right\|_{H^{s-l-2}}^2 + \left\|\partial_x^{l+2} h(r)\right\|_{H^{s-l-2}}^2\right) dr \\ &+ C\int_0^t (1+r)^l \left\|\partial_x^{l+1} U(r)\right\|_{H^{s-l-2}}^2 dr \\ &+ C\int_0^t (1+r)^{l+1} C\|z\|_{L^\infty} \left\|\partial_x^{l+2} z(r)\right\|_{H^{s-l-3}}^2 dr.\end{aligned} \tag{5.35}$$

As before, for $\hat{\lambda}$ small enough, we have by taking (5.34) + $\hat{\lambda}$(5.35)

$$\begin{aligned}
&(1+t)^{l+1} \left\|\partial_x^{l+1} U(t)\right\|_{H^{s-l-2}}^2 + \int_0^t (1+r)^{l+1} \|\partial_x^{l+1} y(r)\|_{H^{s-l-2}}^2 dr \\
&+ \int_0^t (1+r)^{l+1} \left\|\partial_x^{l+2} h(r)\right\|_{H^{s-l-2}}^2 dr + \int_0^t (1+r)^{l+1} \left\|\partial_x^{l+2} U(r)\right\|_{H^{s-l-3}}^2 dr \\
&\leq C\|U_0\|_{H^{s-1}}^2 + C\int_0^t (1+r)^l \left\|\partial_x^{l+1} U(r)\right\|_{H^{s-l-2}}^2 dr \\
&+C\int_0^t (1+r)^{l+1} \|\partial_x U(r)\|_{L^\infty} \|\partial_x^{l+1} U(r)\|_{H^{s-l-2}}^2 dr \\
&+C\int_0^t (1+r)^{l+1} C\|z\|_{L^\infty} \left\|\partial_x^{l+2} z(r)\right\|_{H^{s-l-3}}^2 dr.
\end{aligned} \tag{5.36}$$

The second term on the right-hand side of (5.36) is estimated by the induction hypothesis (5.11) with $j = l$ as

$$\int_0^t (1+r)^l \left\|\partial_x^{l+1} U(r)\right\|_{H^{s-l-2}}^2 dr \leq C\|U_0\|_{H^{s-1}}^2 + C(M_0(t) + M_1(t))\tilde{D}^2(t). \tag{5.37}$$

On the other hand, we have

$$\begin{aligned}
&\int_0^t (1+r)^{l+1} \|\partial_x U(r)\|_{L^\infty} \|\partial_x^{l+1} U(r)\|_{H^{s-l-2}}^2 dr \\
&\leq CM_1(t) \int_0^t (1+r)^l \|\partial_x^{l+1} U(r)\|_{H^{s-l-2}}^2 dr \\
&\leq CM_1(t)\tilde{D}^2(t)
\end{aligned} \tag{5.38}$$

and

$$\begin{aligned}
&\int_0^t (1+r)^{l+1} C\|z\|_{L^\infty} \left\|\partial_x^{l+2} z(r)\right\|_{H^{s-l-3}}^2 dr \\
&\leq CM_0(t) \int_0^t (1+r)^{l+1/2} \left\|\partial_x^{l+2} z(r)\right\|_{H^{s-l-3}}^2 dr \\
&\leq CM_0(t) \int_0^t (1+r)^{l+1} \left\|\partial_x^{l+2} z(r)\right\|_{H^{s-l-3}}^2 dr \\
&\leq CM_0(t)\tilde{D}^2(t).
\end{aligned} \tag{5.39}$$

Inserting the estimates (5.35)-(5.39) into (5.36), we deduce that

$$\begin{aligned}
&(1+t)^{l+1} \left\|\partial_x^{l+1} U(t)\right\|_{H^{s-l-2}}^2 + \int_0^t (1+r)^{l+1} \|\partial_x^{l+1} y(r)\|_{H^{s-l-2}}^2 dr \\
&+ \int_0^t (1+r)^{l+1} \left\|\partial_x^{l+2} h(r)\right\|_{H^{s-l-2}}^2 dr \\
&+ \int_0^t (1+r)^{l+1} \left\|\partial_x^{l+2} U(r)\right\|_{H^{s-l-3}}^2 dr \\
&\leq C\|U_0\|_{H^{s-1}}^2 + C(M_0(t) + M_1(t))\tilde{D}^2(t).
\end{aligned} \tag{5.40}$$

This last estimate shows that (5.11) holds for $j = l + 1$. Consequently, by induction, we have proved that (5.11) holds for all $0 \leq j \leq s-2$. Next, we are going to prove (5.12). Indeed, take $\alpha = k = s-1$ in (5.16), we obtain

$$\begin{aligned}
&(1+t)^{s-1}\left\|\partial_x^{s-1}U(t)\right\|_{L^2}^2 + \int_0^t (1+r)^{s-1}\left\|\partial_x^{s-1}y(r)\right\|_{L^2}^2 dr \\
&+ \int_0^t (1+r)^{s-1}\left\|\partial_x^{s}h(r)\right\|_{L^2}^2 dr \\
&\leq C\left\|\partial_x^{s-1}U_0\right\|_{L^2}^2 + C\int_0^t (1+r)^{s-2}\left\|\partial_x^{s-1}U(r)\right\|_{L^2}^2 dr \\
&+ C\int_0^t (1+r)^{s-1}\left\|\partial_x U(r)\right\|_{L^\infty}\left\|\partial_x^{s-1}U(r)\right\|_{L^2}^2 dr.
\end{aligned} \tag{5.41}$$

Exploiting (5.11), we get for $j = s-2$

$$\int_0^t (1+r)^{s-2}\left\|\partial_x^{s-1}U(r)\right\|_{L^2}^2 dr \leq C\left\|U_0\right\|_{H^{s-1}}^2 + C\left(M_0(t) + M_1(t)\right)\tilde{D}^2(t). \tag{5.42}$$

On the other hand, the last term on the right-hand side of (5.11) is estimated as

$$\begin{aligned}
&\int_0^t (1+r)^{s-1}\left\|\partial_x U(r)\right\|_{L^\infty}\left\|\partial_x^{s-1}U(r)\right\|_{L^2}^2 dr \\
&\leq CM_1(t)\int_0^t (1+r)^{s-2}\left\|\partial_x^{s-1}U(r)\right\|_{L^2}^2 dr \\
&\leq CM_1(t)\tilde{D}^2(t).
\end{aligned} \tag{5.43}$$

Plugging (5.42) and (5.44) into (5.41), we get (5.12). This finishes the proof of Proposition 5.3. □

LEMMA 5.4. *Suppose that $U_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ with $s \geq 2$. Then, we have*

$$M(t) \leq E_0 + C\tilde{M}(t)^2 + CM_0(t)\tilde{E}(t), \tag{5.44}$$

for all $t \in [0,T]$, where C is a positive constant independent of T.

Proof. In order to prove (5.44), it is suffices to establish the estimate

$$\left\|\partial_x^j U(t)\right\|_2 \leq CE_0(1+t)^{-1/4-j/2} + C\left(\tilde{M}(t)^2 + M_0(t)\tilde{E}(t)\right)(1+t)^{-1/4-j/2}, \tag{5.45}$$

for all $t \in [0,T]$ and $0 \leq j \leq s-2$. Using the Duhamel principle, the solution to (5.2) can be expressed as

$$U(t) = e^{t\Theta}U_0 + \int_0^t e^{(t-\tau)\Theta}G(U)_x(\tau)\,d\tau, \tag{5.46}$$

where

$$\left(e^{t\Theta}\omega\right)(x) := \mathcal{F}^{-1}\left[e^{t\hat{\Theta}(i\xi)}\hat{\omega}(\xi)\right](x)$$

with $\hat{\Theta}(i\xi) := -\left(i\xi A + L + \xi^2 B\right)$ and $G(U) := (0,0,0,g(z),0)$.

Let j be a nonnegative integer. Applying ∂_x^j to (5.46), we obtain

$$\begin{aligned}\left\|\partial_x^j U(t)\right\|_{L^2} &\leq \left\|\partial_x^j e^{t\Theta} U_0\right\|_{L^2} + \int_0^t \left\|\partial_x^{j+1} e^{(t-r)\Theta} G(U)(r)\right\|_{L^2} dr \\ &:= \tilde{I}_1 + \tilde{I}_2.\end{aligned}$$

It is clear that from (5.4), we obtain

$$\tilde{I}_1 \leq C E_0 (1+t)^{-1/4-j/2}, \tag{5.47}$$

since $e^{t\Theta} U_0$ is the solution of the linear problem. To estimate $\tilde{I}_2$, we split it into two parts:

$$\begin{aligned}\tilde{I}_2 &= \int_0^{t/2} \left\|\partial_x^{j+1} e^{(t-r)\Theta} G(U)(r)\right\|_{L^2} dr + \int_{t/2}^t \left\|\partial_x^{j+1} e^{(t-r)\Theta} G(U)(r)\right\|_{L^2} dr \\ &= \tilde{J}_1 + \tilde{J}_2.\end{aligned}$$

Using Lemma 2.1, keeping in mind that $g(z) = O\left(z^2\right)$ and applying (5.4) to get

$$\begin{aligned}\tilde{J}_1 &\leq C \int_0^{t/2} (1+r)^{-3/4-j/2} \left\|G(U)(r)\right\|_{L^1} \\ &\quad + C \int_0^{t/2} e^{-c(t-r)} \left\|\partial_x^{j+1} G(U)(r)\right\|_{L^2} dr \\ &\leq C \int_0^{t/2} (1+r)^{-3/4-j/2} \left\|U(r)\right\|_{L^2} dr \\ &\quad + C \int_0^{t/2} e^{-c(t-r)} \left\|\partial_x^{j+1} G(U)(r)\right\|_{L^2} dr \\ &\leq C \tilde{M}(t)^2 \int_0^{t/2} (1+t-r)^{-3/4-j/2} (1+r)^{-1/2} dr \\ &\quad + C \int_0^{t/2} e^{-c(t-r)} \left\|\partial_x^{j+1} G(U)(r)\right\|_{L^2} dr.\end{aligned} \tag{5.48}$$

The first term on the right-hand side of (5.48) can be estimated as

$$\tilde{M}(t)^2 \int_0^{t/2} (1+t-r)^{-3/4-j/2} (1+r)^{-1/2} dr \leq C \tilde{M}(t)^2 (1+t)^{-1/4-j/2}. \tag{5.49}$$

On the other hand, we have $\left\|\partial_x^{j+1} G(U)(r)\right\|_{L^2} \leq \|U\|_{L^\infty} \left\|\partial_x^{j+1} U\right\|_{L^2}$ and for $j+2 \leq s$, we obtain

$$\|U\|_{L^\infty} \left\|\partial_x^{j+1} U\right\|_{L^2} \leq M_0(t) \left\|\partial_x^{j+1} U\right\|_{H^{s-j-2}} \leq M_0(t) \tilde{E}(t) (1+t)^{-1/2-j/2}.$$

Thus, we have, thanks to the above estimate

$$\begin{aligned}&\int_0^{t/2} e^{-c(t-r)} \left\|\partial_x^{j+1} G(U)(r)\right\|_{L^2} dr \\ &\leq C M_0(t) \tilde{E}(t) \int_0^{t/2} e^{-c(t-r)} (1+r)^{-1/2-j/2} dr \\ &\leq C M_0(t) \tilde{E}(t) (1+t)^{-1/2-j/2}.\end{aligned} \tag{5.50}$$

Consequently, using (5.49) and (5.51), we deduce that

$$\tilde{J}_1 \le C\left(\tilde{M}(t)^2 + M_0(t)\,\tilde{E}(t)\right)(1+t)^{-1/4-j/2}. \tag{5.51}$$

Next, $\tilde{J}_2$ can be estimated as follows: applying (5.4) with $j = 1$ and using $\partial_x^j G(U)$ instead of U_0, we obtain

$$\begin{aligned}\tilde{J}_2 &= \int_{t/2}^{t} \left\|\partial_x e^{(t-r)\Theta}\partial_x^j G(U)(r)\right\|_{L^2} dr \\ &\le C\int_{t/2}^{t}(1+t-r)^{-3/4}\left\|\partial_x^j G(U)(r)\right\|_{L^1} dr + C\int_{t/2}^{t} e^{-c(t-r)}\left\|\partial_x^{j+1} G(U)(r)\right\|_{L^2} dr.\end{aligned}$$

On the other hand, we have (see [8, page 1021]):

$$\begin{aligned}\left\|\partial_x^j G(U)\right\|_1 &\le C\|U\|_2\left\|\partial_x^j U\right\|_2 \\ &\le C\tilde{M}^2(t)(1+t)^{-1/2-j/2},\end{aligned}$$

for $j \le s-1$. Thus,

$$\begin{aligned}&C\int_{t/2}^{t}(1+t-r)^{-3/4}\left\|\partial_x^j G(U)(r)\right\|_{L^1} dr \\ &\le C\tilde{M}^2(t)\int_{t/2}^{t}(1+t-r)^{-3/4}(1+r)^{-1/2-j/2} dr \\ &\le C\tilde{M}^2(t)(1+t)^{-1/4-j/2}.\end{aligned} \tag{5.52}$$

Since

$$\begin{aligned}\left\|\partial_x^{j+1} G(U)\right\|_2 &\le C\|U\|_\infty\left\|\partial_x^{j+1} U\right\|_2 \\ &\le CM_0(t)\left\|\partial_x^{j+1} U\right\|_{H^{s-j-1}} \\ &\le CM_0(t)\,\tilde{E}(t)(1+t)^{-1/2-j/2},\end{aligned}$$

we get

$$\begin{aligned}&\int_{t/2}^{t} e^{-c(t-r)}\left\|\partial_x^{j+1} G(U)(r)\right\|_{L^2} dr \\ &\le CM_0(t)\,\tilde{E}(t)\int_{t/2}^{t} e^{-c(t-\zeta)}(1+r)^{-1/2-j/2} dr \\ &\le CM_0(t)\,\tilde{E}(t)(1+t)^{-1/2-j/2}.\end{aligned} \tag{5.53}$$

From (5.53) and (5.54), it follows that

$$\tilde{J}_2 \le C\tilde{M}^2(t)(1+t)^{-1/4-j/2} + CM_0(t)\,\tilde{E}(t)(1+t)^{-1/2-j/2}. \tag{5.54}$$

Therefore, (5.47), (5.48), and (5.54) lead to

$$\left\|\partial_x^j U(t)\right\|_2 \le CE_0(1+t)^{-1/4-j/2} + C\left(\tilde{M}(t)^2 + M_0(t)\,\tilde{E}(t)\right)(1+t)^{-1/4-j/2}, \tag{5.55}$$

for all $0 \le j \le s-1$. The estimate (5.45) is now proved, which concludes the proof of Lemma 5.4. □

Lemma 5.5. *Let $U_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ with $s \geq 4$ and put $E_0 = \|U_0\|_{H^s} + \|U_0\|_1$. Let $T > 0$ and let $U(x,t)$ be the solution of (5.3), satisfying*

$$U \in C\left([0,T]; H^s\right) \cap C^1(0,T]; H^{s-1}).$$

Then, we have the a priori estimates:

$$E^2(T) + D^2(T) \leq CE_s^2, \tag{5.56}$$

$$M\left(T\right) \leq CE_s, \tag{5.57}$$

where C is a positive constant independent of T and E_s.

The proof of Lemma 5.5 can be done as in [8], we omit the details. Consequently, the proof of Theorem 5.2 is a consequence of Proposition 5.3, Lemma 5.4 and Lemma 5.5.

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