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GLOBAL STRONG SOLUTIONS TO THE INHOMOGENEOUS INCOMPRESSIBLE NEMATIC LIQUID CRYSTAL FLOW*

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Abstract. In this paper, we consider the Dirichlet problem to the inhomogeneous incompressible nematic liquid crystal system in bounded smooth domains of two or three dimensions. We prove the global existence and uniqueness of strong solutions to this system, with initial data being of small norm but allowed to have vacuum and large oscillations. More precisely, for the two dimensional case, we only require that the basic energy $\|\sqrt{\rho_0}u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2$ is small, while for the three dimensional case, we ask for the smallness of the production of the basic energy and the quantity $\|\nabla u_0\|_{L^2}^2 + \|\Delta d_0\|_{L^2}^2$. We achieve some suitable time independent a priori estimates on the strong solutions, based on which, one can extend the local strong solution to be a global one.

Key words. Existence and uniqueness, global strong solutions, liquid crystal.

AMS subject classifications. 35D35, 35Q35, 76A15, 76D03.

1. Introduction.

We consider the following hydrodynamic system modeling the flow of nematic liquid crystal materials

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\rho(u_t + (u\cdot\nabla)u) - \nu\Delta u + \nabla p = -\lambda \operatorname{div}(\nabla d \odot \nabla d), \tag{1.2}$$

$$\operatorname{div} u = 0, \tag{1.3}$$

$$d_t + (u\cdot\nabla)d = \gamma(\Delta d + |\nabla d|^2 d), \tag{1.4}$$

in $\Omega \times (0,\infty)$, where Ω is a bounded domain in $\mathbb{R}^N$, $N = 2, 3$, with smooth boundary. Here $u \in \mathbb{R}^N$ represents the velocity field of the flow, $d \in \mathcal{S}^2$, the unit sphere in $\mathbb{R}^3$, represents the macroscopic molecular orientation of the liquid crystal material, $\rho \in \mathbb{R}^+$ and $p \in \mathbb{R}$ are scalar functions, respectively, denoting the density of the fluid and the pressure arising from the usual assumption of incompressibility $\operatorname{div} u = 0$. The positive constants ν, λ and γ represent viscosity, the competition between kinetic energy and potential energy, and microscopic elastic relaxation time or the Dehorah number for the molecular orientation field, respectively. The symbol $\nabla d \odot \nabla d$, which exhibits the property of the anisotropy of the material, denotes the $N \times N$ matrix whose (i, j)-th entry is given by $\partial_i d \cdot \partial_j d$, for $1 \le i, j \le N$.

Noticing that

$$\operatorname{div}(\nabla d \odot \nabla d) = \Delta d \cdot \nabla d + \nabla\left(\frac{|\nabla d|^2}{2}\right),$$

one can rewrite equation (1.2) as

$$\rho(u_t + (u\cdot\nabla)u) - \nu\Delta u + \nabla p = -\lambda\Delta d\cdot\nabla d. \tag{1.5}$$

System (1.1)–(1.4) is a simplified version of the Ericksen-Leslie model, which reduces to the Ossen-Frank model in the static case, for the hydrodynamics of nematic

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liquid crystals developed by Ericksen [5, 6] and Leslie [13] in the 1960's. Both the full Ericksen-Leslie model and the simplified version are the macroscopic continuum description of the time evolution of the materials, under the influence of both the flow velocity field u and the microscopic orientation configurations d of rod-like liquid crystals. A brief account of the Ericksen-Leslie theory and the derivations of several approximate systems can be found in the appendix of [18]. For more details of physics, we refer the readers to the books Gennes-Prost [7] and Chandrasekhar [2].

In the homogeneous case, i.e. the case that $\rho \equiv C$, Lin–Lin [18, 19] initiated the mathematical analysis of the liquid crystal system in the 1990's. More precisely, they considered in [18] the system of variable length, that is replacing $|\nabla d|^2 d$ by the Ginzburg-Landau type approximation term $\frac{1-|d|^2}{\varepsilon^2}d$, to relax the nonlinear constraint $|d| = 1$, and proved the global existence of weak solutions in dimension two or three. They also obtained the global existence and uniqueness of classical solutions in dimension two, and also in dimension three with large enough ν. The partial regularity theorem for suitable weak solutions was proved in [19], which is similar to the classical theorem by Caffarelli-Kohn-Nirenberg [1] for the Navier-Stokes equation. The estimates and arguments in [18, 19] depend on ε, and it's a challenging problem to study the convergence, as ε tends to zero. The two dimensional case of this convergence problem is comparatively easier, and in fact Hong [8] and Hong–Xin [9] have obtained the convergence, as ε goes to zero, up to the first singular time, and consequently established the global existence of weak solutions to the liquid crystal system. One can also establish the global weak solutions directly to the original system, without using the Ginzburg-Landau approximation method, see Lin–Lin–Wang [17]. The uniqueness of weak solutions was later proven in [20]. For three dimensional case, the local or global existence of weak solutions is still an open question.

In the non-homogeneous case, i.e. the density dependent case, the global existence of weak solutions to system (1.1)–(1.4), with $|\nabla d|^2 d$ being replaced by $\frac{1-|d|^2}{\varepsilon^2}d$, the Ginzburg-Laudan type approximation term, was established in [11, 23] and [21], for each $\varepsilon > 0$. They cannot get the uniform estimates in ε, and therefore cannot take the limit, as $\varepsilon \to 0$. It's also a challenging problem to study the convergence, as ε tends to zero, for the non-homogeneous case. If the initial data gains more regularities, the more regular solutions are expected. In fact, Wen–Ding [22] obtained the local existence and uniqueness of strong solutions to system (1.1)–(1.4), with initial density being allowed to have vacuum. If the initial data is small, or satisfies some geometric condition, one can obtain the global existence results: global existence of strong solutions in dimension three, with small initial data, are obtained by Li–Wang in [15] for constant density case, Li–Wang [16] for nonconstant but positive density case, and Ding–Huang–Xia [4] for nonnegative density case; global existence of strong and weak solutions in dimension two, with large initial data, was obtained by Li in [14], under the condition that the third component of the initial direction filed is away from zero.

In the present paper, we establish the global existence of strong solutions to the non-homogeneous system (1.1)–(1.4), subject to the following initial and boundary conditions:

$$(\rho, u, d)|_{t=0} = (\rho_0, u_0, d_0), \quad \text{with } |d_0| = 1, \quad \mathrm{div} u_0 = 0, \quad \text{and } u_0|_{\partial\Omega} = 0, \tag{1.6}$$

$$u(x,t) = 0, \qquad d(x,t) = d_0^*, \quad \text{for } (x,t) \in \partial\Omega \times (0,\infty), \tag{1.7}$$

where d_0^* is a given unit constant vector, and ρ_0 is a given nonnegative function. Compared with the Ginzburg-Landau type approximate system, the term $|\nabla d|^2 d$ in

(1.4) brings us some new difficulties. For example, one can not obtain the a priori L^2 estimates on Δd from the basic energy identity for system (1.1)–(1.4).

Throughout this paper, for any $1 \le p \le \infty$ and any positive integer k, $L^p(\Omega)$ and $H^k(\Omega)$ are the standard Lebessgue and Sobolev spaces, respectively. For convenience, we denote by $\|u\|_p$ the $L^p(\Omega)$ norm of u.

Strong solutions to system (1.1)–(1.4), subject to (1.6)–(1.7), is given by the following definition.

DEFINITION 1.1. *Given a positive time $T \in (0, \infty)$. (ρ, u, d, p) is called a strong solution to system (1.1)–(1.4), subject to (1.6)–(1.7), in $\Omega \times (0,T)$, if it has the regularities*

$$\begin{aligned}
&\rho \in L^\infty(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega)),\\
&u \in L^\infty(0,T;H^2(\Omega)) \cap C([0,T];H_0^1(\Omega)), \quad p \in L^\infty(0,T;H^1(\Omega)),\\
&d \in L^\infty(0,T;H^3(\Omega)) \cap C([0,T];H^2(\Omega)) \cap L^2(0,T;H^4(\Omega)), \quad |d| = 1,\\
&\rho_t \in L^\infty(0,T;L^2(\Omega)), \quad u_t \in L^2(0,T;H_0^1(\Omega)), \quad d_t \in L^2(0,T;H^2(\Omega)),
\end{aligned}$$

satisfies equations (1.1)–(1.4), a.e. in $\Omega \times (0,T)$, and fulfills the initial and boundary conditions (1.6)–(1.7).

DEFINITION 1.2. *(ρ, u, d, p) is called a global strong solution, if it is a strong solution to system (1.1)–(1.4), subject to (1.6)–(1.7), in $\Omega \times (0,T)$, for any finite time T.*

The main result of this paper is the following

THEOREM 1.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N = 2,3$. Assume that $\rho_0 \in H^1(\Omega) \cap L^\infty(\Omega)$, with $0 \le \rho_0(x) \le \bar\rho$ in Ω, for a positive constant $\bar\rho$, $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, with $\operatorname{div} u_0 = 0$ in Ω, $d_0 \in H^3(\Omega)$, with $|d_0| = 1$, and $d_0 = d_0^*$ on $\partial\Omega$, for some constant unit vector d_0^*, and that the following compatible condition holds*

$$-\nu\Delta u_0 - \nabla p_0 - \lambda \mathrm{div}(\nabla d_0 \odot \nabla d_0) = \sqrt{\rho}_0 g_0,$$

in Ω, for some $(p_0, g_0) \in H^1(\Omega) \times L^2(\Omega)$.

Then, there is a positive constant ε_0, which depends only on ν, λ, γ, $\bar\rho$ and Ω, such that if

$$\|\sqrt{\rho_0}u_0\|_2^2 + \|\nabla d_0\|_2^2 < \varepsilon_0, \qquad \textit{for } N = 2,$$

$$(\|\sqrt{\rho_0}u_0\|_2^2 + \|\nabla d_0\|_2^2)(\|\nabla u_0\|_2^2 + \|\Delta d_0\|_2^2) < \varepsilon_0, \quad \textit{for } N = 3,$$

then system (1.1)–(1.4), subject to (1.6)–(1.7), has a unique global strong solution.

REMARK 1.1. *(i) For the 3D case, the same result as Theorem 1.1 also holds true for the whole space case, by the similar arguments used in this paper. Note that, for the whole space case in 3D, the quantity*

$$(\|\sqrt{\rho}u\|_2^2 + \|\nabla d\|_2)(\|\nabla u\|_2^2 + \|\Delta d\|_2^2)$$

is invariant, under the intrinsic transform for system (1.1)–(1.4)

$$\rho_\lambda(x,t) = \rho(\lambda x, \lambda^2 t), \quad p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t),$$

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad d_\lambda(x,t) = d(\lambda x, \lambda^2 t),$$

and thus, our result can be viewed as one on the global existence of strong solutions in critical spaces with vacuum.

(ii) Global existence of strong solutions to system (1.1)–(1.4) in $\mathbb{R}^3$*, with nonnegative density, is also established in [4], where the smallness condition on*

$$\|\sqrt{\rho}_0 u_0\|_2^2 + \|\nabla u_0\|_{H^1} + \|\nabla d_0\|_{H^2} + \|g_0\|_2$$

is required. In our result, for the 3D case, since we only require the smallness on

$$(\|\sqrt{\rho_0}u_0\|_2^2 + \|\nabla d_0\|_2^2)(\|\nabla u_0\|_2^2 + \|\Delta d_0\|_2^2),$$

the quantity $\|g_0\|_2 + \|\nabla^2 u_0\|_2 + \|\nabla^3 d_0\|_2$ *can be arbitrarily large, and moreover the quantity* $\|\nabla u_0\|_2^2 + \|\Delta d_0\|_2^2$ *can be as large as possible, provided the basic energy is accordingly small.*

Since the local existence of strong solutions to system (1.1)–(1.7) has been proven in [22], to establish the global existence result, it suffices to extend the unique local solution to be a global one. To this end, we establish some suitable time independent a priori estimates. Recalling that vacuum is allowed in our paper, these a priori estimates should be independent of the lower bound of the density. The first key estimate is the time independent $L^\infty(0,T;H^1)$ norm of $(u,\nabla d)$. We obtain an energy inequality for $E_1(t)$ (see Section 3 for the expression), which controls the $L^\infty(0,T;H^1)$ norm of $(u,\nabla d)$, with an additional term involving $\|u\|_\infty^2\|\nabla u\|_2^2$, for the two dimensional case. This additional term, for the 2D case, results from the assumption that only the initial basic energy is small; it would disappear, if imposing the same assumption as that for the three dimensional case. On account of such an inequality for $E_1(t)$, one can use the continuity argument to derive the time independent bound of $E_1(t)$, for the 3D case. While for the 2D case, we can employ a logarithmic type Sobolev inequality to treat the term $\|u\|_\infty^2\|\nabla u\|_2^2$, and also obtain the time independent bound of $E_1(t)$, and thus of $L^\infty(0,T;H^1)$ of $(u,\nabla d)$. The next step is to perform the higher order estimates, i.e. the $L^\infty(0,T;H^2)$ norm of $(u,\nabla d)$. After obtaining these estimates, we can use the standard approximation approach to establish the global existence of strong solutions.

The rest of this paper is arranged as follows: in Section 2, we state some preliminary lemmas; in Section 3, we perform some time independent a priori estimates on $(u,\nabla d)$ as well as some time dependent estimates on ρ; in Section 4, we prove the global existence and uniqueness of strong solutions.

Since the exact values of ν, λ and γ play no role in the arguments, we henceforth assume

$$\nu = \lambda = \gamma = 1,$$

thoughout this paper. We denote

$$C_0 = \|\sqrt{\rho_0}u_0\|_2^2 + \|\nabla d_0\|_2^2$$

the basic energy of the initial data in the rest of this paper.

2. Preliminaries. The following result on the transport equations is standard.

LEMMA 2.1 (See [12]). *Let Ω be a Lipschitz domain in $\mathbb{R}^N$, and $v \in L^1(0,T;Lip)$ a solenoidal vector field, such that $v \cdot n = 0$ on $\partial\Omega$, where n denotes the outward normal vector on $\partial\Omega$. Let $\rho_0 \in W^{1,q}(\Omega)$, with $q \in [1,\infty]$.*

Then, the following system

$$\begin{cases} \rho_t + \operatorname{div}(\rho v) = 0, & in\ \Omega \times (0,T), \\ \rho|_{t=0} = \rho_0, & in\ \Omega, \end{cases}$$

has a unique solution in $L^\infty(0,T;W^{1,\infty}(\Omega)) \cap C([0,T];\cap_{1\le r<\infty} W^{1,r}(\Omega))$, if $q = \infty$, and in $C([0,T];W^{1,q}(\Omega))$, if $1 \le q < \infty$.

Besides, the following estimate holds true

$$\|\rho(t)\|_{W^{1,q}(\Omega)} \le e^{\int_0^t \|\nabla v(\tau)\|_\infty d\tau} \|\rho_0\|_{W^{1,q}(\Omega)},$$

for any $t \in [0,T]$. If, in addition, ρ belongs to L^p, for some $p \in [1,\infty]$, then

$$\|\rho(t)\|_p = \|\rho_0\|_p,$$

for all $t \in [0,T]$.

Finally, if $\rho_0(x) \ge \delta$ in Ω, for some positive constant δ, then $\rho(x,t) \ge \delta$, for all $(x,t) \in \Omega \times [0,T]$.

The following lemma states a logarithmic type Sobolev inequality.

LEMMA 2.2 (Logarithmic Sobolev inequality, see [10]). *Let Ω be a bounded smooth domain in $\mathbb{R}^2$, and suppose that $f \in L^2(s,t;H^1(\Omega)) \cap L^2(0,T;W^{1,q}(\Omega))$, for some $q > 2$ and $0 \le s < t \le \infty$.*

Then, it holds that

$$\|f\|_{L^2(s,t;L^\infty(\Omega))} \le C(1 + \|f\|_{L^2(s,t;H^1(\Omega))} (\ln^+ \|f\|_{L^2(s,t;W^{1,q}(\Omega))})^{1/2}),$$

with some constant C depending only on q and Ω, and independent of s,t.

We also need the following local existence result.

LEMMA 2.3 (Local well-posedness, see [22]). *Under the conditions stated in Theorem 1.1 (without the smallness condition), there is a constant $T_* > 0$, such that for any $T < T_*$, system (1.1)–(1.7) has unique solution (ρ,u,p,d), in $\Omega \times (0,T)$.*

3. A priori estimates. In this section, we are concerned with the energy estimates on strong solutions. Given a positive time $T \in (0,\infty)$, and let (ρ,u,p,d) be the strong solution to (1.1)–(1.7), in $\Omega \times [0,T)$, stated in Lemma 2.3. By Lemma 2.1, we have

$$0 \le \rho(x,t) \le \bar\rho, \quad \text{in } Q_T = \Omega \times (0,T).$$

We have the following facts

$$\begin{aligned} &\|f\|_{H^1} \le C\|\nabla f\|_2, && \forall f \in H_0^1(\Omega), \\ &\|f\|_{H^2} \le C\|\Delta f\|_2, && \forall f \in H^2(\Omega) \cap H_0^1(\Omega), \\ &\|f\|_{H^3} \le C(\|\Delta f\|_2 + \|\nabla\Delta f\|_2), && \forall f \in H^3(\Omega) \cap H_0^1(\Omega), \\ &\|f\|_{H^3} \le C(\|\nabla f\|_2 + \|\nabla\Delta f\|_2), && \forall f \in H^3(\Omega) \cap H_0^1(\Omega), \end{aligned}$$

where the first one follows from the Poincaré inequality, the second and third ones follow from the elliptic estimates, while the last one is obtained by combing the third one and the following fact

$$\|\Delta f\|_2 \le C(\|\nabla f\|_2 + \|\nabla\Delta f\|_2), \quad \forall f \in H^3(\Omega)\cap H_0^1(\Omega),$$

which can be proven, in the standard way, by the contradiction argument, and using the elliptic estimates mentioned above, as well as the compact embedding from H^3 to H^2.

Using these facts, recalling that $u|_{\partial\Omega} = 0$ and $d|_{\partial\Omega} = d_0^*$, for a given constant unit vector d_0^*, we have

$$\begin{aligned}
&\|u\|_{H^1} \le C\|\nabla u\|_2, \quad \|u\|_{H^2} \le C\|\Delta u\|_2, \quad \|\nabla d\|_{H^1} \le C\|\Delta d\|_2,\\
&\|\nabla d\|_{H^2} \le C(\|\Delta d\|_2 + \|\nabla\Delta d\|_2), \quad \|\nabla d\|_{H^2} \le C(\|\nabla d\|_2 + \|\nabla\Delta d\|_2),
\end{aligned}$$

from which, for the 3D case, by the Sobolev embedding inequality, we have

$$\begin{aligned}
&\|u\|_6 \le C\|\nabla u\|_2, \quad \|\nabla u\|_6 \le C\|\Delta u\|_2, \quad \|\nabla d\|_6 \le C\|\Delta d\|_2,\\
&\|\nabla^2 d\|_6 \le C(\|\Delta d\|_2 + \|\nabla\Delta d\|_2), \quad \|\nabla^2 d\|_6 \le C(\|\nabla d\|_2 + \|\nabla\Delta d\|_2).
\end{aligned}$$

These estimates are so frequently use that we will not point out when used later.

We introduce the following quantities

$$\begin{aligned}
m(t) &= \sup_{0\le s\le t}(\|\sqrt{\rho}u\|_2^2 + \|\nabla d\|_2^2)\sup_{0\le s\le t}(\|\nabla u\|_2^2 + \|\Delta d\|_2^2),\\
S_1(t) &= \int_\Omega (|u|^2 + |\nabla d|^2)(|\nabla u|^2 + |\nabla^2 d|^2)dx,\\
S_2(t) &= \int_\Omega |\nabla u||\nabla^2 d|^2 dx, \quad S(t) = S_1(t) + S_2(t),
\end{aligned}$$

and energy functionals

$$\begin{aligned}
E_1(t) = &\sup_{0\le s\le t}(\|\nabla u\|_2^2 + \|\Delta d\|_2^2) + \int_0^t (\|\sqrt{\rho}u_t\|_2^2 + \|\Delta u\|_2^2\\
&+ \|\nabla\Delta d\|_2^2 + \|\nabla d_t\|_2^2)ds,\\
E_2(t) = &\sup_{0\le s\le t}(\|\sqrt{\rho}u_t\|_2^2 + \|\Delta u\|_2^2 + \|\nabla p\|_2^2 + \|\nabla\Delta d\|_2^2)\\
&+ \int_0^t (\|\nabla u_t\|_2^2 + \|\Delta d_t\|_2^2)ds,\\
E(t) = &\sup_{0\le s\le t}(\|\nabla\rho\|_2^2 + \|\rho_t\|_2^2 + \|u\|_{H^2}^2 + \|\nabla d\|_{H^2}^2 + \|\nabla p\|_2^2)\\
&+ \int_0^t (\|\nabla u_t\|_2^2 + \|\nabla^2 d_t\|_2^2)ds,
\end{aligned}$$

for any $t \in (0, T)$. It is clear that $m(t)$ is a continuous function on $[0, T]$.

LEMMA 3.1. *There is a positive constant ε_1 depending only on Ω, such that if*

$$m(T) \le \varepsilon_1, \text{ for } N = 3, \quad C_0 \le \varepsilon_1, \text{ for } N = 2,$$

then we have the following estimate

$$\sup_{0\le t\le T}(\|\sqrt{\rho}u\|_2^2 + \|\nabla dd\|_2^2) + \int_0^T (\|\nabla u\|_2^2 + \|\Delta d\|_2^2)dt \le C_0.$$

Proof. Multiplying (1.4) and (1.5) by $-(\Delta d+|\nabla d|^2 d)$ and u, respectively, summing the resulting equations up and integrating over Ω, then it follows from integration by parts and using (1.1) and (1.3) that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega(\rho|u|^2+|\nabla d|^2)dx+\int_\Omega(|\nabla u|^2+|\Delta d+|\nabla d|^2d|^2)dx=0,$$

from which, noticing that

$$|\Delta d+|\nabla d|^2d|^2=|\Delta d|^2-|\nabla d|^4,$$

guaranteed by the condition $|d|=1$, one arrives at

$$\frac{1}{2}\frac{d}{dt}\int_\Omega(\rho|u|^2+|\nabla d|^2)dx+\int_\Omega(|\nabla u|^2+|\Delta d|^2)dx=\int_\Omega|\nabla d|^4dx. \tag{3.8}$$

Noticing that $|\nabla d|^2=-d\cdot\Delta d\le|\Delta d|$, the above inequality implies

$$\frac{1}{2}\frac{d}{dt}\int_\Omega(\rho|u|^2+|\nabla d|^2)dx+\int_\Omega|\nabla u|^2dx\le 0,$$

which implies

$$\sup_{0\le t\le T}(\|\sqrt{\rho}u\|_2^2+\|\nabla d\|_2^2)+\int_0^T\|\nabla u\|_2^2dt\le C_0.$$

By assumption and the Sobolev and Ladyzhenskaya inequalities, we have for $N=3$,

$$\|\nabla d\|_4^4\le C\|\nabla d\|_2\|\nabla d\|_6^3\le C\|\nabla d\|_2\|\Delta d\|_2^3\le C\sqrt{m(t)}\|\Delta d\|_2^2\le C\sqrt{\varepsilon_1}\|\Delta d\|_2^2,$$

and for $N=2$,

$$\|\nabla d\|_4^4\le C\|\nabla d\|_2^2\|\nabla d\|_{H^1}^2\le C\|\nabla d\|_2^2\|\Delta d\|_2^2\le CC_0\|\Delta d\|_2^2\le C\varepsilon_1\|\Delta d\|_2^2.$$

Taking ε_1 sufficiently small, and substituting the above two inequalities into (3.8) yield the conclusion. ∎

LEMMA 3.2. *There is a positive constant $A\ge 1$ depending only on $\bar\rho$ and Ω, such that the following holds*

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\left(\frac{|\nabla u|^2}{2}+\frac{|\Delta d|^2}{2}-\nabla d\odot\nabla d:\nabla u\right)dx\\&+\frac{1}{A}\int_\Omega(\rho|u_t|^2+|\Delta u|^2+|\nabla\Delta d|^2+|\nabla d_t|^2)dx\le C\int_\Omega|\Delta d|^2dx+CS(t),\end{aligned}$$

for any $t\in(0,T)$, where C is a positive constant depending only on $\bar\rho$ and Ω.

Proof. Multiplying (1.2) by u_t, and integrating the resulting equation over Ω, then it follows from integration by parts and the Cauchy inequality that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\frac{|\nabla u|^2}{2}dx+\int_\Omega\rho|u_t|^2dx\\&=\int_\Omega(\nabla d\odot\nabla d:\nabla u_t-\rho(u\cdot\nabla)u\cdot u_t)dx\end{aligned}$$

$$
\begin{aligned}
=&\frac{d}{dt}\int_\Omega \nabla d\odot\nabla d:\nabla u dx-\int_\Omega ((\nabla d\odot\nabla d)_t:\nabla u+\rho(u\cdot\nabla)u\cdot u_t)\,dx\\
\le&\frac{d}{dt}\int_\Omega \nabla d\odot\nabla d:\nabla u dx+\varepsilon\int_\Omega(\rho|u_t|^2+|\nabla d_t|^2)dx+CS(t),
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega\left(\frac{|\nabla u|^2}{2}-\nabla d\odot\nabla d:\nabla u\right)dx+\int_\Omega\rho|u_t|^2dx\\
&\le\varepsilon\int_\Omega(\rho|u_t|^2+|\nabla d_t|^2)dx+CS(t),
\end{aligned}\tag{3.9}
$$

where ε is a sufficient positive constant.

Taking the operator Δ to both sides of equation (1.4), and multiplying the resulting equation by Δd, we deduce

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega\frac{|\Delta d|^2}{2}dx-\int_\Omega\Delta^2 d\cdot\Delta d dx\\
=&\int_\Omega[|\nabla d|^2|\Delta d|^2+2\nabla d:\nabla\Delta d d\cdot\Delta d+2|\nabla^2 d|^2 d\cdot\Delta d\\
&-(\Delta u\cdot\nabla)d\cdot\Delta d-2(\nabla u_i\cdot\partial_i\nabla)d\cdot\Delta d]dx\\
\le&\,\varepsilon\int_\Omega(|\nabla\Delta d|^2+|\Delta u|^2)dx+CS(t),
\end{aligned}\tag{3.10}
$$

where ε is a sufficiently small positive constant. In the above, we have used the fact that $d\cdot\Delta d=-|\nabla d|^2$, guaranteed by $|d|=1$.

Note that $\Delta d|_{\partial\Omega}=|\nabla d|^2 d|_{\partial\Omega}$, guaranteed by equation (1.4) and the boundary condition (1.7). Integration by parts gives

$$
\begin{aligned}
-\int_\Omega\Delta^2 d\cdot\Delta d dx=&-\int_{\partial\Omega}\Delta d\cdot\frac{\partial}{\partial n}\Delta d dS+\int_\Omega|\nabla\Delta d|^2dx\\
=&\int_{\partial\Omega}|\nabla d|^2 d\cdot\frac{\partial}{\partial n}\Delta d dS+\int_\Omega|\nabla\Delta d|^2dx\\
=&\int_{\partial\Omega}\left(|\nabla d|^2\frac{\partial}{\partial n}(d\cdot\Delta d)-|\nabla d|^2\Delta d\cdot\frac{\partial}{\partial n}d\right)dS+\int_\Omega|\nabla\Delta d|^2dx\\
=&-\int_{\partial\Omega}\left(|\nabla d|^2\frac{\partial}{\partial n}|\nabla d|^2+|\nabla d|^2\Delta d\cdot\frac{\partial}{\partial n}d\right)dS+\int_\Omega|\nabla\Delta d|^2dx,
\end{aligned}
$$

which, combined with (3.10) and the trace inequality $\|f\|_{L^1(\partial\Omega)}\le C\|f\|_{W^{1,1}(\Omega)}$, gives

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega\frac{|\Delta d|^2}{2}dx+\int_\Omega|\nabla\Delta d|^2dx\\
\le&\,\varepsilon\int_\Omega(|\nabla\Delta d|^2+|\Delta u|^2)dx+C\int_{\partial\Omega}|\nabla d|^3|\nabla^2 d|dS+CS(t)\\
\le&\,\varepsilon\int_\Omega(|\nabla\Delta d|^2+|\Delta u|^2)dx+C\||\nabla d|^3|\nabla^2 d|\|_{W^{1,1}(\Omega)}+CS(t)\\
\le&\,\varepsilon\int_\Omega(|\nabla\Delta d|^2+|\Delta u|^2)dx+CS(t)\\
&+C\int_\Omega(|\nabla d|^3|\nabla^2 d|+|\nabla d|^2|\nabla^2 d|^2+|\nabla d|^3|\nabla^3 d|)dx
\end{aligned}
$$

$$
\begin{aligned}
&\le \varepsilon \int_\Omega (|\nabla\Delta d|^2 + |\Delta u|^2)dx + CS_1(t) + C\int_\Omega |\nabla d||\Delta d||\nabla^3 d|dx \\
&\le \varepsilon \int_\Omega (|\nabla^3 d|^2 + |\Delta u|^2)dx + CS(t),
\end{aligned}
$$

and thus

$$
\frac{d}{dt}\int_\Omega \frac{|\Delta d|^2}{2}dx + \int_\Omega |\nabla\Delta d|^2 dx \le \varepsilon \int_\Omega (|\nabla^3 d|^2 + |\Delta u|^2)dx + CS(t), \tag{3.11}
$$

where ε is a sufficiently small positive constant.

Applying elliptic estimates for the Stokes equations to (1.5) yields

$$
\|\Delta u\|_2^2 + \|\nabla p\|_2^2 \le C(\|\rho(u_t + u\cdot\nabla u) + \Delta d\cdot\nabla d\|_2^2 \le C\|\sqrt{\rho}u_t\|_2^2 + CS_1(t). \tag{3.12}
$$

It follows equation (1.4) that

$$
\begin{aligned}
\|\nabla d_t\|_2^2 &= \int_\Omega |\nabla(\Delta d + |\nabla d|^2 d - u\cdot\nabla d)|^2 dx \\
&\le 2\int_\Omega (|\nabla\Delta d|^2 + |\nabla(|\nabla d|^2 d + u\cdot\nabla d)|^2)dx \\
&\le 2\|\nabla\Delta d\|_2^2 + C\int_\Omega (|\nabla d|^6 + |\nabla d|^2|\nabla^2 d|^2 + |u|^2|\nabla^2 d|^2 + |\nabla u|^2|\nabla d|^2)dx \\
&\le 2\|\nabla\Delta d\|_2^2 + C\int_\Omega (|\nabla d|^2|\nabla^2 d|^2 + |u|^2|\nabla^2 d|^2 + |\nabla d|^2|\nabla u|^2)dx \\
&\le 2\|\nabla\Delta d\|_2^2 + CS_1(t). \qquad (3.13)
\end{aligned}
$$

Combining the above two inequalities with (3.9) and (3.11), using the fact that

$$
\|\nabla^3 d\|_2^2 \le C(\|\Delta d\|_2^2 + \|\nabla\Delta d\|_2^2),
$$

which has been mentioned at the beginning of this section, and choosing ε small enough, we obtain the conclusion. □

LEMMA 3.3. *Let ε_1 be the positive constant stated in Lemma 3.1. There is a positive constant $\varepsilon_0 \in (0, \frac{\varepsilon_1}{2}]$ depending only on $\bar\rho$ and Ω, such that if*

$$
C_0 \le \varepsilon_0,\ \textit{for } N = 2, \quad C_0(\|\nabla u_0\|_2^2 + \|\Delta d_0\|_2^2) \le \varepsilon_0,\ \textit{for } N = 3,
$$

then, we have the following estimates

$$
\begin{aligned}
&\sup_{0\le t\le T} E_1(t) \le C(1 + \|\nabla u_0\|_2^2 + \|\Delta d_0\|_2^2)^2, &&\textit{for } N = 2, \\
&\sup_{0\le t\le T} E_1(t) \le C(\|\nabla u_0\|_2^2 + \|\Delta d_0\|_2^2), &&\textit{for } N = 3,
\end{aligned}
$$

where C is a positive constant depending only on $\bar\rho$ and Ω.

Proof. Define functions g and G as

$$
\begin{aligned}
g(t) &= \int_\Omega \left(\frac{|\nabla u|^2}{2} + \frac{|\Delta d|^2}{2} - \nabla d\odot\nabla d : \nabla u\right)dx, \\
G(t) &= \frac{1}{A}\int_\Omega (\rho|u_t|^2 + |\Delta u|^2 + |\nabla\Delta d|^2 + |\nabla d_t|^2)dx,
\end{aligned}
$$

for any $t \in (0,T)$. Then, by Lemma 3.2, it follows

$$g'(t) + G(t) \le C\int_\Omega |\Delta d|^2 dx + CS(t), \tag{3.14}$$

for any $t \in (0,T)$.

(i) The two dimensional case. By Lemma 3.1, it follows from the Ladyzhenskaya inequality that

$$\begin{aligned}
\int_\Omega |\nabla d|^2(|\nabla u|^2 + |\nabla^2 d|^2)dx &\le \|\nabla d\|_4^2(\|\nabla u\|_4^2 + \|\nabla^2 d\|_4^2)\\
&\le C\|\nabla d\|_2\|\nabla d\|_{H^1}(\|\nabla u\|_2\|\nabla u\|_{H^1} + \|\nabla^2 d\|_2\|\nabla^2 d\|_{H^1})\\
&\le C\|\nabla d\|_2\|\Delta d\|_2[\|\nabla u\|_2\|\Delta u\|_2 + \|\Delta d\|_2(\|\nabla d\|_2 + \|\nabla\Delta d\|_2)]\\
&\le \varepsilon(\|\nabla\Delta d\|_2^2 + \|\Delta u\|_2^2) + C\|\nabla d\|_2^2(\|\nabla u\|_2^4 + \|\Delta d\|_2^4 + \|\Delta d\|_2^2)\\
&\le \varepsilon(\|\nabla\Delta d\|_2^2 + \|\Delta u\|_2^2) + CC_0(1 + \|\nabla u\|_2^2 + \|\Delta d\|_2^2)(\|\nabla u\|_2^2 + \|\Delta d\|_2^2),
\end{aligned}$$

and

$$\begin{aligned}
S_2(t) = \int_\Omega |\nabla u||\nabla^2 d|^2 dx &\le \|\nabla u\|_2\|\nabla^2 d\|_4^2\\
&\le C\|\nabla u\|_2\|\nabla^2 d\|_2\|\nabla^2 d\|_{H^1} \le C\|\nabla u\|_2\|\Delta d\|_2(\|\nabla d\|_2 + \|\nabla\Delta d\|_2)\\
&\le \varepsilon\|\nabla\Delta d\|_2^2 + C(\|\nabla u\|_2^2\|\Delta d\|_2^2 + \|\nabla d\|_2\|\nabla u\|_2\|\Delta d\|_2)\\
&\le \varepsilon\|\nabla\Delta d\|_2^2 + C(\|\nabla u\|_2^2 + C_0^{1/2})(\|\nabla u\|_2^2 + \|\Delta d\|_2^2),
\end{aligned}$$

and thus

$$\begin{aligned}
S(t) &\le \varepsilon(\|\nabla\Delta d\|_2^2 + \|\Delta u\|_2^2) + C[\|u\|_\infty^2 + \|\nabla u\|_2^2 + C_0^{1/2}\\
&\quad + C_0(1 + \|\nabla u\|_2^2 + \|\Delta d\|_2^2)](\|\nabla u\|_2^2 + \|\Delta d\|_2^2)\\
&\le C[\|u\|_\infty^2 + (1 + C_0)(\|\nabla u\|_2^2 + \|\Delta d\|_2^2)](\|\nabla u\|_2^2 + \|\Delta d\|_2^2)\\
&\quad + C(C_0 + C_0^{1/2})(\|\nabla u\|_2^2 + \|\Delta d\|_2^2) + \varepsilon(\|\nabla\Delta d\|_2^2 + \|\Delta u\|_2^2),
\end{aligned} \tag{3.15}$$

for a sufficiently small positive constant ε.

By assumption, it follows from Lemma 3.1 and the Hölder, Ladyzhenskaya and Cauchy inequalities that

$$\begin{aligned}
&\left|\int_\Omega \nabla d \odot \nabla d : \nabla u dx\right| \le \|\nabla d\|_4^2\|\nabla u\|_2 \le C\|\nabla d\|_2\|\nabla d\|_{H^1}\|\nabla u\|_2\\
&\le C\|\nabla d\|_2\|\Delta d\|_2\|\nabla u\|_2 \le CC_0(\|\Delta d\|_2^2 + \|\nabla u\|_2^2) \le C\varepsilon_0(\|\Delta d\|_2^2 + \|\nabla u\|_2^2),
\end{aligned}$$

from which, taking ε_0 sufficiently small, and recalling the definition of $g(t)$, one obtains

$$\frac{1}{4}(\|\nabla u\|_2^2 + \|\Delta d\|_2^2) \le g(t) \le \|\nabla u\|_2^2 + \|\Delta d\|_2^2. \tag{3.16}$$

On account of (3.15) and (3.16), it follows from (3.14) that

$$g'(t) + \frac{1}{2}G(t) \le w_2(t)g(t) + C(1 + C_0)(\|\nabla u\|_2^2 + \|\Delta d\|_2^2), \tag{3.17}$$

where

$$w_2(t) = C[\|u\|_\infty^2 + (1 + C_0)(\|\nabla u\|_2^2 + \|\Delta d\|_2^2)].$$

It follows from Lemmas 2.2 and 3.1 that

$$\begin{aligned}\int_0^t \|u\|_\infty^2 ds &\le C\left[1+\left(\int_0^t \|u\|_{H^1}^2 ds\right)\ln^+\left(\int_0^t \|u\|_{H^2}^2 ds\right)\right]\\ &\le C\left[1+\left(\int_0^t \|\nabla u\|_2^2 ds\right)\ln^+\left(C\int_0^t \|\Delta u\|_2^2 ds\right)\right]\\ &\le C\left[1+C_0+C_0\ln^+\int_0^t G(s)ds\right],\end{aligned}$$

and thus, using Lemma 3.1 again, we get

$$\int_0^t w_2(s)ds \le C\left[1+C_0^2+C_0\ln^+\int_0^t G(s)ds\right].$$

With the aid of the above inequality, by Lemma 3.1, it follows from the Gronwall inequality and (3.17) that

$$\begin{aligned}g(t)+\int_0^t G(s)ds &\le e^{\int_0^t w_2(s)ds}\left(g(0)+(1+C_0)\int_0^t (\|\nabla u\|_2^2+\|\Delta d\|_2^2)ds\right)\\ &\le e^{C\left[1+C_0^2+C_0\ln^+\int_0^t G(s)ds\right]}[g(0)+(1+C_0)C_0]\\ &\le e^{C\left[1+\varepsilon_0^2+\varepsilon_0\ln^+\int_0^t G(s)ds\right]}[g(0)+(1+\varepsilon_0)\varepsilon_0]\\ &\le e^{C(1+\varepsilon_0^2)}\left(\int_0^t G(s)ds\right)^{C\varepsilon_0}[g(0)+(1+\varepsilon_0)\varepsilon_0],\end{aligned}$$

from which, choosing $\varepsilon_0=\min\left\{\frac{\varepsilon_1}{2},\frac{1}{2C}\right\}$, one obtains

$$\begin{aligned}g(t)+\int_0^t G(s)ds &\le C\left(\int_0^t G(s)ds\right)^{1/2}(g(0)+1)\\ &\le \frac{1}{2}\int_0^t G(s)ds+C(g(0)+1)^2.\end{aligned}$$

Combing the above inequality with (3.16) yields the conclusion for the 2D case.

(ii) The three dimensional case. Let $\varepsilon_2'\in(0,\varepsilon_1]$ be a sufficiently small positive constant, which will be determined later, and define

$$T^*=\max\{t\in(0,T]\,|\,m(t)\le\varepsilon_2'\}.$$

It follows from the Hölder and Cauchy inequalities that

$$\begin{aligned}S_1(t)&=\int_\Omega(|u|^2+|\nabla d|^2)(|\nabla u|^2+|\nabla^2 d|^2)dx\\ &\le C(\|u\|_6^2+\|\nabla d\|_6^2)(\|\nabla u\|_2+\|\nabla^2 d\|_2)(\|\nabla u\|_6+\|\nabla^2 d\|_6)\\ &\le C(\|\nabla u\|_2+\|\Delta d\|_2)^3(\|\Delta u\|_2+\|\nabla d\|_2+\|\nabla\Delta d\|_2)\\ &\le \varepsilon(\|\Delta u\|_2^2+\|\nabla\Delta d\|_2^2)+C[(\|\nabla u\|_2^2+\|\Delta d\|_2^2)^2+\sqrt{m(t)}](\|\nabla u\|_2^2+\|\Delta d\|_2^2)\\ &\le \varepsilon(\|\Delta u\|_2^2+\|\nabla\Delta d\|_2^2)+C[(\|\nabla u\|_2^2+\|\Delta d\|_2^2)^2+\sqrt{\varepsilon_2'}](\|\nabla u\|_2^2+\|\Delta d\|_2^2),\end{aligned}$$

and

$$S_2(t)=\int_\Omega|\nabla u||\nabla^2 d|^2dx\le\|\nabla u\|_6\|\nabla^2 d\|_2\|\nabla^2 d\|_3$$

$$
\begin{aligned}
&\le C\|\Delta u\|_2\|\Delta d\|_2\|\Delta d\|_2^{\frac12}(\|\nabla d\|_2^{\frac12}+\|\nabla\Delta d\|_2^{\frac12})\\
&\le \varepsilon(\|\Delta u\|_2^2+\|\nabla\Delta d\|_2^2)+C(\|\Delta d\|_2^6+\|\nabla d\|_2\|\Delta d\|_2^3)\\
&\le \varepsilon(\|\Delta u\|_2^2+\|\nabla\Delta d\|_2^2)+C(\|\Delta d\|_2^4+\sqrt{m(t)})\|\Delta d\|_2^2\\
&\le \varepsilon(\|\Delta u\|_2^2+\|\nabla\Delta d\|_2^2)+C(\|\Delta d\|_2^4+\sqrt{\varepsilon_2'})\|\Delta d\|_2^2,
\end{aligned}
$$

for any $t\in(0,T^*)$, where ε is a sufficiently small positive constant. Thus, we have

$$
S(t)\le \varepsilon(\|\Delta u\|_2^2+\|\nabla\Delta d\|_2^2)+C[(\|\nabla u\|_2^2+\|\Delta d\|_2^2)^2+\sqrt{\varepsilon_2'}](\|\nabla u\|_2^2+\|\Delta d\|_2^2)],
$$

for any $t\in(0,T^*)$, where ε is a sufficiently small positive constant.

With the aid of the above inequality, it follows from (3.14) that

$$
g'(t)+\frac12 G(t)\le w_3(t)(\|\nabla u\|_2^2+\|\Delta d\|_2^2)+C(1+\varepsilon_2')(\|\nabla u\|_2^2+\|\Delta d\|_2^2), \tag{3.18}
$$

for any $t\in(0,T^*)$, where

$$
w_3(t)=C(\|\nabla u\|_2^2+\|\Delta d\|_2^2)^2.
$$

By assumption, it follows from the Hölder, Sobolev and Young inequalities that

$$
\begin{aligned}
\left|\int_\Omega \nabla d\odot\nabla d:\nabla u dx\right| &\le \|\nabla u\|_2\|\nabla d\|_3\|\nabla d\|_6\le C\|\nabla u\|_2\|\nabla d\|_2^{\frac12}\|\Delta d\|_2^{\frac32}\\
&\le m(t)^{\frac14}\|\nabla u\|_2\|\Delta d\|_2\le C{\varepsilon_2'}^{\frac14}(\|\nabla u\|_2^2+\|\Delta d\|_2^2),
\end{aligned}
$$

for any $t\in(0,T^*)$, from which, choosing $\varepsilon_2'=\min\left\{\varepsilon_1,\left(\frac{1}{4C}\right)^4\right\}$, and recalling the definition of $g(t)$, we have

$$
\frac14(\|\nabla u\|_2^2+\|\Delta d\|_2^2)\le g(t)\le \|\nabla u\|_2^2+\|\Delta d\|_2^2, \tag{3.19}
$$

for any $t\in(0,T^*)$.

Thanks to (3.18) and (3.19), we then obtain

$$
g'(t)+\frac12 G(t)\le w_3(t)g(t)+C(1+\varepsilon_2')(\|\nabla u\|_2^2+\|\Delta d\|_2^2),
$$

from which, by the Gronwall inequality and Lemma 3.1, and noticing that $C_0\le Cg(0)$ and $C_0\le \sup_{0\le s\le t}(\|\sqrt{\rho}u\|_2^2+\|\nabla d\|_2^2)$, we have

$$
\begin{aligned}
g(t)+\frac12\int_0^t G(s)ds &\le e^{\int_0^t w_3(s)ds}\left(g(0)+C\int_0^t(\|\nabla u\|_2^2+\|\Delta d\|_2^2)ds\right)\\
&\le \exp\left\{CC_0\sup_{0\le s\le t}(\|\nabla u\|_2^2+\|\Delta d\|_2^2)\right\}(g(0)+CC_0)\\
&\le e^{Cm(t)}(g(0)+CC_0)\le e^{C\varepsilon_2'}(g(0)+CC_0)\\
&\le C(g(0)+C_0)\le Cg(0),
\end{aligned}
$$

for any $t\in(0,T^*)$. This, combined with (3.19), implies

$$
E_1(t)\le C(\|\nabla u_0\|_2^2+\|\Delta d_0\|_2^2),\quad \forall t\in(0,T^*), \tag{3.20}
$$

from which, by Lemma 3.1 and the assumption, we then obtain

$$
\begin{aligned}
m(t) &\le C_0 \sup_{0\le s\le t}(\|\nabla u\|_2^2+\|\Delta d\|_2^2)\le C_0E_1(t)\\
&\le CC_0(\|\nabla u_0\|_2^2+\|\Delta d_0\|_2^2)\le C\varepsilon_0\le \frac{\varepsilon_2'}{2}<\varepsilon_2',
\end{aligned}
$$

with $\varepsilon_0=\frac{\varepsilon_2'}{2C}$, for any $t\in(0,T^*)$.

We claim that $T^*=T$, and as a result the conclusion follows from (3.20). Suppose, by contradiction, that $T^*\in(0,T)$, then the above inequality implies that there is another time $T^{**}\in(T^*,T]$, such that $m(t)\le\varepsilon_2'$, for any $t\in(0,T^{**}]$, contradicting to the definition of T^*. This prove the conclusion for the 3D case. □

Lemma 3.4. *Under the same conditions as in Lemma 3.3, we have*

$$
\sup_{0\le t\le T}E_2(t)\le C(\|\nabla u_0\|_2^2,\|\Delta d_0\|_2^2,\|g_0\|_2^2,\bar\rho,\Omega).
$$

Proof. Differentiating equation (1.2) in t, multiplying the resulting equation by u_t and integrating over Ω, then it follows from integration by parts and the Young inequality that

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int_\Omega\rho|u_t|^2dx+\int_\Omega|\nabla u_t|^2dx\\
&=\int_\Omega[(\nabla d\odot\nabla d)_t:\nabla u_t-\rho(u_t\cdot\nabla)u\cdot u_t+\operatorname{div}(\rho u)(u_t+(u\cdot\nabla)u)\cdot u_t]dx\\
&=\int_\Omega\{(\nabla d\odot\nabla d)_t:\nabla u_t-\rho(u_t\cdot\nabla)u\cdot u_t-\rho u\cdot\nabla(|u_t|^2)+\rho u\cdot\nabla[(u\cdot\nabla u)\cdot u_t]\}dx\\
&\le C\int_\Omega(|\nabla d||\nabla d_t||\nabla u_t|+\rho|\nabla u||u_t|^2+\rho|u||\nabla u_t||u_t|\\
&\quad+\rho|u||\nabla u|^2|u_t|+\rho|u|^2|\nabla^2u||u_t|+\rho|u|^2|\nabla u||\nabla u_t|)dx\\
&\le\frac12\int_\Omega|\nabla u_t|^2dx+C\int[(|\nabla d|^2+|u|^2)(|\nabla d_t|^2+\rho|u_t|^2\\
&\quad+|\nabla^2u|^2)+\rho(|\nabla u|^4+|u|^8)+\rho|\nabla u||u_t|^2]dx,
\end{aligned}
$$

and thus

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega\rho|u_t|^2dx+\int_\Omega|\nabla u_t|^2dx&\le C\int[(|\nabla d|^2+|u|^2)(|\nabla d_t|^2+\rho|u_t|^2+|\nabla^2u|^2)\\
&\quad+\rho(|\nabla u|^4+|u|^8)+\rho|\nabla u||u_t|^2]dx=:I_1.
\end{aligned}
\tag{3.21}
$$

By the Hölder, Sobolev and Cauchy inequalities, we have

$$
\begin{aligned}
I_1&\le C[(\|\nabla d\|_\infty^2+\|u\|_\infty^2)(\|\nabla d_t\|_2^2+\|\sqrt\rho u_t\|_2^2+\|\nabla^2u\|_2^2)\\
&\quad+\|\nabla u\|_4^4+\|u\|_8^8+\|\sqrt\rho u_t\|_2\|u_t\|_4\|\nabla u\|_4]\\
&\le C[(\|\nabla d\|_{H^2}^2+\|u\|_{H^2}^2)(\|\nabla d_t\|_2^2+\|\sqrt\rho u_t\|_2^2+\|\Delta u\|_2^2)\\
&\quad+\|\nabla u\|_{H^1}^4+\|u\|_8^8+\|\sqrt\rho u_t\|_2\|u_t\|_{H^1}\|\nabla u\|_{H^1}]\\
&\le C[(\|\Delta d\|_2^2+\|\nabla\Delta d\|_2^2+\|\Delta u\|_2^2)(\|\nabla d_t\|_2^2+\|\sqrt\rho u_t\|_2^2+\|\Delta u\|_2^2)\\
&\quad+\|\Delta u\|_2^4+\|u\|_8^8+\|\sqrt\rho u_t\|_2\|\nabla u_t\|_2\|\Delta u\|_2]
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{2}\|\nabla u_t\|_2^2 + C[(\|\Delta d\|_2^2 + \|\nabla\Delta d\|_2^2 + \|\Delta u\|_2^2) \\
&\quad \times (\|\nabla d_t\|_2^2 + \|\sqrt{\rho}u_t\|_2^2 + \|\Delta u\|_2^2) + \|u\|_8^8].
\end{aligned}
$$

For the term $\|u\|_8^8$, by the Sobolev and Poincaré inequalities, we have

$$
\begin{aligned}
&\|u\|_8^8 \le C\|u\|_{H^1}^8 \le C\|\nabla u\|_2^8 \le C\|\nabla u\|_2^7\|\Delta u\|_2, \quad \text{for } N=2,\\
&\|u\|_8^8 \le C\|u\|_6^7\|u\|_{H^2} \le C\|\nabla u\|_2^7\|\Delta u\|_2, \quad \text{for } N=3,
\end{aligned}
$$

and thus, for both $N=2$ and $N=3$, we have

$$
\|u\|_8^8 \le C\|\nabla u\|_2^7\|\Delta u\|_2 \le C\|\nabla u\|_2^6(\|\nabla u\|_2^2 + \|\Delta u\|_2^2).
$$

Thanks to the above inequality, and recalling the estimate for I_1, it follows from (3.21) that

$$
\begin{aligned}
\frac{d}{dt}\|\sqrt{\rho}u_t\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 \le C[(&\|\Delta d\|_2^2 + \|\nabla\Delta d\|_2^2 + \|\Delta u\|_2^2)(\|\nabla d_t\|_2^2 + \|\sqrt{\rho}u_t\|_2^2 \\
&+ \|\Delta u\|_2^2) + \|\nabla u\|_2^6(\|\nabla u\|_2^2 + \|\Delta u\|_2^2)].
\end{aligned} \tag{3.22}
$$

Differentiate equation (1.4) with respect to t, then it has

$$
d_{tt} - \Delta d_t = |\nabla d|^2 d_t - (u\cdot\nabla)d_t + 2(\nabla d : \nabla d_t)d - (u_t\cdot\nabla)d.
$$

Squaring both sides of this equation, and integrating over Ω, then it follows from integration by parts and the Cauchy inequality that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla d_t|^2dx + \int_\Omega (|d_{tt}|^2 + |\Delta d_t|^2)dx \\
&\le C\int_\Omega (|\nabla d|^4|d_t|^2 + |u|^2|\nabla d_t|^2 + |\nabla d|^2|\nabla d_t|^2 + |\nabla d|^2|u_t|^2)dx \\
&\le C\int_\Omega [(|u|^2 + |\nabla d|^2)|\nabla d_t|^2 + |\Delta d|^2|d_t|^2 + |\nabla d|^2|u_t|^2]dx =: I_2.
\end{aligned}
$$

By the Hölder, Sobolev and Poincaré inequalities, and using Lemma 3.3, we can estimate I_2 as follows

$$
\begin{aligned}
I_2 &\le C[(\|u\|_\infty^2 + \|\nabla d\|_\infty^2)\|\nabla d_t\|_2^2 + \|\Delta d\|_4^2\|d_t\|_4^2 + \|\nabla d\|_4^2\|u_t\|_4^2] \\
&\le C[(\|u\|_{H^2}^2 + \|\nabla d\|_{H^2}^2)\|\nabla d_t\|_2^2 + \|\Delta d\|_{H^1}^2\|d_t\|_{H^1}^2 + \|\nabla d\|_{H^1}^2\|u_t\|_{H^1}^2] \\
&\le C(\|\Delta u\|_2^2 + \|\Delta d\|_2^2 + \|\nabla\Delta d\|_2^2)\|\nabla d_t\|_2^2 + C\|\Delta d\|_2^2\|\nabla u_t\|_2^2 \\
&\le C(\|\Delta u\|_2^2 + \|\Delta d\|_2^2 + \|\nabla\Delta d\|_2^2)\|\nabla d_t\|_2^2 + M_0\|\nabla u_t\|_2^2,
\end{aligned}
$$

where

$$
M_0 = C(1 + \|\nabla u_0\|_2^2 + \|\Delta d_0\|_2^2)^2 \ge 1.
$$

Thus, we have

$$
\frac{d}{dt}\|\nabla d_t\|_2^2 + \|\Delta d_t\|_2^2 \le C(\|\Delta u\|_2^2 + \|\Delta d\|_2^2 + \|\nabla\Delta d\|_2^2)\|\nabla d_t\|_2^2 + M_0\|\nabla u_t\|_2^2.
$$

Multiplying (3.22) by $4M_0$, and summing the resulting inequality with the above inequality up yields

$$\begin{aligned}&\frac{d}{dt}(4M_0\|\sqrt{\rho}u_t\|_2^2+\|\nabla d_t\|_2^2)+\|\nabla u_t\|_2^2+\|\Delta d_t\|_2^2\\&\leq CM_0(\|\Delta u\|_2^2+\|\Delta d\|_2^2+\|\nabla\Delta d\|_2^2)(\|\nabla d_t\|_2^2+\|\sqrt{\rho}u_t\|_2^2+\|\Delta u\|_2^2)\\&\quad+CM_0\|\nabla u\|_2^6(\|\nabla u\|_2^2+\|\Delta u\|_2^2).\end{aligned}\tag{3.23}$$

By (3.12), we have $\|\Delta u\|_2^2+\|\nabla p\|_2^2\leq C\|\sqrt{\rho}u_t\|_2^2+CS_1(t)$. Similar to (3.13), one has $\|\nabla\Delta d\|_2^2\leq 2\|\nabla d_t\|_2^2+CS_1(t)$. Thus, we have

$$\|\Delta u\|_2^2+\|\nabla p\|_2^2+\|\nabla\Delta d\|_2^2\leq C(\|\sqrt{\rho}u_t\|_2^2+\|\nabla d_t\|_2^2)+CS_1(t).$$

We estimate $S_1(t)$ as follows. For $N=2$, by the Hölder, Ladyzhenskaya, Poincaré and Young inequalities, we have

$$\begin{aligned}S_1(t)&\leq(\|u\|_4^2+\|\nabla d\|_4^2)(\|\nabla u\|_4^2+\|\nabla^2 d\|_4^2)\\&\leq C(\|u\|_2\|\nabla u\|_2+\|\nabla d\|_2\|\Delta d\|_2)[\|\nabla u\|_2\|\Delta u\|_2+\|\Delta d\|_2(\|\Delta d\|_2+\|\nabla\Delta d\|_2)]\\&\leq\varepsilon(\|\Delta u\|_2^2+\|\nabla\Delta d\|_2^2)+C(\|u\|_2^2+\|\nabla d\|_2^2)(\|\nabla u\|_2^4+\|\Delta d\|_2^4)\\&\quad+C(\|u\|_2+\|\nabla d\|_2)(\|\nabla u\|_2^3+\|\Delta d\|_2^3)\\&\leq\varepsilon(\|\Delta u\|_2^2+\|\nabla\Delta d\|_2^2)+C[(\|\nabla u\|_2^2+\|\Delta d\|_2^2)^3+(\|\nabla u\|_2^2+\|\Delta d\|_2^2)^2]\\&\leq\varepsilon(\|\Delta u\|_2^2+\|\nabla\Delta d\|_2^2)+C(\|\nabla u\|_2^2+\|\Delta d\|_2^2)(E_1(t)+E_1(t)^2),\end{aligned}$$

and for $N=3$, by the Hölder, Sobolev, Poincare and Cauchy inequalities, we have

$$\begin{aligned}S_1(t)&\leq(\|u\|_6^2+\|\nabla d\|_6^2)(\|\nabla u\|_2\|\nabla u\|_6+\|\nabla^2 d\|_2\|\nabla^2 d\|_6)\\&\leq C(\|\nabla u\|_2^2+\|\Delta d\|_2^2)[\|\nabla u\|_2\|\Delta u\|_2+\|\Delta d\|_2(\|\Delta d\|_2+\|\nabla\Delta d\|_2)]\\&\leq\varepsilon(\|\Delta u\|_2^2+\|\nabla\Delta d\|_2^2)+C[(\|\nabla u\|_2^2+\|\Delta d\|_2^2)^3+(\|\nabla u\|_2^2+\|\Delta d\|_2^2)^2]\\&\leq\varepsilon(\|\Delta u\|_2^2+\|\nabla\Delta d\|_2^2)+C(\|\nabla u\|_2^2+\|\Delta d\|_2^2)(E_1(t)+E_1(t)^2).\end{aligned}$$

Thus, for both $N=2$ and $N=3$, we always have

$$\begin{aligned}\|\Delta u\|_2^2+\|\nabla p\|_2^2+\|\nabla\Delta d\|_2^2&\leq C(\|\nabla u\|_2^2+\|\Delta d\|_2^2)(E_1(t)+E_1(t)^2)\\&\quad+C(\|\sqrt{\rho}u_t\|_2^2+\|\nabla d_t\|_2^2).\end{aligned}\tag{3.24}$$

Thanks to the above estimate, it follows from (3.23) that

$$\begin{aligned}&\frac{d}{dt}(4M_0\|\sqrt{\rho}u_t\|_2^2+\|\nabla d_t\|_2^2)+\|\nabla u_t\|_2^2+\|\Delta d_t\|_2^2\\&\leq CM_0(\|\Delta u\|_2^2+\|\Delta d\|_2^2+\|\nabla\Delta d\|_2^2)(\|\nabla d_t\|_2^2+\|\sqrt{\rho}u_t\|_2^2)\\&\quad+CM_0(\|\nabla u\|_2^2+\|\Delta d\|_2^2+\|\Delta u\|_2^2)(E_1(t)+E_1(t)^3),\end{aligned}$$

from which, by the Gronwall inequality, and using Lemmas 3.1 and 3.3, we have

$$\begin{aligned}&\sup_{0\leq s\leq t}(4M_0\|\sqrt{\rho}u_t\|_2^2+\|\nabla d_t\|_2^2)+\int_0^t(\|\nabla u_t\|_2^2+\|\Delta d_t\|_2^2)ds\\&\leq e^{CM_0E_1(t)}[4M_0\|\sqrt{\rho}u_t(0)\|_2^2+\|\nabla d_t(0)\|_2^2+M_0(1+E_1(t)^3)(C_0+E_1(t))]\\&\leq e^{CM_0^2}(4M_0\|\sqrt{\rho}u_t(0)\|_2^2+\|\nabla d_t(0)\|_2^2+M_0^5),\end{aligned}\tag{3.25}$$

where we have used $C_0 \leq CM_0$, by the Poincaré inequality.

By the compatible condition and equations (1.2) and (1.4), one can easily check that

$$\|\nabla\partial_t d(0)\|_2^2 + \|\sqrt{\rho}u_t(0)\|_2^2 \leq C(\|g_0\|_2^2 + M_0),$$

which, combined with (3.24) and (3.25), yields the conclusion. □

Proposition 3.1. *Under the same conditions as in Lemma 3.3, we have*

$$\sup_{0\leq t\leq T} E(t) \leq C(\|\rho_0\|_{H^1}, \|u_0\|_{H^2}, \|\nabla d_0\|_{H^2}, \|g_0\|_2, \overline{\rho}, T, \Omega),$$

where the constant C keeps finite for finite time T.

Proof. Using the Sobolev embedding inequality, and applying elliptic estimates for the Stokes equations, it follows from Lemmas 3.3 and 3.4 that

$$\begin{aligned}\int_0^t \|\nabla^2 u\|_4^2 ds \leq& C\int_0^t (\|\rho u_t\|_4^2 + \|\rho u|\nabla u|\|_4^2 + \|\nabla d|\nabla^2 d|\|_4^2) ds\\ \leq& C\int_0^t (\|\nabla u_t\|_2^2 + \|u\|_\infty^2\|\nabla u\|_{H^1}^2 + \|\nabla d\|_\infty^2\|\nabla^2 d\|_4^2) ds\\ \leq& C\int_0^t (\|\nabla u_t\|_2^2 + \|u\|_{H^2}^2\|\nabla u\|_{H^1}^2 + \|\nabla d\|_{H^2}^2\|\nabla^2 d\|_{H^1}^2) ds\\ \leq& C\int_0^t (\|\nabla u_t\|_2^2 + \|\Delta u\|_2^4 + \|\Delta d\|_2^4 + \|\nabla\Delta d\|_2^4) ds \leq C.\end{aligned}$$

On account of this, by the Sobolev embedding inequality, it follows from Lemma 2.1 that $\|\nabla\rho(t)\|_2^2 \leq C\|\nabla\rho_0\|_2^2$. Using equation (1.1), by the Sobolev embedding inequality and Lemma 3.4, we deduce

$$\|\rho_t\|_2^2 \leq C\|u\|_\infty^2\|\nabla\rho\|_2^2 \leq C\|\Delta u\|_2^2\|\nabla\rho\|_2^2 \leq C,$$

and thus $\sup_{0\leq s\leq t}(\|\nabla\rho\|_2^2 + \|\rho_t\|_2^2) \leq C$. Combining this with Lemmas 3.3 and 3.4 yields the conclusion. □

4. Proof of the main theorem. After obtaining the a priori estimates stated in the previous section, we can now give the proof of our main results.

Proof. We divide the proof into several steps. Let ε_0 be the constant stated in Lemma 3.3, and suppose that

$$\begin{aligned}&\|\sqrt{\rho_0}u_0\|_2^2 + \|\nabla d_0\|_2^2 < \varepsilon_0, \quad \text{if } N = 2,\\ &(\|\sqrt{\rho_0}u_0\|_2^2 + \|\nabla d_0\|_2^2)(\|\nabla u_0\|_2^2 + \|\Delta d_0\|_2^2) < \varepsilon_0, \quad \text{if } N = 3.\end{aligned}$$

Step 1. Global existence of strong solutions without vacuum. Suppose that there is a positive constant $\underline{\rho}$, such that $\rho_0(x) \geq \underline{\rho}$, for all $x \in \Omega$. By Lemma 2.3, there is a unique strong solution to system (1.1)–(1.4), subject to (1.6)–(1.7), in Q_T. Let T_* be the maximal existence time. We claim that $T_* = \infty$. If it's not the case, then $0 < T_* < \infty$. By Proposition 3.1, there is a positive constant K depending only on $\|g_0\|_2$, $\|u_0\|_{H^2}$, $\|\nabla d_0\|_{H^2}$, $\|\rho_0\|_{H^1}$, $\bar{\rho}$ and Ω, such that

$$\sup_{0\leq t\leq T}(\|\nabla d\|_{H^2}^2 + \|\nabla p\|_2^2 + \|u\|_{H^2}^2 + \|\nabla\rho\|_2^2 + \|\rho_t\|_2^2) + \int_0^T (\|d_t\|_{H^2}^2 + \|u_t\|_{H^1}^2) ds \leq K.$$

Since $\rho_0 \geq \underline{\rho}$, by Lemma 2.1, we have $\rho \geq \underline{\rho}$, and thus

$$\sup_{0<t<T_*} \|\rho^{-1/2}(-\Delta u - \operatorname{div}(\nabla d \odot \nabla d))\|_2^2 \leq C,$$

for some constant C. Thus, by Lemma 2.3, we can extend the strong solution to $(0, T^*)$, for another time $T^* > T_*$, which contradicts to the definition of T_*. This contradiction implies that $T_* = \infty$, and thus we obtain the global strong solution.

Step 2. Global existence of strong solutions with vacuum. We prove it by approximation. For $j = 1, 2, \cdots$, define $\rho_0^j = \rho_0 + \frac{1}{j}$, and set

$$g_0^j = (\rho_0^j)^{-1/2}(-\Delta u_0 - \nabla p_0 - \operatorname{div}(\nabla d_0 \odot \nabla d_0)).$$

Then, the compatible condition holds true

$$-\Delta u_0 - \nabla p_0 - \operatorname{div}(\nabla d_0 \odot \nabla d_0) = \sqrt{\rho_0^j} g_0^j,$$

in Ω. It's easy to see that $|g_0^j| \leq |g_0|$ in Ω, and thus $\|g_0^j\|_2^2 \leq \|g_0\|_2^2$, for all j. Then, for j large enough, it has

$$\begin{aligned} &\|\sqrt{\rho_0^j} u_0\|_2^2 + \|\nabla d_0\|_2^2 < \varepsilon_0, \quad \text{if } N = 2, \\ &(\|\sqrt{\rho_0^j} u_0\|_2^2 + \|\nabla d_0\|_2^2)(\|\nabla u_0\|_2^2 + \|\Delta d_0\|_2^2) < \varepsilon_0, \quad \text{if } N = 3. \end{aligned}$$

By step 1, for each j large enough, there is a global strong solution (ρ^j, u^j, p^j, d^j) to system (1.1)–(1.4), subject to (1.6)–(1.7), with initial data $\rho^j(0) = \rho_0^j$ and $u^j(0) = u_0$. Moreover, by Proposition 3.1, (ρ^j, u^j, p^j, d^j) satisfies the following estimates

$$\sup_{0 \leq t \leq T} (\|\nabla d\|_{H^2}^2 + \|\nabla p\|_2^2 + \|u\|_{H^2}^2 + \|\nabla \rho\|_2^2 + \|\rho_t\|_2^2) + \int_0^T (\|d_t\|_{H^2}^2 + \|u_t\|_{H^1}^2) ds \leq K,$$

for a positive constant K independent of j, which keeps finite for any finite time T. By the aid of these uniform estimates with respect to j, we can follow the standard convergence approach to obtain the global strong solution.

Step 3. Uniqueness of global strong solutions. Let (ρ, u, p, d) and $(\hat{\rho}, \hat{u}, \hat{p}, \hat{d})$ be two global strong solutions to system (1.1)–(1.4), subject to the same boundary and initial conditions. We only consider the three dimensional case, and the two dimensional case can be dealt with in the similar way.

Note that $\rho - \hat{\rho}$ satisfies

$$(\rho - \hat{\rho})_t + u \cdot \nabla(\rho - \hat{\rho}) = (\hat{u} - u) \cdot \nabla \hat{\rho},$$

from which we deduce that

$$(|\rho - \hat{\rho}|^{3/2})_t + u \cdot \nabla(|\rho - \hat{\rho}|^{3/2}) \leq \frac{3}{2}|u - \hat{u}||\nabla \hat{\rho}||\rho - \hat{\rho}|^{1/2}.$$

Integrating the above inequality over Ω, it follows from the Sobolev inequality that

$$\frac{d}{dt}\int_\Omega |\rho - \hat{\rho}|^{3/2} dx \leq \frac{3}{2}\int_\Omega |\hat{u} - u||\nabla \hat{\rho}||\rho - \hat{\rho}|^{1/2} dx$$

$$\leq \frac{3}{2}\|u-\hat{u}\|_6\|\nabla\hat{\rho}\|_2\|\rho-\hat{\rho}\|_{3/2}^{1/2} \leq C\|\nabla(u-\hat{u})\|_2\|\nabla\hat{\rho}\|_2\|\rho-\hat{\rho}\|_{3/2}^{1/2},$$

and thus

$$\begin{aligned}\frac{d}{dt}\|\rho-\hat{\rho}\|_{3/2}^2 =&\frac{4}{3}\|\rho-\hat{\rho}\|_{3/2}^{1/2}\frac{d}{dt}\|\rho-\hat{\rho}\|_{3/2}^{3/2}\\ \leq& C\|\nabla(\hat{u}-u)\|_2\|\nabla\hat{\rho}\|_2\|\rho-\hat{\rho}\|_{3/2}\\ \leq&\frac{1}{4}\|\nabla(u-\hat{u})\|_2^2+\alpha(t)\|\rho-\hat{\rho}\|_{3/2}^2,\end{aligned}\tag{4.26}$$

for some nonnegative function $\alpha(t) \in L^1(0,T)$, for any $T>0$.

Using equation (1.2), we deduce

$$\begin{aligned}&\rho(u-\hat{u})_t+\rho u\cdot\nabla(u-\hat{u})-\Delta(u-\hat{u})+\nabla(p-\hat{p})\\ =&\Delta(\hat{d}-d)\cdot\nabla\hat{d}+\Delta d\cdot\nabla(\hat{d}-d)-(\rho-\hat{\rho})(\hat{u}_t+\hat{u}\cdot\nabla\hat{u})-\rho(u-\hat{u})\cdot\nabla\hat{u}.\end{aligned}$$

Multiplying the above equation by $u-\hat{u}$, and integrating the resulting equation over Ω, then it follows from integration by parts that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega\rho|u-\hat{u}|^2dx+\int_\Omega|\nabla(u-\hat{u})|^2dx\\ \leq&\int_\Omega(|\nabla(d-\hat{d})||\nabla\hat{d}||\nabla(u-\hat{u})|+|\nabla(d-\hat{d})||\nabla^2\hat{d}||u-\hat{u}|+|\Delta d||\nabla(\hat{d}-d)||u-\hat{u}|\\ &+|\rho-\hat{\rho}||\hat{u}_t+u\cdot\nabla\hat{u}||u-\hat{u}|+\rho|\nabla\hat{u}||u-\hat{u}|^2)dx,\end{aligned}$$

and consequently it follows from the Hölder, Sobolev and Young inequalities that

$$\begin{aligned}&\frac{d}{dt}\|\sqrt{\rho}(u-\hat{u})\|_2^2+\|\nabla(u-\hat{u})\|_2^2\\ \leq&\frac{1}{4}\|\nabla(u-\hat{u})\|_2^2+C\|\nabla\hat{d}\|_\infty^2\|\nabla(d-\hat{d})\|_2^2\\ &+C\Big(\|\nabla^2\hat{d}\|_3\|\nabla(\hat{d}-d)\|_2\|u-\hat{u}\|_6+\|\Delta d\|_3\|\nabla(d-\hat{d})\|_2\|u-\hat{u}\|_6\\ &+\|\rho-\hat{\rho}\|_{3/2}\|\hat{u}_t+u\cdot\nabla\hat{u}\|_6\|u-\hat{u}\|_6+\|\nabla\hat{u}\|_\infty\|\sqrt{\rho}(u-\hat{u})\|_2^2\Big)\\ \leq&\frac{1}{2}\|\nabla(u-\hat{u})\|_2^2+C\Big[(\|\nabla\hat{d}\|_\infty^2+\|\nabla^2\hat{d}\|_3^2+\|\Delta d\|_3^2)\|\nabla(d-\hat{d})\|_2^2\\ &+\|\hat{u}_t+u\cdot\nabla\hat{u}\|_6^2\|\rho-\hat{\rho}\|_{3/2}^2+\|\nabla\hat{u}\|_\infty\|\sqrt{\rho}(u-\hat{u})\|_2^2\Big],\end{aligned}$$

from which we deduce

$$\begin{aligned}&\frac{d}{dt}\|\sqrt{\rho}(u-\hat{u})\|_2^2+\|\nabla(u-\hat{u})\|_2^2\\ \leq&\beta(t)(\|\nabla(d-\hat{d})\|_2^2+\|\rho-\hat{\rho}\|_{3/2}^2+\|\sqrt{\rho}(u-\hat{u})\|_2^2),\end{aligned}$$

for some nonnegative function $\beta(t)\in L^1(0,T)$, for any $T>0$.

Combining the above inequality with (4.26), we obtain

$$\begin{aligned}&\frac{d}{dt}\Big(\|\rho-\hat{\rho}\|_{3/2}^2+\|\sqrt{\rho}(u-\hat{u})\|_2^2\Big)+\|\nabla(u-\hat{u})\|_2^2\\ \leq&\gamma(t)(\|\nabla(d-\hat{d})\|_2^2+\|\rho-\hat{\rho}\|_{3/2}^2+\|\sqrt{\rho}(u-\hat{u})\|_2^2),\end{aligned}\tag{4.27}$$

for some nonnegative function $\gamma(t) \in L^1(0,T)$, for any $T > 0$.

Using (1.4), it has

$$\begin{aligned}&(d-\hat{d})_t + u\cdot\nabla(d-\hat{d}) - \Delta(d-\hat{d})\\ =&\nabla(d-\hat{d}):\nabla(d+\hat{d})d + |\nabla\hat{d}|^2(d-\hat{d}) - (u-\hat{u})\cdot\nabla\hat{d}.\end{aligned}$$

Multiply the above equation by $-\Delta(d-\hat{d})$, and integrating the resulting equation over Ω, it follows from integration by parts that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla(d-\hat{d})|^2dx + \int_\Omega |\Delta(d-\hat{d})|^2dx\\ =&\int_\Omega \Big[u\cdot\nabla(d-\hat{d}) + (u-\hat{u})\cdot\nabla\hat{d} - \nabla(d-\hat{d}):\nabla(d+\hat{d})d\\ &- |\nabla\hat{d}|^2(d-\hat{d})\Big]\cdot\Delta(d-\hat{d})dx\\ \le&\frac{1}{2}\int_\Omega |\Delta(d-\hat{d})|^2dx + \int_\Omega \Big[(|u|^2 + |\nabla(d+\hat{d})|^2)|\nabla(d-\hat{d})|^2\\ &+ |\nabla\hat{d}|^4|d-\hat{d}|^2 + |\nabla\hat{d}|^2|u-\hat{u}|^2\Big]dx,\end{aligned}$$

from which, by the Poincáre inequality, one has

$$\frac{d}{dt}\|\nabla(d-\hat{d})\|_2^2 + \|\Delta(d-\hat{d})\|_2^2 \le \zeta(t)(\|\nabla(d-\hat{d})\|_2^2 + \|\nabla(u-\hat{u})\|_2^2),$$

for some nonnegative function $\zeta(t) \in L^1(0,T)$, for any $T > 0$.

Combining the above inequality with (4.27), we have

$$\begin{aligned}&\frac{d}{dt}\Big(\|\rho-\hat{\rho}\|_{3/2}^2 + \|\sqrt{\rho}(u-\hat{u})\|_2^2 + \|\nabla(d-\hat{d})\|_2^2\Big) + (\|\nabla(u-\hat{u})\|_2^2 + \|\Delta(d-\hat{d})\|_2^2)\\ \le&\xi(t)(\|\rho-\hat{\rho}\|_{3/2}^2 + \|\sqrt{\rho}(u-\hat{u})\|_2^2 + \|\nabla(d-\hat{d})\|_2^2),\end{aligned}$$

for some nonnegative function $\xi(t) \in L^1(0,T)$, for any $T > 0$. Recalling that $(\rho-\hat{\rho}, u-\hat{u}, d-\hat{d})|_{t=0} = (0,0,0)$, it follows from Gronwall's inequality that $(\rho,u,d) = (\hat{\rho},\hat{u},\hat{d})$, completing the proof. ∎

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