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002

TO THE THEORY OF VISCOSITY SOLUTIONS FOR UNIFORMLY PARABOLIC ISAACS EQUATIONS*

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Abstract. We show how a theorem about the solvability in $W^{1,2}_\infty$ of special parabolic Isaacs equations can be used to obtain the existence and uniqueness of viscosity solutions of general uniformly nondegenerate parabolic Isaacs equations. We apply it also to establish the $C^{1+\chi}$ regularity of viscosity solutions and show that finite-difference approximations have an algebraic rate of convergence. The main coefficients of the Isaacs equations are supposed to be in C^γ with respect to the spatial variables with γ slightly less than 1/2.

Key words. Fully nonlinear equations, viscosity solutions, Hölder regularity of derivatives, numerical approximation rates.

AMS subject classifications. 35K55, 35B65, 65N15.

1. Introduction. The goal of this article is to present a purely PDE exposition of some key results in the theory of viscosity solutions for uniformly nondegenerate parabolic Isaacs equations. We did the same for elliptic equations in [12], and this article is its natural continuation.

Let $\mathbb{R}^d = \{x = (x^1, ..., x^d)\}$ be a d-dimensional Euclidean space. Assume that we are given separable metric spaces A and B, and let, for each $\alpha \in A$ and $\beta \in B$, the following functions on $\mathbb{R}^{d+1} = \{(t, x) : t \in \mathbb{R}, x \in \mathbb{R}^d\}$ be given:

(i) $d \times d$ matrix-valued $a^{\alpha\beta}$,

(ii) $\mathbb{R}^d$-valued $b^{\alpha\beta}$, and

(iii) real-valued functions $c^{\alpha\beta} \geq 0$, $f^{\alpha\beta}$, and g.

Let $\mathbb{S}$ be the set of symmetric $d \times d$ matrices, and for $(u_{ij}) \in \mathbb{S}$, $(u_i) \in \mathbb{R}^d$, and $u \in \mathbb{R}$ introduce

$$F(u_{ij}, u_i, u, t, x) = \sup_{\alpha\in A} \inf_{\beta\in B} [a^{\alpha\beta}_{ij}(t, x)u_{ij} + b^{\alpha\beta}_i(t, x)u_i - c^{\alpha\beta}(t, x)u + f^{\alpha\beta}(t, x)],$$

where and everywhere below the summation convention is enforced and the summations are done inside the brackets.

For sufficiently smooth functions $u = u(t, x)$ introduce

$$L^{\alpha\beta}u(t, x) = a^{\alpha\beta}_{ij}(t, x)D_{ij}u(t, x) + b^{\alpha\beta}_i(t, x)D_iu(t, x) - c^{\alpha\beta}(t, x)u(t, x),$$

where, naturally, $D_i = \partial/\partial x^i$, $D_{ij} = D_iD_j$. Denote

$$\begin{aligned} F[u](t, x) &= F(D^2u(t, x), Du(t, x), u(t, x), t, x) \\ &= \sup_{\alpha\in A} \inf_{\beta\in B} [L^{\alpha\beta}u(t, x) + f^{\alpha\beta}(t, x)], \end{aligned} \tag{1.1}$$

where Du is the gradient of u and D^2u is its Hessian.

Also fix an open bounded subset G of $\mathbb{R}^d$ with C^2 boundary and $T \in (0, \infty)$. We denote the parabolic boundary of the cylinder $Q = (0, T) \times G$ by

$$\partial' Q = (\partial Q) \setminus (\{0\} \times G).$$

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Under appropriate assumptions, which amount to the boundedness and continuity with respect to (t,x) of the coefficients and the free term and uniform nondegeneracy of $a^{\alpha\beta}(t,x)$, the Isaacs equation

$$\partial_t u + F[u] = 0, \tag{1.2}$$

where $\partial_t = \partial/\partial t$, in Q with boundary condition $u = g$ on $\partial' Q$ has a viscosity solution $v \in C(\bar{Q})$, which is proved in Theorem 2.1 of Crandall, Kocan, Lions, and Święch [3] (1999). Recall that $v \in C(\bar{Q})$ is a viscosity solution if for any $\phi(t,x)$ which is smooth in $\bar{Q}$ and any point $(t_0,x_0) \in [0,T) \times G$ at which $\phi - w$ attains

(i) a local (relative to $[0,T)\times G$) maximum which is zero it holds that $\partial_t\phi(t_0,x_0) + F[\phi](t_0,x_0) \le 0$,

(ii) a local minimum which is zero we have $\partial_t\phi(t_0,x_0) + F[\phi](t_0,x_0) \ge 0$.

In [3] also the existence of minimal and maximal continuous viscosity solution is proved. It turns out that this fact holds true also for L_p-viscosity solutions in the framework of the Isaacs equations even if the coefficients and the free terms are just measurable (see, for instance, [10]). Crandall, Kocan, and Święch in [4] (2000) proved the existence of continuous L_p-viscosity solutions when the coefficients are only measurable with respect to (t,x) and $\sup_{\alpha,\beta}|f^{\alpha\beta}| \in L_p$. In the continuous case continuous L_p-viscosity solutions are automatically viscosity solutions and this reproves part of the results in [3].

It seems that in the parabolic case much less is known about uniqueness without convexity assumptions on F with respect to D^2u than in the elliptic case. One could derive uniqueness for parabolic equations considering them as degenerate elliptic ones and using the results in Crandall, Ishii, and Lions [2] (1992), but this would require the coefficients and the free term to be almost Lipschitz in (t,x) (see Section 5.A there). In Lemma 6.2 of [4] the uniqueness even of L_p-viscosity solution is proved for equations which in our terms have coefficients independent of (t,x) and $f^{\alpha\beta}(t,x) = f^{\alpha\beta} + f(t,x)$. We are going to prove uniqueness under the assumption that $a^{\alpha\beta}$ are γ-Hölder continuous in x, with $\gamma < 1/2$. Also all the coefficients and the free term are assumed to be uniformly continuous in (t,x). This result when $a^{\alpha\beta}$ are Lipschitz continuous only with respect to x seems to be available from [18] (see Theorem 3.1 there). In Corollary 3.5 of [18] a comparison result, and hence uniqueness, is stated even for the case of just continuous $a^{\alpha\beta}$ (in our setting).

Concerning regularity of solutions note that the interior $C^{1+\chi}$-regularity was established by Wang [17] (1992) under the assumption that F is almost independent of Du and (in our setting) the coefficients are uniformly sufficiently close to the ones which are uniformly continuous with respect to (t,x) uniformly in α,β. Then Crandall, Kocan, and Święch [4] (see there Theorem 7.3) generalized the result of [17] to the case of full equation and continuous L_p-viscosity solutions assuming the same kind of dependence on (t,x) of $a^{\alpha\beta}(t,x)$ as in [17]. As we know from [11], in the case of the Isaacs equations it is enough to have $a^{\alpha\beta}(t,x)$ in VMO with respect to x. We use this result in the form of Theorem 2.3.

Last issue we are dealing with is the rate of convergence of finite difference approximations of solutions of the Isaacs equations. Caffarelli and Souganidis [1] (2008) gave a method of establishing the rate of convergence for elliptic equations of the form $F(D^2u(x)) = f(x)$. Turanova [15] extended the results of [1] to F's explicitly depending on x and in [16] considered parabolic equations of the form $\partial_t u + F(D^2u) = 0$. We obtain the rate of convergence for the general parabolic Isaacs equations under the same assumptions under which we prove the existence and uniqueness of viscosity solutions additionally assuming that the coefficients and the free term are Hölder continuous with respect to (t,x) with a constant and exponent independent of α,β.

It is worth saying that in all the above cited papers equations much more general than the Isaacs equations are considered and we discuss their results in the case of the Isaacs equations just to be able to compare them with our results. Our methods are quite different from the methods of the above cited articles and are based on an approximation theorem (Theorem 2.1, the proof of which, by the way, does not even require any knowledge of the theory of partial differential equations).

The article is organized as follows. In Section 2 we present our main results and prove all of them apart from Theorem 2.4, assertion (ii) of Theorem 2.5, and Theorem 2.6, which are proved in Sections 4, 5, and 6, respectively, after a rather long Section 3 containing a comparison theorem for smooth functions.

Our equation are considered in C^2 cylinders with $W^{1,2}_\infty$ boundary data. These restrictions can be considerably relaxed and we leave doing that to the interested reader.

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2. Main results. Fix some constants $\delta\in(0,1)$ and $K_0\in[0,\infty)$. Set

$$\mathbb{S}_\delta=\{a\in\mathbb{S}:\delta|\xi|^2\le a_{ij}\xi_i\xi_j\le\delta^{-1}|\xi|^2,\quad\forall\xi\in\mathbb{R}^d\}.$$

In the following assumption the small parameter $\chi\in(0,1)$, which depends only on δ and d, is a constant to be specified in Theorem 2.3 and

$$\gamma=\frac{4-3\chi}{8-4\chi}\quad(<1/2).$$

ASSUMPTION 2.1. (i) The functions $a^{\alpha\beta}$, $b^{\alpha\beta}$, $c^{\alpha\beta}$, and $f^{\alpha\beta}$ are continuous with respect to $\beta\in B$ for each (α,t,x) and continuous with respect to $\alpha\in A$ uniformly with respect to $\beta\in B$ for each (t,x), they are also uniformly continuous with respect to t uniformly with respect to α,β,x, the function g is continuous and

$$\|g\|_{W^{1,2}_\infty(\mathbb{R}^{d+1})}\le K_0,$$

(ii) For all values of the arguments

$$|b^{\alpha\beta}|,|c^{\alpha\beta}|,|f^{\alpha\beta}|\le K_0,$$

(iii) For any $(\alpha,\beta)\in A\times B$ and $(t,x),(t,y)\in\mathbb{R}^{d+1}$ we have

$$\|a^{\alpha\beta}(t,x)-a^{\alpha\beta}(t,y)\|\le K_0|x-y|^\gamma,$$

$$|u^{\alpha\beta}(t,x)-u^{\alpha\beta}(t,y)|\le K_0\omega(|x-y|),$$

where $u=b,c,f$, and ω is a fixed continuous increasing function on $[0,\infty)$, such that $\omega(0)=0$, and for a matrix σ we denote $\|\sigma\|^2=\operatorname{tr}\sigma\sigma^*$,

(iv) For all values of the arguments $a^{\alpha\beta}\in\mathbb{S}_\delta$.

REMARK 2.1. The Hölder exponent in Assumption 2.1 (iii) is certainly not sharp. For instance, if χ is close to one the exponent γ should approach zero.

We wanted to reflect better what is going on when χ is close to zero, since nobody knows how far from zero it is.

In [10] a convex positive homogeneous of degree one function $P(u_{ij})$ is constructed on $(u_{ij}) \in \mathbb{S}$ such that at all points of differentiability of P with respect to (u_{ij}) we have $(P_{u_{ij}}) \in \mathbb{S}_{\hat{\delta}}$, where $\hat{\delta}$ is a constant in $(0, \delta)$ depending only on d and δ. This function is constructed once only d and δ are given and possesses some additional properties to be mentioned and used below. By $P[u]$ we denote $P(D_{ij}u)$.

The following result in what concerns equation (2.1) is part of Theorem 2.1 of [10], in which no regularity assumptions on the coefficients and the free term are imposed. They could be just measurable.

THEOREM 2.1. *For any $K \geq 0$ each of the equations*

$$\partial_t u_K + \max(F[u_K], P[u_K] - K) = 0, \tag{2.1}$$

$$\partial_t u_{-K} + \min(F[u_{-K}], -P[-u_{-K}] + K) = 0, \tag{2.2}$$

in Q with boundary condition $u_{\pm K} = g$ on $\partial' Q$ has a unique solution in the class $W^{1,2}_{\infty,loc}(Q) \cap C(\bar{Q})$. Furthermore, there is a constant N such that for any $K \geq 0$, we have $|\partial_t u_{\pm K}|$, $\rho |D^2 u_{\pm K}|$, $|D u_{\pm K}| \leq N(1+K)$ in Q (a.e.), where $\rho(t,x) = \operatorname{dist}(x, G^c)$.

The result concerning equation (2.2) is obtained just by observing that the operator $-F[-u]$ fits into the scheme of [10] where equations more general than the Isaacs equations are treated.

The functions $u_{\pm K}$ are the central objects of our investigation. Here is a simple property they possess.

LEMMA 2.2. *There exists a constant N, depending only on d, δ, K_0, and G, such that in Q*

$$|u_{\pm K} - g| \leq N\rho, \quad |u_{\pm K}| \leq N.$$

This result for u_K follows from the fact that $|\max(F[0], -K)| \leq |F[0]|$, $g \in W^{1,2}_{\infty}(\mathbb{R}^{d+1})$, $G \in C^2$, and u_K satisfies a linear equation

$$\partial_t u_K + a_{ij} D_{ij} u_K + b_i D_i u_K - c u_K + f = 0,$$

where $(a_{ij}) \in \mathbb{S}_{\hat{\delta}}$, $|(b_i)| \leq K_0$, $K_0 \geq c \geq 0$, and $|f| \leq K_0$. The case of u_{-K} is quite similar.

To characterize some smoothness properties of $u_{\pm K}$ introduce $C^{1+\chi}(Q)$ as the space of functions on Q continuously differentiable with respect to x and with finite norm given by

$$\|u\|_{C^{1+\chi}(Q)} = \sup_Q |u| + \sup_{(t,x),(t,y) \in Q} \frac{|u(t,x) - u(t,y)|}{|x-y|} + [u]_{C^{1+\chi}(Q)},$$

where

$$[u]_{C^{1+\chi}(Q)} = \sup_{(t,x),(s,y) \in Q} \Big[\frac{|u(t,x) - u(s,x)|}{|t-s|^{(1+\chi)/2}}$$

$$+\frac{|Du(t,x)-Du(t,y)|}{|x-y|^{\chi}}+\frac{|Du(t,x)-Du(s,x)|}{|t-s|^{\chi/2}}]$$

(the last term in the brackets can be dropped yielding an equivalent norm as follows from simple interpolation inequalities).

For $\varepsilon>0$ introduce

$$G_{\varepsilon}=\{x\in G:\operatorname{dist}(x,\partial G)>\varepsilon\},\quad Q_{\varepsilon}=(0,T-\varepsilon^{2})\times G_{\varepsilon}.$$

The following result in what concerns u_K is a consequence of Theorem 5.4 of [11], where, so to speak, the leading coefficients are supposed to be in VMO in x rather than Hölder continuous and measurable in t. The remaining terms are assumed to be just measurable. The case of u_{-K} is treated as in Theorem 2.1.

THEOREM 2.3. *There exists a constant $\chi\in(0,1)$, depending only on δ and d, and there exists a constant N, depending only on K_0, δ, d, and G such that for any $\varepsilon>0$ (such that $G_{\varepsilon}\neq\emptyset$)*

$$\|u_{\pm K}\|_{C^{1+\chi}(Q_{\varepsilon})}\leq N\varepsilon^{-1-\chi}. \tag{2.3}$$

Estimate (2.3) without specified dependence on ε of the right-hand side follows from Theorem 7.3 of [4]. We use (2.3) as it is stated in the proof of the following result which is central in the paper. Fix a constant $\tau\in(0,1)$.

THEOREM 2.4. *For $K\to\infty$ we have $|u_K-u_{-K}|\to0$ uniformly in Q. Moreover, if*

$$\omega(t)=t^{\tau}, \tag{2.4}$$

then there exist constants $\xi\in(0,1)$, depending only on τ, d, K_0, and δ, and $N\in(0,\infty)$, depending only on τ, d, K_0, δ, and Q, such that, if $K\geq1$, then

$$|u_K-u_{-K}|\leq NK^{-\xi} \tag{2.5}$$

in Q.

We prove Theorem 2.4 in Section 4 by adapting an argument from Section 5.A of [2] explaining how to prove the comparison principle for $C^{1+\chi}$ subsolutions and supersolutions. We use a quantitative and parabolic version of this argument.

THEOREM 2.5. *(i) The limit*

$$v:=\lim_{K\to\infty}u_K$$

exists,

(ii) The function v is a unique continuous in $\bar{Q}$ viscosity solution of (1.2) *with boundary condition $v=g$ on $\partial'Q$,*

(iii) If condition (2.4) *is satisfied, then for $K\geq1$ we have $|u_K-v|\leq NK^{-\xi}$,*

(iv) For any $\varepsilon\in(0,1]$ (such that $G_{\varepsilon}\neq\emptyset$)

$$\|v\|_{C^{1+\chi}(Q_{\varepsilon})}\leq N\varepsilon^{-1-\chi},$$

where N is the constant from (2.3).

Assertions (i), (iii), and (iv) are simple consequences of Theorems 2.3 and 2.4 and the maximum principle. Indeed, notice that $\partial_t u_K + F[u_K] \leq 0$ and $\partial_t u_{-K} + F[u_{-K}] \geq 0$. Hence by the maximum principle $u_K \geq u_{-K}$. Furthermore, again by the maximum principle u_K decreases and u_{-K} increases as K increases. This takes care of assertions (i), (iii), and (iv).

Assertion (ii) is proved in Section 5.

To state one more result we introduce some new objects. As is well known (see, for instance, [13]), there exists a finite set $\Lambda = \{l_1, ..., l_{d_2}\} \subset \mathbb{Z}^d$ containing all vectors from the standard orthonormal basis of $\mathbb{R}^d$ such that one has the following representation

$$L^{\alpha\beta}u(t,x) = a_k^{\alpha\beta}(t,x)D_{l_k}^2 u(t,x) + \bar{b}_k^{\alpha\beta}(t,x)D_{l_k}u(t,x) - c^{\alpha\beta}(t,x)u(t,x),$$

where $D_{l_k}u(x) = \langle Du, l_k\rangle$, $a_k^{\alpha\beta}$ and $\bar{b}_k^{\alpha\beta}$ are certain bounded functions and $a_k^{\alpha\beta} \geq \delta_1$, with a constant $\delta_1 > 0$. One can even arrange for such representation to have the coefficients $a_k^{\alpha\beta}$ and $\bar{b}_k^{\alpha\beta}$ with the same regularity properties with respect to (t,x) as the original ones $a_{ij}^{\alpha\beta}$ and $b_i^{\alpha\beta}$ (see, for instance, Theorem 3.1 in [8]). Define B as the smallest closed ball centered at the origin containing Λ, and for $h > 0$ set $\mathbb{Z}_h^d = h\mathbb{Z}^d$,

$$G_{(h)} = G \cap \mathbb{Z}_h^d, \quad Q_{(h)} = ((0,T) \cap \mathbb{Z}_{h^2}) \times G_{(h)},$$

$$Q_{(h)}^o = \{(t,x) \in Q_{(h)} : x + hB \subset G, t + h^2 < T\}, \quad \partial_h' Q = Q_{(h)} \setminus Q_{(h)}^o.$$

Next, for $h > 0$ we introduce

$$\delta_{h,t}u(t,x) = \frac{u(t+h^2,x) - u(t,x)}{h^2}, \quad \delta_{h,l_k}u(t,x) = \frac{u(t,x+hl_k) - u(t,x)}{h},$$

$$\Delta_{h,l_k}u(t,x) = \frac{u(t,x+hl_k) - 2u(t,x) + u(t,x-hl_k)}{h^2},$$

$$L_h^{\alpha\beta}u(t,x) = a_k^{\alpha\beta}(t,x)\Delta_{h,l_k}u(t,x) + \bar{b}_k^{\alpha\beta}(t,x)\delta_{h,l_k}u(t,x) - c^{\alpha\beta}(t,x)u(t,x),$$

$$F_h[u](t,x) = \sup_{\alpha\in A}\inf_{\beta\in B}[L_h^{\alpha\beta}u(t,x) + f^{\alpha\beta}(t,x)].$$

It is a simple fact, which may be shown as in [13], that for each sufficiently small h there exists a unique function v_h on $Q_{(h)}$ such that

$$\delta_{h,t}v_h + F_h[v_h] = 0$$

on $Q_{(h)}^o$ and $v_h = 0$ on $\partial_h' Q$.

Here is the result we were talking about above and which is proved in Section 6.

Theorem 2.6. *Suppose that there exists $\gamma_t \in (0,1)$ such that for any $(\alpha,\beta) \in A \times B$ and $(t,x),(s,x) \in \mathbb{R}^{d+1}$ we have*

$$|u^{\alpha\beta}(t,x) - u^{\alpha\beta}(s,x)| \leq K_0|t-s|^{\gamma_t},$$

where $u = a,b,c,f$. Let condition (2.4) be satisfied and let $g = 0$. Then there exist constants N and $\eta > 0$ such that for all sufficiently small $h > 0$ we have on $Q_{(h)}$ that

$$|v_h - v| \leq Nh^\eta.$$

In this theorem we deal with implicit finite-difference schemes and with the time step size rigidly related to the space step size. This is actually irrelevant and explicit or mixed schemes and different step sizes can be treated in the same way. We do not do this just not to overburden the exposition with lots of details of relatively minor importance. The same should be said about zero boundary data.

3. An auxiliary result. In the following theorem G can be just any bounded domain. Below by $C^{1,2}(Q)$ we mean the space of functions $u = u(t,x)$ which are bounded and continuous in $[0,T)\times G$ along with their derivatives $\partial_t u$, $D_{ij}u$, $D_i u$.

THEOREM 3.1. *Let $u, v \in C^{1,2}(Q)\cap C(\bar{Q})$ be such that for a constant $K \geq 1$*

$$\partial_t u + \max(F[u], P[u] - K) \geq 0 \geq \partial_t v + \min(F[v], -P[-v] + K) \tag{3.1}$$

in Q and $v \geq u$ on $\partial' Q$. Also assume that, for a constant $M \in [1,\infty)$,

$$\|u, v\|_{C^{1+\chi}(Q)} \leq M. \tag{3.2}$$

Then there exist a constant $N \in (0,\infty)$, depending only on τ, the diameter of G, d, K_0, and δ, and a constant $\eta \in (0,1)$, depending only on τ, d, and δ, such that, if $K \geq NM^{1/\eta}$ and

$$K \geq T^{-1}, \quad G_{2/\sqrt{K}} \neq \emptyset, \tag{3.3}$$

then in Q

$$u(t,x) - v(t,x) \leq NK^{-\chi/4} + NM\omega(M^{-1/\tau}K^{-1}). \tag{3.4}$$

REMARK 3.1. Observe that for $\omega = t^\tau$ estimate (3.4) becomes $u - v \leq NK^{-\chi/4} + NK^{-\tau}$.

To prove this theorem, we are going to adapt to our situation an argument from Section 5.A of [2]. For that we need a construction and two lemmas. From the start throughout the section we will only concentrate on K satisfying (3.3).

First we introduce $\psi \in C^2(\mathbb{R}^d)$ as a global barrier for G, that is, in G we have $\psi \geq 1$ and

$$a_{ij}D_{ij}\psi + b_i D_i \psi \leq -1$$

for any $(a_{ij}) \in \mathbb{S}_{\hat{\delta}}$, $|(b_i)| \leq K_0$. Such a ψ can be found in the form $\cosh \mu R - \cosh \mu |x|$ for sufficiently large μ and R.

Then we take and fix a radially symmetric with respect to x function $\zeta = \zeta(t,x)$ of class $C_0^\infty(\mathbb{R}^{d+1})$ with support in $(-1,0)\times B_1$. For $\varepsilon > 0$ we define $\zeta_\varepsilon(t,x) = \varepsilon^{-d-2}\zeta(\varepsilon^{-2}t, \varepsilon^{-1}x)$ and for locally summable $u(t,x)$ introduce

$$u^{(\varepsilon)}(t,x) = u(t,x) * \zeta_\varepsilon(t,x). \tag{3.5}$$

Recall some standard properties of parabolic mollifiers in which no regularity properties of G are required: If $u \in C^{1+\chi}(Q)$, then in Q_ε

$$\varepsilon^{-1-\chi}|u - u^{(\varepsilon)}| + \varepsilon^{-\chi}|Du - Du^{(\varepsilon)}| \leq N\|u\|_{C^{1+\chi}(Q)},$$

$$\begin{aligned} &|u^{(\varepsilon)}| + |Du^{(\varepsilon)}| + \varepsilon^{1-\chi}|D^2u^{(\varepsilon)}| + \varepsilon^{1-\chi}|\partial_t u^{(\varepsilon)}| \\ &+\varepsilon^{2-\chi}|D^3u^{(\varepsilon)}| + \varepsilon^{2-\chi}|D\partial_t u^{(\varepsilon)}| + \varepsilon^{3-\chi}|\partial_t D^2 u^{(\varepsilon)}| \leq N\|u\|_{C^{1+\chi}(Q)}. \end{aligned} \tag{3.6}$$

Define the functions

$$\bar{u} = u/\psi, \quad \bar{v} = v/\psi.$$

Replacing M with NM, if necessary, where N depends only on d and the diameter of G, we may assume that (3.2) holds with $\bar{u}, \bar{v}$ in place of u, v.

Next take constants $\nu, \varepsilon_0 \in (0,1)$, introduce

$$\varepsilon = \varepsilon_0 K^{-(1-\gamma)/(2\gamma)},$$

and consider the function

$$W(t,x,y) = \bar{u}(t,x) - \bar{u}^{(\varepsilon)}(t,x) - [\bar{v}(t,y) - \bar{u}^{(\varepsilon)}(t,y)] - \nu K|x-y|^2$$

for $(t,x),(t,y) \in \bar{Q}_\varepsilon$. Note that $Q_\varepsilon \neq \emptyset$ and even $G_{2\varepsilon} \neq \emptyset$ owing to (3.3) and the fact that $1-\gamma > \gamma$.

Denote by $(\bar{t}, \bar{x}, \bar{y})$ a maximum point of W in $[0, T-\varepsilon^2] \times \bar{G}^2_\varepsilon$. Observe that, obviously,

$$\begin{aligned} &\bar{u}(\bar{t},\bar{x}) - \bar{u}^{(\varepsilon)}(\bar{t},\bar{x}) - [\bar{v}(\bar{t},\bar{y}) - \bar{u}^{(\varepsilon)}(\bar{t},\bar{y})] - \nu K|\bar{x}-\bar{y}|^2 \\ &\geq \bar{u}(\bar{t},\bar{x}) - \bar{u}^{(\varepsilon)}(\bar{t},\bar{x}) - [\bar{v}(\bar{t},\bar{x}) - \bar{u}^{(\varepsilon)}(\bar{t},\bar{x})], \end{aligned}$$

which implies that

$$|\bar{x}-\bar{y}| \leq NM/(\nu K). \tag{3.7}$$

where and below by N with indices or without them we denote various constants depending only on d, K_0, δ, and the diameter of G, unless specifically stated otherwise. By the way recall that χ and, hence, γ depend only on d and δ.

Lemma 3.2. *There exist a constant $\nu \in (0,1)$, depending only on d, K_0, δ, and the diameter of G, and a constant $N \in (0,\infty)$ such that, if*

$$K \geq N\varepsilon_0^{(\chi-1)/\eta_1} M^{1/\eta_1}, \tag{3.8}$$

where $\eta_1 = 1 - (1-\chi)(1-\gamma)/(2\gamma)$ (>0), and $\bar{x}, \bar{y} \in G_\varepsilon$ and $\bar{t} < T - \varepsilon^2$, then

(i) We have

$$2\nu K|\bar{x}-\bar{y}| \leq NM\varepsilon^\chi, \quad |\bar{x}-\bar{y}| \leq \varepsilon/2, \tag{3.9}$$

(ii) For any $\xi, \eta \in \mathbb{R}^d$

$$D_{ij}[\bar{u} - \bar{u}^{(\varepsilon)}](\bar{t},\bar{x})\xi^i\xi^j - D_{ij}[\bar{v} - \bar{u}^{(\varepsilon)}](\bar{t},\bar{y})\eta^i\eta^j \leq 2\nu K|\xi-\eta|^2, \tag{3.10}$$

$$\partial_t[\bar{u} - \bar{u}^{(\varepsilon)}](\bar{t},\bar{x}) \leq \partial_t[\bar{v} - \bar{u}^{(\varepsilon)}](\bar{t},\bar{y}), \tag{3.11}$$

(iii) We have

$$\partial_t \bar{u}(\bar{t},\bar{x}) + \sup_{\alpha\in A}\inf_{\beta\in B}\big[a^{\alpha\beta}_{ij}D_{ij}\bar{u} + \hat{b}^{\alpha\beta}_i D_i\bar{u} - \hat{c}^{\alpha\beta}\bar{u} + \hat{f}^{\alpha\beta}\big](\bar{t},\bar{x}) \geq 0, \tag{3.12}$$

$$\partial_t \bar{v}(\bar{t},\bar{y}) + \sup_{\alpha\in A}\inf_{\beta\in B}\big[a^{\alpha\beta}_{ij}D_{ij}\bar{v} + \hat{b}^{\alpha\beta}_i D_i\bar{v} - \hat{c}^{\alpha\beta}\bar{v} + \hat{f}^{\alpha\beta}\big](\bar{t},\bar{y}) \leq 0, \tag{3.13}$$

where

$$\hat{b}_i^{\alpha\beta} = b_i^{\alpha\beta} + 2a_{ij}^{\alpha\beta}\psi^{-1}D_j\psi, \quad \hat{c}^{\alpha\beta} = -\psi^{-1}L^{\alpha\beta}\psi, \quad \hat{f}^{\alpha\beta} = \psi^{-1}f^{\alpha\beta}.$$

Proof. The first inequality in (3.9) follows from (3.6) and the fact that the first derivatives of W with respect to x vanish at $\bar{x}$, that is, $D(\bar{u} - \bar{u}^{(\varepsilon)})(\bar{x}) = 2\nu K(\bar{x} - \bar{y})$. Also the matrix of second-order derivatives of W with respect to (x, y) is nonpositive at $(\bar{t}, \bar{x}, \bar{y})$ as well as its (at least one sided if $\bar{t} = 0$) derivative with respect to t, which yields (ii).

By taking $\eta = 0$ in (3.10) and using the fact that $|D^2\bar{u}^{(\varepsilon)}| \le NM\varepsilon^{\chi-1}$ we see that

$$D^2\bar{u}(\bar{t}, \bar{x}) \le 2\nu K + NM\varepsilon^{\chi-1}.$$

Furthermore

$$D_{ij}u = \psi D_{ij}\bar{u} + (D_i\psi)D_j\bar{u} + (D_i\bar{u})D_j\psi + (D_{ij}\psi)\bar{u},$$

which implies that

$$D^2u(\bar{t}, \bar{x}) \le N(\nu K + M\varepsilon^{\chi-1}).$$

Similarly,

$$D^2v(\bar{t}, \bar{y}) \ge -N(\nu K + M\varepsilon^{\chi-1}),$$

which yields

$$P[u](\bar{t}, \bar{x}) \le N_1(\nu K + M\varepsilon^{\chi-1}), \quad -P[-v](\bar{t}, \bar{y}) \ge -N_1(\nu K + M\varepsilon^{\chi-1}),$$

$$F[u](\bar{t}, \bar{x}) \le N_1(\nu K + M\varepsilon^{\chi-1}), \quad F[v](\bar{t}, \bar{y}) \ge -N_1(\nu K + M\varepsilon^{\chi-1}). \tag{3.14}$$

Also it follows from (3.11) and (3.6) that

$$\partial_t u(\bar{t}, \bar{x}) \le \partial_t v(\bar{t}, \bar{y}) + N_2 M\varepsilon^{\chi-1}. \tag{3.15}$$

Now, if $F[u](\bar{t}, \bar{x}) \le P[u](\bar{t}, \bar{x}) - K$, then at $(\bar{t}, \bar{x})$

$$0 \le \partial_t u + \max(F[u], P[u] - K) \le \partial_t u + N_1(\nu K + M\varepsilon^{\chi-1}) - K,$$

$$\partial_t u \ge K - N_1(\nu K + M\varepsilon^{\chi-1})$$

and at $(\bar{t}, \bar{y})$

$$0 \ge \partial_t v + \min(F[v], -P[-v] + K) \ge K - 2N_1(\nu K + M\varepsilon^{\chi-1}) - N_2 M\varepsilon^{\chi-1}.$$

This is impossible if we choose and fix ν such that

$$(4N_1 + N_2)\nu \le 1/2, \tag{3.16}$$

since, as is easy to see, $M\varepsilon^{\chi-1} \le \nu K$ for K satisfying (3.8) and appropriate N, so that

$$2N_1(\nu K + M\varepsilon^{\chi-1}) + N_2 M\varepsilon^{\chi-1} \le 4N_1\nu K + N_2\nu K \le K/2.$$

It follows that $F[u](\bar t,\bar x)\geq P[u](\bar t,\bar x)-K$,

$$\partial_t u(\bar t,\bar x)+F[u](\bar t,\bar x)\geq 0,$$

and this proves (3.12).

Similarly, if $-P[-v](\bar t,\bar y)+K\leq F[v](\bar t,\bar y)$, then at $(\bar t,\bar y)$

$$0\geq\partial_t v-N_1(\nu K+M\varepsilon^{\chi-1})+K,$$

and at $(\bar t,\bar x)$ we have $\partial_t u\leq N_1(\nu K+M\varepsilon^{\chi-1})-K+N_2M\varepsilon^{\chi-1}$ and

$$0\leq\partial_t u+\max(F[u],P[u]-K)\leq -K+2N_1(\nu K+M\varepsilon^{\chi-1})+N_2M\varepsilon^{\chi-1},$$

which again is impossible with the above choice of ν for K satisfying (3.8). This yields (3.13).

Moreover, not only $M\varepsilon^{\chi-1}\leq\nu K$ for K satisfying (3.8), but we also have $NM\varepsilon^{\chi-1}\leq\nu K$, where N is taken from (3.9), if we increase N in (3.8). This yields the second inequality in (3.9).

The lemma is proved.

LEMMA 3.3. *For any $\mu\geq 0$, there exist a constant $\eta>0$, depending only on d and δ, and a constant N such that, if $K\geq NM^{1/\eta}$ and*

$$W(\bar t,\bar x,\bar y)\geq 2K^{-\chi/4}+\mu M\omega(M^{-1/\tau}K^{-1}), \tag{3.17}$$

then

$$\bar u(\bar t,\bar x)-\bar v(\bar t,\bar y)-\nu K|\bar x-\bar y|^2\geq K^{-\chi/4}+\mu M\omega(M^{-1/\tau}K^{-1}). \tag{3.18}$$

Furthermore, $\bar x,\bar y\in G_{2\varepsilon}$ and $\bar t<T-\varepsilon^2$.

Proof. It follows from (3.7) that (recall that ν is already fixed)

$$|\bar u^{(\varepsilon)}(\bar t,\bar x)-\bar u^{(\varepsilon)}(\bar t,\bar y)|\leq M|\bar x-\bar y|\leq NM^2/K. \tag{3.19}$$

Hence we have from (3.17) that

$$\bar u(\bar t,\bar x)-\bar v(\bar t,\bar y)-\nu K|\bar x-\bar y|^2\geq 2K^{-\chi/4}-NM^2/K+\mu M\omega(M^{-1/\tau}K^{-1}),$$

and (3.18) follows provided that

$$NM^2/K\leq(1/4)K^{-\chi/4}, \tag{3.20}$$

which indeed holds if

$$K\geq NM^{1/\eta_2}, \tag{3.21}$$

with $\eta_2=(4-\chi)/8>0$. Here if $\bar x$ or $\bar y$ are in $\bar G_\varepsilon\setminus G_{2\varepsilon}$, then for appropriate $\hat x\in\partial G$ and $\hat y\in\partial G$ either

$$\bar u(\bar t,\bar x)-\bar v(\bar t,\bar y)\leq 2M\varepsilon+\bar v(\bar t,\hat x)-\bar v(\bar t,\bar y)\leq M(4\varepsilon+|\bar x-\bar y|)$$

or

$$\bar u(\bar t,\bar x)-\bar v(\bar t,\bar y)\leq\bar u(\bar t,\bar x)-\bar u(\bar t,\hat y)+\bar v(\bar t,\hat y)-\bar v(\bar t,\bar y)\leq M(4\varepsilon+|\bar x-\bar y|).$$

In any case in light of (3.18)

$$4\varepsilon M + NM^2/K - \nu K|\bar{x}-\bar{y}|^2 \geq K^{-\chi/4}. \tag{3.22}$$

Notice that (we need the following with a constant N for the future)

$$NM\varepsilon \leq (1/4)K^{-\chi/4} \tag{3.23}$$

for

$$K \geq (4NM)^{1/\eta_3}, \tag{3.24}$$

where $\eta_3 := (1-\gamma)/(2\gamma) - \chi/4 > 0$. This and (3.20) show that (3.22) is impossible for

$$K \geq N(M^{1/\eta_2} + M^{1/\eta_3}). \tag{3.25}$$

Below we assume (3.25) and for such K, we have $\bar{x}, \bar{y} \in G_\varepsilon$.

Furthermore, if $\bar{t} = T - \varepsilon^2$, then (recall (3.19) and that $\bar{u} \leq \bar{v}$ on $\partial' Q$)

$$W(\bar{t},\bar{x},\bar{y}) \leq NM\varepsilon^{1+\chi} + \bar{v}(\bar{t},\bar{x}) - \bar{v}(\bar{t},\bar{y}) + NM^2/K$$

$$\leq NM\varepsilon + M|\bar{x}-\bar{y}| + NM^2/K \leq NM^2/K + NM\varepsilon,$$

which is less than $K^{-\chi/4}$ (cf. (3.23) and (3.20)). This is impossible due to (3.17). Hence, $\bar{t} < T - \varepsilon^2$ and this finishes the proof of the lemma.

Proof of Theorem 3.1. Fix a (large) constant $\mu > 0$ to be specified later as a constant, depending only on d, K_0, δ, and the diameter of G, take ν from Lemma 3.2 and first assume that

$$W(t,x,y) \leq 2K^{-\chi/4} + \mu M\omega(M^{-1/\tau}K^{-1})$$

for $(t,x),(t,y) \in \bar{Q}_\varepsilon$. Observe that for any point $(t,x) \in \bar{Q}$ one can find a point $(s,y) \in \bar{Q}_\varepsilon$ with $|x-y| \leq \varepsilon$ and $|t-s| \leq \varepsilon^2$ and then

$$\bar{u}(t,x) - \bar{v}(t,x) \leq \bar{u}(s,y) - \bar{v}(s,y) + NM\varepsilon$$

$$\leq W(s,y,y) + NM\varepsilon \leq 2K^{-\chi/4} + NM\varepsilon + \mu M\omega(M^{-1/\tau}K^{-1}).$$

In that case, owing to (3.23), (3.4) holds for K satisfying (3.24).

It is clear now that, to prove the theorem, it suffices to find μ such that the inequality (3.17) is impossible if $K \geq NM^{1/\eta}$ with N and η as in the statement of the theorem. Of course, we will argue by contradiction and suppose that (3.17) holds.

Upon combining Lemmas 3.2 and 3.3 we see that there exist $\eta_0 \in (0,1)$, depending only on d and δ, and $N \in (0,\infty)$ such that, if

$$K \geq N\varepsilon_0^{(\chi-1)/\eta_0} M^{1/\eta_0}, \tag{3.26}$$

then (3.18) holds, $\bar{x}, \bar{y} \in G_\varepsilon$ and $\bar{t} < T - \varepsilon^2$, so that we can use the conclusions of Lemma 3.2.

By denoting $\sigma^{\alpha\beta} = (a^{\alpha\beta})^{1/2}$ we may write

$$a^{\alpha\beta}_{ij} = \sigma^{\alpha\beta}_{ik}\sigma^{\alpha\beta}_{jk},$$

and then (3.10) for $\xi^i = \sigma^{\alpha\beta}_{ik}(\bar t, \bar x)$ and $\eta^i = \sigma^{\alpha\beta}_{ik}(\bar t, \bar y)$ implies that

$$a^{\alpha\beta}_{ij}(\bar t, \bar x)D_{ij}\bar u(\bar t, \bar x) \le a^{\alpha\beta}_{ij}(\bar t, \bar y)D_{ij}\bar v(\bar t, \bar y) + I + J,$$

where

$$I = a^{\alpha\beta}_{ij}(\bar t, \bar x)D_{ij}\bar u^{(\varepsilon)}(\bar t, \bar x) - a^{\alpha\beta}_{ij}(\bar t, \bar y)D_{ij}\bar u^{(\varepsilon)}(\bar t, \bar y),$$

$$J = 2\nu K \sum_{i,k=1}^{d} |\sigma^{\alpha\beta}_{ik}(\bar t, \bar x) - \sigma^{\alpha\beta}_{ik}(\bar t, \bar y)|^2.$$

Since $a^{\alpha\beta}$ is uniformly nondegenerate, its square root possesses the same smoothness properties as $a^{\alpha\beta}$ and

$$J \le NK|\bar x - \bar y|^{2\gamma}.$$

We now use (3.9) to get that

$$|\bar x - \bar y|^{2\gamma} \le (M/K)^{2\gamma} N\varepsilon_0^{2\gamma\chi} K^{-\chi(1-\gamma)}.$$

It turns out that

$$-2\gamma - \chi(1-\gamma) = -1 - \chi/4.$$

It follows that (recall that $M \ge 1$)

$$|\bar x - \bar y|^{2\gamma} \le NM^2 K^{-1-\chi/4}\varepsilon_0^{2\gamma\chi},$$

$$J \le NM^2 K^{-\chi/4}\varepsilon_0^{2\gamma\chi}.$$

Also note that

$$\begin{aligned} I &\le [a^{\alpha\beta}_{ij}(\bar t, \bar x) - a^{\alpha\beta}_{ij}(\bar t, \bar y)]D_{ij}\bar u^{(\varepsilon)}(\bar t, \bar x) + N|D^2\bar u^{(\varepsilon)}(\bar t, \bar x) - D^2\bar u^{(\varepsilon)}(\bar t, \bar y)| \\ &\le NM\varepsilon^{\chi-1}|\bar x - \bar y|^{\gamma} + NM\varepsilon^{\chi-2}|\bar x - \bar y| =: I_1 + I_2, \end{aligned}$$

where the last inequality is obtained by the mean-value theorem relying on the fact that $|\bar x - \bar y| \le \varepsilon/2$ and $\bar x, \bar y \in G_{2\varepsilon}$, so that the straight segment connecting these points lies inside G_ε. By looking at the estimate of J we get

$$I_1 \le NM^2\varepsilon^{\chi-1}K^{-1/2-\chi/8}.$$

An easy computation shows that

$$1/2 + \chi/8 - (1-\chi)(1-\gamma)/(2\gamma) = \chi/4 + \theta_1,$$

where $\theta_1 = (1-\gamma)(8\gamma)^{-1}\chi > 0$. Hence,

$$I_1 \le NM^2K^{-\chi/4}[\varepsilon_0^{\chi-1}K^{-\theta_1}].$$

Next,

$$I_2 \le NM\varepsilon_0^{2\chi-2}(M/K)K^{(2-2\chi)(1-\gamma)/(2\gamma)}.$$

Here,

$$(2-2\chi)(1-\gamma)/(2\gamma)-1=-\chi/4-2\theta_1,$$

so that

$$I_2 \le NM^2K^{-\chi/4}[\varepsilon_0^{2\chi-2}K^{-2\theta_1}].$$

It turns out that for

$$K \ge \varepsilon_0^{-\theta_2}, \tag{3.27}$$

where $\theta_2=(2\gamma\chi-\chi+1)/\theta_1$ we have $\varepsilon_0^{\chi-1}K^{-\theta_1},\varepsilon_0^{2\chi-2}K^{-2\theta_1}\le\varepsilon_0^{2\gamma\chi}$ and hence

$$J, I_1, I_2 \le NM^2K^{-\chi/4}\varepsilon_0^{2\gamma\chi}. \tag{3.28}$$

Also (3.11) reads

$$\partial_t\bar u(\bar t,\bar x)\le\partial_t\bar v(\bar s,\bar y)+\partial_t\bar u^{(\varepsilon)}(\bar t,\bar x)-\partial_t\bar u^{(\varepsilon)}(\bar t,\bar y)$$

and as is easy to see

$$|\partial_t\bar u^{(\varepsilon)}(\bar t,\bar x)-\partial_t\bar u^{(\varepsilon)}(\bar t,\bar y)|\le NI_2.$$

It follows that, for K satisfying (3.27) (along with (3.26) and, of course, (3.3)), we have

$$\begin{aligned}&\partial_t\bar u(\bar t,\bar x)+a_{ij}^{\alpha\beta}(\bar t,\bar x)D_{ij}\bar u(\bar t,\bar x)\\&\le\partial_t\bar v(\bar t,\bar y)+a_{ij}^{\alpha\beta}(\bar t,\bar y)D_{ij}\bar v(\bar t,\bar y)+NM^2K^{-\chi/4}\varepsilon_0^{2\gamma\chi}.\end{aligned} \tag{3.29}$$

Next,

$$D_i\bar u(\bar t,\bar x)=2\nu K(\bar x^i-\bar y^i)+D_i\bar u^{(\varepsilon)}(\bar t,\bar x),$$

$$D_i\bar v(\bar t,\bar y)=2\nu K(\bar x^i-\bar y^i)+D_i\bar u^{(\varepsilon)}(\bar t,\bar y),$$

$$D_i\bar u(\bar t,\bar x)-D_i\bar v(\bar t,\bar y)=D_i\bar u^{(\varepsilon)}(\bar t,\bar x)-D_i\bar u^{(\varepsilon)}(\bar t,\bar y),$$

where

$$|D\bar u^{(\varepsilon)}(\bar t,\bar x)-D\bar u^{(\varepsilon)}(\bar t,\bar y)|\le NM\varepsilon^{\chi-1}|\bar x-\bar y|=N\varepsilon I_2.$$

This and the rough estimate

$$|\bar x-\bar y|\le N\varepsilon_0^{\chi}MK^{-1}$$

following from (3.9) lead to

$$\hat b_i^{\alpha\beta}D_i\bar u(\bar t,\bar x)-\hat b_i^{\alpha\beta}D_i\bar v(\bar t,\bar y)\le NM^2K^{-\chi/4}\varepsilon_0^{2\gamma\chi}+NM\omega(N\varepsilon_0^{\chi}MK^{-1}).$$

Finally,

$$\hat f^{\alpha\beta}(\bar t,\bar x)\le \hat f^{\alpha\beta}(\bar t,\bar y)+NM\omega(N\varepsilon_0^{\chi}MK^{-1}),$$

$$\begin{aligned}-\hat c^{\alpha\beta}\bar u(\bar t,\bar x)+\hat c^{\alpha\beta}\bar v(\bar t,\bar y)&=-\hat c^{\alpha\beta}(\bar t,\bar x)[\bar u(\bar t,\bar x)-\bar v(\bar t,\bar y)]+\bar v(\bar t,\bar y)[\hat c^{\alpha\beta}(\bar t,\bar y)-\hat c^{\alpha\beta}(\bar t,\bar x)]\\&\le -\bar c[\bar u(\bar t,\bar x)-\bar v(\bar t,\bar y)]+NM\omega(N\varepsilon_0^{\chi}MK^{-1}),\end{aligned}$$

where $\bar c=\inf_G\psi^{-1}$ and the last inequality follows from the fact that $\hat c^{\alpha\beta}\ge\bar c$ and $\bar u(\bar t,\bar x)-\bar v(\bar t,\bar y)\ge 0$ (see (3.18)).

Therefore we infer from (3.12), (3.13), and the last estimates that

$$\begin{aligned}0\le \sup_{\alpha\in A}\inf_{\beta\in B}&\big[a_{ij}^{\alpha\beta}D_{ij}\bar v+\hat b_i^{\alpha\beta}D_i\bar v-\hat c^{\alpha\beta}\bar v+f^{\alpha\beta}\big)(\bar t,\bar y)\big]\\&-\bar c[\bar u(\bar t,\bar x)-\bar v(\bar t,\bar y)]+N_1M^2\varepsilon_0^{2\gamma\chi}K^{-\chi/4}+N_1M\omega(N\varepsilon_0^{\chi}MK^{-1}),\end{aligned}$$

$$\bar u(\bar t,\bar x)-\bar v(\bar t,\bar y)\le N_1M^2\varepsilon_0^{2\gamma\chi}K^{-\chi/4}+N_1M\omega(N_1\varepsilon_0^{\chi}MK^{-1}).$$

We can certainly assume that $N_1\ge 1$. Then we take $\mu=2N_1$ and take $\kappa\in(0,1)$, depending only on τ, δ, and d, and $\xi\in(0,\infty)$, depending only on τ, K_0, d, δ, and the diameter of G, such that for $\varepsilon_0=\xi M^{-1/\kappa}$ and all $M\ge 1$ we have

$$N_1M^2\varepsilon_0^{2\gamma\chi}\le 1/2,\quad N_1\varepsilon_0^{\chi}M\le M^{-1/\tau}.$$

Then we arrive at a contradiction with (3.18) and, since now (3.27) and (3.26) are satisfied if $K\ge NM^{1/\eta}$ for appropriate η and N, the theorem is proved.

4. Proof of Theorem 2.4. In light of Lemma 2.2 it suffices to prove (2.5) only for $K\ge N$, where N depends only on τ, d, K_0, δ, and Q and satisfies (3.3) with N in place of K.

Fix a sufficiently small $\varepsilon_0>0$ such that $Q_{\varepsilon_0}\ne\emptyset$ and for $\varepsilon\in(0,\varepsilon_0]$ define

$$\xi_{\varepsilon,K}=\partial_t u_K^{(\varepsilon)}+\max(F[u_K^{(\varepsilon)}],P[u_K^{(\varepsilon)}]-K),$$

$$\xi_{\varepsilon,-K}=\partial_t u_{-K}^{(\varepsilon)}+\min(F[u_{-K}^{(\varepsilon)}],-P[-u_{-K}^{(\varepsilon)}]+K),$$

in Q_{ε_0}, where we use notation (3.5). Since the second-order derivatives with respect to x and the first derivatives with respect to t of $u_{\pm K}$ are bounded in Q_{ε_0}, we have $\xi_{\varepsilon,\pm K}\to 0$ as $\varepsilon\downarrow 0$ in any $\mathcal{L}_p(Q_{\varepsilon_0})$ for any K. Furthermore, $\xi_{\varepsilon,\pm K}$ are continuous. Therefore, there exist smooth functions $\zeta_{\varepsilon,K}$ such that

$$-\varepsilon\le\zeta_{\varepsilon,K}-\min(\xi_{\varepsilon,K},-\xi_{\varepsilon,-K})\le 0$$

in G_{ε_0}.

By Theorem 6.4.1 of [7], for any subdomain G' of G_{ε_0} of class C^3, there exists a unique $w_{\varepsilon,K}\in C^{1,2}([0,T-\varepsilon_0^2)\times G')\cap C([0,T-\varepsilon_0^2]\times\bar G')$ satisfying

$$\partial_t w_{\varepsilon,K}+\sup_{a\in\mathbb{S}_{\hat\delta},|b|\le K_0}[a_{ij}D_{ij}w_{\varepsilon,K}+b_iD_iw_{\varepsilon,K}]=\zeta_{\varepsilon,K}\tag{4.1}$$

in $[0,T-\varepsilon_0^2)\times G'$ with zero boundary condition on the parabolic boundary of $[0,T-\varepsilon_0^2)\times G'$. Obviously,

$$\begin{aligned}
&\partial_t(u_K^{(\varepsilon)}-w_{\varepsilon,K})+\max(F[u_K^{(\varepsilon)}-w_{\varepsilon,K}],P[u_K^{(\varepsilon)}-w_{\varepsilon,K}]-K)\\
&\geq \partial_t u_K^{(\varepsilon)}+\max(F[u_K^{(\varepsilon)}],P[u_K^{(\varepsilon)}]-K)-\partial_t w_{\varepsilon,K}\\
&\quad -\sup_{a\in\mathbb{S}_{\hat\delta},|b|\leq K_0}[a_{ij}D_{ij}w_{\varepsilon,K}+b_iD_iw_{\varepsilon,K}]\\
&=\xi_{\varepsilon,K}-\zeta_{\varepsilon,K}\geq 0,
\end{aligned}$$

$$\begin{aligned}
&\partial_t(u_{-K}^{(\varepsilon)}+w_{\varepsilon,K})+\min(F[u_{-K}^{(\varepsilon)}+w_{\varepsilon,K}],-P[-u_{-K}^{(\varepsilon)}-w_{\varepsilon,K}]+K)\\
&\leq \partial_t u_{-K}^{(\varepsilon)}+\min(F[u_{-K}^{(\varepsilon)}],-P[-u_{-K}^{(\varepsilon)}]+K)+\partial_t w_{\varepsilon,K}\\
&\quad +\sup_{a\in\mathbb{S}_{\hat\delta},|b|\leq K_0}[a_{ij}D_{ij}w_{\varepsilon,K}+b_iD_iw_{\varepsilon,K}]\\
&=\xi_{\varepsilon,-K}+\zeta_{\varepsilon,K}\leq 0
\end{aligned}$$

in $[0,T-\varepsilon_0^2)\times G')$. After setting

$$\mu_{\varepsilon,K}=\sup_{\partial'([0,T-\varepsilon_0^2)\times G')}(u_K^{(\varepsilon)}-u_{-K}^{(\varepsilon)}-2w_{\varepsilon,K})_+$$

we conclude by Theorem 3.1 applied to $u_K^{(\varepsilon)}-w_{\varepsilon,K}$ and $u_{-K}^{(\varepsilon)}+w_{\varepsilon,K}+\mu_{\varepsilon,K}$ in place of u and v, respectively, that there exist a constant $N\in(0,\infty)$, depending only on τ, the diameter of G, d, K_0, and δ, and a constant $\eta\in(0,1)$, depending only on τ, d, and δ, such that, if $K\geq NM_{\varepsilon,K}^{1/\eta}$, then

$$u_K^{(\varepsilon)}-u_{-K}^{(\varepsilon)}\leq\mu_{\varepsilon,K}+2w_{\varepsilon,K}+NK^{-\chi/4}+NM\omega(M^{-1/\tau}K^{-1})$$

in $[0,T-\varepsilon_0^2)\times G')$, where $M_{\varepsilon,K}\geq 1$ is any number satisfying

$$M_{\varepsilon,K}\geq\|u_K^{(\varepsilon)}-w_{\varepsilon,K},u_{-K}^{(\varepsilon)}+w_{\varepsilon,K}+\mu_{\varepsilon,K}\|_{C^{1+\chi}([0,T-\varepsilon_0^2)\times G'))}.$$

First we discuss what is happening as $\varepsilon\downarrow 0$. By the $W_p^{1,2}$-theory (see, for instance, Theorem 1.1 of [5]) $w_{\varepsilon,K}\to 0$ in any $W_p^{1,2}$, which by embedding theorems implies that $w_{\varepsilon,K}\to 0$ in $C^{1+\chi}([0,T-\varepsilon_0^2)\times G'))$. Obviously, the constants $\mu_{\varepsilon,K}$ converge in $C^{1+\chi}([0,T-\varepsilon_0^2)\times G'))$ to

$$\sup_{\partial'([0,T-\varepsilon_0^2)\times G')}(u_K-u_{-K})_+.$$

Now Theorem 2.3 implies that for sufficiently small ε one can take $N\varepsilon_0^{-1-\chi}$ as $M_{\varepsilon,K}$, where N depends only on d, δ, G, and K_0. Thus for sufficiently small ε, if $K\geq N\varepsilon_0^{-(1+\chi)/\eta}$, then

$$u_K^{(\varepsilon)}-u_{-K}^{(\varepsilon)}\leq\mu_{\varepsilon,K}+2w_{\varepsilon,K}+NK^{-\chi/4}+N\varepsilon_0^{-1-\chi}\omega(\varepsilon_0^{(1+\chi)/\tau}K^{-1})$$

in $[0,T-\varepsilon_0^2)\times G'$, which after letting $\varepsilon\downarrow 0$ yields

$$u_K-u_{-K}\leq NK^{-\chi/4}+N\varepsilon_0^{-1-\chi}\omega(\varepsilon_0^{(1+\chi)/\tau}K^{-1})$$

in $[0,T-\varepsilon_0^2)\times G'$. The arbitrariness of G' and Lemma 2.2 now allow us to conclude that for any $\varepsilon_0>0$, for which $Q_{\varepsilon_0}\neq\emptyset$,

$$u_K-u_{-K}\leq NK^{-\chi/4}+N\varepsilon_0^{-1-\chi}\omega(\varepsilon_0^{(1+\chi)/\tau}K^{-1})+\sup_{Q\setminus Q_{\varepsilon_0}}(u_K-u_{-K})_+$$

$$\leq NK^{-\chi/4}+N\varepsilon_0^{-1-\chi}\omega(\varepsilon_0^{(1+\chi)/\tau}K^{-1})+N\varepsilon_0 \tag{4.2}$$

in Q provided that

$$K\geq N_1\varepsilon_0^{-(1+\chi)/\eta}. \tag{4.3}$$

This obviously proves the first assertion of the theorem because as is noted in the proof of Theorem 2.5 we have $u_{-K}\leq u_K$.

To prove the second assertion observe that we can certainly assume that (3.3) holds with N_1 in place of K and note that for $\omega=t^\tau$ and $\varepsilon_0=K^{-\eta/(1+\chi)}$ condition (4.3) becomes $K\geq N_1$ and (4.2) becomes

$$u_K-u_{-K}\leq NK^{-\chi/4}+NK^{-\tau}+NK^{-\eta/(1+\chi)}.$$

This yields the desired result and proves the theorem.

5. Proof of assertion (ii) of Theorem 2.5. First we prove uniqueness. Let w be a continuous in $\bar{Q}$ viscosity solution of $\partial_t w+F[w]=0$ with boundary data g. Observe that in the notation from Sections 3 and 4 we have

$$\partial_t u_K^{(\varepsilon)}+F[u_K^{(\varepsilon)}+\bar{w}_{\varepsilon,K}+\kappa\psi]<0$$

in $[0,T-\varepsilon^2)\times G'$ for any $\kappa>0$, where $\bar{w}_{\varepsilon,K}$ is a solution of class $C^{1,2}([0,T-\varepsilon_0^2)\times G')\cap C([0,T-\varepsilon_0^2]\times\bar{G}')$ of (4.1) in the domain $[0,T-\varepsilon_0^2)\times G'$ with zero condition on its parabolic boundary and with a smooth $\zeta_{\varepsilon,K}$ satisfying

$$-\varepsilon\leq\xi_{\varepsilon,K}+\zeta_{\varepsilon,K}\leq 0.$$

This and the definition of viscosity solution imply that the minimum of $u_K^{(\varepsilon)}+\bar{w}_{\varepsilon,K}+\kappa\psi-w$ in $[0,T-\varepsilon^2]\times\bar{G}'$ is either positive or is attained on the parabolic boundary of $[0,T-\varepsilon^2)\times G'$. The same conclusion holds after letting $\varepsilon,\kappa\downarrow 0$ and replacing G' with G_{ε_0}. Hence, in Q

$$u_K-w\geq-\sup_{Q\setminus Q_{\varepsilon_0}}|u_K-w|,$$

which after letting $\varepsilon_0\downarrow 0$ and then $K\to\infty$ yields $w\leq v$. By comparing v with u_{-K} we get $w\geq v$, and hence uniqueness.

To prove that v is a viscosity solution we need the following Lemma 6.1 of [11] derived there from Theorem 3.1 of [6] or Theorem 3.3.9 of [7]. Introduce

$$F_0(u_{ij},t,x)=F(u_{ij},Dv(t,x),v(t,x),t,x)$$

$$C_r(t,x)=(0,r^2)\times\{y\in\mathbb{R}^d:|y|<r\}+(t,x),\quad C_r=C_r(0,0).$$

LEMMA 5.1. *There is a constant N, depending only on d and δ, such that for any $C_r(t,x)$ satisfying $C_r(t,x)\subset Q$ and $\phi\in W^{1,2}_{d+1}(C_r(t,x))\cap C(\bar{C}_r(t,x))$ we have on $\bar{C}_r(t,x)$ that*

$$v\leq\phi+Nr^{d/(d+1)}\|(\partial_t\phi+F_0[\phi])_+\|_{L_{d+1}(C_r(t,x))}+\max_{\partial' C_r(t,x)}(v-\phi)_+. \tag{5.1}$$

$$v \geq \phi - Nr^{d/(d+1)} \|(\partial_t \phi + F_0[\phi])_-\|_{L_{d+1}(C_r(t,x))} - \max_{\partial' C_r(t,x)} (v - \phi)_-. \tag{5.2}$$

Now let $\phi \in C^{1,2}([0,T] \times \bar{G})$ and suppose that $v - \phi$ attains a local maximum at $(t_0, x_0) \in [0,T) \times G$. Without losing generality we may assume that $t_0 = 0, x_0 = 0$, $w(0) - \phi(0) = 0$. Then for $\varepsilon > 0$ and all small $r > 0$ for

$$\phi_{\varepsilon,r}(t,x) = \phi(t,x) + \varepsilon(|x|^2 + t - r^2)$$

we have that

$$\max_{\partial' C_r} (v - \phi_{\varepsilon,r})_+ = 0.$$

Hence, by Lemma 5.1

$$\varepsilon r^2 = (v - \phi_{\varepsilon,r})(0) \leq Nr^{d/(d+1)} \|(\partial_t \phi_{\varepsilon,r} + F_0[\phi_{\varepsilon,r}])_+\|_{L_{d+1}(C_r)}.$$

It follows that

$$\sup_{C_r} [\partial_t \phi_{\varepsilon,r} + F_0[\phi_{\varepsilon,r}]] > 0,$$

which by letting first $r \downarrow 0$ and then $\varepsilon \downarrow 0$ yields

$$0 \leq \partial_t \phi(0) + F_0[\phi](0) = \partial_t \phi(0) + F(D_{ij}\phi(0), D_i\phi(0), \phi(0), 0),$$

where the equality follows from the fact that at 0 the derivatives of $v - \phi$ with respect to x vanish. We have just proved that v is a viscosity subsolution.

Similarly by using (5.2) one proves that v is a viscosity supersolution. This proves the theorem.

6. Proof of Theorem 2.6. We start with the boundary behavior of v_h.

LEMMA 6.1. *There exist a constant N such that for all sufficiently small $h > 0$ we have $|v_h| \leq N\rho$ in $Q_{(h)}$.*

This lemma is easily proved by using standard barriers (see, for instance, Lemma 8.8 in [8]).

We also need the following combination of Theorems 1.9 and 2.3 of [9], which provide a parabolic version of the Fang Hua Lin estimate and in which by $\mathfrak{L}_{\delta,K_0}$ we denote the set of parabolic operators of the form

$$L = \partial_t + a^{ij}(t,x)D_{ij} + b^i(t,x)D_i - c(t,x), \tag{6.1}$$

with the coefficients satisfying Assumptions 2.1 (ii) and (iv) and with $c \geq 0$. Recall that C_r are introduced before Lemma 5.1.

THEOREM 6.2. *Let $u \in C(\bar{C}_1) \cap W^{1,2}_{d+1,loc}(C_1)$. Then there are constants $\theta_0 \in (0,1]$ and N, depending only on δ, d, and K_0, such that for any $\theta \in (0, \theta_0]$ and $L \in \mathfrak{L}_{\delta,K_0}$ we have*

$$\int_{C_1} [|D^2u|^\theta + |Du|^\theta] \, dxdt \leq N \sup_{\partial' C_1} |u|^\theta + N \left(\int_{C_1} |Lu|^{d+1} \, dxdt \right)^{\theta/(d+1)}. \tag{6.2}$$

In [9] estimate (6.2) is derived only with $\theta = \theta_0$. For $\theta \in (0, \theta_0]$ it is obtained by using Hölder's inequality.

COROLLARY 6.3. *There exists a constant $\theta_0 \in (0,1]$, depending only on δ, K_0, d, and G, and there exists a constant N, depending only on δ, K_0, d, T, and G, such that for any $L \in \mathfrak{L}_{\delta,K_0}$, $\theta \in (0,\theta_0]$, and $u \in W^{1,2}_{d,loc}(Q) \cap C(\bar{Q})$ we have*

$$\int_Q [|D^2u|^\theta + |Du|^\theta]\,dxdt \le N\|Lu\|^\theta_{L_{d+1}(Q)} + N\sup_{\partial' Q}|u|^\theta. \tag{6.3}$$

Indeed, one can represent $\bar{Q}$ as the finite union of the closures of cylinders of height one in the t variable with bases of class C^2 each of which (bases) admits a one-to-one C^2 mapping on $\{x : |x| < 1\}$ with C^2 inverse. Then after changing coordinates one can use Theorem 6.2 applied to appropriately changed operator L. For the transformed operator the constants δ and K_0 may change but still will only depend on $\delta, K_0, d,$ and G. Then after combining the results of application of Theorem 6.2 one obtains (6.3) with $\bar{Q}$ in place of $\partial' Q$. However, the parabolic Alexandrov estimate shows that this replacement can be avoided on the account of, perhaps, increasing the first N on the right in (6.3).

The following result is one of our main technical tools. Everywhere below by N we denote generic constants independent of $\varepsilon, \varepsilon_0, K$, h, and the arguments of functions under consideration.

THEOREM 6.4. *There exist constants $N \in (0,\infty)$ and $\theta_1 \in (0,1)$ such that, for all sufficiently small $\varepsilon_0 > 0$, for any $\varepsilon \in [0, \varepsilon_0/2]$, $K \ge 1$ and $|l| = 1$, we have*

$$\|u_K(\varepsilon^2 + \cdot, \varepsilon l + \cdot) - u_K\|^{d+1}_{W^{1,2}_{d+1}(Q_{\varepsilon_0})} \le N\varepsilon^{\theta_1}(K\varepsilon_0^{-1})^{d+1}. \tag{6.4}$$

To specify what we mean by "sufficiently small $\varepsilon_0 > 0$" let us say that a number $\varepsilon > 0$ is sufficiently small if Corollary 6.3 holds with the same θ_0, a constant N which is twice the constant from (6.3), and with Q_ε in place of Q. The fact that the set of sufficiently small ε contains $[0,\alpha)$ with $\alpha > 0$ follows from the way Corollary 6.3 is proved and from the fact that the boundaries of G_ε have the same regularity as that of G if ε is small enough.

Proof. Set $w := u_K(\varepsilon^2 + \cdot, \varepsilon l + \cdot) - u_K$ and observe that the $\mathcal{L}_{d+1}(Q_{\varepsilon_0})$-norm of w and its sup norm are easily estimated since $|\partial_t u_K| + |Du_K| \le NK$ in Q. Next, as is easy to see, there is an operator $L \in \mathfrak{L}_{\hat{\delta},K_0}$ such that

$$Lw + f = 0 \tag{6.5}$$

in Q_ε, where

$$\begin{aligned} f(t,x) &= \max(F[u_K], P[u_K] - K)(s,y) \\ &\quad - \max(F(D^2u_K(s,y), Du_K(s,y), u_K(s,y), t, x), P[u_K](s,y) - K), \end{aligned}$$

$s = \varepsilon^2 + t$, $y = \varepsilon l + x$. It follows that the estimate of $\partial_t w$ can be obtained from (6.5) once D^2w, Dw, and f are properly estimated. By the way, since $|D^2u_K|, |Du_K| \le NK\varepsilon_0^{-1}$ in $Q_{\varepsilon_0/2}$ and the data a, b, c, f are Hölder continuous we have that $|f| \le N\varepsilon^{\tau_1}(K\varepsilon_0^{-1})$ in Q_{ε_0}, where $\tau_1 = \min(\tau, 2\gamma_t, \gamma)$.

We now apply Corollary 6.3 to Q_{ε_0} and w in place of Q and u, respectively. We also use the inequalities like $|u|^{d+1} \le |u|^{\theta_0}\sup|u|^{d+1-\theta_0}$ while estimating the left-hand side of (6.4). Then we obtain

$$\begin{aligned} &\|D^2[u_K(\varepsilon^2 + \cdot, \varepsilon l + \cdot) - u_K]\|^{d+1}_{\mathcal{L}_{d+1}(Q_{\varepsilon_0})} \\ &\quad + \|D[u_K(\varepsilon^2 + \cdot, \varepsilon l + \cdot) - u_K]\|^{d+1}_{\mathcal{L}_{d+1}(Q_{\varepsilon_0})} \le N\varepsilon^{\theta_1}(K\varepsilon_0^{-1})^{d+1} + I \end{aligned}$$

with $\theta_1 = \tau_1\theta_0$, where

$$I = N \sup_{\partial' Q_{\varepsilon_0}} |w|^{d+1}.$$

However, I is dominated by the right-hand side of (6.4) due to Theorem 2.1. The theorem is proved.

LEMMA 6.5. *There exists a constant $N \in (0,\infty)$ such that, if $\varepsilon_0 > 0$ is sufficiently small, then for any $h, \varepsilon \in (0, \varepsilon_0/4]$ and $K \geq 1$ there exists t_0, x_0 with $|x_0| \leq h$ and $0 \leq t_0 \leq h^2$ for which*

$$\sum_{(t,x)\in Q_{(h)}\cap Q_{\varepsilon_0}} I_+^{d+1}(K,\varepsilon,t+t_0,x+x_0,t,x)h^{d+2} \leq N(h^{\tau_1(d+1)} + \varepsilon^{\theta_1})(K\varepsilon_0^{-1})^{d+1}, \tag{6.6}$$

where

$$I(K,\varepsilon,s,y,t,x) = \partial_t u_K^{(\varepsilon)}(s,y) + F(u_K^{(\varepsilon)}(s,y), Du_K^{(\varepsilon)}(s,y), D^2u_K^{(\varepsilon)}(s,y),t,x)$$

and we use notation (3.5).

Proof. Notice that in the left-hand side of (6.6) similarly to the above proof

$$|I(K,\varepsilon,t+t_0,x+x_0,t,x) - \partial_t u_K^{(\varepsilon)}(t+t_0,x+x_0) - F[u_K^{(\varepsilon)}](t+t_0,x+x_0)| \leq Nh^{\tau_1}(K\varepsilon_0^{-1}).$$

This contributes a part of the right-hand side of (6.6). Concerning the remaining part observe that $\partial_t u_K + F[u_K] \leq 0$ and therefore

$$\begin{aligned} &\partial_t u_K^{(\varepsilon)}(t+t_0,x+x_0) + F[u_K^{(\varepsilon)}](t+t_0,x+x_0) \\ &\leq \partial_t[u_K^{(\varepsilon)} - u_K](t+t_0,x+x_0) + F[u_K^{(\varepsilon)}](t+t_0,x+x_0) - F[u_K](t+t_0,x+x_0) \\ &\leq NJ_{\varepsilon,K}(t+t_0,x+x_0), \end{aligned}$$

where

$$J_{\varepsilon,K} = |\partial_t u_K^{(\varepsilon)} - \partial_t u_K| + |D^2u_K^{(\varepsilon)} - D^2u_K| + |Du_K^{(\varepsilon)} - Du_K| + |u_K^{(\varepsilon)} - u_K|.$$

Notice that ($C_r(t,x)$ are introduced before Lemma 5.1)

$$\sum_{(t,x)\in Q_{(h)}\cap Q_{\varepsilon_0}} I_{C_{h/2}(t,x)} \leq I_{Q_{\varepsilon_0/2}}$$

implying that

$$\begin{aligned} &\fint_{C_{h/2}} \sum_{(t,x)\in Q_{(h)}\cap Q_{\varepsilon_0}} J_{\varepsilon,K}^{d+1}(t+t_0,x+x_0)h^{d+2}\,dx_0dt_0 \\ &= N \sum_{(t,x)\in Q_{(h)}\cap Q_{\varepsilon_0}} \int_{C_{h/2}} J_{\varepsilon,K}^{d+1}(t+t_0,x+x_0)\,dx_0dt_0 \\ &= N \sum_{(t,x)\in Q_{(h)}\cap Q_{\varepsilon_0}} \int_{C_{h/2}(t,x)} J_{\varepsilon,K}^{d+1}(t_0,x_0)\,dx_0dt_0 \leq N\int_{Q_{\varepsilon_0/2}} J_{\varepsilon,K}^{d+1}\,dxdt. \end{aligned} \tag{6.7}$$

Furthermore, for $(t,x)\in Q_{\varepsilon_0/2}$ we have

$$|D^2u^{(\varepsilon)}(t,x)-D^2u(t,x)|^{d+1}\leq N\fint_{C_\varepsilon}|D^2u(t+s,x+y)-D^2u(t,x)|^{d+1}\,dyds.$$

Similar relations are true for the first order derivatives and functions themselves. Therefore, in light of Theorem 6.4

$$\int_{Q_{\varepsilon_0/2}}J^{d+1}_{\varepsilon,K}\,dxdt\leq N\fint_{C_\varepsilon}\|u(\cdot+s,\cdot+y)-u\|^{d+1}_{W^{1,2}_{d+1}(Q_{\varepsilon_0/2})}\,dyds$$
$$\leq N_1\varepsilon^{\theta_1}(K\varepsilon_0^{-1})^{d+1}.$$

We conclude that the average of the first integrand in (6.7) over $C_{h/2}$ is less than $N_1\varepsilon^{\theta_1}(K\varepsilon_0^{-1})^{d+1}$ which implies that there is a point $(t_0,x_0)\in C_{h/2}$ at which the integrand itself is less than this quantity and this brings the proof of the lemma to an end.

Proof of Theorem 2.6. Take an $\varepsilon>0$, set $\lambda=\max\{|l_k|\}$, and observe that for any smooth function $w(t,x)$ given in Q_ε, we have in $Q_{\varepsilon+\lambda h}$ that

$$\big|\delta_{h,t}w(t,x)+F_h[w](t,x)-\partial_tw(t,x)-F[w](t,x)\big|$$
$$\leq Nh\sup_{Q_\varepsilon}\big[|\partial_t^2w|+|D^3w|+|D^2w|+|Dw|\big],$$

provided that $Q_{\varepsilon+\lambda h}\neq\emptyset$. We apply this to $w(t,x)=u^{(\varepsilon)}_K(t+t_0,x+x_0)$ for $h,\varepsilon,\varepsilon_0,t_0,x_0$ from Lemma 6.5 also satisfying $\lambda h\leq\varepsilon_0/2$ and $\varepsilon=\varepsilon_0-\lambda h$. Below we only consider $K\geq1$. Denoting

$$f(K,\varepsilon,t,x)=\partial_{h,t}u^{(\varepsilon)}_K(t+t_0,x+x_0)+F_h[u^{(\varepsilon)}_K(\cdot+t_0,\cdot+x_0)](t,x),$$

we conclude from Lemma 6.5 that

$$\sum_{(t,x)\in Q_{(h)}\cap Q_{\varepsilon_0}}f^{d+1}_+(K,\varepsilon,t,x)h^{d+2}\leq N(h^{\tau_1(d+1)}+\varepsilon^{\theta_1})(K\varepsilon_0^{-1})^{d+1}$$
$$+Nh^{d+1}\sup_{Q_{\varepsilon_0/2}}\big[|\partial_t^2u^{(\varepsilon)}_K|+|D^3u^{(\varepsilon)}_K|+|D^2u^{(\varepsilon)}_K|+|Du^{(\varepsilon)}_K|\big]^{d+1},$$

where the last term admits a rough estimate (see Theorems 2.1) by

$$Nh^{d+1}\varepsilon^{-2(d+1)}\sup_{Q_{\varepsilon_0/4}}\big[|\partial_tu_K|+|D^2u_K|+|Du_K|+|u_K|\big]^{d+1}$$
$$\leq Nh^{d+1}\varepsilon^{-2(d+1)}(K\varepsilon_0^{-1})^{d+1}.$$

Also observe that on the discrete parabolic boundary of $Q_{(h)}\cap Q_{\varepsilon_0}$ we have $|v_h|\leq N\varepsilon_0$ and $|u^{(\varepsilon)}_K(\cdot+t_0,\cdot+x_0)|\leq N\varepsilon_0$ owing to Lemmas 2.2 and 6.1. It follows by the discrete maximum principle of Kuo and Trudinger [14] applied to $v_h-u^{(\varepsilon)}_K(\cdot+t_0,\cdot+x_0)$ that in $Q_{(h)}\cap Q_{\varepsilon_0}$ we have

$$v_h\leq u^{(\varepsilon)}_K(\cdot+t_0,\cdot+x_0)+N(h^{\tau_1}+\varepsilon^{\theta_1/(d+1)})K\varepsilon_0^{-1}+Nh\varepsilon^{-2}K\varepsilon_0^{-1}+N\varepsilon_0.$$

By Theorem 2.1 the quantities $|\partial_tu_K|$ and $|Du_K|$ are bounded by NK in Q. Furthermore, the straight segment connecting any point $x\in G_{\varepsilon_0}$ with a point of type

$x + \varepsilon y + x_0$, where $|y| \leq 1$ and $|x_0| \leq h$ lies in $G_{\varepsilon_0/2} \subset G$. It follows that $u_K^{(\varepsilon)}(\cdot + t_0, \cdot + x_0) \leq u_K + NK(\varepsilon + h)$ on G_{ε_0}.

Combining this with Theorem 2.5 yields that for $K \geq 1$ and $h, \varepsilon, \varepsilon_0$ as above

$$v_h \leq v + NK^{-\xi} + N(\varepsilon + h)K\varepsilon_0^{-1} + N(h^{\tau_1} + \varepsilon^{\theta_1/(d+1)})K\varepsilon_0^{-1} + Nh\varepsilon^{-2}K\varepsilon_0^{-1} + N\varepsilon_0,$$

which now holds not only in $Q_{(h)} \cap Q_{\varepsilon_0}$ but also in $Q_{(h)}$ again in light of Lemmas 2.2 and 6.1. By the way, in the third term on the right ε_0^{-1} can be dropped. We introduced it to have K accompanied by ε_0^{-1} in all terms where ε is present.

Now, first we take $\varepsilon = h^{1/3}$. Then we obtain

$$v_h \leq v + NK^{-\xi} + Nh^{\theta_2}K\varepsilon_0^{-1} + N\varepsilon_0,$$

where $\theta_2 = \min(\theta_1/(3d+3), \tau_1)$. Then we take $\varepsilon_0 = h^{\theta_2/2}$ (and only concentrate on h such that $h^{1/3}, h \leq \varepsilon_0/4, \lambda h \leq \varepsilon_0/2$) and get

$$v_h \leq v + NK^{-\xi} + NKh^{\theta_2/2},$$

which for $K = h^{-\eta}$, where $\eta = \theta_2/(2 + 2\xi)$, finally leads to $v_h \leq v + Nh^{\xi\eta}$ in Q_h.

The reader understands that one can prove the inequality $v_h \geq v - Nh^{\xi\eta}$ in Q_h by using u_{-K} in place of u_K. The theorem is proved.

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