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A FUNCTIONAL INEQUALITY AND ITS APPLICATIONS TO A CLASS OF NONLINEAR FOURTH ORDER PARABOLIC EQUATIONS*

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Abstract. In this article we study the initial-boundary value problem for a family of nonlinear fourth order parabolic equations. The classical quantum drift-diffusion model is a member of the family. Two new existence theorems are established. Our approach is based upon a semi-discretization scheme, which generates a sequence of positive approximate solutions, and a functional inequality of the type

$$I_\alpha(u) \equiv \int_\Omega \Delta u u^{\alpha-1}\Delta u^\alpha dx \geq c\int_\Omega |\nabla^2 u^\alpha|^2 dx.$$

We show that a priori estimates that hold for the continuous model under the assumption that solutions are classical are mostly valid for the discretized problems. That is sufficient to justify passing to the limit in the approximation.

Key words. Existence, Nonlinear fourth order parabolic equations, Lipschitz boundaries, Quantum drift-diffusion model, Functional inequalities.

AMS subject classifications. 35D30, 35Q99, 35A01.

1. Introduction.

Let $T > 0$ and Ω be a domain in $\mathbb{R}^N$ with boundary $\partial\Omega$. We consider the existence of a solution to the problem

$$\partial_t u + \operatorname{div}[u\nabla(u^{\alpha-1}\Delta u^\alpha)] - \operatorname{div}(\mathbf{g}u) = 0 \quad \text{in } \Omega_T \equiv \Omega\times(0,T], \tag{1.1}$$

$$\nabla u^\alpha\cdot\nu = [u\nabla(u^{\alpha-1}\Delta u^\alpha) - \mathbf{g}u]\cdot\nu = 0 \quad \text{on } \Sigma_T \equiv \partial\Omega\times(0,T], \tag{1.2}$$

$$u(x,0) = u_0(x) \geq 0 \quad \text{on } \Omega, \tag{1.3}$$

where $\alpha\in(0,\infty)$, $\mathbf{g} = \mathbf{g}(x,t)$, and $u_0 = u_0(x)$ are given data whose precise assumptions will be made subsequently.

Our first result is the following:

THEOREM 1.1. *Assume:*
(H1) Ω *is a bounded domain with Lipschitz boundary;*
(H2) $\frac{1}{2}\leq\alpha\leq 1$*;*
(H3) $\mathbf{g}\in(L^\infty(\Omega_T))^N$*;*
(H4) $u_0\geq 0$, $u_0^\alpha\in W^{1,2}\cap L^\infty(\Omega)$.
Then there is a weak solution to (1.1)-(1.3) in the following sense:
(C1) $u\in C([0,T];L^1(\Omega))$, $u\geq 0$, $u^\alpha\in L^\infty((0,T);W^{1,2}(\Omega))$, $u^{\frac{\alpha}{2}}\in L^2((0,T);W^{1,2}(\Omega))$*;*
(C2) $\Delta u^\alpha\in L^2((0,T);L^{\frac{4N}{3N-2}}(\Omega))$, $u^{\frac{\alpha-1}{2}}\Delta u^\alpha\in L^2(\Omega_T)$, $\nabla\left(u^{\frac{2\alpha-1}{2}}\Delta u^\alpha\right)\in L^1(\Omega_T)$, *and* $\mathbf{S}\equiv -\frac{1}{\alpha}u^{\frac{\alpha-1}{2}}\Delta u^\alpha\nabla u^{\frac{\alpha}{2}} + \nabla\left(u^{\frac{2\alpha-1}{2}}\Delta u^\alpha\right)\in L^2(\Omega_T)$*;*
(C3) $\nabla u^\alpha\cdot\nu = 0$ *on* Σ_T*;*

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(C4) *For each $\xi \in C^\infty(\mathbb{R}^N \times \mathbb{R})$, we have*

$$-\int_{\Omega_T} \partial_t \xi u dxdt + \int_\Omega \xi(x,T)u(x,T)dx - \int_\Omega \xi(x,0)u_0(x)dx$$
$$-\int_{\Omega_T} (\sqrt{u}\mathbf{S} - \mathbf{g}u) \cdot \nabla\xi dxdt = 0. \tag{1.4}$$

Several remarks are in order. First, in spite of **(C2)**, we still do not necessarily have that $\nabla^2 u^\alpha$, the Hessian of u^α, lies in $(L^p(\Omega_T))^{N\times N}$ for some $p \geq 1$ because we only assume that $\partial\Omega$ is Lipschitz. Thus the boundary condition in **(C3)** is understood in the sense

$$\int_{\Omega_T} \nabla u^\alpha \cdot \nabla\xi dxdt = -\int_{\Omega_T} \Delta u^\alpha \xi dxdt \quad \text{for all } \xi \in C^\infty(\mathbb{R}^N \times \mathbb{R}). \tag{1.5}$$

Second, since the set $A_0 \equiv \{u = 0\} = \{(x,t) \in \Omega_T : u(x,t) = 0\}$ may even have positive measure, $u^{\frac{\alpha-1}{2}}\Delta u^\alpha \in L^2(\Omega_T)$ in **(C2)** is understood in the following sense: There is a function η in $L^2(\Omega_T)$ with the property

$$\eta = \begin{cases} u^{\frac{\alpha-1}{2}}\Delta u^\alpha & \text{a.e. on the set } \Omega_T \setminus A_0 \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, it does not really matter what values η takes on the set A_0 since $\mathbf{S}$ is always 0 there. Of course, if it is regular enough, our weak solution is a classical one.

If $\alpha = \frac{1}{2}$, then the resulting problem is often called the quantum drift-diffusion model. Problems of this type have drawn enormous interest. See, e.g., ([4],[6],[7],[8],[13]) and the references therein. In the existing work the assumptions on Ω seem to be stronger than ours. For example, in [8], Ω is assumed to be a torus, while in [4], Ω is taken to be either convex or C^2. The lonely exception is [13], where Ω is also Lipschitz, but it only deals with the case $\alpha = \frac{1}{2}$. Thus Theorem 1.1 generalizes the results in [13]. The initial value problem for the equation (1.1), i.e., $\Omega = \mathbb{R}^N$, has been investigated by D. Matthes et al [10] under the assumptions that $\frac{1}{2} \leq \alpha \leq 1$ and $\mathbf{g} = \lambda x, \lambda \geq 0$. They formulate the problem as gradient flows of a suitably defined functional in a probability space endowed with the L^2-Wasserstein metric, thereby obtaining the existence of a solution and long-time behavior of the solution. They further conjecture that these results should also hold for all $\alpha > \frac{1}{2} - \frac{1}{N}$. In attempting to answer their open question, we obtain the following

Theorem 1.2. *Let* **(H3)** *and* **(H4)** *hold. Assume:*
(H5) *Ω is a bounded convex domain;*
(H6) $\frac{1}{2} - \frac{1}{N+2} < \alpha \leq 1$.
Then there is a weak solution to (1.1)-(1.3) in the following sense:
(C5) *$u \in L^\infty((0,T); L^1(\Omega))$, $u \geq 0$, $u^\alpha \in L^2((0,T); W^{2,2}(\Omega))$;*
(C6) *$\nabla u^\alpha \cdot \nu = 0$ on Σ_T;*
(C7) *For each $\xi \in C^\infty(\mathbb{R}^N \times \mathbb{R})$ with $\xi(x,T) = 0$ and $\nabla\xi \cdot \nu = 0$ on Σ_T , we have*

$$-\int_{\Omega_T} \partial_t \xi u dxdt - \int_\Omega \xi(x,0)u_0(x)dx$$
$$+\int_{\Omega_T} \left(\frac{1}{\alpha}\Delta u^\alpha \nabla u^\alpha \cdot \nabla\xi + u^\alpha \Delta u^\alpha \Delta\xi + u\mathbf{g} \cdot \nabla\xi\right) dxdt = 0. \tag{1.6}$$

A well-known difficulty in the study of fourth-order equations is that the maximum principle is no longer true. In our case, it is the specific nonlinear structure of our equation that enables us to construct a sequence of positive approximate solutions. To gain some further insights into our problem, we proceed to make some formal analysis. That is to say, in the subsequent calculations we will assume that u, u^α, and $u^{\alpha-1}\Delta u^\alpha$ are all sufficiently regular. This implicitly requires that u behave "well" near the set A_0.

Integrate (1.1) to obtain

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx. \tag{1.7}$$

This is the conservation law implied by the equation. Observe that

$$\begin{aligned}\int_\Omega u^{\alpha-1}\Delta u^\alpha \partial_t u dx &= \frac{1}{\alpha}\int_\Omega \Delta u^\alpha \partial_t u^\alpha dx\\ &= -\frac{1}{\alpha}\int_\Omega \nabla u^\alpha \cdot \nabla(\partial_t u^\alpha)dx\\ &= -\frac{1}{2\alpha}\frac{d}{dt}\int_\Omega |\nabla u^\alpha|^2 dx. \end{aligned}\tag{1.8}$$

Use $u^{\alpha-1}\Delta u^\alpha$ as a test function in (1.1), plug (1.8) into the resulting equation, then apply the Hölder inequality appropriately, and integrate to obtain

$$\frac{1}{\alpha}\int_\Omega |\nabla u^\alpha|^2 dx + \int_{\Omega_t} u|\nabla(u^{\alpha-1}\Delta u^\alpha)|^2 dxds \le \frac{1}{\alpha}\int_\Omega |\nabla u_0^\alpha|^2 dx + \int_{\Omega_t} u|\mathbf{g}|^2 dxds, \tag{1.9}$$

where $\Omega_t = \Omega \times (0,t)$. The two estimates (1.7) and (1.9) are the most obvious ones. Unfortunately, they are not enough for the existence assertion, and one must seek new estimates. Our first key estimate is obtained by using $1 + \frac{\alpha}{\alpha-1}u^{\alpha-1}$ as a test function in (1.1) as this leads to the equation

$$\begin{aligned}&\int_\Omega (u + \frac{1}{\alpha-1}u^\alpha)dx + \int_{\Omega_t} u^{\alpha-1}(\Delta u^\alpha)^2 dxds\\ &= \int_\Omega (u_0 + \frac{1}{\alpha-1}u_0^\alpha)dx - \int_{\Omega_t} \mathbf{g}\cdot\nabla u^\alpha dxds.\end{aligned}\tag{1.10}$$

Another key estimate is derived from using $\ln u$ as a test function in (1.1), from which it follows

$$\frac{d}{dt}\int_\Omega u(\ln u - 1)dx + \int_\Omega u^{\alpha-1}\Delta u^\alpha \Delta u dx = -\int_\Omega \mathbf{g}\cdot\nabla u dx. \tag{1.11}$$

The second integral in (1.11) leads us to the study of a class of functional inequalities of the form

$$I_\alpha(u) \equiv \int_\Omega \Delta u u^{\alpha-1}\Delta u^\alpha dx \ge c\int_\Omega |\nabla^2 u^\alpha|^2 dx, \tag{1.12}$$

where c is a positive number independent of u and $|\nabla^2 u^\alpha|^2$ is the sum of squares of all second order partial derivatives of u^α. If this inequality holds, we have from (1.7) and (1.9) that $u^\alpha \in L^2(0,T;W^{2,2}(\Omega))$ for each $T > 0$.

Before we continue, we would like to mention that if $\alpha = \frac{1}{2}, N = 1$ estimates similar to (1.9), (1.10), and (1.11) are used in [2].

Obviously, the validity of (1.12) depends on Ω and α. Define

$$W_\alpha = \{u \geq 0 : u^\alpha \in W^{2,2}(\Omega), \nabla u^\alpha \cdot \nu = 0 \text{ on } \partial\Omega\}, \tag{1.13}$$

where ν is the unit outward normal to $\partial\Omega$. Then the question is: under what conditions on α and Ω do we have that $I_\alpha(u)$ is coercive in W_α in the sense that there is a positive number c such that (1.12) holds for all $u \in W_\alpha$? If $\alpha = 1$ and Ω is bounded and convex, then (1.12) is true for $c = 1$. This result is well-known [3]. In fact, it is the application of this result that yields the $W^{2,2}(\Omega)$-estimate of solutions of second-order elliptic equations with the homogenous Neumann boundary condition. It is known from [1] that if $\alpha = \frac{1}{2}$ then (1.12) holds for a box domain Ω with sides parallel to the coordinate planes. In the case where $\alpha = \frac{1}{2}$ and Ω is bounded and convex, a result in [4] asserts the coercivity of $I_{\frac{1}{2}}(u)$ in $W_{\frac{1}{2}}$. Also see [14]. This result has played a key role in establishing an existence assertion for the initial boundary-value problem for the quantum drift-diffusion model in [4]. For more general α and Ω, we have

Theorem 1.3. *Let Ω be a bounded convex domain in $\mathbb{R}^N$. Then for each $\alpha \in (\frac{1}{2} - \frac{1}{N+2}, 1]$ there exists a positive number $c = c(N, \alpha)$ such that (1.12) holds.*

It is also possible to find $\alpha > 1$ for which (1.12) holds. In this regard, we have

Theorem 1.4. *Let Ω be a bounded convex domain in $\mathbb{R}^N$, then for each $\alpha \in [1, \frac{3}{2})$ there is a positive number $c = c(N, \alpha)$ such that (1.12) holds.*

Theorems 1.3 and 1.4 have been inspired by the estimate (4.11) in [10], where the authors use a measure-theoretic method to obtain a different version of (1.12) for $\Omega = \mathbb{R}^N$. In contrast, our approach here is straightforward and elementary. In a forthcoming paper [9], J-G Liu and this author will consider the more general form

$$\int_\Omega u^{2\gamma-\alpha-\beta} \Delta u^\alpha \Delta u^\beta dx \geq c \int_\Omega (\Delta u^\gamma)^2 dx \tag{1.14}$$

and use it to study the thin film equation. The significance of this type of functional inequalities is that the integrand on the left-hand side of (1.14) is not necessarily non-negative, and thus a cancellation effect has come into play.

Obviously, in the generality considered in Theorems 1.1 and 1.2, the gradient flow theorem is no longer applicable. We will employ the classical approximation scheme of implicit discretization in the time variable. The central issue here is how to obtain discretized versions of the formal estimates mentioned earlier. Unfortunately, (1.7) is lost in our approximating problems. This has given rise to quite a few complications. The discretized version of (1.8) is the inequality

$$\int_\Omega \rho^{\alpha-1} \Delta\rho^\alpha (\rho - f) dx \leq -\frac{1}{2\alpha} \int_\Omega (|\nabla\rho^\alpha|^2 - |\nabla f^\alpha|^2) dx, \tag{1.15}$$

which does not seem to hold for every $\alpha > 0$. In fact, we only have

Theorem 1.5. *Let* **(H1)** *and* **(H2)** *be satisfied. Assume that ρ is a positive solution of the problem*

$$-\Delta\rho^\alpha = G(x) \quad in \quad \Omega, \tag{1.16}$$

$$\nabla\rho^\alpha \cdot \nu = 0 \quad on \quad \partial\Omega \tag{1.17}$$

with the property that $\frac{1}{\rho} \in L^s(\Omega)$ *for each* $s \geq 1$, *where* $G(x) \in L^2(\Omega)$. *Then (1.15) holds for all* $f \geq 0$ *with* $f^\alpha \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$.

By a solution to (1.16)-(1.17), we mean that $\rho^\alpha \in W^{1,2}(\Omega)$ and

$$\int_\Omega \nabla\rho^\alpha\nabla\eta dx = \int_\Omega G\eta dx = -\int_\Omega \Delta\rho^\alpha\eta dx \quad \text{for all } \eta \in W^{1.2}(\Omega). \tag{1.18}$$

Note that the integral on the left hand side of (1.15) makes sense under our assumptions. Also, the restrictions in this theorem account for the assumption **(H2)** in Theorem 1.1. However, we have managed to avoid this theorem in the establishment of Theorem 1.2. In fact, the assumption **(H6)** is totally due to the condition in Theorem 1.3. Naturally, we would expect an existence assertion for (1.1)-(1.3) whenever (1.12) holds. To our surprise, in spite of Theorem 1.4, we have not been able to obtain any existence results for the case $\alpha > 1$, nor have we considered unbounded domains in the context of Theorem 1.2.

The remaining sections of the paper constitute the proof of the five theorems presented earlier. Roughly speaking, Theorem 1.1 is derived from (1.9) and (1.10), while Theorem 1.2 from (1.10), (1.11), and (1.12).

2. Preliminaries. In this section we offer the proof of Theorems 1.3, 1.4 and 1.5. In comparison with the establishment of (4.11) in [10], our proof of Theorems 1.3 and 1.4 is rather elementary.

LEMMA 2.1. *Let* $\alpha > 0$ *and* Ω *be a bounded domain in* $\mathbb{R}^N$ *with Lipschitz boundary* $\partial\Omega$. *If either* $\beta > \alpha$ *or* $\beta < \frac{\alpha}{2}$, *one has*

$$\int_\Omega u^{2\alpha-2\beta}|\nabla^2 u^\beta|^2 dx \geq \frac{4\beta^2(2\beta-\alpha)}{\alpha^2[(N+8)\beta-(N+4)\alpha]}\int_\Omega |\nabla^2 u^\alpha|^2 dx \tag{2.1}$$

for all $u \in W_\alpha$.

Proof. If $\beta = 0$, then (2.1) is trivially true. Thus assume that $\beta \neq 0$. We can also assume, without loss of generality, that u has a positive lower bound. (Otherwise, use $(u^\alpha + \varepsilon)^{\frac{1}{\alpha}}, \varepsilon > 0$, in place of u and then let $\varepsilon \to 0^+$. The same is understood in the subsequent proof.) We compute, for $i, j = 1, \cdots, N$, that

$$\partial_i u^\beta = \partial_i(u^\alpha)^{\frac{\beta}{\alpha}} = \frac{\beta}{\alpha}u^{\beta-\alpha}\partial_i u^\alpha, \tag{2.2}$$

$$\partial_{ij}^2 u^\beta = \frac{\beta(\beta-\alpha)}{\alpha^2}u^{\beta-2\alpha}\partial_i u^\alpha\partial_j u^\alpha + \frac{\beta}{\alpha}u^{\beta-\alpha}\partial_{ij}^2 u^\alpha. \tag{2.3}$$

In particular, we have

$$\Delta u^\beta = \frac{\beta(\beta-\alpha)}{\alpha^2}u^{\beta-2\alpha}|\nabla u^\alpha|^2 + \frac{\beta}{\alpha}u^{\beta-\alpha}\Delta u^\alpha. \tag{2.4}$$

Square both sides of (2.3), multiply through the resulting equation by $\frac{\alpha^2}{\beta^2}u^{2\alpha-2\beta}$, and then sum up i, j to obtain

$$\frac{\alpha^2}{\beta^2}u^{2\alpha-2\beta}|\nabla^2 u^\beta|^2 = |\nabla^2 u^\alpha|^2 + 2\frac{\beta-\alpha}{\alpha}\frac{1}{u^\alpha}\nabla u^\alpha\cdot\nabla^2 u^\alpha\nabla u^\alpha + \left(\frac{\beta-\alpha}{\alpha}\right)^2\frac{1}{u^{2\alpha}}|\nabla u^\alpha|^4. \tag{2.5}$$

By Theorem 3.1 in [4], for each $u \in W_\alpha$ we have that $u^{\frac{\alpha}{2}} \in W^{1,4}(\Omega)$ and

$$4|\nabla u^{\frac{\alpha}{2}}|^4 + \operatorname{div}\left(|\nabla u^{\frac{\alpha}{2}}|^2 \nabla u^\alpha\right) = 2\nabla u^{\frac{\alpha}{2}} \cdot (\nabla^2 u^\alpha \nabla u^{\frac{\alpha}{2}}) + |\nabla u^{\frac{\alpha}{2}}|^2 \Delta u^\alpha. \tag{2.6}$$

Integrate this equation over Ω and keep in mind (1.13) and the inequality

$$(\Delta u^\beta)^2 \le N|\nabla^2 u^\beta|^2 \tag{2.7}$$

to obtain

$$\begin{aligned} 4\int_\Omega |\nabla u^{\frac{\alpha}{2}}|^4 dx &\le 2\left(\int_\Omega |\nabla^2 u^\alpha|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla u^{\frac{\alpha}{2}}|^4 dx\right)^{\frac{1}{2}} \\ &\quad + \left(\int_\Omega |\nabla u^{\frac{\alpha}{2}}|^4 x\right)^{\frac{1}{2}} \left(\int_\Omega |\Delta u^\alpha|^2 dx\right)^{\frac{1}{2}} \\ &\le (2+\sqrt{N})\left(\int_\Omega |\nabla^2 u^\alpha|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla u^{\frac{\alpha}{2}}|^4 dx\right)^{\frac{1}{2}} \end{aligned} \tag{2.8}$$

from whence follows

$$\int_\Omega |\nabla u^{\frac{\alpha}{2}}|^4 dx \le \frac{(2+\sqrt{N})^2}{16} \int_\Omega |\nabla^2 u^\alpha|^2 dx. \tag{2.9}$$

This immediately implies that each term on the right-hand side of (2.5) lies in $L^1(\Omega)$. Note from (2.4) that

$$\begin{aligned} 2\frac{1}{u^\alpha}\nabla u^\alpha \cdot \left(\nabla^2 u^\alpha \nabla u^\alpha\right) &= \frac{1}{u^\alpha}\nabla u^\alpha \cdot \nabla\left(|\nabla u^\alpha|^2\right) \\ &= \operatorname{div}\left(\frac{1}{u^\alpha}|\nabla u^\alpha|^2 \nabla u^\alpha\right) + \frac{1}{u^{2\alpha}}|\nabla u^\alpha|^4 - \frac{1}{u^\alpha}\Delta u^\alpha |\nabla u^\alpha|^2 \\ &= \operatorname{div}\left(\frac{1}{u^\alpha}|\nabla u^\alpha|^2 \nabla u^\alpha\right) + \frac{1}{u^{2\alpha}}|\nabla u^\alpha|^4 \\ &\quad - \left(\frac{\alpha}{\beta}\frac{1}{u^\beta}\Delta u^\beta - \frac{\beta-\alpha}{\alpha}\frac{1}{u^{2\alpha}}|\nabla u^\alpha|^2\right)|\nabla u^\alpha|^2 \\ &= \operatorname{div}\left(\frac{1}{u^\alpha}|\nabla u^\alpha|^2 \nabla u^\alpha\right) + \frac{\beta}{\alpha}\frac{1}{u^{2\alpha}}|\nabla u^\alpha|^4 - \frac{\alpha}{\beta}\frac{1}{u^\beta}\Delta u^\beta |\nabla u^\alpha|^2. \end{aligned} \tag{2.10}$$

Use this in (2.5) and integrate the resulting equation over Ω to obtain

$$\begin{aligned} \frac{\alpha^2}{\beta^2}\int_\Omega u^{2\alpha-2\beta}|\nabla^2 u^\beta|^2 dx &= \int_\Omega |\nabla^2 u^\alpha|^2 dx - \frac{\beta-\alpha}{\beta}\int_\Omega \frac{1}{u^\beta}\Delta u^\beta |\nabla u^\alpha|^2 dx \\ &\quad + \frac{(\beta-\alpha)(2\beta-\alpha)}{\alpha^2}\int_\Omega \frac{1}{u^{2\alpha}}|\nabla u^\alpha|^4 dx. \end{aligned} \tag{2.11}$$

Under our assumptions, we always have

$$(\beta-\alpha)(2\beta-\alpha) > 0.$$

Keeping this in mind, we calculate from (2.11) that

$$\begin{aligned} &\frac{\alpha^2}{\beta^2}\int_\Omega u^{2\alpha-2\beta}|\nabla^2 u^\beta|^2 dx + \frac{\alpha^2(\beta-\alpha)}{4\beta^2(2\beta-\alpha)}\int_\Omega u^{2\alpha-2\beta}(\Delta u^\beta)^2 dx \\ &= \frac{(\beta-\alpha)(2\beta-\alpha)}{\alpha^2}\int_\Omega \left(\frac{1}{u^\alpha}|\nabla u^\alpha|^2 - \frac{\alpha^2}{2\beta(2\beta-\alpha)}u^{\alpha-\beta}\Delta u^\beta\right)^2 dx \\ &\quad + \int_\Omega |\nabla^2 u^\alpha|^2 dx \ge \int_\Omega |\nabla^2 u^\alpha|^2 dx. \end{aligned} \tag{2.12}$$

Use (2.7) in (2.12), combine the two integrals on the left-hand side in the resulting inequality, take note of the fact that the combined coefficient is positive, and thereby obtain the desired result. The proof is complete. □

LEMMA 2.2. *Let $\alpha > 0$ and Ω be a bounded convex domain in $\mathbb{R}^N$. Assume $\alpha \neq \beta$ and $\beta \neq 0$. Then we have*

$$I_\alpha(u) \geq \frac{1+\beta-2\alpha}{(\beta-\alpha)\alpha}\int_\Omega |\nabla^2 u^\alpha|^2 dx + \frac{\beta(1-\alpha)}{\alpha^3}\int_\Omega \frac{1}{u^{2\alpha}}|\nabla u^\alpha|^4 dx + \frac{\alpha(\alpha-1)}{\beta^2(\beta-\alpha)}\int_\Omega u^{2\alpha-2\beta}|\nabla^2 u^\beta|^2 dx \tag{2.13}$$

for all $u \in W_\alpha$.

Proof. Take $\beta = 1$ in (2.4) to obtain

$$\Delta u = \frac{1-\alpha}{\alpha^2}u^{1-2\alpha}|\nabla u^\alpha|^2 + \frac{1}{\alpha}u^{1-\alpha}\Delta u^\alpha. \tag{2.14}$$

If $\alpha = 1$, then (2.13) is a well-known classical result [3]. From now on we assume $\alpha \neq 1$. Multiply through (2.14) by $u^{\alpha-1}\Delta u^\alpha$ and integrate to obtain

$$\int_\Omega u^{-\alpha}|\nabla u^\alpha|^2\Delta u^\alpha dx = \frac{\alpha^2}{1-\alpha}I_\alpha(u) - \frac{\alpha}{1-\alpha}\int_\Omega |\Delta u^\alpha|^2 dx. \tag{2.15}$$

We recall from (2.5) and (2.11)

$$\begin{aligned}\frac{\alpha^2}{\beta^2}\int_\Omega u^{2\alpha-2\beta}|\nabla^2 u^\beta|^2 dx &= \int_\Omega |\nabla^2 u^\alpha|^2 dx + \left(\frac{\beta-\alpha}{\alpha}\right)^2\int_\Omega \frac{1}{u^{2\alpha}}|\nabla u^\alpha|^4 dx \\ &\quad + 2\frac{\beta-\alpha}{\alpha}\int_\Omega \frac{1}{u^\alpha}\nabla u^\alpha \cdot \nabla^2 u^\alpha \nabla u^\alpha dx \\ &= \int_\Omega |\nabla^2 u^\alpha|^2 dx + \left(\frac{\beta-\alpha}{\alpha}\right)^2\int_\Omega \frac{1}{u^{2\alpha}}|\nabla u^\alpha|^4 dx \\ &\quad + \frac{\beta-\alpha}{\alpha}\int_\Omega \left(\frac{1}{u^{2\alpha}}|\nabla u^\alpha|^4 - \frac{1}{u^\alpha}\Delta u^\alpha|\nabla u^\alpha|^2\right) dx \\ &= \int_\Omega |\nabla^2 u^\alpha|^2 dx + \frac{\beta(\beta-\alpha)}{\alpha^2}\int_\Omega \frac{1}{u^{2\alpha}}|\nabla u^\alpha|^4 dx \\ &\quad - \frac{(\beta-\alpha)\alpha}{1-\alpha}I_\alpha(u) + \frac{\beta-\alpha}{1-\alpha}\int_\Omega (\Delta u^\alpha)^2 dx.\end{aligned} \tag{2.16}$$

Remember from [3] that

$$\int_\Omega (\Delta u^\alpha)^2 dx \geq \int_\Omega |\nabla^2 u^\alpha|^2 dx \quad \text{for all } u \in W_\alpha \text{ whenever } \Omega \text{ is bounded and convex.} \tag{2.17}$$

Solve (2.16) for $I_\alpha(u)$, take a note of the above fact and thereby obtain the desired result. □

We are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let $\alpha \in (\frac{1}{2} - \frac{1}{N+2}, 1)$ be given. Note that

$$\int_\Omega \frac{1}{u^{2\alpha}}|\nabla u^\alpha|^4 dx = 16\int_\Omega |\nabla u^{\frac{\alpha}{2}}|^4 dx.$$

For each $\beta < 0$ we have from (2.9), Lemmas 2.1 and 2.2 that

$$
\begin{aligned}
I_\alpha(u) &\geq \left(\frac{1+\beta-2\alpha}{(\beta-\alpha)\alpha} + \frac{4(\alpha-1)(2\beta-\alpha)}{(\beta-\alpha)\alpha[(N+8)\beta-(N+4)\alpha]}\right)\int_\Omega |\nabla^2 u^\alpha|^2 dx \\
&\quad + \frac{(2+\sqrt{N})^2\beta(1-\alpha)}{\alpha^3}\int_\Omega |\nabla^2 u^\alpha|^2 dx \\
&= \frac{1}{\alpha}\left(\frac{(N+8)\beta-(2N+4)\alpha+N}{(N+8)\beta-(N+4)\alpha} + \frac{(2+\sqrt{N})^2\beta(1-\alpha)}{\alpha^2}\right)\int_\Omega |\nabla^2 u^\alpha|^2 dx \\
&\equiv \frac{1}{\alpha}h(\beta)\int_\Omega |\nabla^2 u^\alpha|^2 dx. \qquad (2.18)
\end{aligned}
$$

Our assumptions on α indicate

$$\lim_{\beta\to 0^-} h(\beta) = \frac{-(2N+4)\alpha+N}{-(N+4)\alpha} > 0.$$

By choosing β sufficiently close to 0, we obtain the desired result. The proof is complete.

Proof of Theorem 1.4. We can infer from (2.17) and (2.8) that

$$\int_\Omega |\nabla u^{\frac{\alpha}{2}}|^4 dx \leq \frac{9}{16}\int_\Omega (\Delta u^\alpha)^2 dx.$$

We calculate

$$
\begin{aligned}
I_\alpha(u) &= -\frac{4(\alpha-1)}{\alpha^2}\int_\Omega \Delta u^\alpha |\nabla u^{\frac{\alpha}{2}}|^2 dx + \frac{1}{\alpha}\int_\Omega (\Delta u^\alpha)^2 dx \\
&\geq -\frac{4(\alpha-1)}{\alpha^2}\left(\int_\Omega |\nabla u^{\frac{\alpha}{2}}|^4 dx\right)^{\frac{1}{2}}\left(\int_\Omega (\Delta u^\alpha)^2 dx\right)^{\frac{1}{2}} + \frac{1}{\alpha}\int_\Omega (\Delta u^\alpha)^2 dx \\
&\geq -\frac{4(\alpha-1)}{\alpha^2}\frac{3}{4}\int_\Omega (\Delta u^\alpha)^2 dx + \frac{1}{\alpha}\int_\Omega (\Delta u^\alpha)^2 dx \\
&= \frac{3-2\alpha}{\alpha^2}\int_\Omega (\Delta u^\alpha)^2 dx.
\end{aligned}
$$

Thus the desired result follows.

Finally, we offer the proof of Theorem 1.5.

Proof of Theorem 1.5. Recall the proof of Lemma 2.1, and we can once again assume that ρ is bounded away from 0 below. Let f be given as in the theorem. Setting $\eta = \rho^{\alpha-1}(\rho - f)$ in (1.18), we derive

$$
\begin{aligned}
&\int_\Omega \rho^{\alpha-1}\Delta\rho^\alpha(\rho-f)dx \\
&= -\int_\Omega \nabla\rho^\alpha \cdot \left(\nabla\rho^\alpha - \nabla(\rho^{\alpha-1}f)\right) dx \\
&= -\int_\Omega \left(|\nabla\rho^\alpha|^2 + \frac{(1-\alpha)f}{\alpha\rho}|\nabla\rho^\alpha|^2 - \frac{1}{\alpha}\left(\frac{f}{\rho}\right)^{1-\alpha}\nabla f^\alpha \cdot \nabla\rho^\alpha\right) dx \\
&= -\frac{1}{2\alpha}\int_\Omega (|\nabla\rho^\alpha|^2 - |\nabla f^\alpha|^2) dx \\
&\quad - \int_\Omega \left(\left(\frac{(1-\alpha)f}{\alpha\rho} + \frac{2\alpha-1}{2\alpha}\right)|\nabla\rho^\alpha|^2 - \frac{1}{\alpha}\left(\frac{f}{\rho}\right)^{1-\alpha}\nabla f^\alpha\cdot\nabla\rho^\alpha + \frac{1}{2\alpha}|\nabla f^\alpha|^2\right) dx.
\end{aligned}
$$

Obviously, the lemma follows if we can prove the integrand in the last integral is non-negative. This is true, provided that

$$\frac{1}{\alpha^2}\left(\frac{f}{\rho}\right)^{2-2\alpha} - 4\left(\frac{(1-\alpha)f}{\alpha\rho} + \frac{2\alpha-1}{2\alpha}\right)\frac{1}{2\alpha} \le 0. \tag{2.19}$$

To see this, consider the function

$$\theta(s) = s^{2-2\alpha} - 2(1-\alpha)s - 2\alpha + 1.$$

Recall the condition on α. We can easily show that $\theta(s) \le 0$ for $s \in [0,\infty)$, from which (2.19) follows. The proof is complete.

3. The approximate problems. Throughout this paper, we will assume $N > 2$ for convenience. The first lemma deals with the existence of a solution to the discretized problem.

LEMMA 3.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$. Assume that $\alpha \in (0,1), \mathbf{g} \in (L^\infty(\Omega))^N$ and*

$$p > \max\{\frac{N}{2}, 2\}. \tag{3.1}$$

Then for each $\tau > 0$ and each $f \in L^p(\Omega)$, there is a solution (ρ, F) in the space $\left(W^{1,2}(\Omega) \cap L^\infty(\Omega)\right)^2$ to the problem

$$-div\,[(\rho+\tau)(\nabla F - \mathbf{g})] + \tau F = \frac{\rho - f}{\tau} \quad in \quad \Omega, \tag{3.2}$$

$$-\Delta\rho^\alpha + \tau\rho^p = -F\rho^{1-\alpha} + \tau \quad in \quad \Omega, \tag{3.3}$$

$$\nabla\rho^\alpha\cdot\nu = (\nabla F - \mathbf{g})\cdot\nu = 0 \quad on \quad \partial\Omega. \tag{3.4}$$

Furthermore, we have that $\rho, F \in C^{0,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$ and $\rho \ge c_0$ in Ω for some $c_0 > 0$, where β, c_0 depend on the given data.

Of course, the equations (3.2)-(3.4) are satisfied in the sense of distributions. The last term τ in (3.3) has been added to ensure that ρ cannot be identically 0. Later, we shall see that it is also the main reason why ρ has a positive lower bound.

Proof. A solution will be obtained from the Leray-Schauder Fixed Point Theorem. For the precise statement of the theorem we refer the reader to Theorem 11.3 in [5]. To apply the theorem, we define an operator B from $L^p(\Omega)$ into $L^p(\Omega)$, where p is given as in (3.1), as follows. Given that $\rho \in L^p(\Omega)$, we consider the problem

$$-\text{div}\left[(\rho^+ + \tau)(\nabla F - \mathbf{g})\right] + \tau F = \frac{\rho - f}{\tau} \quad \text{in} \quad \Omega, \tag{3.5}$$

$$(\nabla F - \mathbf{g})\cdot\nu = 0 \quad \text{on} \quad \partial\Omega. \tag{3.6}$$

In spite of the fact that $\rho^+ + \tau$ may not be bounded from above, we can still conclude that there is a unique solution F in the space $W^{1,2}(\Omega)\cap L^\infty(\Omega)$. See, e.g., [12], where the solution to (3.5)-(3.6) is constructed as the limit of a sequence of approximate solutions to problems with the elliptic coefficient $\rho^+ + \tau$ being cut-off from above. Thus we may assume, without any loss of generality, that both ρ^+ and F are bounded. Keeping this in mind and using F as a test function in (3.5) yield

$$\int_\Omega (|\nabla F|^2 + F^2)dx \le c\int_\Omega \rho^2 dx + c. \tag{3.7}$$

Here and in the remaining proof of the lemma the letter c denotes a generic positive number which depends only on the given data in (3.2)-(3.4) such as $\tau, f, \mathbf{g}$. Furthermore, we have

$$\|F\|_\infty \leq c\|\rho\|_p + c. \tag{3.8}$$

The proof of this inequality is a minor modification of the classical arguments in ([5], p.189-190). That is, one assumes $\|F\|_\infty = \|F^+\|_\infty$ and uses $\frac{s^2}{2s-1}(F^+ + A)^{2s-1}$, where $s \geq 1$ and $A = \|\frac{\rho-f}{\tau}\|_p + \|\rho^+ + \tau\|_p^{\frac{1}{2}}$, as a test function in (3.5) to obtain

$$\int_\Omega (\rho^+ + \tau)|\nabla(F^+ + A)^s|^2 dx \leq cs^2 \int_\Omega (\rho^+ + \tau)(F^+ + A)^{2s-2} dx \tag{3.9}$$
$$+ c\frac{s^2}{2s-1}\int_\Omega F^- A^{2s-1} dx + c\frac{s^2}{2s-1}\int_\Omega \frac{\rho - f}{\tau}(F^+ + A)^{2s-1} dx \equiv I_1 + I_2 + I_3.$$

Now we estimate each integral on the right hand-side of (3.10). By the definition of A, we obviously have

$$\|\rho^+ + \tau\|_p \leq A^2 \leq (F^+ + A)^2.$$

Subsequently,

$$\begin{aligned} I_1 &\leq cs^2\|\rho^+ + \tau\|_p \left(\int_\Omega (F^+ + A)^{(2s-2)\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \\ &\leq cs^2 \left(\int_\Omega \|\rho^+ + \tau\|_p^{\frac{p}{p-1}} (F^+ + A)^{(2s-2)\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \\ &\leq cs^2 \left(\int_\Omega (F^+ + A)^{2s\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}}. \end{aligned} \tag{3.10}$$

To estimate I_2, we first integrate (3.5) to obtain

$$\tau\int_\Omega F^- dx = \tau\int_\Omega F^+ dx - \int_\Omega \frac{\rho - f}{\tau} dx.$$

As a result, we have

$$I_2 \leq c\frac{s^2}{2s-1}\int_\Omega (F^+ + A)^{2s} dx + |I_3|. \tag{3.11}$$

Finally, we calculate

$$\begin{aligned} |I_3| &\leq c\frac{s^2}{2s-1}\int_\Omega \left|\frac{\rho - f}{\tau}\right| (F^+ + A)^{2s-1} dx \\ &\leq c\frac{s^2}{2s-1}\left\|\frac{\rho - f}{\tau}\right\|_p \left(\int_\Omega (F^+ + A)^{(2s-1)\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \\ &\leq c\frac{s^2}{2s-1}\left(\int_\Omega (F^+ + A)^{2s\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}}. \end{aligned} \tag{3.12}$$

Collecting (3.10)-(3.12) in (3.10) yields

$$\int_\Omega |\nabla(F^+ + A)^s|^2 dx \leq cs^2 \left(\int_\Omega (F^+ + A)^{2s\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}}. \tag{3.13}$$

By the Sobolev inequality, we have

$$\|(F^+ + A)\|_{2s\frac{N}{N-2}} \leq c^{\frac{1}{2s}} s^{\frac{1}{s}} \|(F^+ + A)\|_{2s\frac{p}{p-1}}. \tag{3.14}$$

Our assumption on p implies that $\chi \equiv \frac{N}{N-2}/\frac{p}{p-1} > 1$. Letting $s = \chi^i, i = 0, 1, \cdots$, in (3.14) and making the same arguments as those in ([5], p.190), we obtain

$$\|F^+ + A\|_\infty \leq c\|F^+ + A\|_2.$$

Then (3.8) follows from (3.7) and the definition of A.

Now we use the function F so-obtained to form the problem

$$-\Delta\psi + \tau|\psi|^{\frac{p}{\alpha}-1}\psi = -F(\rho^+)^{1-\alpha} + \tau \quad \text{in} \quad \Omega, \tag{3.15}$$

$$\nabla\psi \cdot \nu = 0 \quad \text{on} \quad \partial\Omega. \tag{3.16}$$

Obviously, this problem has a unique solution ψ in the space $W^{1,2}(\Omega) \cap L^\infty(\Omega)$. We define

$$B(\rho) = \theta(\psi), \quad \text{where } \theta(s) = |s|^{\frac{1}{\alpha}-1}s.$$

It is easy to see that $B : L^p(\Omega) \to L^p(\Omega)$ is continuous and maps bounded sets into precompact ones. It remains to show that

$$\|\rho\|_p \leq c \tag{3.17}$$

for all $\sigma \in [0, 1]$ and ρ such that $\sigma B(\rho) = \rho$. Without loss of generality, assume $\sigma > 0$. Then the equation $\sigma B(\rho) = \rho$ is equivalent to the problem

$$-\text{div}\left[(\rho^+ + \tau)(\nabla F - \mathbf{g})\right] + \tau F = \frac{\rho - f}{\tau} \quad \text{in} \quad \Omega, \tag{3.18}$$

$$-\Delta\theta^{-1}(\frac{\rho}{\sigma}) + \tau\left|\theta^{-1}(\frac{\rho}{\sigma})\right|^{\frac{p}{\alpha}-1}\theta^{-1}(\frac{\rho}{\sigma}) = -F(\rho^+)^{1-\alpha} + \tau \quad \text{in} \quad \Omega, \tag{3.19}$$

$$\nabla\theta^{-1}(\frac{\rho}{\sigma}) \cdot \nu = (\nabla F - \mathbf{g}) \cdot \nu = 0 \quad \text{on} \quad \partial\Omega. \tag{3.20}$$

Keep in mind that $\rho \in L^p(\Omega)$, and thus (3.7) and (3.8) remain true. Remember that $\alpha < 1$, and thus $\left(\theta^{-1}(\frac{\rho}{\sigma})\right)^- (\rho^+)^{1-\alpha} = 0$ on Ω. Upon using $\left(\theta^{-1}(\frac{\rho}{\sigma})\right)^-$ as a test function in (3.19), we deduce that $\rho \geq 0$ in Ω. Subsequently, we have

$$\theta^{-1}(\frac{\rho}{\sigma}) = \frac{\rho^\alpha}{\sigma^\alpha}.$$

We can rewrite (3.19) as

$$-\frac{1}{\sigma^\alpha}\Delta\rho^\alpha + \frac{\tau}{\sigma^p}\rho^p = -F\rho^{1-\alpha} + \tau \quad \text{in} \quad \Omega. \tag{3.21}$$

Integrate this equation to obtain

$$\begin{aligned}\tau\int_\Omega \rho^p dx &= \sigma^p(\int_\Omega(-F\rho^{1-\alpha} + \tau)dx \\ &\leq \|F\|_\infty\|\rho\|_p^{1-\alpha} + \tau|\Omega| \\ &\leq c\|\rho\|_p^{2-\alpha} + c.\end{aligned} \tag{3.22}$$

The last step is due to (3.8). A simple application of the interpolation inequality

$$ab \le \varepsilon a^p + c(\varepsilon) b^q, \quad \frac{1}{p} + \frac{1}{q} = 1$$

gives (3.17). In the sequel, we will not acknowledge this interpolation inequality again when it is being used.

Applying the proof of (3.8), we can derive from (3.21) that

$$\|\rho^\alpha\|_\infty \le c\|\rho^\alpha\|_2 + c\|\rho^{1-\alpha}\|_p + c \le c. \tag{3.23}$$

By Theorem 8.22 in [5] and a boundary flattening argument [15] , we can conclude that there exists a number $\beta \in (0,1)$, depending only on the given data, such that $F, \rho^\alpha \in C^{0,\beta}(\overline{\Omega})$.

Next, we show

$$\frac{1}{\rho} \in L^s(\Omega) \quad \text{for each } s \ge 1. \tag{3.24}$$

To this end, we use $\frac{1}{(\rho+\delta)^s}$, where $\delta > 0$, as a test function in (3.21) to obtain

$$\tau \int_\Omega \frac{1}{(\rho+\delta)^s} dx \le \frac{\tau}{\sigma^p} \int_\Omega (\rho+\delta)^{p-s} dx + \|F\|_\infty \int_\Omega (\rho+\delta)^{1-\alpha-s} dx$$

from whence follows

$$\int_\Omega \frac{1}{(\rho+\delta)^s} dx \le c \int_\Omega (\rho+\delta)^{p-s} dx + c.$$

If $s \le p$, then we take $\delta \to 0$ in the above inequality to obtain

$$\int_\Omega \frac{1}{\rho^s} dx \le c \int_\Omega \rho^{p-s} dx + c.$$

It is not difficult to see that this inequalities actually holds for each $s > 1$, and thus (3.24) follows.

Now we let $v = \frac{1}{\rho^\alpha+\delta}, \delta > 0$. Then we can easily show that v satisfies the boundary value problem

$$-\Delta v + \frac{2}{v}|\nabla v|^2 = (F\rho^{1-\alpha} - \tau + \frac{\tau}{\sigma^p}\rho^p)\sigma^\alpha v^2 \quad \text{in} \quad \Omega,$$
$$\nabla v \cdot \nu = 0 \quad \text{on} \quad \partial\Omega$$

in the sense of (1.5). By the proof of (3.8), we have

$$\|v\|_\infty \le c\|v\|_2 + c\|(F\rho^{1-\alpha} - \tau + \frac{\tau}{\sigma^p}\rho^p)\sigma^\alpha v^2\|_p \le c.$$

The last step is due to (3.24). This completes the proof of Lemma 3.1. □

Let $T > 0$ be given. We divide the time interval $[0, T]$ into j equal subintervals, $j \in \{1, 2, \cdots\}$. Set

$$\tau = \frac{T}{j}.$$

We discretize and regularize the system (1.1)-(1.3) as follows. For $k = 1, \cdots, j$, solve recursively the systems

$$-\mathrm{div}\,[(\rho_k+\tau)(\nabla F_k-\mathbf{g}_k)]+\tau F_k=\frac{\rho_k-\rho_{k-1}}{\tau}\quad \text{in}\quad \Omega, \tag{3.25}$$

$$-\Delta{\rho_k}^{\alpha}+\tau{\rho_k}^{p}=-F_k{\rho_k}^{1-\alpha}+\tau\quad \text{in}\quad \Omega, \tag{3.26}$$

$$\nabla{\rho_k}^{\alpha}\cdot\nu=(\nabla F_k-\mathbf{g}_k)\cdot\nu=0\quad \text{on}\quad \partial\Omega, \tag{3.27}$$

$$\rho_0(x)=u_0(x), \tag{3.28}$$

where $\mathbf{g}_k(x)=\frac{1}{\tau}\int_{(k-1)\tau}^{k\tau}\mathbf{g}(x,t)dt$. Define the functions

$$\begin{aligned}
\tilde{u}_j(x,t)&=(t-t_{k-1})\frac{\rho_k(x)-\rho_{k-1}(x)}{\tau}+\rho_{k-1}(x),\quad x\in\Omega,\quad t\in(t_{k-1},t_k],\\
\overline{u}_j(x,t)&=\rho_k(x),\quad x\in\Omega,\quad t\in(t_{k-1},t_k],\\
\overline{F}_j(x,t)&=F_k(x),\quad x\in\Omega,\quad t\in(t_{k-1},t_k],\\
\overline{\mathbf{g}}_j(x,t)&=\mathbf{g}_k(x),\quad x\in\Omega,\quad t\in(t_{k-1},t_k],
\end{aligned}$$

where $t_k = k\tau$. We can rewrite the system (3.25)-(3.28) as

$$-\mathrm{div}\left[(\overline{u}_j+\tau)(\nabla\overline{F}_j-\overline{\mathbf{g}}_j)\right]+\tau\overline{F}_j=\frac{\partial\tilde{u}_j}{\partial t}\quad \text{in}\quad \Omega_T, \tag{3.29}$$

$$-\Delta{\overline{u}_j}^{\alpha}+\tau{\overline{u}_j}^{p}=-\overline{F}_j{\overline{u}_j}^{1-\alpha}+\tau\quad \text{in}\quad \Omega_T, \tag{3.30}$$

$$\nabla{\overline{u}_j}^{\alpha}\cdot\nu=(\nabla\overline{F}_j-\overline{\mathbf{g}}_j)\cdot\nu=0\quad \text{on}\quad \Sigma_T, \tag{3.31}$$

$$\overline{u}_j(x,0)=u_0(x)\quad \text{on}\quad \Omega. \tag{3.32}$$

We will show that there are enough a priori estimates to justify passing to the limit in the problem.

Lemma 3.2. *Let the assumptions of Lemma 3.1 hold. Then we have*

$$\begin{aligned}
&\int_{\Omega_t}{\overline{u}_j}^{\alpha-1}(\Delta{\overline{u}_j}^{\alpha})^2dxds+\tau\int_{\Omega_t}{\overline{u}_j}^{p-1}|\nabla{\overline{u}_j}^{\alpha}|^2dxds\\
&\quad+\tau\int_{\Omega_t}{\overline{u}_j}^{-1}|\nabla{\overline{u}_j}^{\alpha}|^2dxds+\tau\int_{\Omega_t}\frac{|K(\overline{u}_j)|}{\overline{u}_j}|\nabla{\overline{u}_j}^{\alpha}|^2dxds\\
&\quad+\tau\int_{\Omega_t}\frac{1}{(\overline{u}_j+\tau){\overline{u}_j}^{1-\alpha}}|\nabla{\overline{u}_j}^{\alpha}|^2dxds+\tau^2\int_{\Omega_t}{\overline{u}_j}^{p+\alpha-1}|K(\overline{u}_j)|dxds\\
&\quad+\tau^2\int_{\Omega_t}{\overline{u}_j}^{\alpha-1}|K(\overline{u}_j)|dxds+\int_{\Omega}\overline{u}_j(x,t)dx\\
&\le c+c\int_{\Omega_t}|\nabla{\overline{u}_j}^{\alpha}|dxds,
\end{aligned}\tag{3.33}$$

where

$$K(r)=\int_1^r\frac{\alpha s^{\alpha-1}}{s+\tau}ds. \tag{3.34}$$

Here and in what follows in the section c denotes a positive constant independent of j.

Proof. Let $K(r)$ be given as in (3.34). We use $-K(\rho_k)$ as a test function in (3.25) to obtain

$$\begin{aligned} &-\int_\Omega \nabla F_k\cdot\nabla\rho_k^\alpha dx-\tau\int_\Omega F_kK(\rho_k)dx+\frac{1}{\tau}\int_\Omega(\rho_k-\rho_{k-1})K(\rho_k)dx\\ &=-\int_\Omega \mathbf{g}_k\cdot\nabla\rho_k^\alpha dx. \end{aligned} \tag{3.35}$$

We proceed to estimate each integral in the above equation. For this purpose, we multiply through (3.26) by $-{\rho_k}^{\alpha-1}$ to yield

$$F_k={\rho_k}^{\alpha-1}\Delta{\rho_k}^\alpha-\tau{\rho_k}^{p+\alpha-1}+\tau{\rho_k}^{\alpha-1}. \tag{3.36}$$

Keeping this in mind, we compute

$$\begin{aligned} -\int_\Omega\nabla F_k\cdot\nabla\rho_k^\alpha dx&=\int_\Omega F_k\Delta\rho_k^\alpha dx\\ &=\int_\Omega\Delta\rho_k^\alpha({\rho_k}^{\alpha-1}\Delta{\rho_k}^\alpha-\tau{\rho_k}^{p+\alpha-1}+\tau{\rho_k}^{\alpha-1})dx\\ &=\int_\Omega{\rho_k}^{\alpha-1}(\Delta{\rho_k}^\alpha)^2dx+\frac{(p+\alpha-1)\tau}{\alpha}\int_\Omega{\rho_k}^{p-1}|\nabla\rho_k^\alpha|^2dx\\ &\quad+\frac{(1-\alpha)\tau}{\alpha}\int_\Omega{\rho_k}^{-1}|\nabla\rho_k^\alpha|^2dx. \end{aligned} \tag{3.37}$$

As for the second integral in (3.35), we have

$$\begin{aligned} -\tau\int_\Omega F_kK(\rho_k)dx&=-\tau\int_\Omega K(\rho_k)({\rho_k}^{\alpha-1}\Delta{\rho_k}^\alpha-\tau{\rho_k}^{p+\alpha-1}+\tau{\rho_k}^{\alpha-1})dx\\ &=-\frac{(1-\alpha)\tau}{\alpha}\int_\Omega\frac{K(\rho_k)}{\rho_k}|\nabla{\rho_k}^\alpha|^2dx+\tau\int_\Omega\frac{1}{(\rho_k+\tau){\rho_k}^{1-\alpha}}|\nabla{\rho_k}^\alpha|^2dx\\ &\quad+\tau^2\int_\Omega{\rho_k}^{p+\alpha-1}K(\rho_k)dx-\tau^2\int_\Omega{\rho_k}^{\alpha-1}K(\rho_k)dx. \end{aligned} \tag{3.38}$$

To deal with the last four integrals in (3.38), we assume $\alpha<1$. First observe that on the set $\{\rho_k\geq 1\}$ we have

$$K(\rho_k)\leq\int_1^{\rho_k}\alpha s^{\alpha-2}ds=\frac{\alpha}{1-\alpha}(1-{\rho_k}^{\alpha-1})\leq\frac{\alpha}{1-\alpha} \tag{3.39}$$

and

$$K(\rho_k)\geq\int_1^{\rho_k}\alpha(s+\tau)^{\alpha-2}ds=\frac{\alpha}{1-\alpha}((1+\tau)^{\alpha-1}-(\rho_k+\tau)^{\alpha-1}), \tag{3.40}$$

while on the set $\{\rho_k\leq 1\}$ we have

$$-K(\rho_k)\leq\int_{\rho_k}^1\alpha s^{\alpha-2}ds=\frac{\alpha}{1-\alpha}({\rho_k}^{\alpha-1}-1)\leq\frac{\alpha}{1-\alpha}{\rho_k}^{\alpha-1}$$

and

$$\begin{aligned} -K(\rho_k)&\geq\int_{\rho_k}^1\frac{\alpha}{(s+\tau)^{2-\alpha}}ds\\ &=\frac{\alpha}{1-\alpha}\left((\rho_k+\tau)^{\alpha-1}-(1+\tau)^{\alpha-1}\right). \end{aligned} \tag{3.41}$$

Keeping in mind the preceding inequalities, we have

$$
\begin{aligned}
{\rho_k}^{p+\alpha-1}K(\rho_k) &= {\rho_k}^{p+\alpha-1}|K(\rho_k)| - 2\chi_{\{\rho_k\le 1\}}{\rho_k}^{p+\alpha-1}|K(\rho_k)| \\
&\ge {\rho_k}^{p+\alpha-1}|K(\rho_k)| - \frac{2\alpha}{1-\alpha}{\rho_k}^{p+2\alpha-2}\chi_{\{\rho_k\le 1\}} \\
&\ge {\rho_k}^{p+\alpha-1}|K(\rho_k)| - c \text{ since } p>2, \qquad (3.42)\\
-{\rho_k}^{\alpha-1}K(\rho_k) &= {\rho_k}^{\alpha-1}|K(\rho_k)| - 2{\rho_k}^{\alpha-1}|K(\rho_k)|\chi_{\{\rho_k\ge 1\}} \\
&\ge {\rho_k}^{\alpha-1}|K(\rho_k)| - c \text{ since } \alpha<1, \text{ and} \qquad (3.43)\\
-\frac{K(\rho_k)}{\rho_k} &= \frac{|K(\rho_k)|}{\rho_k} - 2\frac{K(\rho_k)}{\rho_k}\chi_{\{\rho_k\ge 1\}} \\
&\ge \frac{|K(\rho_k)|}{\rho_k} - \frac{2\alpha}{1-\alpha}\chi_{\{\rho_k\ge 1\}}. \qquad (3.44)
\end{aligned}
$$

Consequently, we can conclude that

$$
\begin{aligned}
&\frac{(p+\alpha-1)\tau}{\alpha}\int_\Omega {\rho_k}^{p-1}|\nabla\rho_k^\alpha|^2dx - \frac{(1-\alpha)\tau}{\alpha}\int_\Omega \frac{K(\rho_k)}{\rho_k}|\nabla{\rho_k}^\alpha|^2dx \\
&\ge \frac{(p-\alpha-1)\tau}{\alpha}\int_\Omega {\rho_k}^{p-1}|\nabla\rho_k^\alpha|^2dx + \frac{(1-\alpha)\tau}{\alpha}\int_\Omega \frac{|K(\rho_k)|}{\rho_k}|\nabla{\rho_k}^\alpha|^2dx \\
&\quad +2\tau\int_\Omega {\rho_k}^{p-1}|\nabla\rho_k^\alpha|^2dx - 2\tau\int_{\{\rho_k\ge 1\}}|\nabla{\rho_k}^\alpha|^2dx \\
&\ge \frac{(p-\alpha-1)\tau}{\alpha}\int_\Omega {\rho_k}^{p-1}|\nabla\rho_k^\alpha|^2dx + \frac{(1-\alpha)\tau}{\alpha}\int_\Omega \frac{|K(\rho_k)|}{\rho_k}|\nabla{\rho_k}^\alpha|^2dx. \qquad (3.45)
\end{aligned}
$$

The left-hand side of (3.35) can be estimated as follows

$$
\begin{aligned}
&-\int_\Omega \nabla F_k\cdot\nabla\rho_k^\alpha dx - \tau\int_\Omega F_kK(\rho_k)dx + \frac{1}{\tau}\int_\Omega(\rho_k-\rho_{k-1})K(\rho_k)dx \\
&\ge \int_\Omega {\rho_k}^{\alpha-1}(\Delta{\rho_k}^\alpha)^2dx + \frac{(p-\alpha-1)\tau}{\alpha}\int_\Omega {\rho_k}^{p-1}|\nabla\rho_k^\alpha|^2dx \\
&\quad +\frac{(1-\alpha)\tau}{\alpha}\int_\Omega {\rho_k}^{-1}|\nabla\rho_k^\alpha|^2dx + \frac{(1-\alpha)\tau}{\alpha}\int_\Omega \frac{|K(\rho_k)|}{\rho_k}|\nabla{\rho_k}^\alpha|^2dx \\
&\quad +\tau\int_\Omega \frac{1}{(\rho_k+\tau){\rho_k}^{1-\alpha}}|\nabla{\rho_k}^\alpha|^2dx + \tau^2\int_\Omega {\rho_k}^{p+\alpha-1}|K(\rho_k)|dx \\
&\quad +\tau^2\int_\Omega {\rho_k}^{\alpha-1}|K(\rho_k)|dx - c + \frac{1}{\tau}\int_\Omega\int_{\rho_{k-1}}^{\rho_k}K(s)dsdx. \qquad (3.46)
\end{aligned}
$$

Plugging (3.46) into (3.35), multiplying through the resulting inequality by τ, and then summing up over k, we obtain

$$\begin{aligned}
&\int_{\Omega_t} \overline{u}_j^{\alpha-1}(\Delta\overline{u}_j^{\alpha})^2 dxds + \tau\int_{\Omega_t} \overline{u}_j^{p-1}|\nabla\overline{u}_j^{\alpha}|^2 dxds \\
&\quad + \tau\int_{\Omega_t} \overline{u}_j^{-1}|\nabla\overline{u}_j^{\alpha}|^2 dxds + \tau\int_{\Omega_t} \frac{|K(\overline{u}_j)|}{\overline{u}_j}|\nabla\overline{u}_j^{\alpha}|^2 dxds \\
&\quad + \tau\int_{\Omega_t} \frac{1}{(\overline{u}_j+\tau)\overline{u}_j^{1-\alpha}}|\nabla\overline{u}_j^{\alpha}|^2 dxds + \tau^2\int_{\Omega_t} \overline{u}_j^{p+\alpha-1}|K(\overline{u}_j)|dxds \\
&\quad + \tau^2\int_{\Omega_t} \overline{u}_j^{\alpha-1}|K(\overline{u}_j)|dxds + c\int_{\Omega}\int_{u_0}^{\overline{u}_j(x,t)} K(s)dsdx \\
&\leq c + c\int_{\Omega_t} |\nabla\overline{u}_j^{\alpha}|dxds \qquad (3.47)
\end{aligned}$$

for $t \in \{t_1, t_2, \cdots, t_j\}$. It is not difficult to see that the above also holds for each $t \in (0, T]$. We claim that

$$\int_{\Omega}\int_{u_0}^{\overline{u}_j} K(s)dsdx \geq c\int_{\Omega} \overline{u}_j dx - c,$$

where $\overline{u}_j = \overline{u}_j(x,t)$. To see this, we estimate

$$\begin{aligned}
\int_1^{\overline{u}_j} K(s)ds &= \chi_{\{\overline{u}_j\geq 1\}}\int_1^{\overline{u}_j} K(s)ds + \chi_{\{\overline{u}_j<1\}}\int_1^{\overline{u}_j} K(s)ds \\
&\geq \chi_{\{\overline{u}_j\geq 1\}}\left(\frac{\alpha}{1-\alpha}(1+\tau)^{\alpha-1}(\overline{u}_j-1) - \frac{1}{1-\alpha}((\overline{u}_j+\tau)^{\alpha}-(1+\tau)^{\alpha})\right) \\
&\quad +\chi_{\{\overline{u}_j<1\}}\left(-\frac{\alpha}{1-\alpha}(1+\tau)^{1-\alpha}(1-\overline{u}_j) + \frac{1}{1-\alpha}((1+\tau)^{\alpha}-(\overline{u}_j+\tau)^{\alpha})\right) \\
&\geq c\overline{u}_j - c.
\end{aligned}$$

This immediately implies

$$\begin{aligned}
\int_{u_0}^{\overline{u}_j} K(s)ds &= \int_1^{\overline{u}_j} K(s)ds + \int_{u_0}^1 K(s)ds \\
&\geq c\overline{u}_j - c.
\end{aligned}$$

This together with (3.47) yields the lemma. The proof is complete. □

Obviously, Lemma 3.2 is the discretized version of (1.10). At this stage, we can not control the right hand side of (3.33), and thus it is useless by itself. To make it work, we need additional estimates, which will require additional assumptions on the data.

4. Proof of Theorem 1.1. Now we are operating under the assumptions of Theorem 1.1. Thus Lemmas 3.1 and 3.2 remain valid. The proof is divided into several lemmas. For simplicity, we assume that $\alpha < 1$.

LEMMA 4.1. *Let $\{\overline{u}_j, \overline{F}_j\}$ be given as in (3.29)-(3.32). Then*

$$\begin{aligned}&\int_{\Omega_t}(\overline{u}_j+\tau)|\nabla\overline{F}_j|^2dxds+\tau\int_{\Omega_t}\overline{F}_j^{\,2}dxds\\&\quad+\int_\Omega|\nabla\overline{u}_j^{\,\alpha}(x,t)|^2dx+\tau\int_\Omega\overline{u}_j^{\,p+\alpha}(x,t)dx\\&\le c+c\int_{\Omega_t}\overline{u}_jdxds.\end{aligned}\tag{4.1}$$

Proof. Using F_k as a test function in (3.25) and applying the Hölder inequality in the resulting equation yield

$$\int_\Omega(\rho_k+\tau)|\nabla F_k|^2dx+\tau\int_\Omega F_k^{\,2}dx-\frac{c}{\tau}\int_\Omega F_k(\rho_k-\rho_{k-1})dx\le c\int_\Omega(\rho_k+\tau)dx.\tag{4.2}$$

Recall that if $\theta(s)$ is an increasing function on the interval $[a,b]$ then the inequality

$$\int_a^b\theta(s)ds\le(b-a)\theta(b)$$

holds while if $\theta(s)$ is a decreasing function on the interval the reverse inequality is true. Keeping this and (1.15) in mind, we have

$$\begin{aligned}\int_\Omega(-F_k)(\rho_k-\rho_{k-1})dx&=\int_\Omega(-\rho_k^{\,\alpha-1}\Delta\rho_k^\alpha+\tau\rho_k^{\,p+\alpha-1}-\tau\rho_k^{\,\alpha-1})(\rho_k-\rho_{k-1})dx\\&\ge\frac{1}{2\alpha}\int_\Omega\left(|\nabla\rho_k^\alpha|^2-|\nabla\rho_{k-1}^\alpha|^2\right)dx+\frac{\tau}{p+\alpha}\int_\Omega\left(\rho_k^{\,p+\alpha}-\rho_{k-1}^{p+\alpha}\right)dx\\&\quad-\frac{\tau}{\alpha}\int_\Omega\left(\rho_k^{\,\alpha}-\rho_{k-1}^\alpha\right)dx.\end{aligned}$$

Plug this into (4.2), multiply through the resulting inequality by τ, then sum up over k, and thereby obtain the lemma. □

LEMMA 4.2. *We have*

$$\begin{aligned}&\int_{\Omega_T}(\overline{u}_j+\tau)|\nabla\overline{F}_j|^2dxds+\tau\int_{\Omega_T}\overline{F}_j^{\,2}dxds\\&\quad+\sup_{0\le t\le T}\int_\Omega|\nabla\overline{u}_j^{\,\alpha}|^2dx+\tau\sup_{0\le t\le T}\int_\Omega\overline{u}_j^{\,p+\alpha}\le c,\\&\int_{\Omega_T}\overline{u}_j^{\,\alpha-1}(\Delta\overline{u}_j^{\,\alpha})^2dxds+\tau\int_{\Omega_T}\overline{u}_j^{\,p-1}|\nabla\overline{u}_j^{\,\alpha}|^2dxds\\&\quad+\tau\int_{\Omega_T}\overline{u}_j^{\,-1}|\nabla\overline{u}_j^{\,\alpha}|^2dxds+\tau\int_{\Omega_T}\frac{|K(\overline{u}_j)|}{\overline{u}_j}|\nabla\overline{u}_j^{\,\alpha}|^2dxds\\&\quad+\tau\int_{\Omega_T}\frac{1}{(\overline{u}_j+\tau)\overline{u}_j^{\,1-\alpha}}|\nabla\overline{u}_j^{\,\alpha}|^2dxds+\tau^2\int_{\Omega_T}\overline{u}_j^{\,p+\alpha-1}|K(\overline{u}_j)|dxds\\&\quad+\tau^2\int_{\Omega_T}\overline{u}_j^{\,\alpha-1}|K(\overline{u}_j)|dxds+\sup_{0\le t\le T}\int_\Omega\overline{u}_jdx\le c.\end{aligned}$$

Proof. We infer from (3.33) and (4.1) that

$$\int_\Omega \overline{u}_j(x,t)dx \le c + c\int_{\Omega_t} |\nabla \overline{u}_j{}^\alpha(x,s)|dxds, \tag{4.3}$$

$$\int_\Omega |\nabla \overline{u}_j{}^\alpha(x,t)|^2 dx \le c\int_{\Omega_t} \overline{u}_j(x,s)dxds + c. \tag{4.4}$$

For each $\varepsilon > 0$ we derive from (4.3) that

$$\begin{aligned}\int_{\Omega_t} \overline{u}_j(x,s)dxds &\le c + c\int_{\Omega_t} |\nabla \overline{u}_j{}^\alpha(x,s)|dxds \\ &\le \varepsilon \int_{\Omega_t} |\nabla \overline{u}_j{}^\alpha(x,s)|^2 dxds + c(\varepsilon).\end{aligned}$$

Use this in (4.4), choose ε suitably small, and thereby yield

$$\int_\Omega |\nabla \overline{u}_j{}^\alpha(x,t)|^2 dx + \int_\Omega \overline{u}_j(x,t)dx \le c. \tag{4.5}$$

This finishes the proof. □

LEMMA 4.3. *$\{\tilde{u}_j\}$ is precompact in $C([0,T];L^s(\Omega))$ and $\{\overline{u}_j\}$ is precompact in $L^2((0,T);L^s(\Omega))$ for each $1 \le s < \frac{N}{N-2}$.*

Proof. Fix $1 < s < \frac{N}{N-2}$. We first show that $\{\tilde{u}_j\}$ is precompact in $C([0,T];L^s(\Omega))$. Since the imbedding $W^{1,\frac{N}{N-1}}(\Omega) \hookrightarrow L^s(\Omega)$ is compact and $L^s(\Omega) \hookrightarrow \left(W^{1,N}(\Omega)\right)^*$ is continuous, the desired result will be a consequence of the boundedness of $\{\tilde{u}_j\}$ in $L^\infty((0,T);W^{1,\frac{N}{N-1}}(\Omega))$ combined with the boundedness of $\{\frac{\partial \tilde{u}_j}{\partial t}\}$ in the space $L^2\left((0,T);\left(W^{1,N}(\Omega)\right)^*\right)$ due to a result in [11]. To seek upper bounds for the two sequences in their respective spaces, we first recall the Sobolev inequality

$$\|\overline{u}_j{}^\alpha\|_{2^*} \le c(\|\nabla \overline{u}_j{}^\alpha\|_2 + \|\overline{u}_j{}^\alpha\|_2).$$

In the interpolation inequality

$$\|h\|_s \le \varepsilon\|h\|_q + c(\varepsilon)\|h\|_r, \quad q > s \ge r > 0,$$

take $h = \overline{u}_j{}^\alpha, s = 2, q = 2^*, r = 1$, choose ε suitably small, and thereby obtain

$$\|\overline{u}_j{}^\alpha\|_{2^*} \le c(\|\nabla \overline{u}_j{}^\alpha\|_2 + \|\overline{u}_j{}^\alpha\|_1) \le c. \tag{4.6}$$

The last step is due to Lemma 4.2 and the fact that $\alpha < 1$. To continue, we invoke the inequality

$$\|fg\|_{\frac{2q}{2+q}} \le \|f\|_2\|g\|_q, \quad f \in L^2(\Omega), \quad g \in L^q(\Omega), \quad q > 2. \tag{4.7}$$

Write $\nabla \overline{u}_j = \frac{1}{\alpha}\overline{u}_j{}^{1-\alpha}\nabla \overline{u}_j{}^\alpha$. Lemma 4.2 asserts that $\{|\nabla \overline{u}_j{}^\alpha|\}$ is bounded in $L^\infty((0,T);L^2(\Omega))$ and $\{\overline{u}_j{}^{1-\alpha}\}$ is bounded in $L^\infty((0,T);L^q(\Omega))$ with $q = \frac{2^*\alpha}{1-\alpha}$. Thus we conclude from (4.7) that

$$\nabla \overline{u}_j \text{ is bounded in } L^\infty((0,T);L^r(\Omega)) \text{ with } r = \tfrac{2\alpha N}{N-2+2\alpha} \ge \tfrac{N}{N-1}. \tag{4.8}$$

It is easy to see

$$\sup_{0\le t\le T}\|\nabla\tilde{u}_j\|_{\frac{N}{N-1}}\le 2\sup_{0\le t\le T}\|\nabla\overline{u}_j\|_{\frac{N}{N-1}}\le c.$$

We estimate

$$\begin{aligned}\int_0^T\left\|(\overline{u}_j+\tau)\nabla\overline{F}_j\right\|^2_{\frac{N}{N-1}}dt &= \int_0^T\left(\int_\Omega|(\overline{u}_j+\tau)\nabla\overline{F}_j|^{\frac{N}{N-1}}dx\right)^{\frac{2(N-1)}{N}}dt\\ &\le c\int_{\Omega_T}(\overline{u}_j+\tau)|\nabla\overline{F}_j|^2dxdt\cdot\sup_{0\le t\le T}\|\overline{u}_j+\tau\|_{\frac{N}{N-2}}\\ &\le c. \end{aligned}\tag{4.9}$$

This together with (3.29) implies that $\{\frac{\partial\tilde{u}_j}{\partial t}\}$ is bounded in the space $L^2\left((0,T);\left(W^{1,N}(\Omega)\right)^*\right)$. This completes the proof of fact that $\{\tilde{u}_j\}$ is precompact in both $C([0,T];L^s(\Omega))$ and $L^2\left((0,T);\left(W^{1,N}(\Omega)\right)^*\right)$. We claim

$$\int_0^T\|\tilde{u}_j-\overline{u}_j\|^2_{(W^{1,N}(\Omega))^*}dt\le c\tau^2.\tag{4.10}$$

To this end, we derive from the definitions of $\overline{u}_j$ and $\tilde{u}_j$ that for each $t\in(t_{k-1},t_k]$ there holds

$$\begin{aligned}\tilde{u}_j-\overline{u}_j &= (t-t_k)\frac{\rho_k-\rho_{k-1}}{\tau}\\ &= (t-t_k)\left[-\mathrm{div}\left[(\rho_k+\tau)(\nabla F_k-\mathbf{g}_k)\right]+\tau F_k\right].\end{aligned}$$

It immediately follows that

$$\|\tilde{u}_j-\overline{u}_j\|_{(W^{1,N}(\Omega))^*}\le c|t-t_k|(\|(\rho_k+\tau)(\nabla F_k-\mathbf{g}_k)\|_{\frac{N}{N-1}}+\|\tau F_k\|_{\frac{N}{N-1}}),\ \ t\in(t_{k-1},t_k].$$

Keeping in mind (4.9) and Lemma 4.2, we calculate

$$\begin{aligned}\int_0^T\|\tilde{u}_j-\overline{u}_j\|^2_{(W^{1,N}(\Omega))^*}dt &= \sum_{k=1}^{k=j}\int_{t_{k-1}}^{t_k}\|\tilde{u}_j-\overline{u}_j\|^2_{(W^{1,N}(\Omega))^*}dt\\ &\le c\sum_{k=1}^{k=j}\tau^3(\|(\rho_k+\tau)(\nabla F_k-\mathbf{g}_k)\|^2_{\frac{N}{N-1}}+\|\tau F_k\|^2_{\frac{N}{N-1}})\\ &\le c\tau^2(\int_0^T\left\|(\overline{u}_j+\tau)\nabla\overline{F}_j\right\|^2_{\frac{N}{N-1}}dt+\int_0^T\|\tau\overline{F}_j\|^2_{\frac{N}{N-1}}dt+c)\\ &\le c\tau^2.\end{aligned}$$

Thus $\{\overline{u}_j\}$ is also precompact in $L^2\left((0,T);\left(W^{1,N}(\Omega)\right)^*\right)$. This, along with (4.8), puts us in a position to apply a result in [11], which yields that $\{\overline{u}_j\}$ is precompact in $L^2((0,T);L^s(\Omega))$. □

LEMMA 4.4. *$\{\Delta\overline{u}_j{}^\alpha\}$ is bounded in $L^2((0,T);L^q(\Omega))$, where $q=\frac{4\alpha N}{\alpha N+N+2\alpha-2}\ge\frac{4N}{3N-2}$.*

Proof. For each $1 < q < 2$, we estimate from Lemma 4.2 that

$$\begin{aligned}\int_\Omega |\Delta \overline{u}_j{}^\alpha|^q dx &= \int_\Omega |\Delta \overline{u}_j{}^\alpha|^q (\overline{u}_j)^{\frac{(\alpha-1)q}{2}} (\overline{u}_j)^{\frac{(1-\alpha)q}{2}} dx \\ &\leq \left(\int_\Omega |\Delta \overline{u}_j{}^\alpha|^2 \overline{u}_j{}^{\alpha-1} dx\right)^{\frac{q}{2}} \left(\int_\Omega \overline{u}_j{}^{\frac{(1-\alpha)q}{2-q}} dx\right)^{\frac{2-q}{2}} \\ &\leq c \left(\int_\Omega |\Delta \overline{u}_j{}^\alpha|^2 \overline{u}_j{}^{\alpha-1} dx\right)^{\frac{q}{2}} \quad \text{if we take } q = \tfrac{4\alpha N}{\alpha N + N + 2\alpha - 2}.\end{aligned}$$

Whence follows the desired result. ▯

Lemma 4.5. $\{(\overline{u}_j)^{\frac{\alpha}{2}}\}$ *is precompact in* $L^2((0,T); W^{1,2}(\Omega))$.

Proof. The key observation here is that for $\sigma > 0$ sufficiently small there hold the inequalities

$$\|\overline{F}_j \overline{u}_j{}^{1-\alpha-\sigma}\|_{\frac{N+2}{N}} \leq c \quad \text{and} \quad \|\overline{u}_j{}^\sigma\|_q \leq c \;\; \text{for some } q > \tfrac{N+2}{2}. \tag{4.11}$$

To see this, we multiply through (3.30) by $-\overline{u}_j{}^{-\sigma}$ to obtain

$$\overline{F}_j \overline{u}_j{}^{1-\alpha-\sigma} = \overline{u}_j{}^{-\sigma} \Delta \overline{u}_j{}^\alpha - \tau \overline{u}_j{}^{p-\sigma} + \tau \overline{u}_j{}^{-\sigma}. \tag{4.12}$$

The first term in the right-hand side of the above equation can be represented in the form

$$\overline{u}_j{}^{-\sigma} \Delta \overline{u}_j{}^\alpha = \overline{u}_j{}^{\frac{\alpha-1}{2}} \Delta \overline{u}_j{}^\alpha \overline{u}_j{}^{\frac{1-\alpha}{2}-\sigma}.$$

By (4.7), this term is bounded in $L^r(\Omega_T)$ with

$$r = \frac{2q}{q+2},$$

where $q = \frac{2^*\alpha}{\frac{1-\alpha}{2}-\sigma}$. A simple calculation shows that

$$r|_{\sigma=0} > \frac{N+2}{N}.$$

Thus we can pick $\sigma > 0$ so small that it satisfies

$$r > \frac{N+2}{N}, \quad \frac{1-\alpha}{\sigma} > \max\{2, \frac{N+2}{N}\} \;\text{ and }\; \sigma\frac{N+2}{2} < \alpha 2^*. \tag{4.13}$$

Let σ be so chosen. Then the second inequality in (4.11) is automatically true, and the first term on the right-hand side of (4.12) is bounded in $L^{\frac{N+2}{N}}(\Omega_T)$. To see that the same is true for the last term there, we estimate, with the aid of Lemma 4.2, that for each $\tau < 1$ there holds

$$\begin{aligned}\tau^2 \int_{\Omega_T} \overline{u}_j{}^{\alpha-1} dxdt &= \tau^2 \int_{\{\overline{u}_j \leq \tau\}} \overline{u}_j{}^{\alpha-1} dxdt + \tau^2 \int_{\{\overline{u}_j > \tau\}} \overline{u}_j{}^{\alpha-1} dxdt \\ &\leq \tau^2 \int_{\{\overline{u}_j \leq \tau\}} \frac{K(\overline{u}_j)}{K(\tau)} \overline{u}_j{}^{\alpha-1} dxdt + c\tau^{\alpha+1} \\ &\leq \frac{c}{|K(\tau)|} + c\tau^{\alpha+1} \to 0 \quad \text{as } \tau \to 0.\end{aligned} \tag{4.14}$$

Here we have used the fact that $K(r)$ is an increasing function and $\lim_{r\to 0} K(r) = -\infty$. Thus $\{\tau \overline{u}_j{}^{-\sigma}\}$ is bounded in $L^{\frac{1-\alpha}{\sigma}}(\Omega_T)$ if (4.13) holds. To estimate the remaining term there, we calculate from the Sobolev inequality that

$$\begin{aligned}
&\int_{\Omega_T} \overline{u}_j{}^{2(\frac{p-1}{2}+\alpha)+\frac{2(p+\alpha)}{N}} dxdt \\
&\le \int_0^T \left(\int_\Omega \overline{u}_j{}^{2(\frac{p-1}{2}+\alpha)\frac{N}{N-2}} dx\right)^{\frac{N-2}{N}} \cdot \left(\int_\Omega \overline{u}_j{}^{p+\alpha} dx\right)^{\frac{2}{N}} dt \\
&\le c\left(\int_{\Omega_T} \overline{u}_j{}^{p-1}|\nabla \overline{u}_j{}^{\alpha}|^2 dxdt + \int_{\Omega_T} \overline{u}_j{}^{p-1+2\alpha} dxdt\right)\left(\sup_{0\le t\le T}\int_\Omega \overline{u}_j{}^{p+\alpha} dx\right)^{\frac{2}{N}}.
\end{aligned}$$

Multiply through this inequality by $\tau^{1+\frac{2}{N}}$, apply Lemma 4.2 in the resulting inequality, and thereby obtain

$$\tau^{1+\frac{2}{N}} \int_{\Omega_T} \overline{u}_j{}^{\frac{(N+2)p}{N}+(2+\frac{2}{N})\alpha-1} dxdt \le c. \tag{4.15}$$

Subsequently, we have

$$\begin{aligned}
\int_{\Omega_T} \left(\tau \overline{u}_j{}^{p-\sigma}\right)^{\frac{N+2}{N}} dxdt &= \tau^{1+\frac{2}{N}} \int_{\Omega_T} \overline{u}_j{}^{\frac{(N+2)p}{N}-\frac{\sigma(N+2)}{N}} dxdt \\
&\le c \quad \text{because } (2+\tfrac{2}{N})\alpha - 1 > 0.
\end{aligned}$$

Hence (4.11) follows.

Therefore, passing to a subsequence if necessary, we have that

$$\overline{u}_j \rightharpoonup u \quad \text{a.e. on } \Omega_T \text{ and weak}^* \text{ in } L^\infty((0,T); L^{\frac{N}{N-2}}(\Omega)), \tag{4.16}$$

$$\overline{F}_j(\overline{u}_j)^{1-\alpha-\sigma} \rightharpoonup G \quad \text{weakly in } L^{\frac{N+2}{N}}(\Omega_T). \tag{4.17}$$

Consequently, we have

$$\Delta \overline{u}_j{}^{\alpha} \rightharpoonup \Delta u^\alpha \quad \text{weakly in } L^2((0,T); L^{\frac{4N}{3N-2}}(\Omega)), \tag{4.18}$$

$$\overline{F}_j \overline{u}_j{}^{1-\alpha} = \overline{F}_j(\overline{u}_j)^{1-\alpha-\sigma}\cdot(\overline{u}_j)^\sigma \rightharpoonup G(u)^\sigma \quad \text{weakly in } L^1(\Omega_T). \tag{4.19}$$

Note that

$$\begin{aligned}
\int_\Omega \tau \overline{u}_j{}^p dx &\le c\tau\left(\int_\Omega \overline{u}_j{}^{p+\alpha} dx\right)^{\frac{p}{p+\alpha}} \\
&\le c\left(\tau^{\frac{p+\alpha}{p}} \int_\Omega \overline{u}_j{}^{p+\alpha} dx\right)^{\frac{p}{p+\alpha}} \\
&\le c\tau^{\frac{\alpha}{p+\alpha}}.
\end{aligned} \tag{4.20}$$

We are ready to take $j \to \infty$ in (3.30) to obtain

$$-\Delta u^\alpha = -Gu^\sigma \quad \text{in } \Omega_T, \tag{4.21}$$

while the boundary condition $\nabla u^\alpha \cdot \nu = 0$ on Σ_T is understood in the sense of (1.5). Let $\delta > 0$. We see from (4.18) that $\ln(u^\alpha + \delta)$ is a legitimate test function. Upon using it, we obtain

$$\int_{\Omega_T} \frac{|\nabla u^\alpha|^2}{u^\alpha + \delta} dxdt = -\int_{\Omega_T} Gu^\sigma \ln(u^\alpha+\delta) dxdt. \tag{4.22}$$

Note that for each $\varepsilon > 0$ there is a positive number $c = c(\varepsilon)$ such that

$$|\ln s| \le cs^{-\varepsilon} \quad \text{on } (0,1] \ \text{ and } \ \ln s \le cs^{\varepsilon} \quad \text{on } [1,\infty). \tag{4.23}$$

Combining these two yields

$$|\ln s| \le c\left(\frac{1}{s^{\varepsilon}} + s^{\varepsilon}\right) \quad \text{on } (0,\infty). \tag{4.24}$$

This together with (4.11) implies that

$$|Gu^{\sigma}\ln(u^{\alpha}+\delta)| \le |G(c + cu^{\sigma+\varepsilon})| \in L^1(\Omega_T)$$

for sufficiently small ε and δ. The Monotone Convergence Theorem and the Lebesgue Convergence Theorem guarantee that we can take $\delta \to 0$ in (4.22) to obtain

$$\int_{\Omega_T} \frac{|\nabla u^{\alpha}|^2}{u^{\alpha}} dxdt = -\int_{\Omega_T} Gu^{\sigma}\ln u^{\alpha} dxdt. \tag{4.25}$$

It follows from (4.11) and (4.24)

$$\{\overline{u}_j{}^{\sigma}\ln \overline{u}_j{}^{\alpha}\} \text{ is bounded in } L^q(\Omega_T) \text{ for some } q > \tfrac{N+2}{2}. \tag{4.26}$$

This together with Lemma 4.3 implies

$$\overline{u}_j{}^{\sigma}\ln \overline{u}_j{}^{\alpha} \to u^{\sigma}\ln u^{\alpha} \quad \text{strongly in } L^2((0,T); L^{\frac{N+2}{2}}(\Omega)). \tag{4.27}$$

For ε sufficiently small, we deduce from (4.24) and Lemma 4.2 that

$$\begin{aligned}\int_{\Omega_T} \tau|\ln \overline{u}_j{}^{\alpha}| dxdt &\le c\left(\tau\int_{\Omega_T}\frac{1}{\overline{u}_j{}^{\alpha\varepsilon}}dxdt + \tau\int_{\Omega_T}\overline{u}_j{}^{\alpha\varepsilon}dxdt\right)\\ &\le c\left(\tau^{1-\frac{2\alpha\varepsilon}{1-\alpha}}\left(\int_{\Omega_T}\tau^2\overline{u}_j{}^{\alpha-1}dxdt\right)^{\frac{\alpha\varepsilon}{1-\alpha}} + \tau\int_{\Omega_T}\overline{u}_j{}^{\alpha\varepsilon}dxdt\right)\\ &\le c\tau^{1-\frac{2\alpha\varepsilon}{1-\alpha}} \to 0 \quad \text{as } j\to\infty. \end{aligned} \tag{4.28}$$

By (4.24) and a calculation similar to (4.20), we have

$$\tau\int_{\Omega_T}\overline{u}_j{}^{p}\ln \overline{u}_j{}^{\alpha}dxdt \to 0.$$

Keeping the preceding results in mind, we have

$$\begin{aligned}&\int_{\Omega_T}\overline{F}_j\overline{u}_j{}^{1-\alpha}\ln \overline{u}_j{}^{\alpha}dxdt\\ &= \int_{\Omega_T}\overline{F}_j\overline{u}_j{}^{1-\alpha-\sigma}\cdot \overline{u}_j{}^{\sigma}\ln \overline{u}_j{}^{\alpha}dxdt \to \int_{\Omega_T}Gu^{\sigma}\ln u^{\alpha}dxdt.\end{aligned} \tag{4.29}$$

In view of the proof of (4.25), we can use $\ln \overline{u}_j{}^{\alpha}$ as a test function in (3.30) to obtain

$$\begin{aligned}&\int_{\Omega_T}\frac{|\nabla \overline{u}_j{}^{\alpha}|^2}{\overline{u}_j{}^{\alpha}}dxdt\\ &= -\int_{\Omega_T}\overline{F}_j\overline{u}_j{}^{1-\alpha}\ln \overline{u}_j{}^{\alpha}dxdt - \tau\int_{\Omega_T}\overline{u}_j{}^{p}\ln \overline{u}_j{}^{\alpha}dxdt + \tau\int_{\Omega_T}\ln \overline{u}_j{}^{\alpha}dxdt\\ &\to -\int_{\Omega_T}Gu^{\sigma}\ln u^{\alpha}dxdt = \int_{\Omega_T}\frac{|\nabla u^{\alpha}|^2}{u^{\alpha}}dxdt.\end{aligned}$$

This is sufficient to conclude the lemma. □

We remark that it does not seem possible to derive any bounds for the sequence $\{\ln \overline{u}_j{}^\alpha\}$. Fortunately, we have that $\{\overline{u}_j{}^\varepsilon \ln \overline{u}_j{}^\alpha\}$ is precompact in certain $L^q(\Omega_T)$ spaces if ε is suitably small. This is what has made our arguments work.

We are ready to prove the main existence theorem.

Proof of Theorem 1.1. According to Lemma 4.2, we may assume that

$$\overline{u}_j{}^{\frac{\alpha-1}{2}} \Delta \overline{u}_j{}^\alpha \rightharpoonup \eta \quad \text{weakly in } L^2(\Omega_T).$$

We claim that

$$\eta = u^{\frac{\alpha-1}{2}} \Delta u^\alpha \ \text{ a.e. on } \quad \{(x,t) \in \Omega_T : u(x,t) > 0\}. \tag{4.30}$$

To see this, we define

$$d_\varepsilon(s) = \begin{cases} 0 & \text{if } s < \varepsilon \\ s - \varepsilon & \text{if } \varepsilon \le s \le \frac{1}{\varepsilon} \\ \frac{1}{\varepsilon} & \text{if } s > \frac{1}{\varepsilon}. \end{cases}$$

Then the sequence $\{d_\varepsilon(\overline{u}_j)\overline{u}_j{}^{\frac{\alpha-1}{2}}\}$ is bounded in $L^\infty(\Omega)$. By (4.16) and (4.18), we have

$$\left(d_\varepsilon(\overline{u}_j)\overline{u}_j{}^{\frac{\alpha-1}{2}}\right) \Delta \overline{u}_j{}^\alpha \rightharpoonup d_\varepsilon(u) u^{\frac{\alpha-1}{2}} \Delta u^\alpha \quad \text{weakly in } L^1(\Omega_T).$$

On the other hand, we obviously have

$$d_\varepsilon(\overline{u}_j)\left(\overline{u}_j{}^{\frac{\alpha-1}{2}} \Delta \overline{u}_j{}^\alpha\right) \rightharpoonup d_\varepsilon(u)\eta \quad \text{weakly in } L^1(\Omega_T).$$

Consequently,

$$d_\varepsilon(u)\eta = d_\varepsilon(u) u^{\frac{\alpha-1}{2}} \Delta u^\alpha$$

for each $\varepsilon > 0$. Taking $\varepsilon \to 0$ yields (4.30).

We are ready to obtain the weak limit of $\{\sqrt{\overline{u}_j}\nabla \overline{F}_j\}$ in $L^2(\Omega_T)$. For this purpose, take any ξ from $\left(C^\infty(\mathbb{R}^N \times \mathbb{R})\right)^N$ with $\xi \cdot \nu = 0$ on Σ_T. We calculate that

$$-\int_{\Omega_T} \sqrt{\overline{u}_j} \nabla \overline{F}_j \cdot \xi dxdt = \int_{\Omega_T} \overline{F}_j \left(\frac{1}{2} \overline{u}_j{}^{-\frac{1}{2}} \nabla \overline{u}_j \cdot \xi + \sqrt{\overline{u}_j} \text{div}\xi \right) dxdt. \tag{4.31}$$

Let us examine the product

$$\begin{aligned} \overline{F}_j \frac{1}{2} \overline{u}_j{}^{-\frac{1}{2}} \nabla \overline{u}_j = & \frac{1}{\alpha} \overline{u}_j{}^{\frac{\alpha-1}{2}} \Delta \overline{u}_j{}^\alpha \nabla \overline{u}_j{}^{\frac{\alpha}{2}} \\ & - \frac{\tau}{2\alpha} \overline{u}_j{}^{p-\frac{1}{2}} \nabla \overline{u}_j{}^\alpha + \frac{\tau}{2\alpha} \overline{u}_j{}^{-\frac{1}{2}} \nabla \overline{u}_j{}^\alpha. \end{aligned} \tag{4.32}$$

By Lemma 4.5, we have

$$\frac{1}{\alpha} \overline{u}_j{}^{\frac{\alpha-1}{2}} \Delta \overline{u}_j{}^\alpha \nabla \overline{u}_j{}^{\frac{\alpha}{2}} \rightharpoonup \frac{1}{\alpha} \eta \nabla u^{\frac{\alpha}{2}} \quad \text{weakly in } L^1(\Omega_T). \tag{4.33}$$

We estimate

$$\tau \int_{\Omega_T} \overline{u}_j^{\,p-\frac{1}{2}} |\nabla \overline{u}_j^{\,\alpha}| dxdt \le \left(\int_{\Omega_T} \tau \overline{u}_j^{\,p-1} |\nabla \overline{u}_j^{\,\alpha}|^2 dxdt \right)^{\frac{1}{2}} \left(\int_{\Omega_T} \tau \overline{u}_j^{\,p} dxdt \right)^{\frac{1}{2}}$$
$$\le c \left(\int_{\Omega_T} \tau \overline{u}_j^{\,p} dxdt \right)^{\frac{1}{2}} \to 0 \quad \text{as } j \to \infty. \tag{4.34}$$

The last step is implied by (4.20). Similarly,

$$\tau \int_{\Omega_T} \overline{u}_j^{\,-\frac{1}{2}} |\nabla \overline{u}_j^{\,\alpha}| dxdt \le c\tau \left(\int_{\Omega_T} \overline{u}_j^{\,-1} |\nabla \overline{u}_j^{\,\alpha}|^2 dxdt \right)^{\frac{1}{2}} \le c\tau^{\frac{1}{2}}. \tag{4.35}$$

In summary, we have

$$\int_{\Omega_T} \overline{F}_j \frac{1}{2} \overline{u}_j^{\,-\frac{1}{2}} \nabla \overline{u}_j \cdot \xi dxdt \to \frac{1}{\alpha} \int_{\Omega_T} \eta \nabla u^{\frac{\alpha}{2}} \cdot \xi dxdt$$
$$= \frac{1}{\alpha} \int_{\Omega_T} u^{\frac{\alpha-1}{2}} \Delta u^\alpha \nabla u^{\frac{\alpha}{2}} \cdot \xi dxdt. \tag{4.36}$$

Note that the integrand in the last integral of the above equation remains the same no matter what values η takes on the set $A_0 = \{u = 0\}$ because $\nabla u^{\frac{\alpha}{2}}$ is zero there. For definiteness, the function η is understood to be 0 on the set $\{u = 0\}$.

Now we consider

$$\overline{F}_j \sqrt{\overline{u}_j} = \overline{u}_j^{\,\frac{\alpha-1}{2}} \Delta \overline{u}_j^{\,\alpha} \overline{u}_j^{\,\frac{\alpha}{2}} - \tau \overline{u}_j^{\,p+\alpha-\frac{1}{2}} + \tau \overline{u}_j^{\,\alpha-\frac{1}{2}}$$
$$\rightharpoonup \eta u^{\frac{\alpha}{2}} = u^{\frac{2\alpha-1}{2}} \Delta u^\alpha. \tag{4.37}$$

Assume

$$\sqrt{\overline{u}_j} \nabla \overline{F}_j \rightharpoonup \mathbf{S} \quad \text{weakly in } L^2(\Omega_T).$$

We can infer from (4.31) that

$$\mathbf{S} = -\frac{1}{\alpha} u^{\frac{\alpha-1}{2}} \Delta u^\alpha \nabla u^{\frac{\alpha}{2}} + \nabla \left(u^{\frac{2\alpha-1}{2}} \Delta u^\alpha \right).$$

Finally, we have

$$(\overline{u}_j + \tau) \nabla \overline{F}_j = \sqrt{\overline{u}_j} \cdot \sqrt{\overline{u}_j} \nabla \overline{F}_j + \tau \nabla \overline{F}_j$$
$$\to \sqrt{u} \mathbf{S} \quad \text{weakly in } (L^1(\Omega_T))^N.$$

Now we have all the ingredients necessary to pass to the limit in (3.29). In doing so, we take a note of the fact that the sequences $\{\tilde{u}_j\}$ and $\{\overline{u}_j\}$ have the same limit due to (4.10) and thereby obtain Theorem 1.1. The proof is complete.

5. Proof of Theorem 1.2. Remember that under the hypotheses of Theorem 1.2 we still have Lemmas 3.1 and 3.2. Throughout this section, we assume that

$$\alpha < 1. \tag{5.1}$$

LEMMA 5.1. *Let $\{\overline{u}_j, \overline{F}_j\}$ be given as in (3.29)-(3.32). If τ is suitably small, then we have*

$$\int_{\Omega_T} |\nabla^2 \overline{u}_j{}^{\alpha}|^2 dxdt + \tau \int_{\Omega_T} \left(\overline{u}_j{}^{p-\alpha} + \overline{u}_j{}^{-\alpha} + \frac{|\ln(\overline{u}_j + \tau)|}{\overline{u}_j} \right) |\nabla \overline{u}_j{}^{\alpha}|^2 dxdt$$
$$+ \tau^2 \int_{\Omega_T} \left(\overline{u}_j{}^{p+\alpha-1} + \overline{u}_j{}^{\alpha-1} \right) |\ln(\overline{u}_j + \tau)| dxdt + \sup_{0\le t\le T} \int_\Omega \overline{u}_j |\ln(\overline{u}_j + \tau)| dx$$
$$\le c. \tag{5.2}$$

Proof. We use $-\ln(\rho_k + \tau)$ as a test function in (3.25) to obtain

$$-\int_\Omega \nabla F_k \cdot \nabla \rho_k dx - \tau \int_\Omega F_k \ln(\rho_k + \tau) dx$$
$$+ \frac{1}{\tau} \int_\Omega (\rho_k - \rho_{k-1}) \ln(\rho_k + \tau) dx = -\int_\Omega \mathbf{g}_k \cdot \nabla \rho_k dx. \tag{5.3}$$

We now proceed to analyze each integral in the above equation. We first calculate

$$\begin{aligned} -\int_\Omega \nabla F_k \cdot \nabla \rho_k dx &= \int_\Omega F_k \Delta \rho_k dx \\ &= \int_\Omega \Delta \rho_k (\rho_k{}^{\alpha-1} \Delta \rho_k{}^{\alpha} - \tau \rho_k{}^{p+\alpha-1} + \tau \rho_k{}^{\alpha-1}) dx \\ &= \int_\Omega \Delta \rho_k \rho_k{}^{\alpha-1} \Delta \rho_k{}^{\alpha} dx + \frac{(p+\alpha-1)\tau}{\alpha^2} \int_\Omega \rho_k{}^{p-\alpha} |\nabla \rho_k{}^{\alpha}|^2 dx \\ &\quad + \frac{(1-\alpha)\tau}{\alpha^2} \int_\Omega \rho_k{}^{-\alpha} |\nabla \rho_k{}^{\alpha}|^2 dx. \end{aligned} \tag{5.4}$$

As for the second integral in (5.3), we have

$$\begin{aligned} &-\tau \int_\Omega F_k \ln(\rho_k + \tau) dx \\ &= -\tau \int_\Omega \ln(\rho_k + \tau)(\rho_k{}^{\alpha-1} \Delta \rho_k{}^{\alpha} - \tau \rho_k{}^{p+\alpha-1} + \tau \rho_k{}^{\alpha-1}) dx \\ &= -\frac{(1-\alpha)\tau}{\alpha} \int_\Omega \frac{\ln(\rho_k + \tau)}{\rho_k} |\nabla \rho_k{}^{\alpha}|^2 dx + \frac{\tau}{\alpha} \int_\Omega \frac{1}{\rho_k + \tau} |\nabla \rho_k{}^{\alpha}|^2 dx \\ &\quad + \tau^2 \int_\Omega \rho_k{}^{p+\alpha-1} \ln(\rho_k + \tau) dx - \tau^2 \int_\Omega \rho_k{}^{\alpha-1} \ln(\rho_k + \tau) dx. \end{aligned} \tag{5.5}$$

We proceed to estimate the last four integrals in (5.5). Taking $\varepsilon = \frac{1-\alpha}{2}$ in (4.23), we deduce

$$\begin{aligned} \int_{\{\rho_k+\tau\ge 1\}} \rho_k{}^{\alpha-1} \ln(\rho_k + \tau) dx &\le c \int_{\{\rho_k+\tau\ge 1\}} \rho_k{}^{\alpha-1} (\rho_k + \tau)^{\frac{1-\alpha}{2}} dx \\ &\le c \int_{\{\rho_k+\tau\ge 1\}} \rho_k{}^{\alpha-1} (\rho_k{}^{\frac{1-\alpha}{2}} + \tau^{\frac{1-\alpha}{2}}) dx \\ &= c \int_{\{\rho_k+\tau\ge 1\}} (\rho_k{}^{-\frac{1-\alpha}{2}} + \tau^{\frac{1-\alpha}{2}} \rho_k{}^{-(1-\alpha)}) dx \\ &\le \frac{c}{(1-\tau)^{1-\alpha}} \le c, \end{aligned}$$

provided that τ is suitably small. Similarly, there holds

$$\begin{aligned}\int_{\{\rho_k+\tau\leq 1\}} {\rho_k}^{p+\alpha-1}|\ln(\rho_k+\tau)|dx &\leq c\int_{\{\rho_k+\tau\leq 1\}} {\rho_k}^{p+\alpha-1}(\rho_k+\tau)^{-\varepsilon}dx\\ &\leq c\int_{\{\rho_k+\tau\leq 1\}} (\rho_k+\tau)^{p+\alpha-1-\varepsilon}dx\\ &\leq c \quad \text{if we choose } \varepsilon<1.\end{aligned}$$

On account of the preceding calculations, we have

$$\begin{aligned}\int_\Omega {\rho_k}^{p+\alpha-1}\ln(\rho_k+\tau)dx &= \int_\Omega {\rho_k}^{p+\alpha-1}|\ln(\rho_k+\tau)|dx\\ &\quad -2\int_{\{\rho_k+\tau\leq 1\}} {\rho_k}^{p+\alpha-1}|\ln(\rho_k+\tau)|dx\\ &\geq \int_\Omega {\rho_k}^{p+\alpha-1}|\ln(\rho_k+\tau)|dx - c, \quad \text{and} \qquad (5.6)\\ -\int_\Omega {\rho_k}^{\alpha-1}\ln(\rho_k+\tau)dx &= \int_\Omega {\rho_k}^{\alpha-1}|\ln(\rho_k+\tau)|dx - 2\int_{\{\rho_k+\tau\geq 1\}} {\rho_k}^{\alpha-1}|\ln(\rho_k+\tau)|dx\\ &\geq \int_\Omega {\rho_k}^{\alpha-1}|\ln(\rho_k+\tau)|dx - c. \qquad (5.7)\end{aligned}$$

Now we consider the function

$$\theta(s) = s^{p-\alpha} - \frac{\ln(s+\tau)}{s} \quad \text{on the interval } [1-\tau,\infty).$$

It is elementary to show that the function is positive if τ is so small that

$$(p-\alpha)(1-\tau)^{p-\alpha-1} - \frac{1}{1-\tau} > 0.$$

This can be done due to (3.1). Let τ be so chosen. Note

$$-\frac{\ln(\rho_k+\tau)}{\rho_k} = \frac{|\ln(\rho_k+\tau)|}{\rho_k} - 2\frac{\ln(\rho_k+\tau)}{\rho_k}\chi_{\{\rho_k+\tau\geq 1\}}.$$

We are ready to estimate the entire left-hand side of (5.3) as follows.

$$\begin{aligned}&-\int_\Omega \nabla F_k\cdot\nabla\rho_k dx - \tau\int_\Omega F_k\ln(\rho_k+\tau)dx + \frac{1}{\tau}\int_\Omega(\rho_k-\rho_{k-1})\ln(\rho_k+\tau)dx\\ &\geq \int_\Omega \Delta\rho_k{\rho_k}^{\alpha-1}\Delta{\rho_k}^{\alpha}dx + \frac{(p+2\alpha^2-\alpha-1)\tau}{\alpha^2}\int_\Omega {\rho_k}^{p-\alpha}|\nabla\rho_k^\alpha|^2dx\\ &\quad + \frac{(1-\alpha)\tau}{\alpha^2}\int_\Omega {\rho_k}^{-\alpha}|\nabla\rho_k^\alpha|^2dx + \frac{(1-\alpha)\tau}{\alpha}\int_\Omega \frac{|\ln(\rho_k+\tau)|}{\rho_k}|\nabla{\rho_k}^\alpha|^2dx\\ &\quad + \frac{2(1-\alpha)\tau}{\alpha}\int_\Omega\left({\rho_k}^{p-\alpha} - \frac{\ln(\rho_k+\tau)}{\rho_k}\chi_{\{\rho_k+\tau\geq 1\}}\right)|\nabla{\rho_k}^\alpha|^2dx\\ &\quad + \frac{\tau}{\alpha}\int_\Omega \frac{1}{\rho_k+\tau}|\nabla{\rho_k}^\alpha|^2dx + \tau^2\int_\Omega {\rho_k}^{p+\alpha-1}|\ln(\rho_k+\tau)|dx\\ &\quad + \tau^2\int_\Omega {\rho_k}^{\alpha-1}|\ln(\rho_k+\tau)|dx + \frac{1}{\tau}\int_\Omega\int_{\rho_{k-1}}^{\rho_k}\ln(s+\tau)dsdx - c. \qquad (5.8)\end{aligned}$$

This immediately implies that

$$\begin{aligned}
&\int_\Omega |\nabla^2 {\rho_k}^\alpha|^2 dx + \tau \int_\Omega \left({\rho_k}^{p-\alpha} + {\rho_k}^{-\alpha} + \frac{|\ln(\rho_k+\tau)|}{\rho_k} \right) |\nabla {\rho_k}^\alpha|^2 dx \\
&\quad + \tau^2 \int_\Omega \left({\rho_k}^{p+\alpha-1} + {\rho_k}^{\alpha-1} \right) |\ln(\rho_k+\tau)| dx + c\frac{1}{\tau} \int_\Omega \int_{\rho_{k-1}}^{\rho_k} \ln(s+\tau) ds dx \\
&\leq -c \int_\Omega \mathbf{g}_k \cdot \nabla \rho_k dx + c. \qquad (5.9)
\end{aligned}$$

Multiplying through this inequality by τ and then summing up over k yield

$$\begin{aligned}
&\int_{\Omega_t} |\nabla^2 {\overline{u}_j}^\alpha|^2 dxds + \tau \int_{\Omega_t} \left({\overline{u}_j}^{p-\alpha} + {\overline{u}_j}^{-\alpha} + \frac{|\ln(\overline{u}_j+\tau)|}{\overline{u}_j} \right) |\nabla {\overline{u}_j}^\alpha|^2 dxds \\
&\quad + \tau^2 \int_{\Omega_t} \left({\overline{u}_j}^{p+\alpha-1} + {\overline{u}_j}^{\alpha-1} \right) |\ln(\overline{u}_j+\tau)| dxds + c \int_\Omega \int_{u_0}^{\overline{u}_j} \ln(s+\tau) ds dx \\
&\leq c \int_{\Omega_t} |\overline{\mathbf{g}}_j \cdot \nabla \overline{u}_j| dxds + c. \qquad (5.10)
\end{aligned}$$

Now recall the interpolation inequality

$$\|\nabla {\overline{u}_j}^\alpha\|_{2^*} \leq c(\|\nabla^2 {\overline{u}_j}^\alpha\|_2 + \|{\overline{u}_j}^\alpha\|_1), \qquad (5.11)$$

while if $q < 2^*, r \geq 1$ then for each $\varepsilon > 0$ there corresponds a positive number c such that

$$\|\nabla {\overline{u}_j}^\alpha\|_q \leq \varepsilon \|\nabla^2 {\overline{u}_j}^\alpha\|_2 + c\|{\overline{u}_j}^\alpha\|_r. \qquad (5.12)$$

With the above in mind, we compute

$$\begin{aligned}
\int_{\Omega_t} |\overline{\mathbf{g}}_j \cdot \nabla \overline{u}_j| dxds &\leq c \int_{\Omega_t} {\overline{u}_j}^{1-\alpha} |\nabla {\overline{u}_j}^\alpha| dxds \\
&\leq c \int_0^t \|{\overline{u}_j}^{1-\alpha}\|_{\frac{2N}{N+2}} \|\nabla {\overline{u}_j}^\alpha\|_{2^*} ds \\
&\leq c \int_0^t \|{\overline{u}_j}^{1-\alpha}\|_{\frac{2N}{N+2}} (\|\nabla^2 {\overline{u}_j}^\alpha\|_2 + \|{\overline{u}_j}^\alpha\|_1) ds \\
&\leq \varepsilon \int_{\Omega_t} |\nabla^2 {\overline{u}_j}^\alpha|^2 dxds + \varepsilon \int_0^t \left(\int_\Omega {\overline{u}_j}^\alpha dx \right)^2 ds \\
&\quad + c \int_0^t \left(\int_\Omega {\overline{u}_j}^{\frac{(1-\alpha)2N}{N+2}} dx \right)^{\frac{N+2}{N}} ds.
\end{aligned}$$

Our assumptions on α imply

$$\frac{(1-\alpha)2N}{N+2} \leq 1.$$

We can infer from Lemma 3.2 that

$$\max_{0 \leq s \leq t} \int_\Omega \overline{u}_j(x,s) dx \leq c \left(\int_{\Omega_t} |\nabla {\overline{u}_j}^\alpha|^2 dxds \right)^{\frac{1}{2}} + c. \qquad (5.13)$$

Using this, we have that

$$\begin{aligned}\int_{\Omega_t} |\mathbf{\overline{g}}_j \cdot \nabla \overline{u}_j| dxds &\le \varepsilon \int_{\Omega_t} |\nabla^2 \overline{u}_j{}^\alpha|^2 dxds + c\int_0^t \left(\int_\Omega \overline{u}_j dx\right)^{2(1-\alpha)} ds \\ &\quad + c\varepsilon \int_0^t \left(\int_\Omega \overline{u}_j dx\right)^{2\alpha} ds + c \\ &\le \varepsilon \int_{\Omega_t} |\nabla^2 \overline{u}_j{}^\alpha|^2 dxds + c\left(\int_{\Omega_t} |\nabla \overline{u}_j{}^\alpha|^2 dxds\right)^{1-\alpha} \\ &\quad + c\varepsilon \left(\int_{\Omega_t} |\nabla \overline{u}_j{}^\alpha|^2 dxds\right)^{\alpha} + c \\ &\le \varepsilon \int_{\Omega_t} |\nabla^2 \overline{u}_j{}^\alpha|^2 dxds + c\int_{\Omega_t} |\nabla \overline{u}_j{}^\alpha|^2 dxds + c \\ &\le 2\varepsilon \int_{\Omega_t} |\nabla^2 \overline{u}_j{}^\alpha|^2 dxds + c\int_{\Omega_t} \overline{u}_j dxds + c.\end{aligned}$$

The last step is due to (5.12) with $r = 1$. Use this in (5.10), remember that

$$\begin{aligned}\int_\Omega \int_{u_0}^{\overline{u}_j} \ln(s+\tau) dsdx &= \int_\Omega (s\ln(s+\tau) - s + \tau\ln(s+\tau)) \,|_{u_0}^{\overline{u}_j}\, dx \\ &\ge \frac{1}{2}\int_\Omega \overline{u}_j |\ln(\overline{u}_j + \tau)| dx - c,\end{aligned}$$

then apply Gronwall's inequality, and thereby obtain the desired result. The proof is complete. □

In all the remaining lemmas we assume that τ is so small that Lemma 5.1 holds.

LEMMA 5.2. $\tau \int_{\Omega_T} \overline{u}_j{}^{p+\alpha+\frac{2}{N}} dxdt \le c.$

Proof. By the Sobolev inequality, we have

$$\begin{aligned}\int_{\Omega_T} \overline{u}_j{}^{\frac{p+\alpha}{2}2+\frac{2}{N}} dxdt &\le c\int_0^T \left(\int_\Omega \overline{u}_j{}^{\frac{p+\alpha}{2}\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}} \left(\int_\Omega \overline{u}_j dx\right)^{\frac{2}{N}} dt \\ &\le c\left(\int_{\Omega_T} |\nabla \overline{u}_j{}^{\frac{p+\alpha}{2}}|^2 dxdt + \int_{\Omega_T} \overline{u}_j{}^{p+\alpha} dxdt\right) \left(\sup_{0\le t\le T} \int_\Omega \overline{u}_j dx\right)^{\frac{2}{N}} \\ &\le c\left(\int_{\Omega_T} \overline{u}_j{}^{p-\alpha} |\nabla \overline{u}_j{}^\alpha|^2 dxdt + \int_{\Omega_T} \overline{u}_j{}^{p+\alpha} dxdt\right) \\ &\le c\int_{\Omega_T} \overline{u}_j{}^{p-\alpha} |\nabla \overline{u}_j{}^\alpha|^2 dxdt + \varepsilon \int_{\Omega_T} \overline{u}_j{}^{p+\alpha+\frac{2}{N}} dxdt + c.\end{aligned}$$

Choosing ε suitably small yields

$$\int_{\Omega_T} \overline{u}_j{}^{p+\alpha+\frac{2}{N}} dxdt \le c\int_{\Omega_T} \overline{u}_j{}^{p-\alpha} |\nabla \overline{u}_j{}^\alpha|^2 dxdt + c.$$

Multiplying through the inequality by τ and taking a note of Lemma 5.1 give the desired result. □

LEMMA 5.3. $\tau \overline{F}_j \to 0$ *strongly in* $L^1(\Omega_T)$.

Proof. Note that

$$\tau\overline{F}_j = \tau\overline{u}_j{}^{\alpha-1}\Delta\overline{u}_j{}^{\alpha} - \tau^2\overline{u}_j{}^{p+\alpha-1} + \tau^2\overline{u}_j{}^{\alpha-1}. \tag{5.14}$$

We estimate from Lemma 5.1 that

$$\begin{aligned}\int_{\Omega_T}\tau^2\overline{u}_j{}^{\alpha-1}dxdt &= \int_{\{\overline{u}_j\le\tau\}}\tau^2\overline{u}_j{}^{\alpha-1}dxdt + \int_{\{\overline{u}_j>\tau\}}\tau^2\overline{u}_j{}^{\alpha-1}dxdt\\ &\le \frac{1}{|\ln(2\tau)|}\int_{\{\overline{u}_j\le\tau\}}\tau^2\overline{u}_j{}^{\alpha-1}|\ln(\overline{u}_j+\tau)|dxdt + c\tau^{1+\alpha}\\ &\le \frac{c}{|\ln(2\tau)|} + c\tau^{1+\alpha} \to 0 \quad \text{as } \tau\to 0. \end{aligned}\tag{5.15}$$

With the aid of Lemma 3.2, we obtain that

$$\begin{aligned}\int_{\Omega_T}\tau\overline{u}_j{}^{\alpha-1}|\Delta\overline{u}_j{}^{\alpha}|dxdt &\le \left(\int_{\Omega_T}\tau^2\overline{u}_j{}^{\alpha-1}dxdt\right)^{\frac12}\left(\int_{\Omega_T}\overline{u}_j{}^{\alpha-1}(\Delta\overline{u}_j{}^{\alpha})^2dxdt\right)^{\frac12}\\ &\le c\left(\int_{\Omega_T}\tau^2\overline{u}_j{}^{\alpha-1}dxdt\right)^{\frac12} \le \frac{c}{|\ln(2\tau)|^{\frac12}}.\end{aligned}\tag{5.16}$$

We deduce from Lemma 5.2 that

$$\begin{aligned}\int_{\Omega_T}\tau^2\overline{u}_j{}^{p+\alpha-1}dxdt &= \int_{\{\overline{u}_j\le 1\}}\tau^2\overline{u}_j{}^{p+\alpha-1}dxdt + \int_{\{\overline{u}_j>1\}}\tau^2\overline{u}_j{}^{p+\alpha-1}dxdt\\ &\le c\tau^2 + \int_{\{\overline{u}_j>1\}}\tau^2\overline{u}_j{}^{p+\alpha+\frac{2}{N}}dxdt\\ &\le c\tau^2 + c\tau \to 0 \quad \text{as } \tau\to 0.\end{aligned}\tag{5.17}$$

This completes the proof. □

Lemma 5.4. *$\{\partial_t\tilde{u}_j\}$ is bounded in $L^1((0,T);(W^{2,\infty}(\Omega))^*)$.*

Proof. Recall that

$$\overline{F}_j\nabla\overline{u}_j = \frac{1}{\alpha}\Delta\overline{u}_j{}^{\alpha}\nabla\overline{u}_j{}^{\alpha} - \frac{\tau}{\alpha}\overline{u}_j{}^{p}\nabla\overline{u}_j{}^{\alpha} + \frac{\tau}{\alpha}\nabla\overline{u}_j{}^{\alpha}. \tag{5.18}$$

The first term on the right-hand side of the above equation is bounded in $(L^1(\Omega_T))^N$, while the last term there is bounded in $(L^2(\Omega_T))^N$. By Lemma 5.2 and an argument similar to (4.20), we obtain

$$\tau\int_{\Omega_T}\overline{u}_j{}^{p+\alpha}dxdt \le c\tau^{\frac{1}{N(p+\alpha)+2}}. \tag{5.19}$$

Thus the remaining term on the right-hand side of (5.18) can be estimated as follows

$$\begin{aligned}\tau\int_{\Omega_T}\overline{u}_j{}^{p}|\nabla\overline{u}_j{}^{\alpha}|dxdt &\le \left(\tau\int_{\Omega_T}\overline{u}_j{}^{p+\alpha}dxdt\right)^{\frac12}\left(\tau\int_{\Omega_T}\overline{u}_j{}^{p-\alpha}|\nabla\overline{u}_j{}^{\alpha}|^2dxdt\right)^{\frac12}\\ &\le c\tau^{\frac{1}{N(p+\alpha)+2}}.\end{aligned}\tag{5.20}$$

We have that $\overline{F}_j\nabla\overline{u}_j$ is bounded in $L^1(\Omega_T)$. We still need to consider the term

$$\overline{F}_j\overline{u}_j = \overline{u}_j{}^{\alpha}\Delta\overline{u}_j{}^{\alpha} - \tau\overline{u}_j{}^{p+\alpha} + \tau\overline{u}_j{}^{\alpha}. \tag{5.21}$$

It is easy to see from Lemma 5.2 that it is also bounded in $L^1(\Omega_T)$. Let ξ be a C^∞ test function with $\nabla \cdot \nu = 0$ on $\partial\Omega$. We have

$$\begin{aligned}(\partial_t \tilde{u}_j, \xi) &= \int_\Omega (\overline{u}_j + \tau)\nabla \overline{F}_j \cdot \nabla \xi dx - \int_\Omega (\overline{u}_j + \tau)\overline{\mathbf{g}}_j \cdot \nabla \xi dx + \tau \int_\Omega \overline{F}_j \xi dx \\ &= -\int_\Omega \overline{F}_j (\nabla \overline{u}_j \cdot \nabla \xi + \overline{u}_j \Delta \xi) dx - \tau \int_\Omega \overline{F}_j \Delta \xi dx \\ &\quad - \int_\Omega (\overline{u}_j + \tau)\overline{\mathbf{g}}_j \cdot \nabla \xi dx + \tau \int_\Omega \overline{F}_j \xi dx, \end{aligned} \tag{5.22}$$

where $(\cdot, \cdot)$ is the duality pairing between $W^{2,\infty}(\Omega)$ and its dual space $(W^{2,\infty}(\Omega))^*$, from which the lemma follows. □

Notice that each term on the right-hand side of (5.18) can be presented as the product of an L^2 function and an L^q function with $q > 2$. The same is true of (5.21). As a result, it is not difficult to see that $\{\partial_t \tilde{u}_j\}$ is actually bounded in $L^1((0,T); (W^{2,r}(\Omega))^*)$ for some r sufficiently large or in $L^s((0,T); (W^{2,\infty}(\Omega))^*)$ for some $s > 1$.

LEMMA 5.5. *$\{\overline{u}_j\}$ is precompact in $L^2((0,T); L^q(\Omega))$, where $q = \frac{2N}{3N - 2\alpha N - 2}$.*

Proof. Let q be given as in the lemma. By our assumption on α, we have $q > 1$. We estimate that

$$\begin{aligned}\int_\Omega |\nabla \overline{u}_j|^q dx &= \frac{1}{\alpha^q} \int_\Omega \overline{u}_j^{\,(1-\alpha)q} |\nabla \overline{u}_j^{\,\alpha}|^q dx \\ &\le c \left(\int_\Omega |\nabla \overline{u}_j^{\,\alpha}|^{2^*} dx \right)^{\frac{q}{2^*}} \left(\int_\Omega \overline{u}_j^{\,\frac{(1-\alpha)q2^*}{2^*-q}} dx \right)^{1-\frac{q}{2^*}}. \end{aligned}$$

Note that $\frac{(1-\alpha)q2^*}{2^*-q} = 1$. Therefore, we obtain

$$\int_0^T \|\nabla \overline{u}_j\|_q^2 dt \le c. \tag{5.23}$$

We can easily deduce from the definition of $\overline{u}_j, \tilde{u}_j$ that

$$\int_0^T \|\nabla \tilde{u}_j\|_q^2 dt \le c \int_0^T \|\nabla \overline{u}_j\|_q^2 dt + c\tau \|\nabla u_0\|_q^2, \quad c = c(q). \tag{5.24}$$

By the proof of (4.10), we have

$$\int_0^T \|\overline{u}_j - \tilde{u}_j\|_{(W^{2,\infty}(\Omega))^*} dt \le c\tau. \tag{5.25}$$

Thus $\{|\nabla \tilde{u}_j|\}$ is bounded in $L^2((0,T); L^q(\Omega))$. Observe that the imbedding $W^{1,q}(\Omega) \hookrightarrow L^q(\Omega)$ is compact and $L^q(\Omega) \hookrightarrow \left(W^{2,\infty}(\Omega)\right)^*$ is continuous. A result of [11] asserts that $\{\tilde{u}_j\}$ is precompact in both $L^2((0,T); L^q(\Omega))$ and $L^1((0,T); (W^{2,\infty}(\Omega))^*)$. According to (5.25), we also have that $\{\overline{u}_j\}$ is precompact in $L^1((0,T); (W^{2,\infty}(\Omega))^*)$. This puts us in a position to apply the results in [11] again, from which the lemmas follows. The proof is complete. □

We are ready to complete the proof of Theorem 1.2. We can extract a subsequence of $\{j\}$, still denoted by $\{j\}$, such that

$$\begin{aligned} \overline{u}_j &\to u \quad \text{strongly in } L^1(\Omega_T) \text{ and a.e.,} \\ \overline{u}_j^{\,\alpha} &\to u^\alpha \quad \text{weakly in } L^2((0,T); W^{2,2}(\Omega)). \end{aligned}$$

With the aid of the Sobolev inequality, we have

$$\int_{\Omega_T} \overline{u}_j^{\,2\alpha+\frac{2}{N}} dxdt \leq c$$

from which it follows

$$\overline{u}_j^{\,\alpha} \to u^\alpha \quad \text{strongly in } L^2((0,T); L^2(\Omega)).$$

Equipped with this, we calculate that

$$\int_{\Omega_T} |\nabla \overline{u}_j^{\,\alpha}|^2 dxdt = -\int_{\Omega_T} \Delta \overline{u}_j^{\,\alpha} \overline{u}_j^{\,\alpha} dxdt \to -\int_{\Omega_T} \Delta u^\alpha u^\alpha dxdt = \int_{\Omega_T} |\nabla u^\alpha|^2 dxdt. \tag{5.26}$$

This implies that

$$\overline{u}_j^{\,\alpha} \to u^\alpha \quad \text{strongly in } L^2((0,T); W^{1,2}(\Omega)). \tag{5.27}$$

Keeping this and (5.20) in mind, we have that

$$\overline{F}_j \nabla \overline{u}_j \rightharpoonup \frac{1}{\alpha} \Delta u^\alpha \nabla u^\alpha \quad \text{weakly in } L^1(\Omega_T).$$

On account of (5.19), we have

$$\overline{F}_j \overline{u}_j \rightharpoonup \Delta u^\alpha u^\alpha \quad \text{weakly in } L^1(\Omega_T).$$

Assume $\xi(x,T) = 0$ in (5.22), integrate it over $(0,T)$, then let $j \to \infty$, and thereby obtain the theorem. The proof is complete.

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