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ON THE LARGE TIME APPROXIMATION OF THE NAVIER-STOKES EQUATIONS IN $\mathbb{R}^n$ BY STOKES FLOWS*

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Abstract. We show, under quite general assumptions, the time asymptotic property $t^{\kappa_{n,q}} \| \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t) \|_{L^q(\mathbb{R}^n)} \to 0$ as $t \to \infty$, for each $2 \le q \le \infty$ and all Leray-Hopf global L^2 solutions $\boldsymbol{u}(\cdot,t)$ of the incompressible Navier-Stokes equations and their associated Stokes flows $\boldsymbol{v}(\cdot,t)$ in $\mathbb{R}^n$ $(n = 2, 3)$, where $\kappa_{n,q} = (n/2)(1 - 1/q) - 1/2$. We use the approximation results to derive several new related results on Stokes flows. Our method is based on classic tools in real analysis and PDE theory like standard Fourier and energy methods.

Key words. Incompressible Navier-Stokes equations, Leray-Hopf (weak) solutions, large time behavior, Leray's problem, Stokes approximation, supnorm estimates.

AMS subject classifications. Primary 35Q30, 76D05; Secondary 76D07.

1. Introduction. We are interested in deriving various fundamental asymptotic properties concerning the so-called Leray-Hopf solutions to the initial value problem for the incompressible Navier-Stokes equations

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \Delta \boldsymbol{u} + \boldsymbol{f}(\cdot,t), \qquad \nabla \cdot \boldsymbol{u}(\cdot,t) = 0, \tag{1.1a}$$

$$\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0 \in L^2(\mathbb{R}^n), \qquad \nabla \cdot \boldsymbol{u}_0 = 0, \tag{1.1b}$$

where $\boldsymbol{f}(\cdot,t) = (f_1(\cdot,t), ..., f_n(\cdot,t)) \in L^1_{\mathtt{loc}}([0,\infty[, L^2(\mathbb{R}^n))$ is a given external force field satisfying the general conditions (1.5), (1.8) below, and $p(\cdot,t)$ denotes the kinematic pressure. We begin with an overview of known results and those obtained here.

Considering the Helmholtz decomposition of $\boldsymbol{f}(\cdot,t)$ in $L^2(\mathbb{R}^n)$, i.e.,

$$\boldsymbol{f}(\cdot,t) = -\nabla\Phi(\cdot,t) + \boldsymbol{g}(\cdot,t), \qquad \nabla \cdot \boldsymbol{g}(\cdot,t) = 0, \tag{1.2a}$$

with $\nabla\Phi(\cdot,t)$, $\boldsymbol{g}(\cdot,t) \in L^1_{\mathtt{loc}}([0,\infty[, L^2(\mathbb{R}^n))$ such that[1]

$$\| \boldsymbol{g}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \le \| \boldsymbol{f}(\cdot,t) \|_{L^2(\mathbb{R}^n)}, \qquad \| D\boldsymbol{g}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \le \| D\boldsymbol{f}(\cdot,t) \|_{L^2(\mathbb{R}^n)}, \tag{1.2b}$$

one can write the equations (1.1) above as

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[1]For the definition of the vector norms involved here, see (1.14), (1.15). If $\boldsymbol{f}(\cdot,t) \notin H^1(\mathbb{R}^n)$, then the second condition in (1.2b) is merely the trivial assertion that $\| D\boldsymbol{g}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \le \infty$.

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla P(\cdot,t) = \Delta\boldsymbol{u} + \boldsymbol{g}(\cdot,t), \qquad \nabla\cdot\boldsymbol{u}(\cdot,t) = 0, \tag{1.3a}$$

$$\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0 \in L^2(\mathbb{R}^n), \qquad \nabla\cdot\boldsymbol{u}_0 = 0, \tag{1.3b}$$

where $P(\cdot,t) = p(\cdot,t) + \Phi(\cdot,t)$ is the modified pressure and $\boldsymbol{g}(\cdot,t) = \mathbb{P}_{H}[\,\boldsymbol{f}(\cdot,t)\,]$ is the divergence-free Helmholtz projection of $\boldsymbol{f}(\cdot,t)$ in $L^2(\mathbb{R}^n)$ introduced in (1.2). Under appropriate conditions on $\boldsymbol{g}(\cdot,t)$, one then may expect to have $\boldsymbol{u}(\cdot,t)$ approximately described (at least for $t \gg 1$) by associated solutions of the related Stokes equations, $\boldsymbol{v}_t + \nabla q(\cdot,t) = \Delta\boldsymbol{v} + \boldsymbol{f}(\cdot,t)$, $\nabla\cdot\boldsymbol{v}(\cdot,t) = 0$ — that is, by (1.2) above (see also [2], p. 293), solutions of the linear heat flow problems

$$\boldsymbol{v}_t = \Delta\boldsymbol{v} + \boldsymbol{g}(\cdot,t), \qquad t > t_0; \qquad \boldsymbol{v}(\cdot,t_0) = \boldsymbol{u}(\cdot,t_0), \tag{1.4}$$

where $t_0 \geq 0$ is some (arbitrary) reference time chosen. [For better clarity, the solution of (1.4) will be denoted by $\boldsymbol{v}(\cdot,t;t_0)$ in the sequel]. In fact, assuming that

$$\int_0^{\infty} \|\,\boldsymbol{g}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)}\,dt < \infty, \tag{1.5}$$

and

$$\limsup_{t\to\infty}\; t^{n/4+1/2}\,\|\,\boldsymbol{g}(\cdot,t)\,\|_{L^n(\mathbb{R}^n)} < \infty \tag{1.6}$$

if $n \geq 3$, Wiegner showed, using a very involved argument, that one effectively has (cf. [27], Theorem (c), p. 305):

$$\lim_{t\to\infty}\; t^{n/4-1/2}\,\|\,\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t;t_0)\,\|_{L^2(\mathbb{R}^n)} = 0 \tag{1.7}$$

for $n \geq 2$ arbitrary. A simpler analysis given by Kato [12], in the case $\boldsymbol{g}(\cdot,t) = \boldsymbol{0}$ and dimension $n = 2, 3$, yields $\lim_{t\to\infty} \|\,\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t;t_0)\,\|_{L^2(\mathbb{R}^2)} = 0$ if $n = 2$, and

$$\lim_{t\to\infty}\; t^{1/4-\epsilon}\,\|\,\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t;t_0)\,\|_{L^2(\mathbb{R}^3)} = 0$$

if $n = 3$, with $\epsilon > 0$ but otherwise arbitrary (cf. [12], Theorem 4′, p. 473).

Here, under different (and neither overlapping nor comparable) assumptions, we give a more elementary derivation of the fundamental estimate (1.7) for $n = 2, 3$. (Note that, besides our restriction on the space dimension, we will also be restricted to solutions as constructed by Leray [18], while a larger class is considered in [27]. This will allow us to obtain important new results, in addition to those by Wiegner.) More specifically, we assume again that $\boldsymbol{g}(\cdot,t) = \mathbb{P}_{H}[\,\boldsymbol{f}(\cdot,t)\,]$ satisfies (1.5) above and, instead of Wiegner's condition (1.6), that

$$\int_{t_1}^{\infty} t^{1/2}\,\|\,D\boldsymbol{g}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)}\,dt < \infty, \tag{1.8}$$

for some $t_1 \geq 0$. We will also use our method to show that, adding the assumption

$$\limsup_{t \to \infty} \; t \, \| \boldsymbol{g}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \; < \; \infty,^2 \tag{1.9}$$

one gets the supnorm estimate

$$\lim_{t \to \infty} \; t^{n/2 - 1/2} \, \| \boldsymbol{u}(\cdot,t) \; - \; \boldsymbol{v}(\cdot,t;t_0) \|_{L^\infty(\mathbb{R}^n)} \; = \; 0. \tag{1.10}$$

Thus, interpolating (1.7) and (1.10), we have, for every $2 \leq q \leq \infty$:

$$\lim_{t \to \infty} \; t^{\frac{n}{2}\left(1 - \frac{1}{q}\right) - \frac{1}{2}} \, \| \boldsymbol{u}(\cdot,t) \; - \; \boldsymbol{v}(\cdot,t;t_0) \|_{L^q(\mathbb{R}^n)} \; = \; 0, \tag{1.11}$$

uniformly in q. This improves earlier results given in [3, 28] in the case $\boldsymbol{g}(\cdot,t) = \mathbf{0}$.

Our derivation of (1.7), (1.10) above follows the basic approach introduced in [13] (see also [30]) to provide an easy derivation of the fundamental Schonbek-Wiegner decay results (cf. [27], Theorem (*b*), p. 305, and [25], p. 679) for the Navier-Stokes solutions $\boldsymbol{u}(\cdot,t)$ and their derivatives. Similarly to [13, 14, 30], we take advantage of the fact that, for $n = 2, 3$, Leray's solutions to (1.1) become eventually strong (see e.g. [8, 12, 15, 18]); together with (1.5) and (1.8), this allows us to quickly obtain

$$\lim_{t \to \infty} \; t^{1/2} \, \| D\boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \; = \; 0, \tag{1.12}$$

from which (1.7), (1.10) can be derived once some preliminaries have been observed. We may summarize our main results in the following way (where, as usual, $L^2_\sigma(\mathbb{R}^n)$ is the set of all $\boldsymbol{u} = (u_1, u_2, ..., u_n) \in L^2(\mathbb{R}^n)$ with $\nabla \cdot \boldsymbol{u} = 0$ in distributional sense):

Theorem A. *Assuming* (1.5) *and* (1.8) *above, let* $\boldsymbol{u}(\cdot,t)$ *be a Leray-Hopf solution of the Navier-Stokes initial-value problem* (1.3), *defined for all* $t > 0$. *Given* $t_0 \geq 0$, *let* $\boldsymbol{v}(\cdot,t;t_0) \in L^\infty([\,t_0,\infty\,[\,, L^2_\sigma(\mathbb{R}^n))$ *be the Stokes flow defined in* (1.4). *Then*

$$\lim_{t \to \infty} \; t^{n/4 - 1/2} \, \| \boldsymbol{u}(\cdot,t) \; - \; \boldsymbol{v}(\cdot,t;t_0) \|_{L^2(\mathbb{R}^n)} \; = \; 0 \qquad (n = 2, 3). \tag{1.13a}$$

Theorem B. *Assuming* (1.5), (1.8) *and* (1.9), *let* $\boldsymbol{u}(\cdot,t)$ *be a Leray-Hopf solution of the Navier-Stokes initial-value problem* (1.3), *defined for all* $t > 0$. *Given* $t_0 \geq 0$, *let* $\boldsymbol{v}(\cdot,t;t_0) \in L^\infty([\,t_0,\infty\,[\,, L^2_\sigma(\mathbb{R}^n))$ *be the Stokes flow defined in* (1.4). *Then*

$$\lim_{t \to \infty} \; t^{n/2 - 1/2} \, \| \boldsymbol{u}(\cdot,t) \; - \; \boldsymbol{v}(\cdot,t;t_0) \|_{L^\infty(\mathbb{R}^n)} \; = \; 0 \qquad (n = 2, 3). \tag{1.13b}$$

The results above may be used to prove some new results summarized in the following remarks (see (1.16), (1.17) and propositions 2.4 and 2.5).

Remark 1.1. By Duhamel's principle, the Stokes flow defined in (1.4) is given by

$$\boldsymbol{v}(\cdot,t;t_0) \; = \; e^{\Delta(t - t_0)} \, \boldsymbol{u}(\cdot,t_0) \; + \int_{t_0}^{t} e^{\Delta(t - s)} \boldsymbol{g}(\cdot,s) \, ds, \qquad t > t_0, \tag{1.14}$$

[2] This means that there exist $C_{\rm g}$, $t_{\rm g} > 0$ such that $t \, \| \boldsymbol{g}(\cdot,t) \|_{L^2(\mathbb{R}^n)} < C_{\rm g}$ for almost all $t > t_{\rm g}$.

where $e^{\Delta t}$ denotes the heat semigroup. With (1.5), we then get $\| \boldsymbol{v}(\cdot,t;t_0) \|_{L^2(\mathbb{R}^n)} \to 0$ as $t \to \infty$,[3] so that Theorem A gives, by assuming (1.5) and (1.8):

$$\lim_{t \to \infty} \| \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = 0 \qquad (n = 2, 3), \tag{1.15}$$

a problem left open by Leray in 1934 ([18], p. 248). This result was first established by Masuda [19] and Kato [12], and finally extended to arbitrary dimension, and for any $\boldsymbol{g}(\cdot,t)$ satisfying (1.5) alone, by Wiegner (cf. [27], Theorem (a), p. 305). On the other hand, regarding pointwise estimates, (1.14) similarly gives us that $t^{n/4} \| \boldsymbol{v}(\cdot,t;t_0) \|_{L^\infty(\mathbb{R}^n)} \to 0$ (as $t \to \infty$) for $n \leq 3$, using (1.5) and (1.9). By Theorem B, we therefore conclude that

$$\lim_{t \to \infty} t^{n/4} \| \boldsymbol{u}(\cdot,t) \|_{L^\infty(\mathbb{R}^n)} = 0 \qquad (n = 2, 3), \tag{1.16}$$

for Leray's solutions, assuming that (1.5), (1.8) and (1.9) hold. This result is new as far as we know.

Remark 1.2. The particular choice of $t_0 \geq 0$ in the Stokes problem (1.4) does not affect the estimates (1.13) to the order of approximation given there. Recalling our practice of denoting the solution of (1.4) by $\boldsymbol{v}(\cdot,t;t_0)$ to explicitly account for its dependence on the reference time, if we consider any two such values (say, $t_0 < \tilde{t}_0$), it follows from Propositions 2.4 and 2.5 in Section 2 below that

$$\limsup_{t \to \infty} t^{n/4} \| \boldsymbol{v}(\cdot,t;t_0) - \boldsymbol{v}(\cdot,t;\tilde{t}_0) \|_{L^2(\mathbb{R}^n)} < \infty, \tag{1.17a}$$

$$\limsup_{t \to \infty} t^{n/2} \| \boldsymbol{v}(\cdot,t;t_0) - \boldsymbol{v}(\cdot,t;\tilde{t}_0) \|_{L^\infty(\mathbb{R}^n)} < \infty. \tag{1.17b}$$

Hence, changing t_0 will only affect $\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t;t_0)$ to higher order as compared to (1.13), thus leaving (1.13) unaffected. The estimates (1.17) are also new.

There is a vast literature on the decay of solutions to the Navier-Stokes equations, especially for L^2-norms: see e.g. [11, 12, 13, 14, 19, 22, 23, 30] for the case $\boldsymbol{f} = \boldsymbol{0}$, and [19, 21, 22, 27] for nonzero $\boldsymbol{f}$. Lower bounds and optimal L^2 estimates are obtained in [1, 9, 20, 23, 24], always under stronger assumptions on the problem data $(\boldsymbol{u}_0, \boldsymbol{f})$. For L^∞ norms, partial results are found in [3, 28], and stronger decay estimates in [1, 2, 6, 10], again assuming stronger properties of the problem data. For a related stability problem concerning strong solutions, see [4].

Notation. We will employ boldface letters to represent vector quantities, as in $\boldsymbol{g}(\cdot,t) = (g_1(\cdot,t), ..., g_n(\cdot,t))$, $\boldsymbol{u}(\cdot,t) = (u_1(\cdot,t), ..., u_n(\cdot,t))$, etc. Vector norms like $\| \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)}$, $\| D\boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)}$, $\| D^2\boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)}$, and so forth, are to be understood in their usual way, that is, setting $D_j \equiv \partial/\partial x_j$:

$$\| \boldsymbol{u}(\cdot,t) \|^2_{L^2(\mathbb{R}^n)} = \sum_{i=1}^{n} \int_{\mathbb{R}^n} | u_i(x,t) |^2 \, dx, \tag{1.18}$$

[3] Using Fourier transforms, one sees that imposing $\boldsymbol{g}(\cdot,t) \in L^r([\,0,\infty\,[\,, L^2(\mathbb{R}^n))$ for *all* $r > 1$ is *not* enough to get $\| \boldsymbol{v}(\cdot,t;t_0) \|_{L^2(\mathbb{R}^n)} \to 0$ as $t \to \infty$. Thus, (1.5) is the natural condition here.

$$\| D\boldsymbol{u}(\cdot,t) \|^2_{L^2(\mathbb{R}^n)} = \sum_{i,\,j\,=\,1}^{n} \int_{\mathbb{R}^n} |\, D_j u_i(x,t)\,|^2\, dx, \tag{1.19}$$

$$\| D^2\boldsymbol{u}(\cdot,t) \|^2_{L^2(\mathbb{R}^n)} = \sum_{i,\,j,\,\ell\,=\,1}^{n} \int_{\mathbb{R}^n} |\, D_j D_\ell u_i(x,t)\,|^2\, dx. \tag{1.20}$$

2. Some mathematical preliminaries. In this section, we recall some basic results which will play an important role in Sections 3 and 4, where the proof of (1.13) is carried out. For the Leray's construction of global weak solutions $\boldsymbol{u}(\cdot,t)$ to the Navier-Stokes equations (1.3), see e.g. [8, 18]. One has $\boldsymbol{u}(\cdot,t) \in L^\infty([\,0,T], L^2_\sigma(\mathbb{R}^n)) \cap L^2([\,0,T], H^1(\mathbb{R}^n))$ for each $T > 0$, with $\boldsymbol{u}(\cdot,t)$ weakly continuous in $L^2(\mathbb{R}^n)$, $\|\, \boldsymbol{u}(\cdot,t) - \boldsymbol{u}_0 \,\|_{L^2(\mathbb{R}^n)} \to 0$ as $t \searrow 0$ and satisfying the energy inequality

$$\begin{aligned} &\|\, \boldsymbol{u}(\cdot,t) \,\|^2_{L^2(\mathbb{R}^n)} + 2\int_0^t \|\, D\boldsymbol{u}(\cdot,s) \,\|^2_{L^2(\mathbb{R}^n)}\, ds \\ \le\;& \|\, \boldsymbol{u}_0 \,\|^2_{L^2(\mathbb{R}^n)} + 2\int_0^t \|\, \boldsymbol{u}(\cdot,s) \,\|_{L^2(\mathbb{R}^n)} \,\|\, \boldsymbol{g}(\cdot,s) \,\|_{L^2(\mathbb{R}^n)}\, ds \end{aligned} \tag{2.1}$$

for all $t \geq 0$. Moreover, taking (any) $G \in C_0^\infty(\mathbb{R}^n)$ nonnegative with $\int_{\mathbb{R}^n} G(x)\, dx = 1$ and setting $\bar{\boldsymbol{u}}_{0,\,\delta}(\cdot)$, $\bar{\boldsymbol{g}}_{\delta}(\cdot,t) \in C^\infty(\mathbb{R}^n)$ by convolving $\boldsymbol{u}_0(\cdot)$, $\boldsymbol{g}(\cdot,t)$, respectively, with $G_\delta(x) = \delta^{-n} G(x/\delta)$, $\delta > 0$, if we define $\boldsymbol{u}_\delta(\cdot,t)$, $P_\delta(\cdot,t) \in C^\infty(\mathbb{R}^n)$ as the globally defined L^2 solutions of the regularized equations

$$\frac{\partial}{\partial t}\boldsymbol{u}_\delta + \bar{\boldsymbol{u}}_\delta(\cdot,t)\cdot\nabla \boldsymbol{u}_\delta + \nabla P_\delta = \Delta \boldsymbol{u}_\delta + \bar{\boldsymbol{g}}_\delta(\cdot,t), \qquad \nabla\cdot \boldsymbol{u}_\delta(\cdot,t) = 0, \tag{2.2a}$$

$$\boldsymbol{u}_\delta(\cdot,0) = \bar{\boldsymbol{u}}_{0,\,\delta} := G_\delta * \boldsymbol{u}_0 \in \bigcap_{m\,=\,1}^{\infty} H^m(\mathbb{R}^n), \tag{2.2b}$$

where $\bar{\boldsymbol{u}}_\delta(\cdot,t) := \boldsymbol{u}_\delta(\cdot,t)$ if $n = 2$, $\bar{\boldsymbol{u}}_\delta(\cdot,t) := G_\delta * \boldsymbol{u}_\delta(\cdot,t)$ if $n = 3$, we then have, for a suitable sequence $\delta' \to 0$, the weak convergence property

$$\boldsymbol{u}_{\delta'}(\cdot,t) \rightharpoonup \boldsymbol{u}(\cdot,t) \quad \text{as } \delta' \to 0, \qquad \forall\; t \geq 0, \tag{2.3}$$

that is, $\boldsymbol{u}_{\delta'}(\cdot,t) \to \boldsymbol{u}(\cdot,t)$ weakly in $L^2(\mathbb{R}^n)$, for each $t \geq 0$ (see e.g. [18], p. 237). More sophisticated versions of Leray's regularization (2.2) have been (and continue to be) developed as a fundamental step in the design of many high resolution numerical schemes for both viscous and inviscid flows, particularly in turbulent regimes, see e.g. [7, 16, 17] and references therein. While the Leray regularization (2.2) is not really useful for numerical approximations, it is very important for analytical purposes, as solutions to (2.2) can be more easily studied due to their regularity (see e.g. Propositions 2.1, 2.4 and 2.5 below).

Proposition 2.1. *For all $t \geq 0$, we have*

$$\|\, \boldsymbol{u}(\cdot,t) \,\|^2_{L^2(\mathbb{R}^n)} + 2\int_0^t \|\, D\boldsymbol{u}(\cdot,s) \,\|^2_{L^2(\mathbb{R}^n)}\, ds \;\le\; E_0(t)^2, \tag{2.4a}$$

where

$$E_0(t) \;=\; \|\, \boldsymbol{u}_0 \,\|_{L^2(\mathbb{R}^n)} \;+ \int_0^t \|\, \boldsymbol{g}(\cdot, s) \,\|_{L^2(\mathbb{R}^n)} \, ds. \tag{2.4b}$$

Proof. For lack of a reference and also for later use, we give a detailed derivation below. Taking $\boldsymbol{u}_\delta(\cdot, t)$ given by (2.2) above, we obtain, using that $\nabla\cdot\, \boldsymbol{u}_\delta(\cdot, t) \,=\, 0$,

$$\begin{aligned} &\|\, \boldsymbol{u}_\delta(\cdot, t) \,\|_{L^2(\mathbb{R}^n)}^2 \;+\; 2\int_0^t \|\, D\boldsymbol{u}_\delta(\cdot, s) \,\|_{L^2(\mathbb{R}^n)}^2 \, ds \\ &\leq\; \|\, \bar{\boldsymbol{u}}_{0,\,\delta} \,\|_{L^2(\mathbb{R}^n)}^2 \;+\; 2\int_0^t \|\, \boldsymbol{u}_\delta(\cdot, s) \,\|_{L^2(\mathbb{R}^n)} \,\|\, \bar{\boldsymbol{g}}_\delta(\cdot, s) \,\|_{L^2(\mathbb{R}^n)} \, ds \end{aligned}$$

for all $t \,\geq\, 0$, with $\|\, \boldsymbol{u}_\delta(\cdot, t) \,\|_{L^2(\mathbb{R}^n)}$ continuous. Setting $\mathrm{w}(t) := \|\, \boldsymbol{u}_\delta(\cdot, t) \,\|_{L^2(\mathbb{R}^n)}^2$, and recalling that $\|\, \bar{\boldsymbol{u}}_{0,\,\delta} \,\|_{L^2(\mathbb{R}^n)} \leq \|\, \boldsymbol{u}_0 \,\|_{L^2(\mathbb{R}^n)}$, $\|\, \bar{\boldsymbol{g}}_\delta(\cdot, s) \,\|_{L^2(\mathbb{R}^n)} \leq \|\, \boldsymbol{g}(\cdot, s) \,\|_{L^2(\mathbb{R}^n)}$, we then have

$$\mathrm{w}(t) \;\leq\; \|\, \boldsymbol{u}_0 \,\|_{L^2(\mathbb{R}^n)}^2 \;+\; 2\int_0^t \|\, \boldsymbol{g}(\cdot, s) \,\|_{L^2(\mathbb{R}^n)} \mathrm{w}(s)^{1/2} \, ds, \qquad t \geq 0.$$

In particular, letting $\mathrm{v} \in C^0([\,0, \infty\,[\,)$ be defined by

$$\mathrm{v}(t) \;=\; \|\, \boldsymbol{u}_0 \,\|_{L^2(\mathbb{R}^n)}^2 + 2\int_0^t \|\, \boldsymbol{g}(\cdot, s) \,\|_{L^2(\mathbb{R}^n)} \mathrm{v}(s)^{1/2} \, ds, \qquad t \geq 0,$$

or, equivalently,

$$\mathrm{v}(t) \;=\; \Big[\; \|\, \boldsymbol{u}_0 \,\|_{L^2(\mathbb{R}^n)} + \int_0^t \|\, \boldsymbol{g}(\cdot, s) \,\|_{L^2(\mathbb{R}^n)} \, ds \;\Big]^2, \qquad t \geq 0,$$

we obtain $\mathrm{w}(t) \leq \mathrm{v}(t)$ for all $t \geq 0$, that is,

$$\|\, \boldsymbol{u}_\delta(\cdot, t) \,\|_{L^2(\mathbb{R}^n)} \;\leq\; \|\, \boldsymbol{u}_0 \,\|_{L^2(\mathbb{R}^n)} + \int_0^t \|\, \boldsymbol{g}(\cdot, s) \,\|_{L^2(\mathbb{R}^n)} \, ds, \qquad t \geq 0, \qquad \delta > 0.$$

Taking $\delta = \delta'$ as in (2.3), we then obtain, letting $\delta' \to 0$,

$$\|\, \boldsymbol{u}(\cdot, t) \,\|_{L^2(\mathbb{R}^n)} \;\leq\; \|\, \boldsymbol{u}_0 \,\|_{L^2(\mathbb{R}^n)} \;+ \int_0^t \|\, \boldsymbol{g}(\cdot, s) \,\|_{L^2(\mathbb{R}^n)} \, ds \;=\; E_0(t), \tag{2.5}$$

since one has $\|\, \boldsymbol{u}(\cdot, t) \,\|_{L^2(\mathbb{R}^n)} \leq \liminf_{\delta' \to 0} \|\, \boldsymbol{u}_{\delta'}(\cdot, t) \,\|_{L^2(\mathbb{R}^n)}$ (see [29], Theorem 1(*ii*), p. 120). In particular, by (2.1), we get

$$\begin{aligned} \mathrm{z}(t) \;:=\; &\|\, \boldsymbol{u}(\cdot, t) \,\|_{L^2(\mathbb{R}^n)}^2 \;+\; 2\int_0^t \|\, D\boldsymbol{u}(\cdot, s) \,\|_{L^2(\mathbb{R}^n)}^2 \, ds \\ \leq\; &\|\, \boldsymbol{u}_0 \,\|_{L^2(\mathbb{R}^n)}^2 \;+\; 2\int_0^t \|\, \boldsymbol{u}(\cdot, s) \,\|_{L^2(\mathbb{R}^n)} \,\|\, \boldsymbol{g}(\cdot, s) \,\|_{L^2(\mathbb{R}^n)} \, ds \\ \leq\; &\|\, \boldsymbol{u}_0 \,\|_{L^2(\mathbb{R}^n)}^2 \;+\; 2\, E_0(t) \int_0^t \|\, \boldsymbol{g}(\cdot, s) \,\|_{L^2(\mathbb{R}^n)} \, ds \qquad \text{(by (2.5))} \\ \leq\; &2\, \|\, \boldsymbol{u}_0 \,\|_{L^2(\mathbb{R}^n)}^2 \;+\; 3\, \Big[\int_0^t \|\, \boldsymbol{g}(\cdot, s) \,\|_{L^2(\mathbb{R}^n)} \, ds \,\Big]^2 \end{aligned}$$

for all $t \geq 0$. Actually, more is true: defining $\mathrm{v}_\eta, \zeta \in C^0([0,\infty[)$ by

$$\zeta(t) \;:=\; \|\,\boldsymbol{u}_0\,\|^2_{L^2(\mathbb{R}^n)} \,+\, 2\int_0^t \|\,\boldsymbol{g}(\cdot,s)\,\|_{L^2(\mathbb{R}^n)}\, \mathrm{z}(s)^{1/2}\, ds,$$

$$\mathrm{v}_\eta(t) \;:=\; \left[\, \Big\{ \|\,\boldsymbol{u}_0\,\|^2_{L^2(\mathbb{R}^n)} + \eta^2 \Big\}^{1/2} + \int_0^t \|\,\boldsymbol{g}(\cdot,s)\,\|_{L^2(\mathbb{R}^n)}\, ds \,\right]^2,$$

where $\eta > 0$, we have

$$\mathrm{v}_\eta(t) \;=\; \|\,\boldsymbol{u}_0\,\|^2_{L^2(\mathbb{R}^n)} + \eta^2 \,+\, 2\int_0^t \|\,\boldsymbol{g}(\cdot,s)\,\|_{L^2(\mathbb{R}^n)}\, \mathrm{v}_\eta(s)^{1/2}\, ds, \qquad t \geq 0,$$

so that $\zeta(t) < \mathrm{v}_\eta(t)$ for all $t \geq 0$ (and any $\eta > 0$). Letting $\eta \to 0$, we get $\zeta(t) \leq E_0(t)^2$, for each $t \geq 0$. This shows (2.4a), since, by their definitions, $\mathrm{z}(t) \leq \zeta(t)$ for all $t \geq 0$. □

Assuming the condition (1.5) above, it follows from (2.4) that

$$\int_0^\infty \|\, D\boldsymbol{u}(\cdot,s)\,\|^2_{L^2(\mathbb{R}^n)}\, ds \;<\; \infty. \tag{2.6}$$

Hence, Leray's solutions verify $\boldsymbol{u}(\cdot,t) \in L^\infty([0,\infty[, L^2_\sigma(\mathbb{R}^n)) \cap L^2([0,\infty[, H^1(\mathbb{R}^n))$ if (1.5) is valid, in view of Proposition 2.1. According to Wiegner [27], assumption (1.5) also ensures that $\|\,\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \to 0$ as $t \to \infty$ (see [27], Theorem (a), p. 305), but to keep our analysis simple and self-contained, we will not use his result, proceeding instead along the much easier lines in [13, 14] (cf. also [30]). Introducing $\boldsymbol{Q}(\cdot,t)$ defined by

$$\boldsymbol{Q}(\cdot,t) \;=\; -\,\boldsymbol{u}(\cdot,t)\cdot\nabla\,\boldsymbol{u}(\cdot,t) \;-\; \nabla P(\cdot,t), \tag{2.7}$$

where $P(\cdot,t)$ is the modified pressure given in equation (1.3a), the following basic estimate will play a key role in our discussion, as it did in [13, 14, 30] before.

Proposition 2.2. *For almost every $s > 0$, we have*

$$\|\, e^{\Delta(t-s)}\,\boldsymbol{Q}(\cdot,s)\,\|_{L^2(\mathbb{R}^n)} \leq\; K_n\,(t-s)^{-n/4}\, \|\,\boldsymbol{u}(\cdot,s)\,\|_{L^2(\mathbb{R}^n)}\, \|\, D\boldsymbol{u}(\cdot,s)\,\|_{L^2(\mathbb{R}^n)} \tag{2.8a}$$

and

$$\|\, e^{\Delta(t-s)}\,\boldsymbol{Q}(\cdot,s)\,\|_{L^\infty(\mathbb{R}^n)} \leq\; K_n\,(t-s)^{-n/4}\, \|\,\boldsymbol{u}(\cdot,s)\,\|_{L^\infty(\mathbb{R}^n)}\, \|\, D\boldsymbol{u}(\cdot,s)\,\|_{L^2(\mathbb{R}^n)} \tag{2.8b}$$

for all $t > s$, where $K_n = (8\pi)^{-n/4}$ and $\|\,\mathbf{u}\,\|_{L^\infty(\mathbb{R}^n)} = \operatorname{ess\,sup}\{\, |\,\mathbf{u}(x)\,|_2 : x \in \mathbb{R}^n \,\}$.

Proof. We begin with (2.8a), reviewing the proof in [13, 14, 30], as it will be needed later. Letting $\mathbb{F}[f] \equiv \hat{f}$ denote the Fourier transform of a given function $f \in L^1(\mathbb{R}^n)$, namely,

$$\mathbb{F}[f](k) \;\equiv\; \hat{f}(k) \;:=\; (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-\,\mathring{\imath}\, k\cdot x} f(x)\, dx, \qquad k \in \mathbb{R}^n \tag{2.9}$$

(where $\mathring{\imath}^2 = -1$), and, given some field $\mathbf{v}(\cdot, s) = (\mathrm{v}_1(\cdot, s), ..., \mathrm{v}_n(\cdot, s)) \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, we obtain, using Parseval's identity,

$$\begin{aligned} \| e^{\Delta(t-s)} \mathbf{v}(\cdot, s) \|^2_{L^2(\mathbb{R}^n)} &= \| \mathbb{F}[e^{\Delta(t-s)} \mathbf{v}(\cdot, s)] \|^2_{L^2(\mathbb{R}^n)} \\ &= \int_{\mathbb{R}^n} e^{-2|k|_2^2 (t-s)} | \hat{\mathbf{v}}(k, s) |_2^2 \, dk \\ &\le \| \hat{\mathbf{v}}(\cdot, s) \|^2_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} e^{-2|k|_2^2 (t-s)} \, dk \\ &= \Big(\frac{\pi}{2} \Big)^{n/2} (t-s)^{-n/2} \| \hat{\mathbf{v}}(\cdot, s) \|^2_{L^\infty(\mathbb{R}^n)}, \end{aligned}$$

that is,

$$\| e^{\Delta(t-s)} \mathbf{v}(\cdot, s) \|_{L^2(\mathbb{R}^n)} \le \Big(\frac{\pi}{2} \Big)^{n/4} (t-s)^{-n/4} \| \hat{\mathbf{v}}(\cdot, s) \|_{L^\infty(\mathbb{R}^n)}, \tag{2.10}$$

where $|\cdot|_2$ denotes the Euclidean norm in $\mathbb{R}^n$ and

$$\| \hat{\mathbf{v}}(\cdot, s) \|_{L^\infty(\mathbb{R}^n)} = \sup_{k \in \mathbb{R}^n} | \hat{\mathbf{v}}(k, s) |_2. \tag{2.11}$$

As we will see below, bound (2.8a) follows from a direct application of inequality (2.10) to $\mathbf{v}(\cdot, s) = \boldsymbol{Q}(\cdot, s)$, for $s > 0$ such that $\| D\boldsymbol{u}(\cdot, s) \|_{L^2(\mathbb{R}^n)} < \infty$. So, we estimate $\| \hat{\boldsymbol{Q}}(\cdot, s) \|_{L^\infty(\mathbb{R}^n)}$ as follows: Since $\mathbb{F}[\nabla P(\cdot, s)](k) = \mathring{\imath}\, \hat{P}(k, s)\, k$ and $\sum_{j=1}^n k_j \hat{Q}_j(k, s) = 0$ (in view of $\nabla \cdot \boldsymbol{Q}(\cdot, s) = 0$), the vectors $\mathbb{F}[\nabla P(\cdot, s)](k)$ and $\hat{\boldsymbol{Q}}(k, s)$ are orthogonal in $\mathbb{C}^n$, for all $k \in \mathbb{R}^n$. Recalling that, from (2.7), we have $\hat{\boldsymbol{Q}}(k, s) + \mathbb{F}[\nabla P(\cdot, s)](k) = -\mathbb{F}[\boldsymbol{u}(\cdot, s) \cdot \nabla \boldsymbol{u}(\cdot, s)](k)$, it follows that $| \hat{\boldsymbol{Q}}(k, s) |_2 \le | \mathbb{F}[\boldsymbol{u}(\cdot, s) \cdot \nabla \boldsymbol{u}(\cdot, s)](k) |_2$ for each k, so that we get

$$\| \hat{\boldsymbol{Q}}(\cdot, s) \|_{L^\infty(\mathbb{R}^n)} \le \| \mathbb{F}[\boldsymbol{u} \cdot \nabla \boldsymbol{u}](\cdot, s) \|_{L^\infty(\mathbb{R}^n)}. \tag{2.12a}$$

Now, we have, for each $1 \le i \le n$,

$$\begin{aligned} | \mathbb{F}[\boldsymbol{u}(\cdot, s) \cdot \nabla u_i(\cdot, s)](k) | &\le \sum_{j=1}^n | \mathbb{F}[u_j(\cdot, s) D_j u_i(\cdot, s)](k) | \\ &\le (2\pi)^{-n/2} \sum_{j=1}^n \| u_j(\cdot, s) D_j u_i(\cdot, s) \|_{L^1(\mathbb{R}^n)} \\ &\le (2\pi)^{-n/2} \| \boldsymbol{u}(\cdot, s) \|_{L^2(\mathbb{R}^n)} \| \nabla u_i(\cdot, s) \|_{L^2(\mathbb{R}^n)}, \end{aligned}$$

by the Cauchy-Schwarz inequality. (Here, as before, $D_j = \partial/\partial x_j$.) This gives

$$\| \mathbb{F}[\boldsymbol{u} \cdot \nabla \boldsymbol{u}](\cdot, s) \|_{L^\infty(\mathbb{R}^n)} \le (2\pi)^{-n/2} \| \boldsymbol{u}(\cdot, s) \|_{L^2(\mathbb{R}^n)} \| D\boldsymbol{u}(\cdot, s) \|_{L^2(\mathbb{R}^n)}. \tag{2.12b}$$

The estimate (2.8a) now follows directly from (2.10), (2.12a) and (2.12b), which concludes the first part of the proof. The derivation of (2.8b) follows a similar path, using again that $| \hat{\boldsymbol{Q}}(k, s) |_2 \le | \mathbb{F}[\boldsymbol{u}(\cdot, s) \cdot \nabla \boldsymbol{u}(\cdot, s)](k) |_2$ for all $k \in \mathbb{R}^n$. Recalling the elementary estimate $\| e^{\Delta \tau} \mathrm{u} \|_{L^\infty(\mathbb{R}^n)} \le K_n \, \tau^{-n/4} \| \mathrm{u} \|_{L^2(\mathbb{R}^n)}$ for the heat semigroup, where $\tau > 0$ is arbitrary and $K_n = (8\pi)^{-n/4}$, we get, for any $s > 0$ such that $\| \boldsymbol{u}(\cdot, s) \|_{L^\infty(\mathbb{R}^n)}$ and $\| D\boldsymbol{u}(\cdot, s) \|_{L^2(\mathbb{R}^n)}$ are finite,

$$\begin{aligned} \| e^{\Delta(t-s)} \boldsymbol{Q}(\cdot, s) \|_{L^\infty(\mathbb{R}^n)} &\leq K_n (t-s)^{-n/4} \| \boldsymbol{Q}(\cdot, s) \|_{L^2(\mathbb{R}^n)} \\ &\leq K_n (t-s)^{-n/4} \| \boldsymbol{u}(\cdot, s) \cdot \nabla \boldsymbol{u}(\cdot, s) \|_{L^2(\mathbb{R}^n)} \\ &\leq K_n (t-s)^{-n/4} \| \boldsymbol{u}(\cdot, s) \|_{L^\infty(\mathbb{R}^n)} \| D\boldsymbol{u}(\cdot, s) \|_{L^2(\mathbb{R}^n)} \end{aligned}$$

for all $t > s$, by Parseval's identity. This shows (2.8*b*), and the proof is now complete. □

Remark 2.1. Applying the argument above to solutions of the regularized Navier-Stokes equations (2.2), we obtain, in a completely similar way,

$$\begin{aligned} &\| e^{\Delta(t-s)} \boldsymbol{Q}_\delta(\cdot, s) \|_{L^2(\mathbb{R}^n)} \\ &\leq K_n (t-s)^{-n/4} \| \boldsymbol{u}_\delta(\cdot, s) \|_{L^2(\mathbb{R}^n)} \| D\boldsymbol{u}_\delta(\cdot, s) \|_{L^2(\mathbb{R}^n)} \end{aligned} \tag{2.13a}$$

and

$$\begin{aligned} &\| e^{\Delta(t-s)} \boldsymbol{Q}_\delta(\cdot, s) \|_{L^\infty(\mathbb{R}^n)} \\ &\leq K_n (t-s)^{-n/4} \| \boldsymbol{u}_\delta(\cdot, s) \|_{L^\infty(\mathbb{R}^n)} \| D\boldsymbol{u}_\delta(\cdot, s) \|_{L^2(\mathbb{R}^n)} \end{aligned} \tag{2.13b}$$

for all $t > s > 0$, where $\boldsymbol{Q}_\delta(\cdot, s) = -\bar{\boldsymbol{u}}_\delta(\cdot, s) \cdot \nabla \boldsymbol{u}_\delta(\cdot, s) - \nabla P_\delta(\cdot, s)$, $K_n = (8\pi)^{-n/4}$.

Proposition 2.3. *We have*

$$\| \boldsymbol{u}_\delta(\cdot, t) \|^2_{L^2(\mathbb{R}^n)} + 2\int_0^t \| D\boldsymbol{u}_\delta(\cdot, s) \|^2_{L^2(\mathbb{R}^n)} \, ds \leq E_0(t)^2 \tag{2.14}$$

for all $t > 0$, $\delta > 0$, *where* $E_0(t)$ *is given by* (2.4*b*) *above.*

Proof. The proof is similar to that of (2.4). Taking the dot product of the first equation in (2.2*a*) with $\boldsymbol{u}_\delta(\cdot, t)$ and integrating on $\mathbb{R}^n \times [0, t]$, we get

$$\begin{aligned} &\| \boldsymbol{u}_\delta(\cdot, t) \|^2_{L^2(\mathbb{R}^n)} + 2\int_0^t \| D\boldsymbol{u}_\delta(\cdot, s) \|^2_{L^2(\mathbb{R}^n)} \, ds \\ &\leq \| \bar{\boldsymbol{u}}_{0,\delta} \|^2_{L^2(\mathbb{R}^n)} + 2\int_0^t \| \boldsymbol{u}_\delta(\cdot, s) \|_{L^2(\mathbb{R}^n)} \| \bar{\boldsymbol{g}}_\delta(\cdot, s) \|_{L^2(\mathbb{R}^n)} \, ds \\ &\leq \| \boldsymbol{u}_0 \|^2_{L^2(\mathbb{R}^n)} + 2\int_0^t \| \boldsymbol{u}_\delta(\cdot, s) \|_{L^2(\mathbb{R}^n)} \| \boldsymbol{g}(\cdot, s) \|_{L^2(\mathbb{R}^n)} \, ds, \end{aligned}$$

so that we have, for $\mathrm{v}_\delta(t) := \| \boldsymbol{u}_\delta(\cdot, t) \|^2_{L^2(\mathbb{R}^n)} + 2\int_0^t \| D\boldsymbol{u}_\delta(\cdot, s) \|^2_{L^2(\mathbb{R}^n)} \, ds$, the inequality

$$\mathrm{v}_\delta(t) \leq \| \boldsymbol{u}_0 \|^2_{L^2(\mathbb{R}^n)} + 2\int_0^t \| \boldsymbol{g}(\cdot, s) \|_{L^2(\mathbb{R}^n)} \mathrm{v}_\delta(s)^{1/2} \, ds, \qquad t \geq 0.$$

This gives, as in the proof of Proposition 2.1, that $\mathrm{v}_\delta(t) \leq E_0(t)^2$, as claimed. □

We are now in good standing to show that the particular value of $t_0 \geq 0$ chosen in defining the Stokes approximations (1.4) is not relevant in regard to (1.13).

Proposition 2.4. *Given* $\tilde{t}_0 > t_0 \geq 0$, *let* $\boldsymbol{v}(\cdot,t\,;t_0)$, $\boldsymbol{v}(\cdot,t\,;\tilde{t}_0)$ *be the solutions of the Stokes equations* (1.4) *with Cauchy data* $\boldsymbol{v}(\cdot,t_0;t_0)=\boldsymbol{u}(\cdot,t_0)$, $\boldsymbol{v}(\cdot,\tilde{t}_0;\tilde{t}_0)=\boldsymbol{u}(\cdot,\tilde{t}_0)$, *respectively, where* $\boldsymbol{u}(\cdot,t)$ *is any given Leray solution of* (1.3), *defined for all* $t>0$. *Then*

$$\| \boldsymbol{v}(\cdot,t;t_0) - \boldsymbol{v}(\cdot,t;\tilde{t}_0) \|_{L^2(\mathbb{R}^n)} \leq \frac{K_n}{\sqrt{2}}\, E_0(\tilde{t}_0)^2\, (\tilde{t}_0 - t_0)^{1/2}\, (t-\tilde{t}_0)^{-n/4} \tag{2.15}$$

for all $t>\tilde{t}_0$, *where* $K_n=(8\pi)^{-n/4}$ *and* $E_0(\cdot)$ *is defined in* (2.4*b*) *above.*

Proof. Let us set $\boldsymbol{v}(\cdot,t) := \boldsymbol{v}(\cdot,t\,;t_0)$, $\tilde{\boldsymbol{v}}(\cdot,t) := \boldsymbol{v}(\cdot,t\,;\tilde{t}_0)$. By the integral representation (1.14), we have, for $t>t_0$,

$$\boldsymbol{v}(\cdot,t) = e^{\Delta(t-t_0)}[\,\boldsymbol{u}(\cdot,t_0)-\boldsymbol{u}_\delta(\cdot,t_0)\,] + e^{\Delta(t-t_0)}\boldsymbol{u}_\delta(\cdot,t_0) + \int_{t_0}^{t} e^{\Delta(t-s)}\boldsymbol{g}(\cdot,s)\,ds,$$

with $\boldsymbol{u}_\delta(\cdot,t)$ given in (2.2), $\delta>0$. Since

$$\boldsymbol{u}_\delta(\cdot,t_0) = e^{\Delta t_0}\bar{\boldsymbol{u}}_{0,\,\delta} + \int_0^{t_0} e^{\Delta(t_0-s)}\bar{\boldsymbol{g}}_\delta(\cdot,s)\,ds + \int_0^{t_0} e^{\Delta(t_0-s)}\boldsymbol{Q}_\delta(\cdot,s)\,ds,$$

where $\boldsymbol{Q}_\delta(\cdot,s) = -\,\bar{\boldsymbol{u}}_\delta(\cdot,s)\cdot\nabla\boldsymbol{u}_\delta(\cdot,s) - \nabla P_\delta(\cdot,s)$, we get

$$\begin{aligned}\boldsymbol{v}(\cdot,t) &= e^{\Delta(t-t_0)}[\,\boldsymbol{u}(\cdot,t_0)-\boldsymbol{u}_\delta(\cdot,t_0)\,] + \int_0^{t_0} e^{\Delta(t-s)}[\,\bar{\boldsymbol{g}}_\delta(\cdot,s)-\boldsymbol{g}(\cdot,s)\,]\,ds \\ &\quad + e^{\Delta t}\bar{\boldsymbol{u}}_{0,\,\delta} + \int_0^{t} e^{\Delta(t-s)}\boldsymbol{g}(\cdot,s)\,ds + \int_0^{t_0} e^{\Delta(t-s)}\boldsymbol{Q}_\delta(\cdot,s)\,ds,\end{aligned}$$

for $t>t_0$. Similarly, we have, for $t>\tilde{t}_0$,

$$\begin{aligned}\tilde{\boldsymbol{v}}(\cdot,t) &= e^{\Delta(t-\tilde{t}_0)}[\,\boldsymbol{u}(\cdot,\tilde{t}_0)-\boldsymbol{u}_\delta(\cdot,\tilde{t}_0)\,] + \int_0^{\tilde{t}_0} e^{\Delta(t-s)}[\,\bar{\boldsymbol{g}}_\delta(\cdot,s)-\boldsymbol{g}(\cdot,s)\,]\,ds \\ &\quad + e^{\Delta t}\bar{\boldsymbol{u}}_{0,\,\delta} + \int_0^{t} e^{\Delta(t-s)}\boldsymbol{g}(\cdot,s)\,ds + \int_0^{\tilde{t}_0} e^{\Delta(t-s)}\boldsymbol{Q}_\delta(\cdot,s)\,ds,\end{aligned}$$

so that we have, for $t>\tilde{t}_0$:

$$\begin{aligned}\tilde{\boldsymbol{v}}(\cdot,t) - \boldsymbol{v}(\cdot,t) &= e^{\Delta(t-\tilde{t}_0)}[\,\boldsymbol{u}(\cdot,\tilde{t}_0)-\boldsymbol{u}_\delta(\cdot,\tilde{t}_0)\,] - e^{\Delta(t-t_0)}[\,\boldsymbol{u}(\cdot,t_0)-\boldsymbol{u}_\delta(\cdot,t_0)\,] \\ &\quad + \int_{t_0}^{\tilde{t}_0} e^{\Delta(t-s)}[\,\bar{\boldsymbol{g}}_\delta(\cdot,s)-\boldsymbol{g}(\cdot,s)\,]\,ds + \int_{t_0}^{\tilde{t}_0} e^{\Delta(t-s)}\boldsymbol{Q}_\delta(\cdot,s)\,ds\end{aligned} \tag{2.16}$$

Therefore, given any $\mathbb{K}\subset\mathbb{R}^n$ compact, we get, for each $t>\tilde{t}_0$, $\delta>0$,

$$\begin{aligned}&\| \tilde{\boldsymbol{v}}(\cdot,t) - \boldsymbol{v}(\cdot,t) \|_{L^2(\mathbb{K})} \\ &\leq J_\delta(t) + \int_{t_0}^{\tilde{t}_0} \| e^{\Delta(t-s)}\boldsymbol{Q}_\delta(\cdot,s) \|_{L^2(\mathbb{K})}\,ds \\ &\leq J_\delta(t) + K_n\int_{t_0}^{\tilde{t}_0} (t-s)^{-n/4}\, \| \boldsymbol{u}_\delta(\cdot,s) \|_{L^2(\mathbb{R}^n)} \| D\boldsymbol{u}_\delta(\cdot,s) \|_{L^2(\mathbb{R}^n)}\,ds \\ &\leq J_\delta(t) + \frac{K_n}{\sqrt{2}}\,(\tilde{t}_0-t_0)^{\frac{1}{2}}\,E_0(\tilde{t}_0)^2\,(t-\tilde{t}_0)^{-\frac{n}{4}}\end{aligned}$$

by (2.4b), (2.13a) and (2.14), where

$$
\begin{aligned}
J_\delta(t) &= \| e^{\Delta(t-\tilde{t}_0)} [\, \boldsymbol{u}(\cdot,\tilde{t}_0) - \boldsymbol{u}_\delta(\cdot,\tilde{t}_0) \,] \|_{L^2(\mathbb{K})} \\
&\quad + \| e^{\Delta(t-t_0)} [\, \boldsymbol{u}(\cdot,t_0) - \boldsymbol{u}_\delta(\cdot,t_0) \,] \|_{L^2(\mathbb{K})} \\
&\quad + \int_{t_0}^{\tilde{t}_0} \| e^{\Delta(t-s)} [\, \bar{\boldsymbol{g}}_\delta(\cdot,s) - \boldsymbol{g}(\cdot,s) \,] \|_{L^2(\mathbb{K})}\, ds \\
&\le \| e^{\Delta(t-\tilde{t}_0)} [\, \boldsymbol{u}(\cdot,\tilde{t}_0) - \boldsymbol{u}_\delta(\cdot,\tilde{t}_0) \,] \|_{L^2(\mathbb{K})} \\
&\quad + \| e^{\Delta(t-t_0)} [\, \boldsymbol{u}(\cdot,t_0) - \boldsymbol{u}_\delta(\cdot,t_0) \,] \|_{L^2(\mathbb{K})} \\
&\quad + \int_{t_0}^{\tilde{t}_0} \| \bar{\boldsymbol{g}}_\delta(\cdot,s) - \boldsymbol{g}(\cdot,s) \|_{L^2(\mathbb{R}^n)}\, ds.
\end{aligned}
$$

Taking $\delta = \delta' \to 0$, according to (2.3) we get $J_\delta(t) \to 0$, since $\bar{\boldsymbol{g}}_\delta(\cdot,s) \to \boldsymbol{g}(\cdot,s)$ in $L^2(\mathbb{R}^n)$ and, by (2.3) and the Lebesgue's Dominated Convergence Theorem, we get, for any σ, $\tau > 0$:

$$\| e^{\Delta\tau} [\, \boldsymbol{u}(\cdot,\sigma) - \boldsymbol{u}_{\delta'}(\cdot,\sigma) \,] \|_{L^2(\mathbb{K})} \to 0 \qquad \text{as} \quad \delta' \to 0,$$

since $\mathbb{K}$ has finite measure. Hence, we obtain

$$\| \tilde{\boldsymbol{v}}(\cdot,t) - \boldsymbol{v}(\cdot,t) \|_{L^2(\mathbb{K})} \le \frac{K_n}{\sqrt{2}} (\tilde{t}_0 - t_0)^{1/2} E_0(\tilde{t}_0)^2 (t - \tilde{t}_0)^{-n/4}$$

for each $t > \tilde{t}_0$, and for *any* compact set $\mathbb{K} \subset \mathbb{R}^n$. This is clearly equivalent to (2.15). ▯

Proposition 2.5. *Given* $\tilde{t}_0 > t_0 \ge 0$, *let* $\boldsymbol{v}(\cdot,t\,;t_0)$, $\boldsymbol{v}(\cdot,t\,;\tilde{t}_0)$ *be the solutions of the Stokes equations* (1.4) *with Cauchy data* $\boldsymbol{v}(\cdot,t_0;t_0) = \boldsymbol{u}(\cdot,t_0)$, $\boldsymbol{v}(\cdot,\tilde{t}_0;\tilde{t}_0) = \boldsymbol{u}(\cdot,\tilde{t}_0)$, *respectively, where* $\boldsymbol{u}(\cdot,t)$ *is any given Leray solution of* (1.3), *defined for all* $t > 0$. *Then*

$$\| \boldsymbol{v}(\cdot,t;t_0) - \boldsymbol{v}(\cdot,t;\tilde{t}_0) \|_{L^\infty(\mathbb{R}^n)} \le \frac{\Gamma_n}{\sqrt{2}} E_0(\tilde{t}_0)^2 (\tilde{t}_0 - t_0)^{1/2} (t - \tilde{t}_0)^{-n/2} \tag{2.17}$$

for all $t > \tilde{t}_0$, *where* $\Gamma_n = (4\pi)^{-n/2}$ *and* $E_0(\cdot)$ *is defined in* (2.4b) *above.*

Proof. For simplicity, let us write (again) $\boldsymbol{v}(\cdot,t) \equiv \boldsymbol{v}(\cdot,t;t_0)$, $\tilde{\boldsymbol{v}}(\cdot,t) \equiv \boldsymbol{v}(\cdot,t;t_0)$. Taking $\mathbb{K} \subset \mathbb{R}^n$ compact and $2 < q < \infty$ arbitrary, we get, for each $t > \tilde{t}_0$, $\delta > 0$, recalling (2.16):

$$
\begin{aligned}
&\| \tilde{\boldsymbol{v}}(\cdot,t) - \boldsymbol{v}(\cdot,t) \|_{L^q(\mathbb{K})} \\
&\le J_{\delta,q}(t) + \int_{t_0}^{\tilde{t}_0} \| e^{\Delta(t-s)} \boldsymbol{Q}_\delta(\cdot,s) \|_{L^q(\mathbb{R}^n)}\, ds \\
&\le J_{\delta,q}(t) + \int_{t_0}^{\tilde{t}_0} \big[4\pi (t-s) \big]^{-\frac{n}{4}\left(1-\frac{2}{q}\right)} \| e^{\frac{1}{2}\Delta(t-s)} \boldsymbol{Q}_\delta(\cdot,s) \|_{L^2(\mathbb{R}^n)}\, ds \\
&\le J_{\delta,q}(t) + \gamma_q \int_{t_0}^{\tilde{t}_0} (t-s)^{-\frac{n}{2}\left(1-\frac{1}{q}\right)} \| \boldsymbol{u}_\delta(\cdot,s) \|_{L^2(\mathbb{R}^n)} \| D\boldsymbol{u}_\delta(\cdot,s) \|_{L^2} ds \\
&\le J_{\delta,q}(t) + \frac{\gamma_q}{\sqrt{2}} (\tilde{t}_0 - t_0)^{\frac{1}{2}} E_0(\tilde{t}_0)^2 (t - \tilde{t}_0)^{-\frac{n}{2}\left(1-\frac{1}{q}\right)}
\end{aligned}
$$

by (2.13a), (2.14) and standard heat kernel L^q-L^2 estimates, where $\gamma_q = (4\pi)^{-\frac{n}{2}\left(1-\frac{1}{q}\right)}$ and

$$\begin{aligned}
J_{\delta,q}(t) &= \| e^{\Delta(t-\tilde{t}_0)} [\boldsymbol{u}(\cdot,\tilde{t}_0) - \boldsymbol{u}_\delta(\cdot,\tilde{t}_0)] \|_{L^q(\mathbb{K})} \\
&\quad + \| e^{\Delta(t-t_0)} [\boldsymbol{u}(\cdot,t_0) - \boldsymbol{u}_\delta(\cdot,t_0)] \|_{L^q(\mathbb{K})} \\
&\quad + \int_{t_0}^{\tilde{t}_0} \| e^{\Delta(t-s)} [\bar{\boldsymbol{g}}_\delta(\cdot,s) - \boldsymbol{g}(\cdot,s)] \|_{L^q(\mathbb{R}^n)}\, ds \\
&\le \| e^{\Delta(t-\tilde{t}_0)} [\boldsymbol{u}(\cdot,\tilde{t}_0) - \boldsymbol{u}_\delta(\cdot,\tilde{t}_0)] \|_{L^q(\mathbb{K})} \\
&\quad + \| e^{\Delta(t-t_0)} [\boldsymbol{u}(\cdot,t_0) - \boldsymbol{u}_\delta(\cdot,t_0)] \|_{L^q(\mathbb{K})} \\
&\quad + \big(8\pi(t-\tilde{t}_0)\big)^{-\frac{n}{4}\left(1-\frac{2}{q}\right)} \int_{t_0}^{\tilde{t}_0} \| \bar{\boldsymbol{g}}_\delta(\cdot,s) - \boldsymbol{g}(\cdot,s) \|_{L^2(\mathbb{R}^n)}\, ds
\end{aligned}$$

Taking $\delta = \delta' \to 0$, according to (2.3) we get $J_{\delta,q}(t) \to 0$, since, by Lebesgue's Dominated Convergence Theorem and (2.3), we have $\| e^{\Delta\tau} [\boldsymbol{u}(\cdot,\sigma) - \boldsymbol{u}_{\delta'}(\cdot,\sigma)] \|_{L^q(\mathbb{K})} \to 0$ as $\delta' \to 0$, for each σ, $\tau > 0$. Hence, letting $\delta = \delta' \to 0$, we obtain

$$\| \tilde{\boldsymbol{v}}(\cdot,t) - \boldsymbol{v}(\cdot,t) \|_{L^q(\mathbb{K})} \le \frac{\gamma_q}{\sqrt{2}} (\tilde{t}_0 - t_0)^{\frac{1}{2}} E_0(\tilde{t}_0)^2 (t - \tilde{t}_0)^{-\frac{n}{2}\left(1-\frac{1}{q}\right)}$$

for each $t > \tilde{t}_0$, $q > 2$. This gives, letting $q \to \infty$,

$$\| \tilde{\boldsymbol{v}}(\cdot,t) - \boldsymbol{v}(\cdot,t) \|_{L^\infty(\mathbb{K})} \le \frac{\Gamma_n}{\sqrt{2}} (\tilde{t}_0 - t_0)^{\frac{1}{2}} E_0(\tilde{t}_0)^2 (t - \tilde{t}_0)^{-\frac{n}{2}}$$

for each $t > \tilde{t}_0$, with $\mathbb{K} \subset \mathbb{R}^n$ compact arbitrary. This estimate is equivalent to (2.17). □

We can finally begin the derivation of our main results, which will be carried out in the following two sections. In dimension $n = 2$, the estimates (1.13a), (1.13b) are much easier to obtain, so this case will be considered first (see Section 3 below).

3. Derivation of (1.13): $\boldsymbol{n = 2}$. As in [13, 14], we begin with the following estimate, which is important on its own and makes the derivation of (1.13) very simple.

Proposition 3.1. *Assuming* (1.5) *and* (1.8), *we have*

$$\lim_{t\to\infty} t^{1/2} \| D\boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^2)} = 0. \tag{3.1}$$

Proof. Let $t_* > t_1$, where t_1 is given in (1.8). Taking the curl of the first equation in the Navier-Stokes system (1.3a), we have

$$\omega_t + \boldsymbol{u}(\cdot,t)\cdot\nabla\omega(\cdot,t) = \Delta\omega(\cdot,t) + h(\cdot,t), \qquad h(\cdot,t) = D_1 g_2(\cdot,t) - D_2 g_1(\cdot,t),$$

where $\omega(x,t) = D_1 u_2(x,t) - D_2 u_1(x,t)$ is the vorticity, and $h(x,t) = D_1 g_2(x,t) - D_2 g_1(x,t)$ is the curl of $\boldsymbol{g}(\cdot,t)$. Multiplying this equation by

$2\,(t-t_*)\,\omega(\cdot,t)$ and integrating the result on $\mathbb{R}^2\times[\,t_*,\,t]$, $t>t_*$, we then obtain

$$(t-t_*)\,\|\,\omega(\cdot,t)\,\|^2_{L^2(\mathbb{R}^2)} + 2\int_{t_*}^{t}(s-t_*)\,\|\,D\omega(\cdot,s)\,\|^2_{L^2(\mathbb{R}^2)}\,ds$$
$$\leq \int_{t_*}^{t}\|\,\omega(\cdot,s)\,\|^2_{L^2(\mathbb{R}^2)}\,ds + 2\int_{t_*}^{t}(s-t_*)\,\|\,\omega(\cdot,s)\,\|_{L^2(\mathbb{R}^2)}\,\|\,h(\cdot,s)\,\|_{L^2(\mathbb{R}^2)}\,ds,$$

or, since $\|\,\omega(\cdot,s)\,\|_{L^2(\mathbb{R}^2)} = \|\,D\boldsymbol{u}(\cdot,s)\,\|_{L^2(\mathbb{R}^2)}$, $\|\,h(\cdot,s)\,\|_{L^2(\mathbb{R}^2)} = \|\,D\boldsymbol{g}(\cdot,s)\,\|_{L^2(\mathbb{R}^2)}$,

$$(t-t_*)\,\|\,D\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^2)} + 2\int_{t_*}^{t}(s-t_*)\,\|\,D^2\boldsymbol{u}(\cdot,s)\,\|^2_{L^2(\mathbb{R}^2)}\,ds$$
$$\leq \int_{t_*}^{t}\|\,D\boldsymbol{u}(\cdot,s)\,\|^2_{L^2(\mathbb{R}^2)}\,ds + 2\int_{t_*}^{t}(s-t_*)\,\|\,D\boldsymbol{u}(\cdot,s)\,\|_{L^2(\mathbb{R}^2)}\,\|\,D\boldsymbol{g}(\cdot,s)\,\|_{L^2(\mathbb{R}^2)}\,ds.$$

In terms of $\mathrm{w}(t) := (t-t_*)\,\|\,D\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^2)}$, this gives

$$\mathrm{w}(t) \leq \int_{t_*}^{t}\|\,D\boldsymbol{u}(\cdot,s)\,\|^2_{L^2(\mathbb{R}^2)}\,ds + 2\int_{t_*}^{t}(s-t_*)^{1/2}\,\|\,D\boldsymbol{g}(\cdot,s)\,\|_{L^2(\mathbb{R}^2)}\,\mathrm{w}(s)^{1/2}\,ds,$$

for any $t>t_*$. As in the proof of Proposition 2.1, we then get

$$\mathrm{w}(t)^{1/2} \leq \left[\int_{t_*}^{t}\|\,D\boldsymbol{u}(\cdot,s)\,\|^2_{L^2(\mathbb{R}^2)}\,ds\right]^{1/2} + \int_{t_*}^{t}(s-t_*)^{1/2}\,\|\,D\boldsymbol{g}(\cdot,s)\,\|_{L^2(\mathbb{R}^2)}\,ds,$$

so that we have, setting $\bar{\lambda} := \limsup_{t\to\infty} t^{1/2}\,\|\,D\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^2)}$,

$$\bar{\lambda} \leq \left[\int_{t_*}^{\infty}\|\,D\boldsymbol{u}(\cdot,s)\,\|^2_{L^2(\mathbb{R}^2)}\,ds\right]^{1/2} + \int_{t_*}^{\infty}s^{1/2}\,\|\,D\boldsymbol{g}(\cdot,s)\,\|_{L^2(\mathbb{R}^2)}\,ds.$$

Since $t_*>t_1$ is arbitrary, this shows (3.1), in view of (1.8) and (2.6) above. ◻

Once (3.1) has been obtained, Theorems A and B follow very readily, recalling the basics in Section 2. The easier result is Theorem A, which is derived next.

Proposition 3.2. *Assuming* (1.5) *and* (1.8), *with* $n=2$, *let* $\boldsymbol{u}(\cdot,t)$ *be the globally defined Leray-Hopf solution of the Navier-Stokes problem* (1.3). *Given* $t_0\geq 0$, *let* $\boldsymbol{v}(\cdot,t;t_0)\in L^\infty([t_0,\infty[\,,L^2_\sigma(\mathbb{R}^2))$ *be the Stokes approximation defined in* (1.4). *Then*

$$\lim_{t\to\infty}\|\,\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t;t_0)\,\|_{L^2(\mathbb{R}^2)} = 0. \tag{3.2}$$

Proof. By Proposition 2.4, we have only to show the result for $t_0\geq 0$ sufficiently large; we choose $t_0>t_1$, with $t_1\geq 0$ given in (1.8). We then have, for $t>t_0$,

$$\boldsymbol{u}(\cdot,t) = e^{\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0) + \int_{t_0}^{t}e^{\Delta(t-s)}\boldsymbol{g}(\cdot,s)\,ds + \int_{t_0}^{t}e^{\Delta(t-s)}\boldsymbol{Q}(\cdot,s)\,ds,$$

where $\boldsymbol{Q}(\cdot,s)$ is given in (2.7), and

$$\boldsymbol{v}(\cdot,t;t_0) = e^{\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0) + \int_{t_0}^{t}e^{\Delta(t-s)}\boldsymbol{g}(\cdot,s)\,ds, \qquad t>t_0.$$

Therefore, by (2.4) and (2.8a),

$$\begin{aligned}\| \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t;t_0) \|_{L^2(\mathbb{R}^2)} &\le \int_{t_0}^{t} \| e^{\Delta(t-s)} \boldsymbol{Q}(\cdot,s) \|_{L^2(\mathbb{R}^2)} ds \\ &\le K_2 \int_{t_0}^{t} (t-s)^{-1/2} \| \boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^2)} \| D\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^2)} ds \\ &\le K_2\, E_0(\infty) \int_{t_0}^{t} (t-s)^{-1/2} \| D\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^2)} ds,\end{aligned}$$

where $K_2 = (8\pi)^{-1/2}$ and $E_0(\infty) = \| \boldsymbol{u}_0 \|_{L^2(\mathbb{R}^2)} + \int_0^\infty \| \boldsymbol{g}(\cdot,s) \|_{L^2(\mathbb{R}^2)} ds < \infty$ [by (1.5)]. Now, given $\epsilon > 0$, let $t_\epsilon > t_0$ be large enough such that

$$t^{1/2} \| D\boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^2)} \le \frac{\epsilon}{6 K_2 E_0(\infty)}, \qquad \forall\; t > t_\epsilon \qquad (\text{by } (3.1)). \tag{3.3}$$

We then have, using (3.3):

$$\begin{aligned}&\| \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t;t_0) \|_{L^2(\mathbb{R}^2)} \\ &\le K_2\, E_0(\infty) \int_{t_0}^{t} (t-s)^{-1/2} \| D\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^2)} ds \\ &\le K_2\, E_0(\infty) \int_{t_0}^{t_\epsilon} (t-s)^{-1/2} \| D\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^2)} ds + \frac{\epsilon}{6} \int_{t_\epsilon}^{t} (t-s)^{-1/2} s^{-1/2} ds \\ &\le K_2\, E_0(\infty)\, (t - t_\epsilon)^{-1/2} \int_{t_0}^{t_\epsilon} \| D\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^2)} ds + \frac{2}{3}\epsilon\end{aligned}$$

for all $t > t_\epsilon$. Recalling (2.6), this gives $\| \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t;t_0) \|_{L^2(\mathbb{R}^2)} \le \epsilon$ for all $t \gg t_\epsilon$. ☐

Another simple derivation of (3.2) can be given using (2.8a), (2.15), (3.1) and the next lemma, along the lines of the proof of Proposition 3.3 below.

LEMMA 3.1. *Given* $0 < \alpha < 1$, *if* $\mathrm{w} \in L^\infty(t_0, T)$ *is nonnegative and satisfies*

$$\mathrm{w}(t) \le \mathrm{w}_0 + K \int_{t_0}^{t} (t-s)^{-\alpha} (s-t_0)^{-1+\alpha}\, \mathrm{w}(s)\, ds, \qquad \forall\; t_0 \le t \le T, \tag{3.4a}$$

for some constant $0 < K < \pi^{-1} \sin(\alpha\pi)$, *then*

$$\mathrm{w}(t) \le \frac{\sin(\alpha\pi)}{\sin(\alpha\pi) - K\pi}\, \mathrm{w}_0, \qquad \forall\; t_0 \le t \le T. \tag{3.4b}$$

Proof. Setting $\mathrm{W}(t) := \sup\{ \mathrm{w}(s) \colon t_0 \le s \le t \}$, we obtain from (3.4$a$) above that $\mathrm{w}(t) \le \mathrm{w}_0 + K\,\mathrm{W}(t) \int_{t_0}^{t} (t-s)^{-\alpha} (s-t_0)^{-1+\alpha} ds = \mathrm{w}_0 + K\,\mathrm{W}(t)\, \pi / \sin(\alpha\pi)$ for all $t_0 \le t \le T$. This gives $(1 - K\pi/\sin(\alpha\pi))\, \mathrm{W}(t) \le \mathrm{w}_0$ for all $t_0 \le t \le T$, which is the same as (3.4b). ☐

Similarly, if a nonnegative function $\mathrm{w} \in L^\infty(t_0, T)$ happened to satisfy

$$\mathrm{w}(t) \le \mathrm{w}_0 + K \int_{\mu(t)}^{t} (t-s)^{-\alpha} (s-t_0)^{-1+\alpha}\, \mathrm{w}(s)\, ds, \qquad \forall\; t_0 \le t \le T, \tag{3.5a}$$

for $0 < \alpha < 1$, $0 < K < 1 - \alpha$ constant, with $\mu(t) = (t + t_0)/2$, then we would have

$$\mathrm{w}(t) \;\le\; \frac{1-\alpha}{1-\alpha-K}\;\mathrm{w}_0, \qquad \forall\; t_0 \le t \le T. \tag{3.5b}$$

Another elementary estimate which will be needed in our proof of (1.13b) is the following. Assuming (1.9), let us take C_{g}, $t_{\mathrm{g}} > 0$ such that $t \,\|\, \boldsymbol{g}(\cdot,t) \,\|_{L^2(\mathbb{R}^n)} < C_{\mathrm{g}}$ for almost all $t > t_{\mathrm{g}}$. Then, it follows from (1.14) that, in dimension $n \le 3$, the Stokes approximations $\boldsymbol{v}(\cdot,t;t_0) \in L^\infty([\,t_0,\infty\,[, L^2_\sigma(\mathbb{R}^n))$ satisfy the bound

$$\|\, \boldsymbol{v}(\cdot,t;t_0) \,\|_{L^\infty(\mathbb{R}^n)} \;\le\; (\Gamma_n E_0(\infty) + C_{\mathrm{g}})\,(t - t_0)^{-\,n/4}, \qquad \forall\; t > t_0 \ge t_{\mathrm{g}}, \tag{3.6}$$

where $\Gamma_n = (4\pi)^{-\,n/4}$ and $E_0(\cdot)$ is given in (2.4b).

Proposition 3.3. *Assuming* (1.5), (1.8), (1.9), *with* $n = 2$, *let* $\boldsymbol{u}(\cdot,t)$ *be the globally defined Leray-Hopf solution of the Navier-Stokes problem* (1.3). *Given* $t_0 \ge 0$, *let* $\boldsymbol{v}(\cdot,t;t_0) \in L^\infty([\,t_0,\infty\,[, L^2_\sigma(\mathbb{R}^2))$ *be the Stokes approximation defined in* (1.4). *Then*

$$\lim_{t\to\infty}\; t^{1/2}\,\|\, \boldsymbol{u}(\cdot,t) \;-\; \boldsymbol{v}(\cdot,t;t_0) \,\|_{L^\infty(\mathbb{R}^2)} \;=\; 0. \tag{3.7}$$

Proof. By Proposition 2.5, we have only to show the result for $t_0 \ge 0$ sufficiently large; we choose $t_0 > \max\{t_1, t_{\mathrm{g}}\}$, with t_1, t_{g} defined in (1.8), (3.6). Now, given any $0 < \epsilon \le 1$, we may proceed as follows. Taking $t_\epsilon > t_0$ large enough so that, by (3.1) above, we have

$$t^{1/2}\,\|\, D\boldsymbol{u}(\cdot,t) \,\|_{L^2(\mathbb{R}^2)} \;<\; \epsilon, \qquad \forall\; t > t_\epsilon, \tag{3.8}$$

we will consider for the moment the Stokes flow $\boldsymbol{v}(\cdot,t;t_\epsilon)$, $t > t_\epsilon$. Let $\mu(t) := (t + t_\epsilon)/2$. We have, by (2.8$a$) and (3.8),

$$\begin{aligned}
&\int_{t_\epsilon}^{\mu(t)} \|\, e^{\Delta(t-s)}\,\boldsymbol{Q}(\cdot,s) \,\|_{L^\infty(\mathbb{R}^2)}\, ds \\
&\le \frac{1}{\sqrt{4\pi}} \int_{t_\epsilon}^{\mu(t)} (t-s)^{-\,1/2}\,\|\, e^{\frac{1}{2}\Delta(t-s)}\,\boldsymbol{Q}(\cdot,s) \,\|_{L^2(\mathbb{R}^2)}\, ds \\
&\le \frac{1}{4\pi} \int_{t_\epsilon}^{\mu(t)} (t-s)^{-\,1}\,\|\, \boldsymbol{u}(\cdot,s) \,\|_{L^2(\mathbb{R}^2)}\,\|\, D\boldsymbol{u}(\cdot,s) \,\|_{L^2(\mathbb{R}^2)}\, ds \\
&\le \frac{\sqrt{2}}{4\pi}\,\epsilon\,(t - t_\epsilon)^{-\,1/2} \int_{t_\epsilon}^{\mu(t)} (t-s)^{-\,1/2}\, s^{-\,1/2}\,\|\, \boldsymbol{u}(\cdot,s) \,\|_{L^2(\mathbb{R}^2)}\, ds
\end{aligned}$$

for all $t > t_\epsilon$, so that, by (2.4),

$$\int_{t_\epsilon}^{\mu(t)} \|\, e^{\Delta(t-s)}\,\boldsymbol{Q}(\cdot,s) \,\|_{L^\infty(\mathbb{R}^2)}\, ds \;\le\; \frac{1}{\pi\sqrt{2}}\; E_0(\infty)\,\epsilon\,(t - t_\epsilon)^{-\,1/2}. \tag{3.9}$$

Similarly, by (2.8b), (3.6) and (3.8),

$$\begin{aligned}
&\int_{\mu(t)}^{t} \| \, e^{\Delta(t-s)} \, \boldsymbol{Q}(\cdot, s) \, \|_{L^\infty(\mathbb{R}^2)} ds \\
&\leq \; \frac{1}{\sqrt{8\pi}} \, \epsilon \int_{\mu(t)}^{t} (t-s)^{-1/2} \, s^{-1/2} \, \| \, \boldsymbol{u}(\cdot, s) \, \|_{L^\infty(\mathbb{R}^2)} ds \\
&\leq \; \frac{1}{\sqrt{8\pi}} \, \epsilon \int_{\mu(t)}^{t} (t-s)^{-1/2} \, s^{-1/2} \, \| \, \boldsymbol{u}(\cdot, s) - \boldsymbol{v}(\cdot, s; t_\epsilon) \, \|_{L^\infty(\mathbb{R}^2)} ds \\
&\quad + \frac{1}{\sqrt{8\pi}} \, \epsilon \int_{\mu(t)}^{t} (t-s)^{-1/2} \, s^{-1/2} \, \| \, \boldsymbol{v}(\cdot, s; t_\epsilon) \, \|_{L^\infty(\mathbb{R}^2)} ds \\
&\leq \; \frac{1}{\sqrt{4\pi}} \, \epsilon \, (t - t_\epsilon)^{-1/2} \int_{\mu(t)}^{t} (t-s)^{-1/2} \, \| \, \boldsymbol{u}(\cdot, s) - \boldsymbol{v}(\cdot, s; t_\epsilon) \, \|_{L^\infty(\mathbb{R}^2)} ds \\
&\quad + \frac{1}{\sqrt{4\pi}} \, C_2 \, \epsilon \, (t - t_\epsilon)^{-1/2} \int_{\mu(t)}^{t} (t-s)^{-1/2} \, (s - t_\epsilon)^{-1/2} \, ds
\end{aligned}$$

for all $t > t_\epsilon$, where $C_2 = (4\pi)^{-1/2} E_0(\infty) + C_{\mathrm{g}}$, with $E_0(\cdot)$ given in (2.4b). This gives

$$\begin{aligned}
&\int_{\mu(t)}^{t} \| \, e^{\Delta(t-s)} \, \boldsymbol{Q}(\cdot, s) \, \|_{L^\infty(\mathbb{R}^2)} ds \\
&\leq \; \frac{1}{\sqrt{\pi}} \, C_2 \, \epsilon \, (t - t_\epsilon)^{-1/2} \\
&\quad + \frac{1}{\sqrt{4\pi}} \, \epsilon \, (t - t_\epsilon)^{-1/2} \int_{\mu(t)}^{t} (t-s)^{-1/2} \, \| \, \boldsymbol{u}(\cdot, s) - \boldsymbol{v}(\cdot, s; t_\epsilon) \, \|_{L^\infty(\mathbb{R}^2)} ds
\end{aligned} \tag{3.10}$$

so that we have, from (3.9) and (3.10),

$$\begin{aligned}
&\| \, \boldsymbol{u}(\cdot, t) - \boldsymbol{v}(\cdot, t; t_\epsilon) \, \|_{L^\infty(\mathbb{R}^2)} \\
&\leq \int_{t_\epsilon}^{t} \| \, e^{\Delta(t-s)} \, \boldsymbol{Q}(\cdot, s) \, \|_{L^\infty(\mathbb{R}^2)} ds \\
&\leq \frac{1}{\pi\sqrt{2}} \, E_0(\infty) \, \epsilon \, (t - t_\epsilon)^{-1/2} + \frac{1}{\sqrt{\pi}} \, C_2 \, \epsilon \, (t - t_\epsilon)^{-1/2} \\
&\quad + \frac{1}{\sqrt{4\pi}} \, \epsilon \, (t - t_\epsilon)^{-1/2} \int_{\mu(t)}^{t} (t-s)^{-1/2} \, \| \, \boldsymbol{u}(\cdot, s) - \boldsymbol{v}(\cdot, s; t_\epsilon) \, \|_{L^\infty(\mathbb{R}^2)} ds
\end{aligned}$$

for all $t > t_\epsilon$. In terms of $\mathrm{w}(t) := (t - t_\epsilon)^{1/2} \, \| \, \boldsymbol{u}(\cdot, t) - \boldsymbol{v}(\cdot, t; t_\epsilon) \, \|_{L^\infty(\mathbb{R}^2)}$, this reads

$$\mathrm{w}(t) \; \leq \; \frac{1}{\pi\sqrt{2}} \, E_0(\infty) \, \epsilon \; + \; \frac{1}{\sqrt{\pi}} \, C_2 \, \epsilon \; + \; \frac{1}{\sqrt{4\pi}} \, \epsilon \int_{\mu(t)}^{t} (t-s)^{-1/2} \, (s - t_\epsilon)^{-1/2} \, \mathrm{w}(s) \, ds$$

for all $t > t_\epsilon$. By (3.5), it follows that $\mathrm{w}(t) < \epsilon \cdot (E_0(\infty) + C_{\mathrm{g}})/(\sqrt{\pi} - 1)$ for all $t > t_\epsilon$, that is, $(t - t_\epsilon)^{1/2} \, \| \, \boldsymbol{u}(\cdot, t) - \boldsymbol{v}(\cdot, t; t_\epsilon) \, \|_{L^\infty(\mathbb{R}^2)} < K\epsilon$, with

$K = (E_0(\infty) + C_{\mathrm{g}})/(\sqrt{\pi} - 1)$, if $t > t_\epsilon$. By Proposition 2.5, it follows that

$$t^{1/2} \| \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t;t_0) \|_{L^\infty(\mathbb{R}^2)} < K\epsilon, \qquad \forall\ t > \hat{t}_\epsilon,$$

for some $\hat{t}_\epsilon > t_\epsilon$ sufficiently large. Since $0 < \epsilon \le 1$ is arbitrary, this shows (3.7). ☐

4. Derivation of (1.13): $\boldsymbol{n = 3}$. We now follow a similar approach to derive (1.13) in dimension $n = 3$. This case requires some extra work due to difficulties introduced by vortex stretching effects. Again, we explore the fact that $D\boldsymbol{g}(\cdot,t)$, $D\boldsymbol{u}(\cdot,t)$ are very regular for $t \gg 1$. In fact, by (1.8), we have $D\boldsymbol{g}(\cdot,t) \in L^1([\,t_1, \infty\,[, L^2(\mathbb{R}^3))$, and, as it will become clear from the proof of Proposition 4.1 below, taking $t_2 > t_1$ large enough that (by (1.8), (2.6))

$$(1 + t_2) \| D\boldsymbol{u}(\cdot,t_2) \|^2_{L^2(\mathbb{R}^3)} \le \frac{1}{8}\, E_0(\infty)^{-2}, \tag{4.1a}$$

$$\int_{t_2}^{\infty} \| D\boldsymbol{u}(\cdot,t) \|^2_{L^2(\mathbb{R}^3)} dt \le \frac{1}{8}\, E_0(\infty)^{-2}, \tag{4.1b}$$

and

$$\int_{t_2}^{\infty} (1 + t)^{1/2} \| D\boldsymbol{g}(\cdot,t) \|_{L^2(\mathbb{R}^3)} dt \le \frac{1}{2}\, E_0(\infty)^{-1}, \tag{4.1c}$$

where $E_0(\infty) = \| \boldsymbol{u}_0 \|_{L^2(\mathbb{R}^3)} + \int_0^\infty \| \boldsymbol{g}(\cdot,s) \|_{L^2(\mathbb{R}^3)} ds$ (see (2.4b)), then $\| D\boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^3)}$ is continuous for $t \ge t_2$, and

$$(1 + t) \| D\boldsymbol{u}(\cdot,t) \|^2_{L^2(\mathbb{R}^3)} \le E_0(\infty)^{-2}, \qquad \forall\ t \ge t_2. \tag{4.2}$$

As in the proof of Proposition 4.1 below, the derivation of (4.2) makes use of two basic Sobolev-Nirenberg-Gagliardo (SNG) inequalities for arbitrary $\mathrm{u} \in H^2(\mathbb{R}^3)$:

$$\| \mathrm{u} \|_{L^\infty(\mathbb{R}^3)} \le K_0 \| \mathrm{u} \|^{1/4}_{L^2(\mathbb{R}^3)} \| D^2 \mathrm{u} \|^{3/4}_{L^2(\mathbb{R}^3)}, \qquad K_0 < 0.678, \tag{4.3}$$

see e.g. ([26], Proposition 2.4, p. 13), and the elementary inequality

$$\| D\mathrm{u} \|_{L^2(\mathbb{R}^3)} \le K_1 \| \mathrm{u} \|^{1/2}_{L^2(\mathbb{R}^3)} \| D^2 \mathrm{u} \|^{1/2}_{L^2(\mathbb{R}^3)}, \qquad K_1 < 1.317, \tag{4.4}$$

which follows directly from integration by parts.

Proposition 4.1. *Assuming* (1.5) *and* (1.8), *with* $n = 3$, *we have*

$$\lim_{t \to \infty} t^{1/2} \| D\boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^3)} = 0. \tag{4.5}$$

Proof. The following argument is adapted from [13], Lemma 2.2. Let $t_* > t_2$, with $t_2 \gg 1$ given in (4.1). Applying $D_\ell = \partial/\partial x_\ell$ to the first equation in (1.3a), taking the dot product with $(1 + t) D_\ell \boldsymbol{u}(\cdot,t)$ and integrating on $\mathbb{R}^3 \times [\,t_*, t\,]$, we get, summing over $1 \leq \ell \leq 3$,

$$\begin{aligned}&(1+t)\,\|\,D\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^3)} + \,2\int_{t_*}^{t}(1+s)\,\|\,D^2\boldsymbol{u}(\cdot,s)\,\|^2_{L^2(\mathbb{R}^3)}\,ds\\ &= (1+t_*)\,\|\,D\boldsymbol{u}(\cdot,t_*)\,\|^2_{L^2(\mathbb{R}^3)} + \int_{t_*}^{t}\|\,D\boldsymbol{u}(\cdot,s)\,\|^2_{L^2(\mathbb{R}^3)}\,ds\\ &+2\sum_{i,\,j,\,\ell}\int_{t_*}^{t}(1+s)\int_{\mathbb{R}^3}u_i(x,s)\,D_{\ell}u_j(x,s)\,D_jD_{\ell}u_i(x,s)\,dx\,ds\\ &+2\sum_{i,\,\ell}\int_{t_*}^{t}(1+s)\int_{\mathbb{R}^3}D_{\ell}u_i(x,s)\,D_{\ell}\,g_i(x,s)\,dx\,ds.\end{aligned}$$

This gives, recalling (1.19) and (1.20),

$$\begin{aligned}&(1+t)\,\|\,D\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^3)} + \,2\int_{t_*}^{t}(1+s)\,\|\,D^2\boldsymbol{u}(\cdot,s)\,\|^2_{L^2(\mathbb{R}^3)}\,ds\\ &\leq (1+t_*)\,\|\,D\boldsymbol{u}(\cdot,t_*)\,\|^2_{L^2(\mathbb{R}^3)} + \int_{t_*}^{t}\|\,D\boldsymbol{u}(\cdot,s)\,\|^2_{L^2(\mathbb{R}^3)}\,ds\\ &+2\int_{t_*}^{t}(1+s)\,\|\,\boldsymbol{u}(\cdot,s)\,\|_{L^\infty(\mathbb{R}^3)}\,\|\,D\boldsymbol{u}(\cdot,s)\,\|_{L^2(\mathbb{R}^3)}\,\|\,D^2\boldsymbol{u}(\cdot,s)\,\|_{L^2(\mathbb{R}^3)}\,ds\\ &+2\int_{t_*}^{t}(1+s)\,\|\,D\boldsymbol{u}(\cdot,s)\,\|_{L^2(\mathbb{R}^3)}\,\|\,D\boldsymbol{g}(\cdot,s)\,\|_{L^2(\mathbb{R}^3)}\,ds,\end{aligned}$$

for all $t > t_*$, where $\|\,\boldsymbol{u}(\cdot,s)\,\|_{L^\infty(\mathbb{R}^3)} = \sup_{x\,\in\,\mathbb{R}^3}|\,\boldsymbol{u}(x,s)\,|_2$. Now, by (4.3) and (4.4), we have

$$\begin{aligned}&\|\,\boldsymbol{u}(\cdot,s)\,\|_{L^\infty(\mathbb{R}^3)}\,\|\,D\boldsymbol{u}(\cdot,s)\,\|_{L^2(\mathbb{R}^3)}\\ &\leq K_0\,\|\,\boldsymbol{u}(\cdot,s)\,\|^{1/4}_{L^2(\mathbb{R}^3)}\,\|\,D\boldsymbol{u}(\cdot,s)\,\|_{L^2(\mathbb{R}^3)}\,\|\,D^2\boldsymbol{u}(\cdot,s)\,\|^{3/4}_{L^2(\mathbb{R}^3)}\\ &= \|\,D^2\boldsymbol{u}(\cdot,s)\,\|_{L^2(\mathbb{R}^3)}\Big[\,K_0\,\|\,\boldsymbol{u}(\cdot,s)\,\|^{1/4}_{L^2(\mathbb{R}^3)}\,\|\,D\boldsymbol{u}(\cdot,s)\,\|_{L^2(\mathbb{R}^3)}\,\|\,D^2\boldsymbol{u}(\cdot,s)\,\|^{-1/4}_{L^2(\mathbb{R}^3)}\Big]\\ &\leq \|\,D^2\boldsymbol{u}(\cdot,s)\,\|_{L^2(\mathbb{R}^3)}\Big[\,K_0\,K_1^{1/2}\,\|\,\boldsymbol{u}(\cdot,s)\,\|^{1/2}_{L^2(\mathbb{R}^3)}\,\|\,D\boldsymbol{u}(\cdot,s)\,\|^{1/2}_{L^2(\mathbb{R}^3)}\Big]\end{aligned}$$

where K_0, K_1 are the same constants given in (4.3) and (4.4) above. This gives

$$\|\,\boldsymbol{u}(\cdot,s)\,\|_{L^\infty(\mathbb{R}^3)}\,\|\,D\boldsymbol{u}(\cdot,s)\,\|_{L^2(\mathbb{R}^3)} \leq\; E_0(\infty)^{1/2}\,\|\,D\boldsymbol{u}(\cdot,s)\,\|^{1/2}_{L^2(\mathbb{R}^3)}\,\|\,D^2\boldsymbol{u}(\cdot,s)\,\|_{L^2(\mathbb{R}^3)}$$

by Proposition 2.1, since $K_0\,K_1^{1/2} < 1$.

Here, $E_0(\infty) = \|\,\boldsymbol{u}_0\,\|_{L^2(\mathbb{R}^3)} + \int_0^\infty \|\,\boldsymbol{g}(\cdot,s)\,\|_{L^2(\mathbb{R}^3)}\,ds$, which is finite (by (1.5)). Therefore, we have

$$\begin{aligned}
&(1+t)\,\|\,D\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^3)} + 2\int_{t_*}^{t}(1+s)\,\|\,D^2\boldsymbol{u}(\cdot,s)\,\|^2_{L^2(\mathbb{R}^3)}ds \\
&\leq (1+t_*)\,\|\,D\boldsymbol{u}(\cdot,t_*)\,\|^2_{L^2(\mathbb{R}^3)} + \int_{t_*}^{t}\|\,D\boldsymbol{u}(\cdot,s)\,\|^2_{L^2(\mathbb{R}^3)}ds \\
&\quad + 2\int_{t_*}^{t}(1+s)\Big[\,E_0(\infty)^{1/2}\,\|\,D\boldsymbol{u}(\cdot,s)\,\|^{1/2}_{L^2(\mathbb{R}^3)}\Big]\,\|\,D^2\boldsymbol{u}(\cdot,s)\,\|^2_{L^2(\mathbb{R}^3)}ds \\
&\quad + 2\int_{t_*}^{t}(1+s)\,\|\,D\boldsymbol{u}(\cdot,s)\,\|_{L^2(\mathbb{R}^3)}\,\|\,D\boldsymbol{g}(\cdot,s)\,\|_{L^2(\mathbb{R}^3)}ds
\end{aligned} \tag{4.6}$$

for all $t > t_*$. It is now very simple to get (4.5) above: given $0 < \epsilon \leq E_0(\infty)^{-2}$ arbitrary, let $t_* > t_2$ be large enough such that

$$(1+t_*)\,\|\,D\boldsymbol{u}(\cdot,t_*)\,\|^2_{L^2(\mathbb{R}^3)} \;\leq\; \frac{1}{8}\,\epsilon, \tag{4.7a}$$

$$\int_{t_*}^{\infty}\|\,D\boldsymbol{u}(\cdot,s)\,\|^2_{L^2(\mathbb{R}^3)}ds \;\leq\; \frac{1}{8}\,\epsilon, \tag{4.7b}$$

and

$$\int_{t_*}^{\infty}(1+s)^{1/2}\,\|\,D\boldsymbol{g}(\cdot,s)\,\|_{L^2(\mathbb{R}^3)}ds \;\leq\; \frac{1}{2}\sqrt{\epsilon}, \tag{4.7c}$$

by (1.8) and (2.6). From (4.7a), it follows, in particular, that $E_0(\infty)\,\|\,D\boldsymbol{u}(\cdot,t_*)\,\|_{L^2(\mathbb{R}^3)} < 1$. Actually, more is true: one has

$$E_0(\infty)\,\|\,D\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^3)} \;<\; 1, \qquad \forall\; t \geq t_*. \tag{4.8}$$

In fact, if (4.8) were false, there would exist $t_3 > t_*$ such that $E_0(\infty)\,\|\,D\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^3)} < 1$ for all $t_* \leq t < t_3$, while $E_0(\infty)\,\|\,D\boldsymbol{u}(\cdot,t_3)\,\|_{L^2(\mathbb{R}^3)} = 1$. This would then give, using (4.6), (4.7a) and (4.7b),

$$\begin{aligned}
(1+t)\,\|\,D\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^3)} \;&\leq\; (1+t_*)\,\|\,D\boldsymbol{u}(\cdot,t_*)\,\|^2_{L^2(\mathbb{R}^3)} + \int_{t_*}^{t}\|\,D\boldsymbol{u}(\cdot,s)\,\|^2_{L^2(\mathbb{R}^3)}ds \\
&\quad + 2\int_{t_*}^{t}(1+s)\,\|\,D\boldsymbol{u}(\cdot,s)\,\|_{L^2(\mathbb{R}^3)}\,\|\,D\boldsymbol{g}(\cdot,s)\,\|_{L^2(\mathbb{R}^3)}ds \\
&\leq\; \frac{\epsilon}{4} + 2\int_{t_*}^{t}(1+s)\,\|\,D\boldsymbol{u}(\cdot,s)\,\|_{L^2(\mathbb{R}^3)}\,\|\,D\boldsymbol{g}(\cdot,s)\,\|_{L^2(\mathbb{R}^3)}ds
\end{aligned}$$

for all $t_* \leq t \leq t_3$. In terms of $\mathrm{w}(t) := (1+t)\,\|\,D\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^3)}$, we would then have

$$\mathrm{w}(t) \;\leq\; \frac{\epsilon}{4} + 2\int_{t_*}^{t}(1+s)^{1/2}\,\|\,D\boldsymbol{g}(\cdot,s)\,\|_{L^2(\mathbb{R}^3)}\,\mathrm{w}(s)^{1/2}\,ds, \qquad \forall\; t_* \leq t \leq t_3.$$

As in the proof of Proposition 2.1, this would give

$$\mathrm{w}(t)^{1/2} \;\leq\; \frac{\sqrt{\epsilon}}{2} + \int_{t_*}^{t}(1+s)^{1/2}\,\|\,D\boldsymbol{g}(\cdot,s)\,\|_{L^2(\mathbb{R}^3)}ds, \qquad \forall\; t_* \leq t \leq t_3,$$

so that, by (4.7c), $\mathrm{w}(t) \leq \epsilon$ for every $t \in [t_*, t_3]$. In particular, we would get

$$E_0(\infty)\, \| D\boldsymbol{u}(\cdot, t_3) \|_{L^2(\mathbb{R}^3)} \;\leq\; \frac{1}{\sqrt{1+t_3}}\, E_0(\infty)\sqrt{\epsilon} \;<\; 1,$$

while we had $E_0(\infty)\, \| D\boldsymbol{u}(\cdot, t_3) \|_{L^2(\mathbb{R}^3)} = 1$. This contradiction shows (4.8), as claimed.

By (4.6), (4.7a), (4.7b) and (4.8), we therefore have

$$\begin{aligned}(1+t)\, \| D\boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^3)}^2 \;\leq\;& (1+t_*)\, \| D\boldsymbol{u}(\cdot,t_*) \|_{L^2(\mathbb{R}^3)}^2 + \int_{t_*}^{t} \| D\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)}^2\, ds \\ &+ 2\int_{t_*}^{t} (1+s)\, \| D\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)} \| D\boldsymbol{g}(\cdot,s) \|_{L^2(\mathbb{R}^3)}\, ds \\ \leq\;& \frac{\epsilon}{4} + 2\int_{t_*}^{t} (1+s)\, \| D\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)} \| D\boldsymbol{g}(\cdot,s) \|_{L^2(\mathbb{R}^3)}\, ds\end{aligned}$$

for all $t \geq t_*$, or, introducing once more $\mathrm{w}(t) := (1+t)\, \| D\boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^3)}^2$, as we did above,

$$\mathrm{w}(t) \;\leq\; \frac{\epsilon}{4} + 2\int_{t_*}^{t} (1+s)^{1/2}\, \| D\boldsymbol{g}(\cdot,s) \|_{L^2(\mathbb{R}^3)}\, \mathrm{w}(s)^{1/2}\, ds, \qquad \forall\; t \geq t_*.$$

By (4.7c), this gives, again, $\mathrm{w}(t) \leq \epsilon$ for all $t \geq t_*$, that is, $(1+t)\, \| D\boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^3)}^2 \leq \epsilon$ for all $t > t_2$ sufficiently large. Since $0 < \epsilon \leq E_0(\infty)^{-2}$ is arbitrary, this shows (4.5). ☐

We will now conclude our discussion with the proof of (1.13) in dimension $n = 3$. We begin with (1.13a):

Proposition 4.2. *Assuming* (1.5) *and* (1.8), *with* $n = 3$, *let* $\boldsymbol{u}(\cdot,t)$ *be a globally defined Leray-Hopf solution of the Navier-Stokes problem* (1.3). *Given* $t_0 \geq 0$, *let* $\boldsymbol{v}(\cdot,t;t_0) \in L^\infty([\,t_0,\infty[\,, L^2_\sigma(\mathbb{R}^3))$ *be the Stokes approximation defined in* (1.4). *Then*

$$\lim_{t\to\infty} t^{1/4}\, \| \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t;t_0) \|_{L^2(\mathbb{R}^3)} \;=\; 0. \tag{4.9}$$

Proof. (Again, we could show this result along the lines of the proof of Proposition 3.3, but we choose instead the slightly shorter procedure below.) By (2.15), we have only to prove (4.9) for $t_0 \geq 0$ sufficiently large; we choose $t_0 > t_2$, where $t_2 \gg 1$ is given in (4.1). We then have, for $t > t_0$:

$$\boldsymbol{u}(\cdot,t) \;=\; e^{\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0) + \int_{t_0}^{t} e^{\Delta(t-s)}\boldsymbol{g}(\cdot,s)\, ds + \int_{t_0}^{t} e^{\Delta(t-s)}\boldsymbol{Q}(\cdot,s)\, ds,$$

where $\boldsymbol{Q}(\cdot,s)$ is given in (2.7), and

$$\boldsymbol{v}(\cdot,t;t_0) \;=\; e^{\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0) + \int_{t_0}^{t} e^{\Delta(t-s)}\boldsymbol{g}(\cdot,s)\, ds, \qquad t > t_0.$$

Therefore, by (2.4) and (2.8a),

$$\begin{aligned}\| \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t;t_0) \|_{L^2(\mathbb{R}^3)} &\leq \int_{t_0}^{t} \| e^{\Delta(t-s)} \boldsymbol{Q}(\cdot,s) \|_{L^2(\mathbb{R}^3)} ds \\ &\leq K_3 \int_{t_0}^{t} (t-s)^{-3/4} \| \boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)} \| D\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)} ds \\ &\leq K_3 E_0(\infty) \int_{t_0}^{t} (t-s)^{-3/4} \| D\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)} ds,\end{aligned}$$

where $K_3 = (8\pi)^{-3/4}$ and $E_0(\infty) = \| \boldsymbol{u}_0 \|_{L^2(\mathbb{R}^3)} + \int_0^\infty \| \boldsymbol{g}(\cdot,s) \|_{L^2(\mathbb{R}^3)} ds < \infty$ (by (1.5)). Now, given $\epsilon > 0$, let $t_\epsilon > t_0$ be large enough so that

$$t^{1/2} \| D\boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^2)} \leq \frac{\epsilon}{18 K_3 E_0(\infty)}, \qquad \forall\ t > t_\epsilon,$$

in view of Proposition 4.1. We then have

$$\begin{aligned}&\| \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t;t_0) \|_{L^2(\mathbb{R}^3)} \\ &\leq K_3 E_0(\infty) \int_{t_0}^{t} (t-s)^{-3/4} \| D\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)} ds \\ &\leq K_3 E_0(\infty) \int_{t_0}^{t_\epsilon} (t-s)^{-3/4} \| D\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)} ds + \frac{\epsilon}{18} \int_{t_\epsilon}^{t} (t-s)^{-3/4} s^{-1/2} ds \\ &\leq K_3 E_0(\infty) (t-t_\epsilon)^{-3/4} \int_{t_0}^{t_\epsilon} \| D\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)} ds + \frac{2}{3} (t-t_\epsilon)^{-1/4} \epsilon\end{aligned}$$

for all $t > t_\epsilon$, where we have used that $\int_{t_\epsilon}^{t} (t-s)^{-3/4} s^{-1/2} ds \leq 6 \left(\frac{t-t_\epsilon}{2} \right)^{-1/4}$. Therefore,

$$\begin{aligned}&(t-t_\epsilon)^{1/4} \| \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t;t_0) \|_{L^2(\mathbb{R}^3)} \\ &\leq K_3 E_0(\infty) (t-t_\epsilon)^{-1/2} \int_{t_0}^{t_\epsilon} \| D\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)} ds + \frac{2}{3} \epsilon\end{aligned}$$

for all $t > t_\epsilon$, so that, by (2.6), $\limsup_{t\to\infty} t^{1/4} \| \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t;t_0) \|_{L^2(\mathbb{R}^3)} < \epsilon$, as claimed. □

We now conclude our analysis by establishing (1.13b) in space dimension $n = 3$. Let us mention in passing the striking and much finer pointwise estimates recently obtained in [1] under special, strong assumptions on the problem data $(\boldsymbol{u}_0, \boldsymbol{f})$.

PROPOSITION 4.3. *Assuming* (1.5), (1.8), (1.9), *with* $n = 3$, *let* $\boldsymbol{u}(\cdot,t)$ *be a globally defined Leray-Hopf solution of the Navier-Stokes problem* (1.3). *Given* $t_0 \geq 0$, *let* $\boldsymbol{v}(\cdot,t;t_0) \in L^\infty([t_0,\infty[, L^2_\sigma(\mathbb{R}^3))$ *be the Stokes approximation defined in* (1.4). *Then*

$$\lim_{t\to\infty} t \| \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t;t_0) \|_{L^\infty(\mathbb{R}^3)} = 0. \tag{4.10}$$

Proof. Again, by Proposition 2.5, we have only to show (4.10) for $t_0 \geq 0$ sufficiently large; we choose $t_0 > \max\{1, t_2, t_{\mathtt{g}}\}$, with $t_{\mathtt{g}}, t_2$ given in (3.6), (4.1) above. Let then $0 < \epsilon \leq 1$ be given. Taking $t_\epsilon > t_0$ large enough so that, by Proposition 4.1 above, we have

$$t^{1/2} \, \| \, D\boldsymbol{u}(\cdot, t) \, \|_{L^2(\mathbb{R}^3)} < \epsilon, \qquad \forall \; t > t_\epsilon, \tag{4.11}$$

we will once again consider for the most part of the argument the Stokes solution $\boldsymbol{v}(\cdot, t; t_\epsilon)$, (defined for $t \geq t_\epsilon$), instead of $\boldsymbol{v}(\cdot, t; t_0)$. Let $\mu(t) = (t + t_\epsilon)/2$. By(2.8$a$) and (4.11),

$$\begin{aligned}
&\int_{t_\epsilon}^{\mu(t)} \| \, e^{\Delta(t-s)} \, \boldsymbol{Q}(\cdot, s) \, \|_{L^\infty(\mathbb{R}^3)} \, ds \\
&\leq \frac{1}{(4\pi)^{3/4}} \int_{t_\epsilon}^{\mu(t)} (t-s)^{-3/4} \, \| \, e^{\frac{1}{2}\Delta(t-s)} \, \boldsymbol{Q}(\cdot, s) \, \|_{L^2(\mathbb{R}^3)} \, ds \\
&\leq \frac{1}{(4\pi)^{3/2}} \int_{t_\epsilon}^{\mu(t)} (t-s)^{-3/2} \, \| \, \boldsymbol{u}(\cdot, s) \, \|_{L^2(\mathbb{R}^3)} \, \| \, D\boldsymbol{u}(\cdot, s) \, \|_{L^2(\mathbb{R}^3)} \, ds \\
&\leq \frac{1}{4\pi^{3/2}} \, \epsilon \, (t - t_\epsilon)^{-1} \int_{t_\epsilon}^{\mu(t)} (t-s)^{-1/2} \, s^{-1/2} \, \| \, \boldsymbol{u}(\cdot, s) \, \|_{L^2(\mathbb{R}^3)} \, ds
\end{aligned}$$

for all $t > t_\epsilon$, so that, by (2.4),

$$\int_{t_\epsilon}^{\mu(t)} \| \, e^{\Delta(t-s)} \, \boldsymbol{Q}(\cdot, s) \, \|_{L^\infty(\mathbb{R}^3)} \, ds \;\leq\; \frac{1}{2\pi^{3/2}} \, E_0(\infty) \, \epsilon \, (t - t_\epsilon)^{-1}. \tag{4.12a}$$

Similarly, by (2.8b), (3.6) and (4.11),

$$\begin{aligned}
&\int_{\mu(t)}^{t} \| \, e^{\Delta(t-s)} \, \boldsymbol{Q}(\cdot, s) \, \|_{L^\infty(\mathbb{R}^3)} \, ds \\
&\leq \frac{1}{(8\pi)^{3/4}} \, \epsilon \int_{\mu(t)}^{t} (t-s)^{-3/4} \, s^{-1/2} \, \| \, \boldsymbol{u}(\cdot, s) \, \|_{L^\infty(\mathbb{R}^3)} \, ds \\
&\leq \frac{1}{(8\pi)^{3/4}} \, \epsilon \int_{\mu(t)}^{t} (t-s)^{-3/4} \, s^{-1/2} \, \| \, \boldsymbol{u}(\cdot, s) - \boldsymbol{v}(\cdot, s; t_\epsilon) \, \|_{L^\infty(\mathbb{R}^3)} \, ds \\
&\quad + \frac{1}{(8\pi)^{3/4}} \, \epsilon \int_{\mu(t)}^{t} (t-s)^{-3/4} \, s^{-1/2} \, \| \, \boldsymbol{v}(\cdot, s; t_\epsilon) \, \|_{L^\infty(\mathbb{R}^3)} \, ds \\
&\leq \frac{1}{2^{7/4} \pi^{3/4}} \, \epsilon \, (t + t_\epsilon)^{-1/2} \int_{\mu(t)}^{t} (t-s)^{-3/4} \, \| \, \boldsymbol{u}(\cdot, s) - \boldsymbol{v}(\cdot, s; t_\epsilon) \, \|_{L^\infty(\mathbb{R}^3)} \, ds \\
&\quad + \frac{1}{2^{5/4} \pi^{3/4}} \, C_3 \, \epsilon \, (t - t_\epsilon)^{-1} \int_{\mu(t)}^{t} (t-s)^{-3/4} \, (s - t_\epsilon)^{-1/4} \, ds
\end{aligned}$$

for $t > t_\epsilon$, where $C_3 = (4\pi)^{-3/4} E_0(\infty) + C_{\mathtt{g}}$,
$E_0(\infty) = \| \, \boldsymbol{u}_0 \, \|_{L^2(\mathbb{R}^3)} + \int_0^\infty \| \, \boldsymbol{g}(\cdot, s) \, \|_{L^2(\mathbb{R}^3)} \, ds$ (note that $E_0(\infty) < \infty$, by (1.5)). This

gives

$$\begin{aligned}
&\int_{\mu(t)}^{t} \| \, e^{\Delta(t-s)} \, \boldsymbol{Q}(\cdot,s) \, \|_{L^\infty(\mathbb{R}^3)} \, ds \\
&\leq \frac{2^{3/4}}{\pi^{3/4}} \, C_3 \, \epsilon \, (t - t_\epsilon)^{-1} \\
&\quad + \frac{1}{2^{7/4} \pi^{3/4}} \, \epsilon \, (t + t_\epsilon)^{-1/2} \int_{\mu(t)}^{t} (t-s)^{-3/4} \, \| \, \boldsymbol{u}(\cdot,s) - \boldsymbol{v}(\cdot,s;t_\epsilon) \, \|_{L^\infty(\mathbb{R}^3)} \, ds
\end{aligned} \tag{4.12b}$$

so that we have, by (4.12*a*) and (4.12*b*),

$$\begin{aligned}
&\| \, \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t;t_\epsilon) \, \|_{L^\infty(\mathbb{R}^3)} \\
&\leq \int_{t_\epsilon}^{t} \| \, e^{\Delta(t-s)} \, \boldsymbol{Q}(\cdot,s) \, \|_{L^\infty(\mathbb{R}^3)} \, ds \\
&\leq \frac{1}{2\pi^{3/2}} \, E_0(\infty) \, \epsilon \, (t - t_\epsilon)^{-1} + \frac{2^{3/4}}{\pi^{3/4}} \, C_3 \, \epsilon \, (t - t_\epsilon)^{-1} \\
&\quad + \frac{1}{2^{7/4} \pi^{3/4}} \, \epsilon \, (t + t_\epsilon)^{-1/2} \int_{\mu(t)}^{t} (t-s)^{-3/4} \, \| \, \boldsymbol{u}(\cdot,s) - \boldsymbol{v}(\cdot,s;t_\epsilon) \, \|_{L^\infty(\mathbb{R}^3)} \, ds
\end{aligned}$$

for all $t > t_\epsilon$. In terms of $\mathrm{w}(t) := (t - t_\epsilon) \, \| \, \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t;t_\epsilon) \, \|_{L^\infty(\mathbb{R}^3)}$, this reads

$$\mathrm{w}(t) \;\leq\; \frac{E_0(\infty)}{2\pi^{3/2}} \, \epsilon + \frac{2^{3/4}}{\pi^{3/4}} \, C_3 \, \epsilon + \frac{1}{2\pi^{3/4}} \, \frac{\epsilon}{(t + t_\epsilon)^{1/4}} \int_{\mu(t)}^{t} (t-s)^{-3/4} \, (s - t_\epsilon)^{-1/4} \, \mathrm{w}(s) \, ds$$

for all $t > t_\epsilon$. By (3.5), it follows that $\mathrm{w}(t) < K\epsilon$ for all $t > t_\epsilon$, with $K < 3\,(E_0(\infty) + C_{\mathrm{g}})$, so that $(t - t_\epsilon) \, \| \, \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t;t_\epsilon) \, \|_{L^\infty(\mathbb{R}^3)} < K\epsilon$ for $t > t_\epsilon$. By Proposition 2.5, this gives $(t - t_\epsilon) \, \| \, \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t;t_0) \, \|_{L^\infty(\mathbb{R}^3)} < K\epsilon$ for all $t \gg 1$. □

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