METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 23, No. 4, pp. 317–328, December 2016
002

ON THE CR-CURVATURE OF LEVI DEGENERATE TUBE HYPERSURFACES*

ALEXANDER ISAEV†

Abstract. In article [I2] we studied tube hypersurfaces in $\mathbb{C}^3$ that are 2-nondegenerate and uniformly Levi degenerate of rank 1. In particular, we discovered that for the CR-curvature of such a hypersurface to vanish it suffices to require that only two coefficients (called Θ^2_{21} and Θ^2_{10}) in the expansion of a certain component of the CR-curvature form be identically zero. In this paper, we show that, surprisingly, the vanishing of the entire CR-curvature is in fact implied by the vanishing of a single quantity derived from Θ^2_{10}. This result strengthens the main theorem of [I2] and also leads to a remarkable system of partial differential equations. Furthermore, we explicitly characterize the class of not necessarily CR-flat tube hypersurfaces given by the vanishing of Θ^2_{21}.

Key words. CR-curvature, the Monge-Ampère equation, the Monge equation.

AMS subject classifications. 32V05, 32V20, 35J96, 34A05, 34A26.

1. Introduction. This paper extends our earlier article [I2], and the reader will be extensively referred to [I2] in what follows. In particular, a detailed review of all the necessary CR-geometric concepts is contained in [I2, Section 2], and we will utilize those concepts without further reference.

We consider connected C^∞-smooth real hypersurfaces in the complex vector space $\mathbb{C}^n$ with $n \geq 2$. Specifically, we study *tube hypersurfaces*, or simply *tubes*, i.e, locally closed real submanifolds of the form

$$M = \mathcal{S} + iV,$$

where $\mathcal{S}$ is a hypersurface in $\mathbb{R}^n \subset \mathbb{C}^n$ called the base of M. Two tube hypersurfaces are called affinely equivalent if there exists an affine transformation of $\mathbb{C}^n$ given by

$$z \mapsto Az + b, \quad A \in \mathrm{GL}_n(\mathbb{R}), \quad b \in \mathbb{C}^n \tag{1.1}$$

that maps one hypersurface onto the other (this occurs if and only if the bases of the tubes are affinely equivalent as submanifolds of $\mathbb{R}^n$).

There has been a substantial effort to relate the CR-geometric and affine-geometric aspects of the study of tubes (see [I2, Section 1] for an extensive bibliography). Specifically, the following question has attracted much attention:

(∗) when does local or global CR-equivalence of tubes imply affine equivalence?

Until recently, a reasonable answer to the above question has only existed for Levi nondegenerate tube hypersurfaces that are also CR-flat, i.e., have identically vanishing CR-curvature (see monograph [I1] for an up-to-date exposition of the existing theory). In an attempt to relax the Levi nondegeneracy requirement, in [I2] we set out to investigate question (∗) for a class of Levi degenerate 2-nondegenerate hypersurfaces while still assuming CR-flatness. As part of our considerations, we analyzed CR-curvature for this class, and in the present paper we improve on that analysis.

*Received September 8, 2016; accepted for publication October 3, 2016.
†Mathematical Sciences Institute, Australian National University, Acton, ACT 2601, Australia (alexander.isaev@anu.edu.au).

Notice that CR-curvature is only defined in situations when the CR-structures in question are reducible to absolute parallelisms with values in some Lie algebra $\mathfrak{g}$. Indeed, let $\mathfrak{C}$ be a class of CR-manifolds. Then the CR-structures in $\mathfrak{C}$ are said to reduce to $\mathfrak{g}$-valued absolute parallelisms if to every $M \in \mathfrak{C}$ one can assign a fiber bundle $\mathcal{P}_M \to M$ and an absolute parallelism ω_M on $\mathcal{P}_M$ such that for every $p \in M$ the parallelism establishes an isomorphism between $T_p(M)$ and $\mathfrak{g}$ and for any $M_1, M_2 \in \mathfrak{C}$ the following holds:
(i) every CR-isomorphism $f : M_1 \to M_2$ can be lifted to a diffeomorphism $F : \mathcal{P}_{M_1} \to \mathcal{P}_{M_2}$ satisfying

$$F^*\omega_{M_2} = \omega_{M_1}, \tag{1.2}$$

and
(ii) any diffeomorphism $F : \mathcal{P}_{M_1} \to \mathcal{P}_{M_2}$ satisfying (1.2) is a bundle isomorphism that is a lift of a CR-isomorphism $f : M_1 \to M_2$.

In this situation one introduces the $\mathfrak{g}$-valued *CR-curvature form*

$$\Omega_M := d\omega_M - \frac{1}{2}\left[\omega_M, \omega_M\right],$$

and the condition of the CR-flatness of M means that Ω_M identically vanishes on the bundle $\mathcal{P}_M$.

Reducing CR-structures (as well as other geometric structures) to absolute parallelisms goes back to É. Cartan who showed that reduction takes place for all 3-dimensional Levi nondegenerate CR-hypersurfaces (see [C]). Since then there have been many developments incorporating the assumption of Levi nondegeneracy (see [I2, Section 1] for references). On the other hand, reducing the CR-structures of Levi degenerate CR-manifolds has proved to be rather difficult, and the first result for a large class of Levi degenerate manifolds only appeared in 2013 in our paper [IZ]. Specifically, we considered the class $\mathfrak{C}_{2,1}$ of connected 5-dimensional CR-hypersurfaces that are 2-nondegenerate and uniformly Levi degenerate of rank 1 and showed that the CR-structures in this class reduce to $\mathfrak{so}(3,2)$-valued parallelisms (see [MS], [Poc] for alternative constructions and [Por] for a reduction in the 7-dimensional case). In particular, in [IZ] we prove that a manifold $M \in \mathfrak{C}_{2,1}$ is CR-flat (with respect to our reduction) if and only if near its every point M is CR-equivalent to an open subset of the tube hypersurface over the future light cone in $\mathbb{R}^3$:

$$M_0 := \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid (\operatorname{Re} z_1)^2 + (\operatorname{Re} z_2)^2 - (\operatorname{Re} z_3)^2 = 0,\ \operatorname{Re} z_3 > 0\right\}.$$

Now, the main result of [I2] (see Theorem 1.1 therein) asserts that every CR-flat tube hypersurface in $\mathfrak{C}_{2,1}$ is affinely equivalent to an open subset of M_0. This conclusion is a complete answer to question (∗) in the situation at hand and is in stark contrast to the Levi nondegenerate case where the CR-geometric and affine-geometric classifications differ even in low dimensions.

In fact, in [I2] we obtain a stronger result. Namely, for the assertion of [I2, Theorem 1.1] to hold, it suffices to require that only two coefficients (called Θ^2_{21} and Θ^2_{10}) in the expansion of a single component of the CR-curvature form Ω_M (called Θ^2) be identically zero on $\mathcal{P}_M$ (see [I2, Theorem 3.1]). Our argument is local, and for every $x \in M$ we only utilize the vanishing of Θ^2_{21} and Θ^2_{10} on a particular section γ of $\mathcal{P}_M$ over a neighborhood of x:

$$\begin{cases} \Theta^2_{21}|_\gamma = 0, \\ \Theta^2_{10}|_\gamma = 0. \end{cases} \tag{1.3}$$

Each of the two conditions in system (1.3) can be expressed as a partial differential equation on the local defining function of the hypersurface M near x. These equations are quite complicated; for example, the first identity in (1.3) is equivalent to (2.7). The expression for $\Theta^2_{10}|_\gamma$ is especially hard to find, and in our computation of $\Theta^2_{10}|_\gamma$ in [I2] some of its terms were only calculated under the simplifying assumption $\Theta^2_{21}|_\gamma = 0$. This was sufficient for our purposes as we were only interested in solving system (1.3). Indeed, denoting by $\boldsymbol{\Theta}^2_{10}$ the quantity arising from the constrained calculation of $\Theta^2_{10}|_\gamma$, we see that the system of equations

$$\begin{cases} \Theta^2_{21}|_\gamma = 0, \\ \boldsymbol{\Theta}^2_{10} = 0 \end{cases} \tag{1.4}$$

is equivalent to (1.3). Interestingly, if in suitable coordinates the base of M is given locally as the graph of a function of two variables, the second equation in (1.4) becomes the well-known Monge equation on this function with respect to one of the variables (see (2.8)).

To write system (1.4) more explicitly, recall that M is uniformly Levi degenerate of rank 1 and 2-nondegenerate. Due to Levi degeneracy, the graphing function of M satisfies the homogeneous Monge-Ampère equation (see (2.3)). Thus, the detailed form of (1.4) is

$$\begin{cases} \Theta^2_{21}|_\gamma = 0, \\ \text{The Monge equation w.r.t. one variable: } \boldsymbol{\Theta}^2_{10} = 0, \\ \text{The Monge-Ampère equation,} \end{cases} \tag{1.5}$$

where we additionally assume that certain quantities responsible for the Levi form to have rank precisely 1 and for 2-nondegeneracy are everywhere nonzero (see (2.4) and (2.6), respectively). System (1.5) is the centerpiece of the proof of [I2, Theorem 3.1]. In [I2] we explicitly solved (1.5) and observed that every solution of this system defines a tube hypersurface affinely equivalent to an open subset of M_0. As M_0 is CR-flat, this shows, in particular, that conditions (1.4) imply the vanishing of the CR-curvature form Ω_M on an open subset of the bundle $\mathcal{P}_M$ over a neighborhood of x. Hence if both Θ^2_{21} and Θ^2_{10} are identically zero on $\mathcal{P}_M$, so is the entire form Ω_M.

The main theorem of the present paper establishes a surprising dependence between the two local conditions in (1.4). We will now state the theorem in general terms, with the detailed formulation postponed until the next section (see Theorem 2.1).

Theorem 1.1. *Let M be a tube hypersurface in $\mathbb{C}^3$ and assume that $M \in \mathfrak{C}_{2,1}$. Fix $x \in M$ and a suitable section γ of $\mathcal{P}_M$ over a neighborhood of x. Then the condition $\boldsymbol{\Theta}^2_{10} = 0$ implies $\Theta^2_{21}|_\gamma = 0$.*

We stress that although the quantity $\boldsymbol{\Theta}^2_{10}$ was computed in part under the assumption $\Theta^2_{21}|_\gamma = 0$, it is not at all clear *a priori* why the vanishing of $\boldsymbol{\Theta}^2_{10}$ should imply that of $\Theta^2_{21}|_\gamma$.

Together with results of [I2], Theorem 1.1 yields:

Corollary 1.2. *Let M be a tube hypersurface in $\mathbb{C}^3$ and assume that $M \in \mathfrak{C}_{2,1}$. Fix $x \in M$ and a suitable section γ of $\mathcal{P}_M$ over a neighborhood of x. If $\boldsymbol{\Theta}^2_{10} = 0$, then M near x is affinely equivalent to an open subset of M_0; in particular, the*

CR-curvature form Ω_M vanishes on an open subset of $\mathcal{P}_M$ over a neighborhood of x.

The above result is rather unexpected as it has been believed for some time now that CR-flatness for manifolds in the class $\mathfrak{C}_{2,1}$ should be controlled by two conditions rather than one (cf. Remark 3.4).

By Theorem 1.1, system (1.5) reduces to a system of two equations:

$$\begin{cases} \text{The Monge equation w.r.t. one variable: } \mathbf{\Theta}^2_{10} = 0, \\ \text{The Monge-Ampère equation,} \end{cases} \tag{1.6}$$

where we assume in addition that (2.4) and (2.6) are satisfied. This system is truly remarkable. Indeed, by Corollary 1.2 it has a clear geometric meaning as it locally describes all CR-flat tubes in the class $\mathfrak{C}_{2,1}$. Moreover, all solutions of this system can be explicitly found, and every solution yields a tube hypersurface affinely equivalent to an open subset of M_0. Next, each of the two equations in (1.6) has its own geometric interpretation: the classical single-variable Monge equation describes all planar conics (see, e.g., [Lan, pp. 51–52], [Las]), whereas the graphs of the solutions of the Monge-Ampère equation are exactly the surfaces in $\mathbb{R}^3$ with degenerate second fundamental form. Finally—and quite curiously—both equations in (1.6) happen to be named after Gaspard Monge. It is rather satisfying to see that the invariants constructed in [IZ] lead to an object so abundantly filled with geometric features. This indicates that the theory of the class $\mathfrak{C}_{2,1}$ is rich and deserves further exploration.

The paper is organized as follows. In Section 2 we state and prove Theorem 2.1, which is the detailed variant of Theorem 1.1. Further, in Section 3 we investigate the converse implication, namely the question whether the vanishing of $\Theta^2_{21}|_\gamma$ implies that of $\mathbf{\Theta}^2_{10}$. The answer to this question turns out to be negative, and in Propositions 3.1, 3.2 we write the general form of a solution of the system

$$\begin{cases} \Theta^2_{21}|_\gamma = 0, \\ \text{The Monge-Ampère equation,} \end{cases} \tag{1.7}$$

where, as before, we assume that (2.4) and (2.6) are satisfied. Unlike (1.6), system (1.7) describes a class of not necessarily CR-flat tubes in $\mathfrak{C}_{2,1}$, and Propositions 3.1, 3.2 show that this interesting class can be effectively characterized as well.

Acknowledgements. This work is supported by the Australian Research Council. The author is grateful to Boris Kruglikov for useful discussions.

2. The main result. Let M be any tube hypersurface in $\mathbb{C}^3$. For $x \in M$, a tube neighborhood of x in M is an open subset U of M that contains x and has the form $M \cap (\mathcal{U} + i\mathbb{R}^3)$, where $\mathcal{U}$ is an open subset of $\mathbb{R}^3$. It is easy to see that for every point $x \in M$ there exists a tube neighborhood U of x in M and an affine transformation of $\mathbb{C}^3$ as in (1.1) that maps x to the origin and establishes affine equivalence between U and a tube hypersurface of the form

$$\Gamma_\rho := \{(z_1, z_2, z_3) : z_3 + \bar{z}_3 = \rho(z_1 + \bar{z}_1, z_2 + \bar{z}_2)\} = \left\{(z_1, z_2, z_3) : \operatorname{Re} z_3 = \frac{1}{2}\,\rho(2\operatorname{Re} z_1, 2\operatorname{Re} z_2)\right\}, \tag{2.1}$$

where $\rho(t_1, t_2)$ is a smooth function defined in a neighborhood of 0 in $\mathbb{R}^2$ with

$$\rho(0) = 0, \quad \rho_1(0) = 0, \quad \rho_2(0) = 0 \tag{2.2}$$

(here and below subscripts 1 and 2 indicate partial derivatives with respect to t_1 and t_2). In what follows, Γ_ρ will be analyzed locally near the origin, thus we will only be interested in the germ of ρ at 0 and the domain of ρ will be allowed to shrink if necessary.

Let now M be uniformly Levi degenerate of rank 1. Then the Hessian matrix of ρ has rank 1 at every point, hence ρ is a solution of the homogeneous Monge-Ampère equation

$$\rho_{11}\rho_{22} - \rho_{12}^2 = 0, \tag{2.3}$$

where one can additionally assume

$$\rho_{11} > 0 \text{ everywhere.} \tag{2.4}$$

In [I2, Section 3] we showed that for ρ satisfying (2.3), (2.4), the hypersurface Γ_ρ is 2-nondegenerate if and only if the function

$$S := \left(\frac{\rho_{12}}{\rho_{11}}\right)_1 \tag{2.5}$$

vanishes nowhere. Thus, assuming that M is 2-nondegenerate, we have

$$S \neq 0 \text{ everywhere.} \tag{2.6}$$

Next, consider the fiber bundle $\mathcal{P}_{\Gamma_\rho} \to \Gamma_\rho$ arising from the reduction to absolute parallelisms achieved in [IZ] for CR-hypersurfaces in the class $\mathfrak{C}_{2,1}$, and let γ_0 be the section of $\mathcal{P}_{\Gamma_\rho}$ given in suitable coordinates by [I2, formula (4.21)]. In [I2, formula (4.27)] we computed the restriction of the curvature coefficient Θ^2_{21} to γ_0. The condition $\Theta^2_{21}|_{\gamma_0} = 0$ can be then written as the equation

$$\begin{gathered} 2\sqrt{\rho_{11}}\left[\rho_{12}\left(\frac{S_1}{\sqrt{\rho_{11}}S}\right)_1 - \rho_{11}\left(\frac{S_1}{\sqrt{\rho_{11}}S}\right)_2\right] - \\ 2\sqrt{\rho_{11}}\left[\rho_{12}\left(\frac{\rho_{111}}{\sqrt{\rho_{11}^3}}\right)_1 - \rho_{11}\left(\frac{\rho_{111}}{\sqrt{\rho_{11}^3}}\right)_2\right] - 11S_1\,\rho_{11} - S\,\rho_{111} = 0 \end{gathered} \tag{2.7}$$

(cf. [I2, formula (4.28)]). Further, in [I2, formula (4.46)] we found the expression for the restriction of Θ^2_{10} to γ_0 in which some of the terms were computed under the simplifying assumption that equation (2.7) holds. If we denote the quantity resulting from this calculation by $\boldsymbol{\Theta}^2_{10}$, then one observes that the condition $\boldsymbol{\Theta}^2_{10} = 0$ can be written as the equation

$$9\rho^{(\mathrm{V})}\rho_{11}^2 - 45\rho^{(\mathrm{IV})}\rho_{111}\rho_{11} + 40\rho_{111}^3 = 0, \tag{2.8}$$

where $\rho^{(\mathrm{IV})} := \partial^4\rho/\partial t_1^4$, $\rho^{(\mathrm{V})} := \partial^5\rho/\partial t_1^5$ (cf. [I2, formula (4.47)]). Notice that, remarkably, (2.8) is the Monge equation with respect to the first variable.

We are now ready to state and prove the detailed variant of Theorem 1.1:

THEOREM 2.1. *Let ρ be a smooth function satisfying* (2.2)–(2.4) and (2.6), *where S is defined in* (2.5). *Then condition* (2.8) *implies condition* (2.7).

REMARK 2.2. We emphasize that although the quantity $\mathbf{\Theta}^2_{10}$ was computed partly under the assumption $\Theta^2_{21}|_{\gamma_0} = 0$, the fact that the vanishing of $\mathbf{\Theta}^2_{10}$ implies the vanishing of $\Theta^2_{21}|_{\gamma_0}$ is not at all obvious and is actually quite surprising.

Proof. We start by recalling classical facts concerning solutions of the homogeneous Monge-Ampère equation (2.3). For details the reader is referred to paper [U], which treats this equation in somewhat greater generality.

Let us make the following change of variables near the origin:

$$\begin{aligned} v &= \rho_1(t_1, t_2), \\ w &= t_2 \end{aligned} \tag{2.9}$$

and set

$$\begin{aligned} p(v, w) &:= \rho_2(t_1(v, w), w), \\ q(v) &:= t_1(v, 0). \end{aligned} \tag{2.10}$$

Equation (2.3) immediately implies that p is independent of w, so we write p as a function of the variable v alone. Furthermore, we have

$$q'(v) = \frac{1}{\rho_{11}(t_1(v, 0), 0)}. \tag{2.11}$$

Clearly, (2.2), (2.4), (2.9), (2.10), (2.11) yield

$$p(0) = 0, \quad q(0) = 0, \quad q' > 0 \text{ everywhere.} \tag{2.12}$$

In terms of p and q, the inverse of (2.9) is written as

$$\begin{aligned} t_1 &= q(v) - w\, p'(v), \\ t_2 &= w, \end{aligned} \tag{2.13}$$

and the solution ρ in the variables v, w is given by

$$\rho(t_1(v, w), w) = vq(v) - \int_0^v q(\tau)d\tau + w(p(v) - vp'(v)). \tag{2.14}$$

In particular, we see that the general smooth solution of the homogeneous Monge-Ampère equation (2.3) satisfying conditions (2.2), (2.4) is parametrized by a pair of arbitrary smooth functions satisfying (2.12).

We will now rewrite equation (2.7) in the variables v, w introduced in (2.9). First of all, from (2.9), (2.13) we compute

$$\begin{aligned} \rho_{11}(t_1(v, w), w) &= \frac{1}{q' - w\, p''}, \\ \rho_{12}(t_1(v, w), w) &= \frac{p'}{q' - w\, p''}, \\ \rho_{111}(t_1(v, w), w) &= -\frac{q'' - w\, p'''}{(q' - w\, p'')^3}. \end{aligned} \tag{2.15}$$

Next, from formulas (2.5), (2.13) and the first two identities in (2.15) we obtain

$$\begin{aligned} S(t_1(v,w),w) &= \frac{p''}{q' - w\,p''}, \\ S_1(t_1(v,w),w) &= \frac{p'''q' - p''q''}{(q' - w\,p'')^3}. \end{aligned} \tag{2.16}$$

Now, plugging the expressions from (2.15), (2.16) into (2.7), we see that the latter simplifies to the equation

$$p'''q' - p''q'' = 0, \tag{2.17}$$

that is, to the condition $S_1 = 0$. Since S vanishes nowhere, the first identity in (2.16) implies that p'' does not vanish either (this condition characterizes 2-nondegeneracy). Then, dividing (2.17) by $(p'')^2$, one obtains

$$q'/p'' = \text{const}. \tag{2.18}$$

Thus, we see that after passing to the variables v, w the complicated equation (2.7) turns into the simple condition (2.18).

Further, we will rewrite equation (2.8) in the variables v, w. From (2.13) and (2.15) one computes

$$\begin{aligned} \rho^{(\mathrm{IV})}(t_1(v,w),w) = -\frac{1}{(q'-w\,p'')^5}\Big[&(q'''-w\,p^{(\mathrm{IV})})(q'-w\,p'')- \\ &3(q''-w\,p''')^2\Big], \\ \rho^{(\mathrm{V})}(t_1(v,w),w) = -\frac{1}{(q'-w\,p'')^7}\Big[&\Big((q^{(\mathrm{IV})}-w\,p^{(\mathrm{V})})(q'-w\,p'')- \\ 5(q''-w\,p''')(q'''-w\,p^{(\mathrm{IV})})\Big)&(q'-w\,p'')- \\ 5\Big((q'''-w\,p^{(\mathrm{IV})})(q'-w\,p'')-3(q''-w\,p''')^2\Big)&(q''-w\,p''')\Big]. \end{aligned} \tag{2.19}$$

Plugging expressions from (2.15), (2.19) into (2.8) and collecting coefficients at w^k for $k = 0, 1, 2, 3$ in the resulting formula, we see that (2.8) is equivalent to the following system of four ordinary differential equations:

$$\begin{cases} 9p^{(\mathrm{V})}(p'')^2 - 45p^{(\mathrm{IV})}p'''p'' + 40(p''')^3 = 0, \\ 6p^{(\mathrm{V})}p''q' + 3(p'')^2q^{(\mathrm{IV})} - 15(p^{(\mathrm{IV})}p'''q' + p^{(\mathrm{IV})}p''q'' + p'''p''q''')+ \\ \qquad\qquad 40(p''')^2q'' = 0, \\ 3p^{(\mathrm{V})}(q')^2 + 6p''q^{(\mathrm{IV})}q' - 15(p^{(\mathrm{IV})}q''q' + p'''q'''q' + p''q'''q'')+ \\ \qquad\qquad 40p'''(q'')^2 = 0, \\ 9q^{(\mathrm{IV})}(q')^2 - 45q'''q''q' + 40(q'')^3 = 0. \end{cases} \tag{2.20}$$

Thus, in order to prove the theorem, we need to show that system (2.20) implies condition (2.18).

Notice that the first equation in (2.20) is the Monge equation and that the last one yields the Monge equation for any primitive of the function q. Also observe that all the equations in (2.20) reduce to the first one if condition (2.18) is satisfied.

Recall now that the Monge equation describes planar conics and can be solved explicitly. Indeed, assuming that $p''>0$ we calculate

$$\frac{1}{(p'')^{11/3}}\left(9p^{(\mathrm{V})}(p'')^2-45p^{(\mathrm{IV})}p'''p''+40(p''')^3\right)=$$

$$\left(\frac{9p^{(\mathrm{IV})}}{(p'')^{5/3}}-\frac{15(p''')^2}{(p'')^{8/3}}\right)'=9\left(\frac{p'''}{(p'')^{5/3}}\right)''=-\frac{27}{2}\left((p'')^{-2/3}\right)'''.$$

Similarly, for $p''<0$ we have

$$\frac{1}{(-p'')^{11/3}}\left(9p^{(\mathrm{V})}(p'')^2-45p^{(\mathrm{IV})}p'''p''+40(p''')^3\right)=\frac{27}{2}\left((-p'')^{-2/3}\right)'''.$$

Thus, the first equation in (2.20) yields

$$p''=\pm P^{-3/2}, \tag{2.21}$$

where P is a polynomial with $\deg P\le 2$ and $P(0)>0$. Similarly, taking into account (2.12), from the last equation in (2.20) we see

$$q'=Q^{-3/2}, \tag{2.22}$$

where Q is a polynomial with $\deg Q\le 2$ and $Q(0)>0$.

Next, from (2.21), (2.22) we calculate

$$\begin{aligned}
p''' &= \mp\frac{3}{2}P^{-5/2}P',\\
p^{(\mathrm{IV})} &= \pm\frac{15}{4}P^{-7/2}(P')^2\mp\frac{3}{2}P^{-5/2}P'',\\
p^{(\mathrm{V})} &= \mp\frac{105}{8}P^{-9/2}(P')^3\pm\frac{45}{4}P^{-7/2}P''P',\\
q'' &= -\frac{3}{2}Q^{-5/2}Q',\\
q''' &= \frac{15}{4}Q^{-7/2}(Q')^2-\frac{3}{2}Q^{-5/2}Q'',\\
q^{(\mathrm{IV})} &= -\frac{105}{8}Q^{-9/2}(Q')^3+\frac{45}{4}Q^{-7/2}Q''Q'.
\end{aligned} \tag{2.23}$$

Plugging (2.21), (2.22), (2.23) into the second and third equations in (2.20) and simplifying the resulting expressions, we obtain, respectively,

$$\begin{aligned}
&7P^3(Q')^3-6P^3Q''Q'Q-(P')^3Q^3+9(P')^2PQ'Q^2-6P''P'PQ^3-\\
&\qquad 15P'P^2(Q')^2Q+6P'P^2Q''Q^2+6P''P^2Q'Q^2=0,\\
&7(P')^3Q^3-6P''P'PQ^3-P^3(Q')^3+9P'P^2(Q')^2Q-6P^3Q''Q'Q-\\
&\qquad 15(P')^2PQ'Q^2+6P''P^2Q'Q^2+6P'P^2Q''Q^2=0.
\end{aligned} \tag{2.24}$$

Subtracting the second identity in (2.24) from the first one, we arrive at

$$8(PQ'-P'Q)^3=0.$$

It then follows that $Q=\mathrm{const}\,P$, and therefore condition (2.18) holds as required. The proof is complete. □

3. A class of nonflat tubes. In this section we investigate the question of whether for a hypersurface of the form (2.1) that is uniformly Levi degenerate of rank 1 and 2-nondegenerate the condition $\Theta^2_{21}|_{\gamma_0} = 0$ yields CR-flatness. Equivalently,

($**$) given a smooth function ρ satisfying (2.2)–(2.4) and (2.6), does condition (2.7) imply condition (2.8)?

If one looks at (2.7) and (2.8) in their original complicated PDE form, this question may appear to be hard. Luckily, after passing to the variables v, w as in (2.9), both equations simplify: (2.7) becomes condition (2.18), whereas (2.8) turns into the system of four ordinary differential equations (2.20). In fact, as we remarked in the proof of Theorem 2.1 in the preceding section, (2.18) forces all the equations in (2.20) to be identical to the Monge equation on the function p, thus, assuming that (2.18) holds, (2.8) actually reduces to a single ODE. Therefore, for any p that is *not* a solution of the Monge equation and such that

$$p(0) = 0, \quad p'' \neq 0 \text{ everywhere}, \tag{3.1}$$

and for

$$q := C(p' - p'(0)), \text{ with } Cp'' > 0, \tag{3.2}$$

formula (2.14) provides a counterexample to question ($**$) (see, e.g., Example 3.3 below).

In fact, for p having properties (3.1) and q chosen as in (3.2), formula (2.14) significantly simplifies:

Proposition 3.1. *Let ρ be a smooth function satisfying* (2.2)–(2.4), (2.6) *and* (2.7)*. Then formula* (2.14) *becomes*

$$\rho(t_1(v, w), w) = (w - C)(p(v) - vp'(v)). \tag{3.3}$$

Proof. Substituting (3.2) into (2.14) we calculate

$$\rho(t_1(v, w), w) = vC(p'(v) - p'(0)) - \int_0^v C(p'(\tau) - p'(0))d\tau + w(p(v) - vp'(v)) =$$
$$vC(p'(v) - p'(0)) - C(p(v) - p'(0)v) + w(p(v) - vp'(v)) = (w - C)(p(v) - vp'(v))$$

as required. □

Thus, all hypersurfaces of the form (2.1) that are 2-nondegenerate, uniformly Levi degenerate of rank 1, and for which $\Theta^2_{21}|_{\gamma_0} = 0$, are described by formula (3.3). This is an interesting class of not necessarily CR-flat tubes, and it is quite useful to have an explicit characterization for it. Notice, however, that although formula (3.3) is very simple, it is written in the variables v, w, whereas the expression for ρ in the original variables t_1, t_2 (which is what we are really interested in) may turn out to be more complicated. This expression was found in [I2, Lemma 4.1], and, as the argument is quite short, we repeat it here for the sake of the completeness of our exposition.

Let ζ be the inverse of the function $p'(0) - p'$ near the origin. Define

$$\chi(\tau) := \frac{1}{\tau} \int_0^\tau \zeta(\sigma)d\sigma. \tag{3.4}$$

Clearly, χ is smooth near 0 and satisfies

$$\chi(0) = 0, \quad \chi'(0) = -\frac{1}{2p''(0)}. \tag{3.5}$$

Now set

$$\tilde{\rho}(t_1, t_2) := (t_1 + p'(0)t_2)\chi\left(\frac{t_1 + p'(0)t_2}{t_2 - C}\right). \tag{3.6}$$

PROPOSITION 3.2. *One has $\rho = \tilde{\rho}$.*

Proof. From (3.6) we compute:

$$\begin{aligned}
\tilde{\rho}_1 &= \chi\left(\frac{t_1 + p'(0)t_2}{t_2 - C}\right) + \frac{t_1 + p'(0)t_2}{t_2 - C}\chi'\left(\frac{t_1 + p'(0)t_2}{t_2 - C}\right) = \zeta\left(\frac{t_1 + p'(0)t_2}{t_2 - C}\right),\\
\tilde{\rho}_2 &= p'(0)\,\chi\left(\frac{t_1 + p'(0)t_2}{t_2 - C}\right) + \\
&\qquad \frac{t_1 + p'(0)t_2}{t_2 - C}\left(p'(0) - \frac{t_1 + p'(0)t_2}{t_2 - C}\right)\chi'\left(\frac{t_1 + p'(0)t_2}{t_2 - C}\right),\\
\tilde{\rho}_{11} &= \frac{2}{t_2 - C}\chi'\left(\frac{t_1 + p'(0)t_2}{t_2 - C}\right) + \frac{t_1 + p'(0)t_2}{(t_2 - C)^2}\chi''\left(\frac{t_1 + p'(0)t_2}{t_2 - C}\right).
\end{aligned} \tag{3.7}$$

Formulas (3.5), (3.6), (3.7) imply

$$\tilde{\rho}(0) = 0, \quad \tilde{\rho}_1(0) = 0, \quad \tilde{\rho}_2(0) = 0, \quad \tilde{\rho}_{11} > 0.$$

Also, it is easy to observe that $\tilde{\rho}$ satisfies the Monge-Ampère equation (2.3). Hence, $\tilde{\rho}$ is fully determined by a pair of functions $\tilde{p}$, $\tilde{q}$ as in formulas (2.13), (2.14). These functions satisfy

$$\tilde{p}(0) = 0, \quad \tilde{q}(0) = 0, \quad \tilde{q}\,' > 0 \text{ everywhere}$$

(cf. conditions (2.12)).

Let us make a change of coordinates near the origin analogous to (2.9):

$$\begin{aligned}
\tilde{v} &= \tilde{\rho}_1(t_1, t_2),\\
\tilde{w} &= t_2.
\end{aligned} \tag{3.8}$$

Then by the first identity in (3.7) we have

$$\tilde{v} = (p'(0) - p')^{-1}\left(\frac{t_1 + p'(0)t_2}{t_2 - C}\right)$$

and therefore, taking into account (3.2), we see that (3.8) is inverted as

$$\begin{aligned}
t_1 &= C(p'(\tilde{v}) - p'(0)) - \tilde{w}p'(\tilde{v}) = q(\tilde{v}) - \tilde{w}p'(\tilde{v}),\\
t_2 &= \tilde{w}.
\end{aligned}$$

On the other hand, as in (2.13) we have

$$t_1 = \tilde{q}(\tilde{v}) - \tilde{w}\tilde{p}\,'(\tilde{v}).$$

Hence, it follows that $\tilde{q} = q$ and, since $\tilde{p}(0) = p(0) = 0$, one also has $\tilde{p} = p$. Therefore, $\tilde{\rho} = \rho$, and the proof is complete. □

We will now demonstrate how Propositions 3.1, 3.2 work for a particular example.

EXAMPLE 3.3. Let $p(v) = e^v - 1$. Clearly, conditions (3.1) hold for this choice of p. Then by formula (3.3) we compute

$$\rho(t_1(v, w), w) = (w - C)((1 - v)e^v - 1), \tag{3.9}$$

where $C > 0$. To rewrite ρ in the variables t_1, t_2, we can either directly invert formula (2.13) or use Proposition 3.2. To invert formula (2.13), we notice that for our choice of p and q it becomes

$$\begin{aligned} t_1 &= (C - w)e^v - C, \\ t_2 &= w. \end{aligned}$$

We then obtain

$$\begin{aligned} v &= \log\left(\frac{t_1 + C}{C - t_2}\right), \\ w &= t_2, \end{aligned} \tag{3.10}$$

and plugging (3.10) into (3.9) yields

$$\rho(t_1, t_2) = (t_1 + C)\log\left(\frac{t_1 + C}{C - t_2}\right) - (t_1 + t_2). \tag{3.11}$$

Rather than inverting formula (2.13), let us now utilize Proposition 3.2 in order to determine $\rho(t_1, t_2)$. We have $\zeta(\sigma) = \log(1 - \sigma)$, and therefore by (3.4) we see

$$\chi(\tau) = \frac{\tau - 1}{\tau}\log(1 - \tau) - 1.$$

Then after a short calculation formula (3.6) leads to expression (3.11) as well.

Note that, since the function $p(v)$ in this example does not satisfy the Monge equation, the corresponding tube hypersurface Γ_ρ defined by (2.1) is not CR-flat, or, equivalently, the quantity $\Theta^2_{10}|_{\gamma_0} = \boldsymbol{\Theta}^2_{10}$ does not identically vanish.

REMARK 3.4. For any real hypersurface M in $\mathbb{C}^3$ in the class $\mathfrak{C}_{2,1}$, paper [Poc] introduces a pair of expressions, called J and W, in terms of a local defining function that vanish simultaneously on M if and only if M is locally CR-equivalent to M_0. The expressions are rather complicated, but in the tube case it is not very hard to see that the condition $W = 0$ is identical to equation (2.7) (i.e., to the vanishing of $\Theta^2_{21}|_{\gamma_0}$) and the condition $J = 0$ calculated under the assumption $W = 0$ to equation (2.8) (i.e., to the vanishing of $\Theta^2_{10}|_{\gamma_0}$ calculated in part under the assumption $\Theta^2_{21}|_{\gamma_0} = 0$ as in [I2]). It would be interesting to see whether an analogue of Theorem 1.1 holds for J and W in place of $\Theta^2_{10}|_{\gamma_0}$, $\Theta^2_{21}|_{\gamma_0}$, respectively, if the hypersurface is no longer assumed to be tube, i.e., whether it is possible to find a reasonable single condition characterizing CR-flatness for the entire class $\mathfrak{C}_{2,1}$.

REFERENCES

[C] É. Cartan, *Sur la géometrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes: I*, Ann. Math. Pura Appl., 11 (1933), pp. 17–90; II, Ann. Scuola Norm. Sup. Pisa, 1 (1932), pp. 333–354.

[I1] A. V. Isaev, *Spherical Tube Hypersurfaces*, Lecture Notes in Mathematics, 2020, Springer, New York, 2011.

[I2] A. Isaev, *Affine rigidity of Levi degenerate tube hypersurfaces*, J. Differential Geom., 104 (2016), pp. 111–141.

[IZ] A. Isaev and D. Zaitsev, *Reduction of five-dimensional uniformly Levi degenerate CR structures to absolute parallelisms*, J. Geom. Anal., 23 (2013), pp. 1571–1605.

[Lan] J. M. Landsberg, *Differential-geometric characterizations of complete intersections*, J. Differential Geom., 44 (1996), pp. 32–73.

[Las] J. W. Lasley, Jr., *On Monge's differential equation*, Amer. Math. Monthly, 43 (1936), pp. 284–286.

[MS] C. Medori and A. Spiro, *The equivalence problem for five-dimensional Levi degenerate CR manifolds*, Int. Math. Res. Not. (IMRN), (2014), pp. 5602–5647.

[Poc] S. Pocchiola, *Explicit absolute parallelism for 2-nondegenerate real hypersurfaces $M^5 \subset \mathbb{C}^3$ of constant Levi rank*, preprint, available from `http://arxiv.org/abs/1312.6400`.

[Por] C. Porter, *The local equivalence problem for 7-dimensional 2-nondegenerate CR manifolds whose cubic form is of conformal unitary type*, preprint, available from `http://arxiv.org/abs/1511.04019`.

[U] V. Ushakov, *The explicit general solution of trivial Monge-Ampère equation*, Comment. Math. Helv., 75 (2000), pp. 125–133.