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COHERENCE OF LOCAL AND GLOBAL HULLS*

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Let R be a Noetherian, normal, integral domain and M a finite R-module. Its *reflexive hull* is the R-module $M^{**} := \operatorname{Hom}_R\bigl(\operatorname{Hom}_R(M,R),R\bigr)$. The map $\tau : M \to M^{**}$ is characterized by 3 properties

- M^{**} is finite and S_2,
- $\ker\tau = \operatorname{tors} M$, the torsion submodule of M and
- $\tau_P : (M/\operatorname{tors} M)_P \to M^{**}_P$ is an isomorphism for all height 1 primes $P \subset R$.

One can see that $M \mapsto M^{**}$ is a left exact functor from finite R-modules to finite, torsion free, S_2 R-modules and it is the "best" such functor; see [Sta15, Tag 0AUY].

The concept of hull generalizes this to coherent sheaves over Noetherian schemes. The notion of torsion subsheaf is problematic in general, especially if the support of a sheaf F is not pure dimensional. It seems best to assume instead that $\ker\tau$ is the subsheaf of F generated by those local sections whose support is nowhere dense in $\operatorname{Supp} F$; we denote it by $\operatorname{emb} F$. Thus $F/\operatorname{emb} F$ has no embedded associated primes and it is the largest quotient of F that is S_1.

DEFINITION 1. Let X be a Noetherian scheme and F a coherent sheaf on X. The *hull* of F is a quasi-coherent sheaf $F^{(**)}$ together with a morphism $\tau : F \to F^{(**)}$ such that

(1) $F^{(**)}$ is S_2,
(2) $\ker\tau = \operatorname{emb} F$,
(3) $\operatorname{Supp} F^{(**)} = \operatorname{Supp} F$ and
(4) $\tau : F/\operatorname{emb} F \to F^{(**)}$ is an isomorphism at all codimension ≤ 1 points of $\operatorname{Supp} F$.

Note that if $\dim \operatorname{Supp} F \leq 1$ then $F^{(**)} \cong F/\operatorname{emb} F$.

Let $Z \subset X$ be a closed subscheme. We also define a local version of the hull; it is the "best" sheaf that one can associate to F without changing $F|_{X\setminus Z}$. Thus the *local hull* of F centered at Z is a quasi-coherent sheaf $F_Z^{(**)}$ together with a morphism $\tau_Z : F \to F_Z^{(**)}$ such that

(1') $\operatorname{depth}_Z F_Z^{(**)} \geq 2$,
(2') $\ker\tau_Z = \operatorname{tors}_Z F$, the largest subsheaf of F supported on Z,
(3') $\operatorname{Supp} F^{(**)}$ is the closure of $\operatorname{Supp}(F|_{X\setminus Z})$ and
(4') $\tau_Z : F \to F_Z^{(**)}$ is an isomorphism over $X \setminus Z$.

(The depth for non-coherent sheaves is defined in [Gro68, Exp.III]. By that definition (1.1') is equivalent to (1.5') below; see Lemma 14. I believe that the above is the only sensible definition of a local hull if $Z \cap \operatorname{Supp} F$ has codimension ≥ 2 in $\operatorname{Supp} F$; this is the main case that we use.)

If $x \in \operatorname{Supp} F$ is a point, we let $X_x := \operatorname{Spec} \mathcal{O}_{X,x}$ denote the corresponding local scheme, $F_x^{(**)}$ the hull of the localization F_x centered at $\{x\} \subset X_x$, and call the latter

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the *punctual hull* of F at x.

It is easy to see that local and global hulls always exists. In the local case

$$F_Z^{(**)} = j_*\big(F|_{X\setminus Z}\big), \tag{1.5'}$$

where $j : X \setminus Z \hookrightarrow X$ is the natural injection. (See Lemma 14 for condition (1').) In the global case

$$F^{(**)} = \varinjlim\, (F/\operatorname{emb} F)_Z^{(**)}, \tag{1.5}$$

where Z ranges through all codimension ≥ 2 subsets of $\operatorname{Supp} F$.

The formulas (1.5–5') show that local and global hulls are quasi-coherent. Our aim is to understand when local and global hulls are coherent, using

- $\overline{\operatorname{ass}}(F)$, the set of closures $W_i \subset X$ of the associated points of F and
- properties of the localizations F_x for all points $x \in Z$.

The local case turns out to be the easier. The following theorem is a generalization of [Gro60, IV.5.11.1].

Theorem 2. *Let X be a Noetherian scheme, $Z \subset X$ a closed subscheme and F a coherent sheaf on X such that $\operatorname{tors}_Z F = 0$. The following are equivalent.*

(1) *The local hull $F_Z^{(**)}$ is coherent.*
(2) *For every $x \in Z$, the punctual hull $F_x^{(**)}$ is coherent.*
(3) *For every $x \in Z$, $H^1_x(X_x, F_x)$ has finite length over $k(x)$.*
(4) *For every $x \in Z$, the completion $\hat{F}_x$ has no 1-dimensional associated primes.*
(5) *For every $W \in \overline{\operatorname{ass}}(F)$ the local hull $(\mathcal{O}_W)^{(**)}_{Z\cap W}$ is coherent.*
(6) *For every $W \in \overline{\operatorname{ass}}(F)$ and $x \in Z \cap W$, the punctual hull $(\mathcal{O}_W)^{(**)}_x$ is coherent.*
(7) *For every $W \in \overline{\operatorname{ass}}(F)$ and $x \in Z \cap W$, $H^1_x(W_x, \mathcal{O}_{W_x})$ has finite length over $k(x)$.*
(8) *For every $W \in \overline{\operatorname{ass}}(F)$ and $x \in Z \cap W$, the completion $\hat{\mathcal{O}}_{W,x}$ has no 1-dimensional associated primes.*

For the coherence of global hulls, there are some obvious restrictions.

3 (Necessary conditions). Let X be a Noetherian scheme and F a coherent, S_2 sheaf on X. We prove in (16) that its support satisfies the following *codimension 1 purity condition.*

(1) If $x \in \operatorname{Supp} F$ has codimension 1 in some irreducible component of $\operatorname{Supp} F$ then x has codimension 1 in $\operatorname{Supp} F$.

Note that the condition is vacuous if $\operatorname{Supp} F$ is irreducible, but it is meaningful if $\operatorname{Supp} F$ is not pure dimensional. For instance, the union of a plane and an intersecting line can not be the support of a coherent, S_2 sheaf. (It is, however, the support of a quasi-coherent, S_2 sheaf.)

Since a coherent sheaf F and its hull $F^{(**)}$ have the same support by definition, if the hull is coherent then $\operatorname{Supp} F$ also satisfies condition (1).

Another condition is the following. If $F^{(**)}$ is coherent then $\tau : (F/\operatorname{emb} F) \to F^{(**)}$ is an isomorphism over a dense open subset of $\operatorname{Supp} F$, hence

(2) there is a dense open subset $U \subset \operatorname{Supp} F$ such that $(F/\operatorname{emb} F)|_U$ is S_2.

This condition turns out to be automatic if X is "nice" (for instance excellent or N-2) but not always; see Example 18.

Thus the key question is to understand when condition (3.1) is also sufficient for the existence of coherent hulls. The answer is given by the following.

Theorem 4. *For a Noetherian scheme X the following are equivalent.*

(1) *A coherent sheaf F has a coherent hull iff the codimension 1 purity condition (3.1) holds for* $\operatorname{Supp} F$.
(2) $\mathcal{O}_W$ *has a coherent hull for every integral subscheme $W \subset X$.*
(3) *For every integral subscheme $W \subset X$*
  (a) *there is an open, dense subset $W^0 \subset W$ that is S_2 and*
  (b) *for every point $x \in W$ of codimension ≥ 2, the completion $\hat{\mathcal{O}}_{W,x}$ has no 1-dimensional associated primes.*

5. The relationship between the global and local hulls is clear in most cases. Assume that F is coherent, S_1 and let $Z \subset \operatorname{Supp} F$ be a closed subscheme of codimension ≥ 2. Then

(1) $F_Z^{(**)} \subset F^{(**)}$,
(2) $F_Z^{(**)} = F^{(**)}$ iff $F|_{X\setminus Z}$ is S_2 and
(3) in general $F_Z^{(**)} \subset F^{(**)}$ is determined by the exact sequence

$$0 \to F \to F_Z^{(**)} \to \operatorname{tors}_Z\bigl(F^{(**)}/F\bigr) \to 0.$$

So there is no conflict between the local and global notions. However, if $\operatorname{codim}_X Z = 1$ then the local hull $F_Z^{(**)}$ is neither coherent nor a subsheaf of the hull $F^{(**)}$.

Definition 6 (Pure hulls). If X is a finite type scheme, more generally if X admits a dimension function (cf. [Sta15, Tag 02I8]), then one can define another version of the hull—denoted by $F^{[**]}$—where the kernel of $F \to F^{[**]}$ is not $\operatorname{emb} F$ but the subsheaf of sections whose support has dimension $< \dim F$. Thus $F^{[**]}$ has pure dimensional support and so it could be called the *pure hull* of F.

This was the definition adopted in [Kol08] and [Kol17] and it is better suited to the applications there.

The two notions agree if $\operatorname{Supp} F$ is irreducible, more generally, if $\operatorname{Supp} F$ is pure dimensional. Thus Theorem 4 takes the following form for pure hulls.

Theorem 7. *Let X be a Noetherian scheme with a dimension function. The following are equivalent.*

(1) *Every coherent sheaf F has a coherent pure hull $F^{[**]}$.*
(2) $\mathcal{O}_W$ *has a coherent hull for every integral subscheme $W \subset X$.*
(3) *For every integral subscheme $W \subset X$*
  (a) *there is an open, dense subset $W^0 \subset W$ that is S_2 and*
  (b) *for every point $x \in W$ of codimension ≥ 2, the completion $\hat{\mathcal{O}}_{W,x}$ has no 1-dimensional associated primes.* □

8 (What is new in this note?). Most of the results in this note are contained in—or can be obtained by a careful contemplation of—[Gro60, IV.5.11.1], which essentially says that local hulls are coherent if the completed local rings $\hat{\mathcal{O}}_{W,x}$ are reduced and pure dimensional.

Our observation is that only 1-dimensional associated primes of the completed local rings $\hat{\mathcal{O}}_{W,x}$ cause problems, and this way one obtains necessary and sufficient conditions. The key technical point, following [Kol16], is the systematic use of punctual hulls.

1. Proof of Theorem 2.

9 (Left exactness). Since push-forward is a left exact functor, the formula (1.5') shows that the local hull $F \mapsto F_Z^{(**)}$ is also left exact.

The situation is more delicate for the hull. The problem is that, even if $F_1 \hookrightarrow F_2$ is an injection, it can happen that a subscheme Z has codimension ≥ 2 in $\operatorname{Supp} F_2$ but codimension 1 in $\operatorname{Supp} F_1$. This is, however, the only obstruction. The following special case is especially useful.

Claim 9.1. Let F be a coherent, S_1 sheaf and $G \subset F$ a subsheaf. Assume that $\operatorname{Supp} F$ satisfies the purity condition (3.1). Then $G^{(**)} \subset F^{(**)}$. □

We start the proof of Theorem 2 with the implications (2.1) ⇔ (2.5).

10 (Comparing hulls). Let X be a Noetherian, affine, integral scheme and F a torsion-free coherent sheaf on X. If F has generic rank r then there are injections

$$\mathcal{O}_X^r \hookrightarrow F \quad \text{and} \quad F \hookrightarrow \mathcal{O}_X^r. \tag{10.1}$$

Thus we get injections

$$\big(\mathcal{O}_X^{(**)}\big)^r \hookrightarrow F^{(**)} \quad \text{and} \quad F^{(**)} \hookrightarrow \big(\mathcal{O}_X^{(**)}\big)^r. \tag{10.2}$$

If $Z \subset X$ is a closed, nowhere dense subscheme, we also get injections

$$\big((\mathcal{O}_X)_Z^{(**)}\big)^r \hookrightarrow F_Z^{(**)} \quad \text{and} \quad F_Z^{(**)} \hookrightarrow \big((\mathcal{O}_X)_Z^{(**)}\big)^r. \tag{10.2$'$}$$

Thus we conclude that

Claim 10.3. $F^{(**)}$ (resp. $F_Z^{(**)}$) is coherent iff $\mathcal{O}_X^{(**)}$ (resp. $(\mathcal{O}_X)_Z^{(**)}$) is. □

Let us now drop the assumptions that X is integral and F is torsion free. Let $W_i \subset X$ be the closures of the associated points of F.

Claim 10.4. Let F be a coherent sheaf such that $\operatorname{Supp} F$ satisfies the purity condition (3.1). Then $F^{(**)}$ (resp. $F_Z^{(**)}$) is coherent iff each $\mathcal{O}_{W_i}^{(**)}$ (resp. $(\mathcal{O}_{W_i})_{Z\cap W_i}^{(**)}$) is.

Proof. By dévissage (see, for instance, [Sta15, Tag 01YC]) F admits a finite filtration $0 = G_0 \subset G_1 \cdots \subset G_m = F$ such that every graded piece G_{r+1}/G_r is torsion free over some W_i. By induction we obtain that if each $\mathcal{O}_{W_i}^{(**)}$ is coherent then so is $F^{(**)}$. Conversely, for each i there is an injection $\mathcal{O}_{W_i} \hookrightarrow F$. Thus if $F^{(**)}$ is coherent then so is $\mathcal{O}_{W_i}^{(**)}$. The proof of the local version is the same. □

Next we show some cases when the structure sheaf has a coherent hull, using some better known conditions of the theory of commutative rings. Note that (11.1) is known as the N-1 condition [Sta15, Tag 0BI1]. If, in addition, W is universally catenary, then $\pi : W^{\mathrm{n}} \to W$ preserves codimension [Sta15, Tag 02II]. In particular, if X is excellent then both of these conditions hold for every integral subscheme $W \subset X$, but the conditions (11.1–2) are strictly weaker than excellence.

Proposition 11. *Let W be a Noetherian, integral scheme such that*
(1) *the normalization $\pi : W^{\mathrm{n}} \to W$ is finite and*
(2) *if $w' \in W^{\mathrm{n}}$ has codimension 1 then $\pi(w') \in W$ also has codimension 1.*

Then the hull $\mathcal{O}_W^{(**)}$ is coherent.

Proof. Since $\pi : W^{\mathrm{n}} \to W$ is finite, $\pi_*(\mathcal{O}_W^{\mathrm{n}})$ is coherent and torsion free. Let $w \in W$ be point of codimension ≥ 2. Then $\pi^{-1}(w)$ has codimension ≥ 2 by assumption (2), so W^{n} has depth ≥ 2 at $\pi^{-1}(w)$ since W^{n} is normal. Thus $\pi_*(\mathcal{O}_W^{\mathrm{n}})$ has depth ≥ 2 at w and hence $\pi_*(\mathcal{O}_W^{\mathrm{n}})$ is S_2. Therefore

$$\mathcal{O}_W^{(**)} \subset \big(\pi_*(\mathcal{O}_W^{\mathrm{n}})\big)^{(**)} = \pi_*(\mathcal{O}_W^{\mathrm{n}}) \quad \text{is coherent.} \quad \square$$

Next we show the implications (2.2) ⇔ (2.3) ⇔ (2.4).

12 (Punctual hulls). Let (x, X) be a Noetherian, local scheme and F a coherent sheaf on X. Set $U := X \setminus \{x\}$. We have an exact sequence

$$0 \to H^0_x(X, F) \to H^0(X, F) \to H^0(U, F|_U) \to H^1_x(X, F) \to H^1(X, F) = 0.$$

Thus $H^0(U, F|_U)$ is a finite $H^0(X, \mathcal{O}_X)$-module iff $H^1_x(X, F)$ is. Since $F_x^{(**)}$ is the sheaf corresponding to $H^0(U, F|_U)$, we have proved the following.

Claim 12.2. $F_x^{(**)}$ is coherent iff $H^1_x(X, F)$ has finite length. $\square$

Assume next that $F_x^{(**)}$ is coherent and consider its completion $\widehat{F_x^{(**)}}$. Its depth at x is ≥ 2 and $\widehat{F_x^{(**)}}/\hat{F} \cong F_x^{(**)}/F$ is Artinian. Thus $\widehat{F_x^{(**)}}$ is the hull of $\hat{F}$. Conversely, if $F_x^{(**)}$ is not coherent, then it is a limit of an infinite increasing sequence of coherent sheaves $F \subsetneq F_1 \subsetneq \cdots \subsetneq F_x^{(**)}$ and taking completions shows that the hull of $\hat{F}$ is also not coherent. Thus we get the following.

Claim 12.3. $F_x^{(**)}$ is coherent iff $(\hat{F})_x^{(**)}$ is and then $(\hat{F})_x^{(**)} = \widehat{F_x^{(**)}}$. $\square$

It remains to understand when the punctual hull is coherent when (x, X) is local and complete. Let $W \subset X$ be an integral subscheme. Since X is complete so is W, hence X and W are excellent (cf. [Sta15, Tag 07QW]). Thus, by Proposition 11, the global hull $\mathcal{O}_W^{(**)}$ is coherent. As we noted in Paragraph 5, the punctual hull $(\mathcal{O}_W)_x^{(**)} \subset \mathcal{O}_W^{(**)}$ is also coherent provided $\dim W \geq 2$.

Thus, if $\hat{F}$ has no 1-dimensional associated primes then $F_x^{(**)}$ is coherent by (10.4).

13 (Modification and localization). Let X be a scheme, F a quasi-coherent sheaf on X and $Z \subset X$ a closed, nowhere dense subscheme. A *modification* of F *centered* at Z is a quasi-coherent sheaf G together with a map of sheaves $q : F \to G$ such that none of the associated primes of G is contained in Z and q is an isomorphism over $X \setminus Z$. A modification is called coherent if G is coherent. If $\dim Z = 0$ then a modification of F centered at Z is called a *punctual modification* of F.

Let $j : X \setminus Z \hookrightarrow X$ be the natural injection. There is a one-to-one correspondence between modifications and quasi-coherent sheaves

$$F/\operatorname{tors}_Z F \subset G \subset j_*\big(F|_{X\setminus Z}\big).$$

Let $\pi : Y \to X$ be a flat morphism and set $W := \pi^{-1}(Z)$. Since j_* commutes with flat base change, we obtain that

$$(\pi^* F)_W^{(**)} \cong \pi^*\big(F_Z^{(**)}\big).$$

In particular, the local hull is a sheaf in the Zariski topology and commutes with arbitrary localizations.

Next let $x \in Z$ be a point. By localizing we obtain $Z_x \subset X_x$ and F_x. Let $p_x : F_x \to G_x$ be a coherent modification centered at Z_x. First we can extend $p_x : F_x \to G_x$ to a coherent modification $p^0 : F^0 \to G^0$ defined over some open neighborhood $x \in X^0 \subset X$ and then, using [Har77, Exrc.II.5.15], to a coherent modification $p : F \to G$. The following special case is especially useful.

Claim 13.1. Let X be a Noetherian scheme, $x \in X$ a point and F a coherent sheaf on X. Then every coherent modification $p_x : F_x \to G_x$ centered at $x \in X_x$ can be extended to a coherent modification $p : F \to G$ centered at $\bar{x} \in X$. □

The following lemma is a special case of the assertion that the definition of depth given in [Gro68, Exp.III] is equivalent to the usual definition.

Lemma 14. *Let X be a Noetherian scheme and $Z \subset X$ a closed, nowhere dense subscheme. Let $F \to G$ be a coherent modification centered at Z. Then $G = F_Z^{(**)}$ iff $\operatorname{depth}_Z G \geq 2$.*

Proof. As we noted in Paragraph 5, neither condition holds if $Z \cap \operatorname{Supp} F$ has codimension 1 in $\operatorname{Supp} F$.

We may assume that $\operatorname{tors}_Z F = \operatorname{tors}_Z G = 0$, so we have injections $F \subset G \subset F_Z^{(**)}$ and $G_Z^{(**)} = F_Z^{(**)}$. We need to show that $j_Z^{(**)} : G \to G_Z^{(**)}$ is an isomorphism iff $\operatorname{depth}_Z G \geq 2$. For this we may assume that X is affine.

Pick any $x \in Z$ and let $s \in \mathcal{O}_X$ be an equation of Z that is not a zero divisor on G. Note that $\operatorname{depth}_x G \geq 2$ iff x is not an associated point of G/sG.

If $\operatorname{depth}_x G = 1$ then let $sG \subsetneq G' \subset G$ be a subsheaf such that G'/sG is supported on $\bar{x}$. Then $G \to s^{-1}G'$ is a non-identity modification of G centered at $\bar{x} \subset Z$, thus $G \neq G_Z^{(**)}$.

Conversely, if $G \subset G_Z^{(**)}$ is not an isomorphism then there is a non-identity coherent modification $G \to G'$ centered at Z. Let x be a generic point of $\operatorname{Supp}(G'/G)$. Then $s^m G' \subset G$ for some $m > 0$ and $s^m G \subset s^m G' \subset G$ shows that $\operatorname{depth}_x G = 1$. □

15 (Proof of Theorem 2). We may assume that X is affine. The equivalence of (2.1) and (2.5) is proved in (10.4) and (2.2) $\Leftrightarrow$ (2.3) $\Leftrightarrow$ (2.4) is proved in Paragraph 12. Note further that the equivalences of (2.5), (2.6), (2.7) and (2.8) with each other are special cases of the equivalences of (2.1), (2.2), (2.3) and (2.4) with each other. Thus it remains to show that (2.1) and (2.2) are equivalent.

Assume (2.1) and set $T = \bar{x}$. There is a natural map $F_T^{(**)} \to F_Z^{(**)}$ whose kernel is supported on Z. Since $W_i \not\subset Z$ for every i, $F_T^{(**)} \to F_Z^{(**)}$ is injective. Therefore $F_T^{(**)}$ is coherent. Since $F_x^{(**)}$ is a localization of $F_T^{(**)}$, this shows that (2.1) $\Rightarrow$ (2.2).

Next assume that (2.2) holds. By Proposition 17 there are only finitely many points $x_i \in Z$ such that $\operatorname{depth}_{x_i} F = 1$. Let $D_Z(F) \subset Z$ denote the union of their closures.

Let x be a generic point of $D_Z(F)$. By our assumption (2.2), $F_x \to F_x^{(**)}$ is a coherent modification centered at x; in particular $\operatorname{codim}_X x \geq \operatorname{depth}_x F \geq 2$ by Lemma 14. By (13.2) we can extend $F_x^{(**)}$ to a coherent modification $p : F \to F_1$ centered at $\bar{x}$.

By (10.3) our assumption (2.2) also holds for F_1 and we claim that $D_Z(F_1) \subsetneq D_Z(F)$. To see the latter, pick any point $x' \in Z$ such that $\operatorname{depth}_{x'} F_1 \leq 1$. If $x' \notin \bar{x}$ then p is an isomorphism near x', hence $\operatorname{depth}_{x'} F \leq 1$. From Lemma 14 we obtain that $\operatorname{depth}_x F_1 \geq 2$, hence if $x' \in \bar{x}$ then it is a non-generic point. Thus $D_Z(F_1) \subsetneq D_Z(F)$.

We can thus repeat this process and eventually obtain a coherent modification $F \to F_m$ such that $D_Z(F_m) = \emptyset$. This means that $\operatorname{depth}_Z F_m \geq 2$ hence $F_m = F_Z^{(**)}$ by Lemma 14. □

2. Coherence of global hulls.

16 (Depth of submodules). Let (R, m) be a Noetherian local ring and M a finite R-module. Then $\operatorname{depth}_m M \geq 1$ iff M has no nonzero submodule $0 \neq N \subset M$ such that $mN = 0$. Thus $\operatorname{depth}_m M' \geq 1$ for every submodule $M' \subset M$.

Assume next that $\operatorname{depth}_m M \geq 2$ and $M' \subset M$ is m-saturated, meaning that there is no submodule $M' \subsetneq N \subset M$ such that $mN \subset M'$. Thus there is an $r \in m$ that is not a zero divisor on M/M' and so we get an injection $M'/rM' \hookrightarrow M/rM$. Therefore $\operatorname{depth}_m M'/rM' \geq 1$ and so $\operatorname{depth}_m M' \geq 2$.

Let next $P \subset R$ be an associated prime of M and $\operatorname{tors}_P M \subset M$ the torsion submodule corresponding to P. Then $\operatorname{tors}_P M$ is m-saturated, hence $\dim \operatorname{tors}_P M \geq \operatorname{depth}_m \operatorname{tors}_P M \geq 2$. In particular, every associated prime of M has dimension ≥ 2.

Let now X be a Noetherian scheme, F a coherent, S_2 sheaf on X and $x \in \operatorname{Supp} F$ a point of codimension ≥ 2. Then $\operatorname{depth}_x F \geq 2$. Therefore, if W is an irreducible component of $\operatorname{Supp} F$ that contains x then $\dim \mathcal{O}_{W.x} \geq 2$. Equivalently, x also has codimension ≥ 2 in W. This is exactly the codimension 1 purity condition claimed in (3.1).

Proposition 17. *Let X be a Noetherian scheme, F a coherent sheaf on X and $Z \subset X$ a closed, nowhere dense subscheme that does not contain any of the associated points of F. Then there are only finitely many points $x \in Z$ such that* $\operatorname{depth}_x F \leq 1$.

Proof. The question is local so we may assume that X is affine. By our assumptions there is a Cartier divisor $(g = 0)$ containing Z and $\operatorname{depth}_x X \leq 1$ iff x is an associated point of F/gF. Since X is Noetherian, there are only finitely many such points. □

Example 18. The assumption that Z should not contain any of the associated points of F is necessary. For instance, set $X = \operatorname{Spec} k[x, y, z, t]/(tx, ty, t^2)$. The associated primes are (0), (x, y, t) and $\operatorname{depth}_p X = 1$ at every point $p \in V(x, y, t)$.

The assumption that Z should be nowhere dense is also necessary. The following example is modeled on [Nag62, A.1]. Start with $R_1 := k[x_1, y_1, x_2, y_2, \dots]$. Let $R_2 \subset R_1$ be the subring generated by all monomials of degree ≥ 2 and R_3 the ring obtained by inverting every element not contained in any of the ideals $m_i := R_2 \cap (x_i, y_i)R_1$. Then R_3 is Noetherian, has dimension 2 and its maximal ideals are $m_i R_3$. These form a Zariski dense subset of $\operatorname{Spec} R_3$ and R_3 has depth 1 at all of the maximal ideals.

The next results says that a generically S_2 sheaf is always S_2 outside a codimension ≥ 2 subset.

Corollary 19. *Let X be a Noetherian scheme, F a coherent, S_1 sheaf on X and $Z \subset \operatorname{Supp} F$ a closed, nowhere dense subscheme such that $F|_{X \setminus Z}$ is S_2. Then there is a closed subscheme $W \subset Z$ such that*

(1) *W has codimension ≥ 2 in* $\operatorname{Supp} F$ *and*
(2) *$F|_{X \setminus W}$ is S_2.*

Proof. By Proposition 17, there are only finitely many points $x \in Z$ such that $\operatorname{depth}_x F \leq 1$. We can thus take $W := \cup \bar{x}$ where x runs through all points of codimension ≥ 2 for which $\operatorname{depth}_x F \leq 1$. □

20 (Proof of Theorem 4). The equivalence of (4.1) and (4.2) follows from (10.4).

Assume next that (4.2) holds. As we noted in (3.2), if $\mathcal{O}_W$ has a hull then there is an open dense subset $W^0 \subset W$ that is S_2 and (2.5) ⇒ (2.8) shows that for every point $x \in W$ of codimension ≥ 2, the completion $\hat{\mathcal{O}}_{W,x}$ has no 1-dimensional associated primes. These are (4.3.a–b).

Conversely, if (4.3.a) holds then by Corollary 19 there is a closed subscheme $W \subset X$ of codimension ≥ 2 such that $\mathcal{O}_{X \setminus W}$ is S_2. Thus $\mathcal{O}_X^{(**)} = (\mathcal{O}_X)_W^{(**)}$ by (5.2) and $(\mathcal{O}_X)_W^{(**)}$ is coherent by (2.8) ⇒ (2.5). □

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REFERENCES

[Gro60] A. Grothendieck, *Éléments de géométrie algébrique. I–IV.*, Inst. Hautes Études Sci. Publ. Math. (1960), no. 4,8,11,17,20,24,28,32.

[Gro68] ———, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux* (*SGA* 2), North-Holland Publishing Co., Amsterdam, 1968, Augmenté d'un exposé par Michèle Raynaud, Séminaire de Géométrie Algébrique du Bois-Marie, 1962, Advanced Studies in Pure Mathematics, Vol. 2. MR 0476737 (57 #16294)

[Har77] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157 (57 #3116)

[Kol08] J. Kollár, *Hulls and husks*, 2008. ArXiv:0805.0576

[Kol16] ———, *Variants of normality for Noetherian schemes*, Pure and Applied Mathematics Quarterly, 12:1 (2016), pp. 1–31.

[Kol17] ———, *Moduli of varieties of general type*, (book in preparation), 2017.

[Nag62] M. Nagata, *Local rings*, Interscience Tracts in Pure and Applied Mathematics, No. 13, Interscience Publishers a division of John Wiley & Sons New York-London, 1962. MR MR0155856 (27 #5790)

[Sta15] The Stacks Project Authors, *Stacks Project*, http://stacks.math.columbia.edu, 2015.