CERTAIN NORMAL SURFACE SINGULARITIES OF GENERAL TYPE∗

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Dedicated to Professor Henry B. Laufer on his seventieth birthday

Abstract. Koyama’s inequality for normal surface singularities gives the upper bound on the self-intersection number of the canonical cycle in terms of the arithmetic genus. For those singularities of fundamental genus two attaining the bound, a formula for computing the geometric genus is shown and the resolution dual graphs are roughly classified. In Gorenstein case, the multiplicity and the embedding dimension are also computed.

Key words. Even singularity, canonical cycle, Yau cycle, maximal ideal cycle.

AMS subject classifications. 14J17, 14B05.

Introduction. Various studies have done on normal singular points of a complex surface. Among others, most fruitful achievements are beautiful theorems on classes of small genera such as rational and elliptic singular points.

In this article, we study a class of normal (numerically) Gorenstein surface singularities of fundamental genus $\geq 2$, which may be regarded as “singularities on the Noether line” on the analogy of the geography of global surfaces of general type. To be more precise, let $(V, o)$ be a normal surface singularity and $\pi : X \to V$ a resolution. The arithmetic genus of the fundamental cycle is called the fundamental genus of $(V, o)$ and we denote it by $p_f(V, o)$. The arithmetic genus $p_a(V, o)$ of $(V, o)$ is the maximum of the arithmetic genus of effective divisors with support in $\pi^{-1}(o)$. Let $Z_K$ be the canonical cycle on $\pi^{-1}(o)$, that is, the $\mathbb{Q}$-divisor numerically equivalent to $-K_X$. Then Yoichi Koyama obtained the inequality $-Z_K^2 \geq 8p_a(V, o) - 8$ around 1984 (unpublished). If the equality sign holds here, then one has $Z_K = 2D$ with the unique effective $\mathbb{Z}$-divisor $D$ computing $p_a(V, o)$ and, in particular, $(V, o)$ is numerically Gorenstein. We call such a singular point even and study those of fundamental genus 2 in this paper.

In §1, we summarize mostly known results about normal surface singularities for the later use. Some facts are reproved from the point of view of [5]. We discuss in §2 Koyama’s inequality and introduce the notion of even singularities. Then we restrict ourselves to those of fundamental genus 2. In this case, a half of the canonical cycle is nothing more than the Yau cycle introduced in [6]. Using such information, we can roughly classify the resolution dual graphs. A general picture of the exceptional set is the core part attached two ADE-brances of $(-2)$-curves. Here, the core part is the minimally even cycle, i.e., the minimal model of the fundamental cycle $Z$, which itself is the fundamental cycle of an even singular point with $p_f = p_a = 2$. Similarly as in the case of elliptic singularities, minimally even cycles of genus 2 is closely related to the singular fibers in pencils of curves of genus 2 (cf. [9]). Our classification is much coarser than [9] and the cycles fall into five classes (0), (i.a), (i.b), (ii.a) and (ii.b) according to the numerical connectivity and the base locus of the canonical linear

∗Received January 31, 2016; accepted for publication October 13, 2016.
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system. In §3, we establish a formula computing the geometric genus that is modeled on Okuma’s formula for elliptic singularities [11]. In §4 and §5, we compute the numerical invariants of \((V,\alpha)\), such as the multiplicity and the embedding dimension, when \((V,\alpha)\) is Gorenstein.

In the course of the study, basic results for 1-connected curves or chain-connected curves are used freely. For these, we refer the readers to [1, Appendix] and [5].

1. Preliminaries. In this paper, a curve means a non-zero effective divisor (with support compact) on a smooth surface. For a curve \(D\), the arithmetic genus \(p_a(D)\) is defined by \(p_a(D) = 1 - \chi(D, \mathcal{O}_D)\), where \(\chi(D, \mathcal{O}_D) = h^0(D, \mathcal{O}_D) - h^1(D, \mathcal{O}_D)\). Then \(2p_a(D) - 2 = K_XD + D^2\) by the adjunction formula.

1.1. Chain-connected curves. A curve \(D\) is said to be chain-connected if \(\mathcal{O}_{D-\Gamma}(-\Gamma)\) is not nef for any proper subcurve \(\Gamma < D\). If \(D\) is chain-connected and \(p_a(D) > 0\), then there exists the unique chain-connected subcurve \(D_{\min}\) of \(D\) such that \(p_a(D_{\min}) = p_a(D)\) and \(K_{D_{\min}}\) is nef. We call it the minimal model of \(D\).

A maximal chain-connected subcurve of a curve is called a chain-connected component. Every curve \(D\) decomposes into a sum of chain-connected curves as \(D = \Gamma_1 + \cdots + \Gamma_n\), where \(\Gamma_i\) is a chain-connected component of \(D - \sum_{j<i} \Gamma_j\). Then, \(\mathcal{O}_{\Gamma_i}(-\Gamma_i)\) is nef when \(i < j\), which implies that either \(\text{Supp}(\Gamma_i) \cap \text{Supp}(\Gamma_j) = \emptyset\) or \(\Gamma_j \leq \Gamma_i\). Such an ordered decomposition is essentially unique and is called a chain-connected component decomposition (a CCCD for short) of \(D\). For the properties, see [5].

Lemma 1.1. Let \(D\) be a chain-connected curve on a smooth surface \(X\).

(1) If \(\mathcal{O}_D(-\Gamma)\) is nef and \(p_a(D) \leq 1\), then \(p_a(\Gamma) \leq p_a(D)\) for any curve \(\Gamma\) with support in \(\text{Supp}(D)\).

(2) Let \(p \in D\) and put \(\nu = \min\{\mu \mid A \leq D\text{ is a component, } p \in A\}\) If \(\rho : \tilde{X} \to X\) is the blowing-up at \(p\) and \(E = \rho^{-1}(p)\), then \(\rho^*D - kE\) is chain-connected for \(0 \leq k \leq \nu\).

Proof. (1) Assume first that \(\Gamma\) is chain-connected. Since \(\mathcal{O}_\Gamma(-\Gamma)\) is nef, \(\Gamma \leq D\). Then \(p_a(\Gamma) \leq h^1(\mathcal{O}_\Gamma) \leq h^1(\mathcal{O}_D) = p_a(D)\). Consider the general case and let \(\Gamma = \sum_{i=1}^n \Gamma_i\) be a CCCD. Then \(p_a(\Gamma) - 1 = \sum_{i=1}^n (p_a(\Gamma_i) - 1) + \sum_{i<j} \Gamma_i \Gamma_j\). Since \(\Gamma_i \Gamma_j \leq 0\) when \(i < j\), and \(p_a(\Gamma_i) \leq p_a(D)\) by the first step, we get \(p_a(\Gamma) \leq p_a(D)\)

when \(p_a(D) = 0, 1\).

(2) We fix a component \(A\) satisfying \(\mu(A) = \nu\) and take a connecting chain (cf. [5]) \(D_0 = A, \ldots, D_N = D\) starting from \(A\), where \(A_i = D_i - D_{i-1}\) is an irreducible curve with \(A_i D_{i-1} > 0\). Let \(\tilde{A}\) be the proper transform of \(A\). Then \(\rho^*A = \tilde{A} + \nu E\). Put \(\tilde{D}_i = \rho^*D_i - \nu E\) for any \(i\). Since \(\tilde{D}_i - \tilde{D}_{i-1} = \rho^*A_i\), \(\tilde{D}_i\)'s are effective and non-zero for any \(i\). We construct a connecting chain for \(\rho^*D - \nu E\) by refining \(\{\tilde{D}_i\}\). For this purpose, it suffices to consider \(i\) such that \(\rho^*A_i\) is reducible, and to construct a connecting chain from \(\tilde{D}_{i-1}\) to \(\tilde{D}_i\). Since \(\rho^*A_i\) is reducible, \(p \in A_i\). If \(\tilde{A}_i\) denotes the proper transform of \(A_i\) by \(\rho\), then \(\rho^*A_i = \tilde{A}_i + \nu_i E\), where \(\nu_i = \mu(A_i)\). We have \(\nu_i \geq \nu\) by the definition of \(\nu\). Put

\[
\Gamma_j = \begin{cases} 
\tilde{D}_{i-1} + jE, & \text{if } 0 \leq j \leq \nu, \\
\rho^*D_{i-1} + \tilde{A}_i, & \text{if } j = \nu + 1, \\
\Gamma_{\nu+1} + (j - \nu - 1)E, & \text{if } \nu + 2 \leq j \leq \nu_i + 1 \text{ and } \nu_i > \nu.
\end{cases}
\]

Then \(\Gamma_0 = \tilde{D}_{i-1}, \Gamma_1, \ldots, \Gamma_{\nu_i+1} = \tilde{D}_i\) is the desired connecting chain, since \(\Gamma_j - \Gamma_{j-1}\), which is either \(E\) or \(\tilde{A}_i\), is irreducible and \(\Gamma_{i-1} \Gamma_{i-1} > 0\). Therefore, \(\rho^*D - \nu E\)
is chain-connected. Since \((\rho^*D - (v - i)E)E > 0\) when 0 \(\leq i < v\), we see inductively that \(\rho^*D - (v - i - 1)E\) is chain-connected by [5, Proposition 1.5].

**Proposition 1.2.** Let \(D\) be a chain-connected curve with \(p_a(D) > 0\) and \(D_{\min}\) its minimal model. Assume that \(\text{Supp}(D - D_{\min})\) contracts to several rational double points and that \(-D\) is nef on \(\text{Supp}(D - D_{\min})\). Put \((D - D_{\min})^2 = -2n\). Then the following hold.

1. \(-(D - D_{\min})\) is nef on its support.
2. \(D - D_{\min}\) decomposes as \(\Gamma_1 + \cdots + \Gamma_n\), where each \(\Gamma_i\) is the fundamental cycle on its support, \(-\Gamma_i^2 = 2\), \(D_{\min}\Gamma_i = 1\) for any \(i\) and \(\mathcal{O}_{\Gamma_i}(-\Gamma_i) \cong \mathcal{O}_{\Gamma_j}\) for any pair \((i, j)\) with \(i < j\). In particular, either \(\Gamma_j < \Gamma_i\) or \(\text{Supp}(\Gamma_i) \cap \text{Supp}(\Gamma_j) = \emptyset\) for \(i < j\).
3. If \(n \geq 2\) and \(\Gamma_2 < \Gamma_1\) in (2), then \((\text{type}(\Gamma_1), \text{type}(\Gamma_2)) = (A_{l+2}, A_l)\) \((l \geq 1)\), \((D_{l+2}, D_l)\) \((l \geq 3)\), \((D_1, A_1)\) \((l \geq 4)\), \((E_5, A_5)\), \((E_7, D_6)\) or \((E_8, E_7)\), with an obvious convention that \(D_3 = A_3\).

**Proof.** (1) Assume that there exists an irreducible component \(C \cong D - D_{\min}\) such that \(C(D - D_{\min}) > 0\). By the assumption, we have \(CD \leq 0\) and, hence, \(CD_{\min} < 0\). In particular, \(C\) is a component of \(D_{\min}\). Then, since \(K_{D_{\min}}\) is nef, we have \(0 \leq \deg K_{D_{\min}}|_C = \deg K_C + C(D_{\min} - C)\), from which we get the contradiction that \(CD_{\min} \geq 0\) by \(\deg K_C = C^2 = -2\). Therefore, \(-(D - D_{\min})\) is nef on \(\text{Supp}(D - D_{\min})\).

(2) By [5, Lemma 3.6], \(D - D_{\min}\) decomposes as \(D - D_{\min} = \Gamma_1 + \cdots + \Gamma_n\), where \(\Gamma_i\) is a chain-connected curve, \(\mathcal{O}_{\Gamma_i}(-\Gamma_i)\) is numerically trivial for \(i < j\), and \(D_{\min} + \Gamma_i\) is a chain-connected curve satisfying \(D_{\min}\Gamma_i = 1 - p_a(\Gamma_i)\) for \(i \in \{1, 2, \ldots, n\}\). In our case, \(p_a(\Gamma_i) = 0\) and, hence, \(D_{\min}\Gamma_i = 1\), \(\Gamma_i^2 = 2\) and \(\mathcal{O}_{\Gamma_i}(-\Gamma_i) \cong \mathcal{O}_{\Gamma_j}\) for \(i < j\). Furthermore, \((D - D_{\min})^2 = \sum_{i=1}^n \Gamma_i^2 = \sum_{i=1}^n \Gamma_i^2 = -2n\). So, it suffices to see that every \(\Gamma_i\) is the fundamental cycle on its support.

Since \(-(D - D_{\min})\) is nef by (1), every chain-connected component of \(D - D_{\min}\) is the fundamental cycle on a connected component of \(\text{Supp}(D - D_{\min})\). Hence so is \(\Gamma_1\). Assume that \(\Gamma_1\) is the fundamental cycle on its support for \(i \leq k\). Now, \(\Gamma_{k+1}\) is a chain-connected component of \(D - D_{\min} - \sum_{i=1}^k \Gamma_i\). Since \(-(D - D_{\min} - \sum_{i=1}^k \Gamma_i)\) is nef on its support by the property \(\mathcal{O}_{\Gamma_i}(-\Gamma_i) \cong \mathcal{O}_{\Gamma_j}\) for \(i < j\), we see that \(\Gamma_{k+1}\) is the fundamental cycle on its support. Therefore, we are done by induction.

(3) We have \(\Gamma_2 < \Gamma_1\) and \((\Gamma_1 - \Gamma_2)\Gamma_2 = 2\). Let \(C\) be a component of \(\Gamma_1\) with \(C\) as above which is the “second” component. Furthermore, \(\Gamma_1 - 2C\) has connected support unless \(\Gamma_1\) is of type \(D_l\). Then one can determine the possible type of \(\Gamma_2\) by examining the weighted dual graph of \(\Gamma_1\). We leave the detail to the reader.

### 1.2. Cycles over a normal surface singularity.

Let \((V, o)\) be a normal surface singular point and \(\pi : X \to V\) a resolution. Let \(Z\) be the fundamental cycle on \(\pi^{-1}(o)\), that is, the smallest curve with support in \(\pi^{-1}(o)\) such that \(-Z\) is nef on \(\pi^{-1}(o)\). It is chain-connected. Its arithmetic genus is called the fundamental genus and denoted by \(p_f(V, o)\). The arithmetic genus of \((V, o)\) which we denote by \(p_a(V, o)\) is the maximum of \(p_a(D) := D(K_X + D)/2 + 1\) when \(D\) runs over all curves with
support in $\pi^{-1}(o)$. Clearly, we have $p_f(V,o) \leq p_a(V,o)$. The geometric genus is defined by $p_g(V,o) = \dim_{\mathbb{C}}(R^1\pi_\ast\mathcal{O}_X)_o$. Then $p_a(V,o) \leq p_g(V,o)$. When $p_g(V,o) = 0$ (resp. $p_a(V,o) = 1$), $(V,o)$ is called rational (resp. elliptic). It is known (and follows from Lemma 1.1) that $(V,o)$ is rational (resp. elliptic) if and only if $p_f(V,o) = 0$ (resp. 1).

According to [12], we put

$$Z_1 = \min\{D \mid h^1(D,\mathcal{O}_D) = p_g(V,o), \text{ } D \text{ is a curve}\}$$

and call it the cohomological cycle on $\pi^{-1}(o)$. If $p_g(V,o) > 0$, then we have $h^1(D,\mathcal{O}_D) = p_g(V,o)$ when $Z_1 \leq D$, and $h^1(D,\mathcal{O}_D) < p_g(V,o)$ when $Z_1 \nleq D$.

Since the intersection form is negative definite on $\pi^{-1}(o)$, there is a $\mathbb{Q}$-divisor $Z_K$ with support in $\pi^{-1}(o)$ such that $-Z_K$ is numerically equivalent to a canonical divisor $K_X$. We call $Z_K$ the canonical cycle. If it is integral, then $(V,o)$ is said to be numerically Gorenstein. In this case, it is shown in [12] that $Z_1 \leq Z_K$ on the minimal resolution (see also [3]). In particular, we have $p_a(Z_K) = 1$ and $h^1(Z_K,\mathcal{O}_{Z_K}) = h^0(Z_K,\mathcal{O}_{Z_K}) = p_g(V,o)$. The singular point is Gorenstein if and only if $-Z_K$ is linearly equivalent to $K_X$. When $p_f(V,o) > 0$, $Z_1 = Z_K$ holds if and only if $(V,o)$ is Gorenstein (see, e.g., [3]).

We remark the following:

**Lemma 1.3.** Let $\pi : X \to V$ be a resolution of a normal surface singular point $(V,o)$ with $p_g(V,o) > 0$ and $Z$ the fundamental cycle on $\pi^{-1}(o)$. Let $L$ be a line bundle on $X$ that is numerically trivial on $\pi^{-1}(o)$. Then $L$ is trivial if and only if the restriction map $H^0(X,L) \to H^0(Z,L)$ is not the zero map. In particular, when $(V,o)$ is numerically Gorenstein and $\pi$ is minimal, $(V,o)$ is Gorenstein if and only if $\omega_{Z_K} \cong \mathcal{O}_{Z_K}$.

**Proof.** One direction is trivial. So, we assume that $L$ is non-trivial and show that $H^0(X,L) \to H^0(Z,L)$ is the zero map. We may assume that $H^0(X,L) \neq 0$. Take any non-zero $s \in H^0(X,L)$ and write its zero divisor as $(s) = A + B$, where $\text{Supp}(A) \subseteq \pi^{-1}(o)$ and $\text{Supp}(B) \cap \pi^{-1}(o)$ is at most a finite set. Since $L$ is numerically trivial and $B$ is nef on $\pi^{-1}(o)$, we see that $-A$ is nef on $\pi^{-1}(o)$. Note that $A$ is a non-effective divisor, because $A = 0$ implies $\text{Supp}(B) \cap \pi^{-1}(o) = \emptyset$ and, thus, $L$ is trivial. Then, by the nefness of $-A$, we have $Z \leq A$. Hence $s$ vanishes identically on $Z$.

In order to see the last assertion, we apply the above argument to $L = K_X + Z_K$. Note that $\mathcal{O}_{Z_K}(L) \cong \omega_{Z_K}$ and $H^0(X,L) \to H^0(Z_K,L)$ is surjective by $H^1(X,K_X) = 0$. Since $Z$ is the chain-connected component of $Z_K$, the restriction map $H^0(Z_K,L) \to H^0(Z,L)$ is not the zero map if and only if $\mathcal{O}_{Z_K}(L) \cong \mathcal{O}_{Z_K}$.

The following is a special case of [13, (2.11)].

**Lemma 1.4.** If $(V,o)$ is Gorenstein and $p_g(V,o) \geq 2$, then $p_a(V,o) < p_g(V,o)$.

**Proof.** We work on the minimal resolution. We assume that there exists a curve $D$ with support in $\pi^{-1}(o)$ satisfying $p_a(D) = p_g(V,o)$, and show that this eventually leads us to a contradiction.

Recall the general inequality $p_a(D) \leq h^1(D,\mathcal{O}_D) \leq p_g(V,o)$. Hence, in the present case, $p_a(D) = h^1(D,\mathcal{O}_D) = p_g(V,o)$ and it follows $h^0(D,\mathcal{O}_D) = 1$. Furthermore, since $h^1(D,\mathcal{O}_D) = p_g(V,o)$, $D$ contains the cohomological cycle $Z_1$ as a subcurve. Since $(V,o)$ is Gorenstein, we have $Z_1 = Z_K$. So, $Z_K \leq D$. Note that we
cannot have \( D = Z_K \), because \( h^0(D, \mathcal{O}_D) = 1 \) but \( h^0(Z_K, \mathcal{O}_{Z_K}) = p_g(V, o) > 1 \). We consider the cohomology long exact sequence for

\[
0 \to \mathcal{O}_{Z_K}(K_{Z_K}) \to \mathcal{O}_D(K_D) \to \mathcal{O}_{D-Z_K}(K_D) \to 0.
\]

We have \( \mathcal{O}_{Z_K}(K_{Z_K}) \cong \mathcal{O}_{Z_K} \) and \( H^0(Z_K, K_{Z_K}) \to H^0(D, K_D) \) is an isomorphism. Furthermore, \( H^1(Z_K, K_{Z_K}) \to H^1(D, K_D) \) is non-trivial, being the dual of the restriction map \( H^0(D, \mathcal{O}_D) \to H^0(Z_K, \mathcal{O}_{Z_K}) \). Hence \( h^0(D - Z_K, K_D) = p_g(V, o) - 1 > 0 \). On the other hand, one has \( \mathcal{O}_{D-Z_K}(K_D) \cong \mathcal{O}_{D-Z_K}(D - Z_K) \) by the adjunction formula. Since the intersection form is negative definite on \( \pi^{-1}(o) \), we get \( H^0(D - Z_K, D - Z_K) = 0 \). This contradiction implies that we cannot have \( p_a(D) = p_g(V, o) \). Therefore, \( p_a(V, o) < p_g(V, o) \).

The Yau sequence for \( Z \) is the longest sequence of curves

\[
D_m < D_{m-1} < \cdots < D_2 < D_1 = Z,
\]

where \( D_{i+1} \) is the biggest (non-trivial) subcurve of \( D_i \) such that \( p_a(D_{i+1}) = p_a(D_i) \) and \( \mathcal{O}_{D_{i+1}}(-D_i) \) is numerically trivial (1 \( \leq i \leq m - 1 \)). If \( Z_{\text{min}} \) denotes the minimal model of \( Z \), then \( Z_{\text{min}} \preceq D_m \) and \( Z_{\text{min}}D_m < 0 \). It is shown in [6] that each \( D_i \) is the fundamental cycle on its support and

\[
\text{Supp}(D_k) \cap \text{Supp}(D_i - D_j) = \emptyset \text{ for } i < j < k.
\]

We put \( Y = \sum_{i=1}^m D_i \) and call it the Yau cycle. When \( p_f(V, o) = 2 \), we have \( p_a(V, o) = p_a(Y) = m + 1 \) by [6, Corollary 2.5].

**Lemma 1.5.** Put \( Y_\nu = \sum_{i=\nu}^m D_i \) and let \( L \in \text{Pic}^0(Y_\nu) \). If \( H^0(Y_\nu, L) \neq 0 \), then \( \mathcal{O}_{Y_i}(L) \cong \mathcal{O}_{Y_i}(Y_\nu - Y_i) \) for some \( i, \nu \leq i \leq m \).

**Proof.** Take a non-zero section \( s \in H^0(Y_\nu, L) \). If \( s \) does not vanish identically on any components, then \( L \) is trivial and the assertion holds with \( i = \nu \). Assume not and let \( C_s \) be the biggest subcurve of \( Y_\nu \) on which \( s \) vanishes identically. We may assume that there are no sections \( f \) such that \( C_s < C_f \). Then \( \mathcal{O}_{Y_\nu-C_s}(L - C_s) \) is nef, because \( s \) induces a section of it that does not vanish identically on any components. Applying [6, Lemma 1.3] to \( \Delta = Y_\nu - C_s \), we see \( \Delta = Y_i \) for some \( \nu < i \leq m \). Since \( \mathcal{O}_{Y_i}(L - (Y_\nu - Y_i)) \) is numerically trivial, the section induced by \( s \) is nowhere vanishing and we get \( \mathcal{O}_{Y_i}(L) \cong \mathcal{O}_{Y_i}(Y_\nu - Y_i) \).

**1.3. Koyama’s inequality.** The author was informed by Masataka Tomari that Yoichi Koyama obtained the following inequality around 1984:

**Proposition 1.6 (Koyama’s inequality).** Let \((V, o)\) be a normal surface singular point with \( p_f(V, o) > 1 \) and \( \pi : X \to V \) a resolution. Then

\[
-Z_K^2 \geq 8p_a(V, o) - 8
\]

with equality holding if and only if there exists a curve \( D \) such that \( Z_K = 2D \) and \( p_a(D) = p_a(V, o) \). In particular, \((V, o)\) is a numerically Gorenstein singularity if \(-Z_K^2 = 8p_a(V, o) - 8\).

**Proof.** Let \( D \) be any curve with support in \( \pi^{-1}(o) \). We have \((D - Z_K/2)^2 \leq 0\), because the intersection form is negative definite. Hence,

\[
2p_a(D) - 2 = K_XD + D^2 = -Z_KD + D^2 = (D - Z_K/2)^2 - Z_K^2/4 \leq -Z_K^2/4
\]

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and we get the inequality $-Z_K^2 \geq 8p_a(V, o) - 8$ as wished. We have the equality sign if and only if $D - Z_K/2 = 0$ and $p_a(D) = p_a(V, o)$. Then $Z_K = 2D$ is a $Z$-divisor and, hence $(V, o)$ is numerically Gorenstein. 

**Definition 1.7.** A normal surface singularity $(V, o)$ with $\rho_f(V, o) > 1$ is called an even singularity if there exist a resolution $\pi : X \to V$ and a curve $D$ on $X$ such that $Z_K = 2D$. For an even singularity, the minimal model (cf. [5]) of the fundamental cycle (on $X$) is called the minimally even cycle.

We collect some fundamental properties of even singularities.

**Lemma 1.8.** Let $(V, o)$ be an even singular point and $\pi : X \to V$ a resolution on which $Z_K = 2D$ holds for a curve $D$. Then the following hold.

1. $D$ is the unique curve satisfying $p_a(D) = p_a(V, o)$.
2. The self-intersection number $E^2$ is even for any divisor $E$ with support in $\pi^{-1}(o)$. In particular, $\pi$ is the minimal resolution.
3. If $L$ is a line bundle on $X$ such that $L - K_X$ is nef, then $L$ is generated by its global sections.

**Proof.** (1) is clear from the proof of Proposition 1.6. (2) Since $K_XE + E^2 = 2DE + E^2$ is even, so is $E^2$. In particular, we have no $(-1)$-curves on $X$. Hence $\pi$ is the minimal resolution. (3) follows from (2), because, if the linear system $|L|$ has a base point $p$, then there exists a curve $\Delta$ passing through $p$ and $\Delta^2 = -1$ by [4, Proposition 5.1].

**Lemma 1.9.** Let $(V, o)$ be an even singular point with $p_a(V, o) = \rho_f(V, o) > 1$. Then $Z_K = 2Z$ holds on the minimal resolution space, where $Z_K$ and $Z$ are the canonical cycle and the fundamental cycle, respectively. Furthermore, $Z$ is a minimally even cycle and, either $p_g(V, o) = \rho_f(V, o) + 1$ or $p_g(V, o) = \rho_f(V, o)$ holds according to whether $(V, o)$ is Gorenstein or not.

**Proof.** The first assertion is clear from the previous lemma, since $p_a(V, o) = p_a(Z)$. As to the second, we consider the exact sequence

$$0 \to \omega_Z \to \omega_{Z_K} \to O_Z(K_{Z_K}) \to 0$$

noting that $Z_K = 2Z$. By Lemma 1.3, $H^0(Z_K, K_{Z_K}) \to H^0(Z, K_{Z_K})$ is not the zero map if and only if $(V, o)$ is Gorenstein. Hence, $h^0(\omega_{Z_K})$ equals $p_a(Z) + 1$ when $(V, o)$ is Gorenstein and, otherwise, it equals $p_a(Z)$. Furthermore, $Z$ is the minimally even cycle, because $K_Z$ is nef being numerically equivalent to $-Z$. 

**Example 1.10.** Let $C$ be a non-singular projective curve of genus $g \geq 2$. Take an invertible sheaf $\mathcal{L}$ on $C$ with $\deg \mathcal{L} = 2g - 2$. We consider the $\mathbb{P}^1$-bundle $X := \mathbb{P}(O_C \oplus \mathcal{L})$ over $C$. Let $\varphi : X \to C$ be the projection and $H$ a tautological divisor on $X$ such that $\varphi_*\mathcal{O}_X(H) \simeq O_C \oplus \mathcal{L}$. We choose the unique element $Z \in |H - \varphi^*\mathcal{L}|$. Then $Z$ is isomorphic to $C$ and $Z^2 = -\deg \mathcal{L} = 2 - 2g < 0$. We contract $Z$ to obtain a normal surface singularity $(V, o)$. Since the canonical bundle of $X$ is induced by $-2H + \varphi^*(K_C + \mathcal{L})$ and it is numerically equivalent to $-2Z$ on $Z$, we see that $(V, o)$ is an even singularity. More precisely, we have $O_Z(K_X + 2Z) = O_Z(\varphi^*(K_C - \mathcal{L}))$. Hence, $(V, o)$ is Gorenstein if and only if $\mathcal{L} \simeq O_C(K_C)$. On the other extreme, it tells us that $O_Z(K_X + Z_K)$ is not necessarily of finite order.
Lemma 1.11. Let \((V, o)\) be an even singularity with \(p_f(V, o) = 2\). Then \(Z_K = 2Y\) on the minimal resolution, where \(Y = \sum_{i=1}^{m} D_i\) denotes the Yau cycle; \(D_1 = Z\), \(m = p_a(V, o) - 1\). Furthermore, the following hold.

(1) \(D_i\) is a 1-connected curve with \(D_i^2 = -2\) for \(1 \leq i \leq m\).

(2) \(D_m = Z_{\text{min}}\). Any component \(A \leq D_m\) satisfying \(AD_m < 0\) is not a \((-2)\)-curve.

(3) Put \(Y_o := \sum_{i=1}^{m} D_i\) for \(\nu \in \{1, 2, \ldots, m\}\). Then \(Y_o\) and \(2Y_o\) are the Yau cycle and the canonical cycle on \(\text{Supp}(Y_o)\), respectively. In particular, the singularity \((V_o, o_o)\) obtained by contracting \(Y_o\) is an even singularity with \(p_f(V_o, o_o) = 2\) and \(p_a(V_o, o_o) = m - \nu + 2\).

Proof. Since \(Y\) computes the arithmetic genus of \((V, o)\) as \(p_a(V, o) = p_a(Y) = m + 1\) by [6], we have \(Z_K = 2Y\) by the uniqueness of such a curve.

(1) For any \(i\), we have \(p_a(D_i) = 2\) by the definition of the Yau sequence. Hence 
\[2 = K_X D_i + D_i^2 = -2Y D_i + D_i^2 = -D_i^2,\]
because we have \(D_i D_j = 0\) when \(j \neq i\). Let \(\Gamma\) be any non-trivial subcurve of \(D_i\). Then 
\[-2 = D_i^2 = \Gamma^2 + (D_i - \Gamma)^2 + 2(D_i - \Gamma)\Gamma.\]
Since \(\Gamma^2\) and \((D_i - \Gamma)^2\) are negative even integers, we get \((D_i - \Gamma)\Gamma > 0\). Hence \(D_i\) is numerically 1-connected.

(2) For \(i < m\), \(O_{D_m}(-D_i)\) is numerically trivial. So, \(K_X + D_m\), which is numerically equivalent to \(-2Y + D_m\) on \(\pi^{-1}(o)\), is numerically equivalent to \(-D_m\) on \(D_m\). Since \(D_m\) is the fundamental cycle on its support, this shows that \(K_{D_m}\) is nef. Therefore, \(D_m\) is the minimal model of \(Z = D_1\). Assume that \(A\) is a \((-2)\)-curve and \(AD_m < 0\). It follows that \(D_m\) is reducible and we have \(0 < A(D_m - A) = AD_m + 2\) by the 1-connectivity of \(D_m\). Then \(A(D_m - A) = 1\), which implies that \(A\) is a \((-1)D_m\)-curve, contradicting that \(D_m\) is the minimal model.

(3) Since \(O_{D_j}(-D_i)\) is numerically trivial for \(i < j\), \(K_X\) is numerically equivalent to \(-2Y_o\) on \(\text{Supp}(Y_o)\). This is sufficient to imply the assertion. \(\square\)

2. Resolution dual graphs. In the sequel, unless otherwise stated explicitly, we let \((V, o)\) be an even singular point of fundamental genus 2. We denote by \(Z, Y\) and \(Z_K\) the fundamental cycle, the Yau cycle and the canonical cycle, respectively, on the minimal resolution \(\pi : X \to V\) of \((V, o)\).

The purpose of this section is to classify the weighted dual graph of the fundamental cycle \(Z\) (modulo some 2-connected curves).

Recall that \(Z\) is numerically 1-connected by Lemma 1.11 (1). Numerical decompositions and the canonical algebra of such a curve were extensively studied in [8]. If \(Z\) is 2-connected, then the canonical linear system \(|K_Z|\) is free from base points by [1, Proposition (A.7)]. If \(Z\) is not 2-connected, then, letting \(\Delta\) be a minimal subcurve of \(Z\) with respect to the condition that \(\Delta(Z - \Delta) = 1\), we see that \(\Delta\) is 2-connected and \(Z - \Delta\) is 1-connected ([1, Lemma (A.4)]). We have \(\Delta^2 = (Z - \Delta)^2 = -2\) by \(Z^2 = -2\), and \(p_a(\Delta) = p_a(Z - \Delta) = 1\). In particular, \(\omega_\Delta \simeq O_\Delta\) and it follows that \(O_\Delta(-\Delta)\) is nef. Then, \(\Delta\) is the fundamental cycle of a minimally elliptic singular point (cf. [7]).

2.1. Minimally even cycles. We first consider the case \(p_a(V, o) = 2\). Then, \(Z\) is a minimally even cycle (of fundamental genus 2) and vice versa, by Lemmas 1.9 and 1.11.

Lemma 2.1. Let \((V, o)\) be an even singularity with \(p_f(V, o) = p_a(V, o) = 2\). Then, \(O_X(-Z)\) is \(\pi\)-free if and only if \((V, o)\) is Gorenstein and \(Z\) is 2-connected.
Proof. Since $Z_K = 2Z$ and $H^1(X, -Z_K) = 0$ by the vanishing theorem, we see from the cohomology long exact sequence for

$$0 \to \mathcal{O}_X(-Z_K) \to \mathcal{O}_X(-Z) \to \mathcal{O}_Z(-Z) \to 0$$

that the restriction map $H^0(X, -Z) \to H^0(Z, -Z)$ is surjective. Since $\mathcal{O}_Z(-Z)$ is numerically equivalent to $K_Z$ and $Z$ is a 1-connected curve with $p_a(Z) = 2$, one easily see that the following three conditions are equivalent: (i) $h^0(Z, -Z) > 1$, (ii) $\mathcal{O}_Z(-Z) = \omega_Z$, (iii) $(V, o)$ is Gorenstein. [To show (ii) $\Rightarrow$ (iii), one may need the exact sequence

$$0 \to \omega_Z \to \omega_{Z_K} \to \mathcal{O}_Z(K_X + 2Z) \to 0$$

and Lemma 1.3.] So, we may assume that $(V, o)$ is Gorenstein.

If $Z$ is 2-connected, then $|\omega_Z|$ is free from base points by [1, Proposition (A.7)]. It follows that $\mathcal{O}_X(-Z)$ is $\pi$-free.

Assume that $Z$ is not 2-connected and take a minimal subcurve $\Delta$ of $Z$ with respect to the condition that $(Z - \Delta)\Delta = 1$. Then $\Delta$ is 2-connected and $Z - \Delta$ is 1-connected by [1, Lemma (A.4)]. Since $\mathcal{O}_\Delta(-Z)$ is nef of degree 1 and $\Delta$ is 2-connected with $p_a(\Delta) = 1$, we see that any non-zero element in $H^0(\Delta, -Z) \cong \mathbb{C}$ vanishes at one non-singular point $p \in \Delta$ by [1, Proposition (A.5)]. This implies that $p$ is a base point of $|\mathcal{O}_Z(-Z)|$. It also means that $p$ is a base point of $|\mathcal{O}_X(-Z)|$.

Suppose that $Z$ is not 2-connected and take a subcurve $\Delta$ as above. Since $\omega_\Delta \simeq \mathcal{O}_\Delta$, we have $\mathcal{O}_\Delta(K_X) \simeq \mathcal{O}_\Delta(-\Delta)$. From

$$0 \to \omega_{Z - \Delta} \to \omega_Z \to \mathcal{O}_\Delta(K_X + Z) \to 0,$$

$h^0(\omega_Z) = 2$ and $h^0(\omega_{Z - \Delta}) = 1$, the restriction map $H^0(Z, \omega_Z) \to H^0(\Delta, K_X + Z)$ is non-trivial. Since $\mathcal{O}_\Delta(K_X + Z) \simeq \mathcal{O}_\Delta(Z - \Delta)$ is nef of degree 1 and $\Delta$ is 2-connected, we see that $\mathcal{O}_\Delta(Z - \Delta) \simeq \mathcal{O}_\Delta(x)$ with a non-singular point $x$ of $\Delta$. Then, either $\Delta \cap (Z - \Delta) = \{x\}$ or $\Delta \prec Z - \Delta$. Note that we have $\mathcal{O}_\Delta(Z - \Delta) \equiv \mathcal{O}_\Delta(-Z)$, where $\equiv$ means the numerical equivalence. Let $A$ be the unique irreducible component of $\Delta$ through $x$.

**Lemma 2.2.** Let the notation be as above. Either $\Delta$ is irreducible and reduced, or $A$ is a $(−4)$-curve and $\Delta − A$ is the fundamental cycle of a rational double point. In the latter case, an irreducible component $B$ of $\Delta − A$ satisfies $AB > 0$ if and only if $B(\Delta − A) < 0$.

**Proof.** Since $K_X \equiv -2Z$, we have $\mathcal{O}_\Delta(K_X) \equiv \mathcal{O}_\Delta(2x)$. Then, $K_X A = -2ZA = 2$. It follows that either $p_a(A) = 1$ and $A^2 = -2$ (so $\Delta = A$), or $A$ is a $(−4)$-curve. Assume the latter. Then $\Delta A = -2$, $K_X(\Delta - A) = 0$ and $(\Delta - A)^2 = -2$. Hence $\Delta - A$ is a 1-connected curve consisting of $(−2)$-curves. Since $\mathcal{O}_{\Delta - A}(-(\Delta - A)) \simeq \mathcal{O}_{\Delta - A}(A)$ is nef, $\Delta - A$ is the fundamental cycle on its support. The last assertion for $B$ is clear from $\mathcal{O}_{\Delta - A}(\Delta) \simeq \mathcal{O}_{\Delta - A}$. Since $(\Delta - A)A = 2$, there are at most two irreducible components of $\Delta - A$ meeting $A$. [7]

**Remark 2.3.** The possible dual graphs of $\Delta$’s are in one-to-one correspondence with Kodaira’s list of non-multiple fibers in elliptic surfaces [2, Theorem 6.2], with an obvious modification in the reducible case: choose a component of multiplicity one and replace it by a $(−4)$-curve. See also [7].

We turn our attention to the 2-disconnected $Z$ itself.
The A-D-E classification, we see that $\Gamma = 0$. So, if $\Gamma = 0$ for a non-singular point $y \in \Delta$. We have $\Delta \Delta = 0.1$. If $\Delta \Delta = 1$, then $\Gamma^2 = 0$ and, thus, $\Gamma = 0$. In this case, we have $Z = \Delta + \Delta_1$ and $x = y$. If $\Delta_1 = 0$, then $\Delta \cup \Delta_1 = \emptyset$ and $\Gamma^2 = -2$. Since $K_X \Gamma = 0$, $\Gamma$ is a $(-2)$-cycle. We have $\Delta \cap \Gamma = \{x\}$ and $\Delta_1 \cap \Gamma = \{y\}$. Since $O_\Gamma(-\Gamma) \simeq O_\Gamma(-Z + \Delta + \Delta_1) \simeq O_\Gamma(x + y)$, $\Gamma$ is the fundamental cycle of a rational double point of type $A_\ell$ for some $\ell$. The dual graph of $Z$ would be as in Fig. 1 if we consider $\Delta$ and $\Delta_1$ as if they were irreducible curves. It is clear that $\Gamma$ is the fixed part of $|K_Z|$, and that $O_{Z - \Gamma}(-Z - \Gamma) \simeq O_{Z - \Gamma}, h^0(Z - \Gamma, O_{Z - \Gamma}) = 2$.

**Fig. 1.**

**Case 2.** Assume that $\Delta < Z - \Delta$. Then $\Delta$ is the minimal model of $Z - \Delta$. We have $(Z - 2\Delta)^2 = -6$. It follows from $K_X(Z - 2\Delta) = -2Z(Z - 2\Delta) = 0$ that $Z - 2\Delta$ consists of $(-2)$-curves. Note also that $O_{\Delta - A}(-\Delta) \simeq O_{\Delta - A}$. In particular, we see that $\Delta$ is nef on $Z - 2\Delta$, even when $Z - 2\Delta$ has a common component with $\Delta$. Then, by Proposition 1.2, $Z - 2\Delta$ decomposes as $\Gamma_1 + \Gamma_2 + \Gamma_3$, where each $\Gamma_i$ is the fundamental cycle on its support, $\Gamma_i^2 = -2$, $\Delta \Gamma_i = 1$ and $O_{\Gamma_i}(-\Gamma_i) \simeq O_{\Gamma_i}$ when $i < j$. We have the following three cases after re-labeling if necessary:

(a) $\Gamma_1, \Gamma_2$ and $\Gamma_3$ are mutually disjoint,
(b) $\Gamma_2 \prec \Gamma_1$ and $\Gamma_1 \cap \Gamma_3 = \emptyset$,
(c) $\Gamma_3 \prec \Gamma_2 \prec \Gamma_1$.

**Lemma 2.4.** For each $i = 1, 2, 3$, $\Gamma_i$ is the fundamental cycle of a rational double point. If $\Gamma_i$ is a minimal curve (with respect to $\preceq$) in $\{\Gamma_1, \Gamma_2, \Gamma_3\}$, then it is a $(-2)$-curve that is not a component of $\Delta$. The possible Dynkin types (type$(\Gamma_1)$, type$(\Gamma_2)$, type$(\Gamma_3)$) are as follows:

(a) $(A_1, A_1, A_1)$.
(b) $(A_3, A_1, A_1)$ or $(D_1, A_1, A_1)$, $i \geq 4$.
(c) $(A_5, A_3, A_1)$ or $(D_6, D_4, A_1)$.

**Proof.** For each $i$, since $O_{\Gamma_i}(\Delta)$ is nef of degree 1, there exists a unique component $C_i$ of $\Gamma_i$, along which $\Gamma_i$ is of multiplicity one, such that $C_i \Delta = 1$ and $\Delta(\Gamma_i - C_i) = 0$. Recall that $Z - 2\Delta = \Gamma_1 + \Gamma_2 + \Gamma_3$, $O_{\Gamma_i}(Z) \simeq O_{\Gamma_i}$ and $O_{\Gamma_i}(-\Gamma_i) \simeq O_{\Gamma_j}$ when $i < j$. So, if $\Gamma_i$ is a minimal curve in $\{\Gamma_1, \Gamma_2, \Gamma_3\}$, then $O_{\Gamma_i}(2\Delta) \simeq O_{\Gamma_i}(-\Gamma_i)$. Hence $C_i \Gamma_i = -2$ and it follows $\Gamma_i = C_i$, since $(\Gamma_i - C_i)^2 = 0$. Note then that $\Gamma_i$ is not a component of $\Delta$, since $\Delta$ is the fundamental cycle and $\Delta \Gamma_i = 1$.

We restrict ourselves to the case (c), since the other cases are similar and easier. In this case, we have $C_1 = C_2 = C_3$, and $\Gamma_3 = C_3$ as we saw above. We have $O_{\Gamma_2}(-\Gamma_2) \simeq O_{\Gamma_2}(2\Delta + \Gamma_3)$ which is trivial on $\Gamma_3$. So, $\Gamma_2$ has a non-multiple component $\Gamma_3$ and any component $B$ with $B \Gamma_2 < 0$ has to meet $\Gamma_3$. From this property and the A-D-E classification, we see that $\Gamma_2$ is of type $A_3$ or $D_1$. Similarly, we have $O_{\Gamma_1}(-\Gamma_1) \simeq O_{\Gamma_1}(2\Delta + \Gamma_2 + \Gamma_3)$ which is trivial on $\Gamma_2$. Then $\Gamma_1$ has a non-multiple component $\Gamma_3$ and any component $B$ with $B \Gamma_1 < 0$ satisfies $B(\Gamma_2 + \Gamma_3) > 0$. From
this, we see that \( \Gamma_1 \) is of type \( A_5 \) when \( \Gamma_2 \) is of type \( A_3 \). When \( \Gamma_2 \) is of type \( D_1 \), it is not so hard to see that \( \Gamma_2 \) is of type \( D_4 \) and \( \Gamma_1 \) is of type \( D_6 \). \( \Box \)

Since \( \mathcal{O}_D(-Z) \equiv \mathcal{O}_D(x) \) and \( \mathcal{O}_D(-\Delta) \equiv \mathcal{O}_D(2x) \), we get \( \mathcal{O}_D(\Gamma_1 + \Gamma_2 + \Gamma_3) \equiv \mathcal{O}_D(3x) \). So, \( A(\Gamma_1 + \Gamma_2 + \Gamma_3) = 3 \) and \( \mathcal{O}_{D-A}(\Gamma_1 + \Gamma_2 + \Gamma_3) \simeq \mathcal{O}_{D-A} \). We have either \( (\Delta - A) \cap (\Gamma_1 + \Gamma_2 + \Gamma_3) = \emptyset \) or \( \Delta - A \preceq \Gamma_1 + \Gamma_2 + \Gamma_3 \), because \( \Delta - A \) is 1-connected even if \( \Delta \neq A \). When \( (\Delta - A) \cap (\Gamma_1 + \Gamma_2 + \Gamma_3) = \emptyset \), the dual graph of \( Z \) would be as in Fig. 2 if we treat \( \Delta \) as if it were a single elliptic curve.

Assume that \( \Delta - A \preceq \Gamma_1 + \Gamma_2 + \Gamma_3 \) (\( \Delta \neq A \)). We cannot have the case (a), because, if (a) is the case, each \( \Gamma_i \) is a \((-2\)-curve which is not a component of \( \Delta \) by Lemma 2.4.

If (b) is the case, then Lemma 2.4 implies that \( \Gamma_2 \) and \( \Gamma_3 \) are \((-2\)-curves not contained in \( \Delta \), because they are minimal curves in \( \{ \Gamma_i \}_{i=1}^3 \). So, \( \Delta - A \preceq \Gamma_1 + \Gamma_2 \). We have \( \partial \Gamma_3 = 1 \) by \( (\Delta - A) \cap \Gamma_3 \subseteq (\Gamma_1 - \Gamma_2) \cap \Gamma_3 = \emptyset \) and \( \Delta \Gamma_3 = 1 \). Since \( 2 = (\Delta - A)A \preceq (\Gamma_1 - \Gamma_2)A \) and \( A(\Gamma_1 + \Gamma_2) = 2 \), we get \( \partial \Gamma_1 = 2 \), \( \partial \Gamma_2 = 0 \). So, \( A \cap \Gamma_2 = \emptyset \) and \( (\Delta - A)\Gamma_2 = 1 \). There exists a component \( B \preceq \Delta - A \) of multiplicity one with \( B\Gamma_2 = 1 \) and \( B\Gamma_1 = -1 \). Since \( \mathcal{O}_{\Gamma_1 - \Gamma_2}(\Delta) \) is nef of degree 0, any chain-connected subcurve of \( \Gamma_1 - \Gamma_2 \) is either a subcurve of \( \Delta \) or disjoint from \( \Delta \). This implies that \( \Delta - A \) is nothing but a chain-connected component of \( \Gamma_1 - \Gamma_2 \). By using this fact, one can easily see the following by considering the CCCD of \( \Gamma_1 - \Gamma_2 \): If \( \Gamma_1 \) is of type \( A_3 \), then \( \Delta - A = B \) is a \((-2\)-curve and, either \( A \) meets \( B \) normally at two distinct points or \( A \) contacts \( B \) at a point to the second order. If \( \Gamma_1 \) is of type \( D_1 \), then \( \Delta - A \) is either \( B \) or \( \Gamma_1 - \Gamma_2 - B \) which is of type \( D_{-1} \). Here, we cannot have \( \Delta - A = B \), because, if so, we have \( \partial A(Z) = 2 \), \( \partial B(Z) = 2\partial B(\Delta) + \partial B(\Gamma_1) = 4 \) and \( AB = 2 \) which leads us to a contradiction: \( AZ = A(2A + 4B + 3A^3) = 1 \).

If (c) is the case, then \( \Delta - A \preceq (\Gamma_1 - \Gamma_3) + (\Gamma_2 - \Gamma_3) \) and it follows \( A(\Gamma_1 - \Gamma_3) + A(\Gamma_2 - \Gamma_3) \geq A(\Delta - A) = 2 \). From this and \( A(\Gamma_1 + \Gamma_2 + \Gamma_3) = 3 \), we infer that \( \partial \Gamma_1 = 2 \), \( \partial \Gamma_2 = 1 \) and \( \partial \Gamma_3 = 0 \). Also in this case, any chain-connected component of \( \Gamma_1 - \Gamma_3 \) is a subcurve of \( \Delta \) or disjoint from \( \Delta \). So, one can conclude similarly as in the previous case: If \( \Gamma_1 \) is of type \( A_5 \), then \( \Delta - A \) is of type \( A_2 \). If \( \Gamma_1 \) is of type

![Fig. 2. Dual graph of Z when gcd(\( \Delta \), \( Z - 2\Delta \)) = 0](image)
D_6, then \( \Delta - A \) is of type A_5.

In each case, the possible dual graph of \( Z \) is as in Fig. 3 in which two circles connected by a line with 2 means that the corresponding two curves contact at a point to the second order.

\[
\begin{array}{c}
\text{(b)} \\
\begin{array}{c}
\text{\includegraphics{diagram}}
\end{array}
\end{array}
\]

**Fig. 3. Dual graph of \( Z \) when \( \gcd(\Delta, Z - 2\Delta) \neq 0 \)**

We continue to assume that \( \Delta < Z - \Delta \) and study the base locus of \( |\omega_Z| \) closely. It follows from

\[
0 \to \omega_\Delta \to \omega_Z \to \mathcal{O}_{Z - \Delta}(K_X + Z) \to 0
\]

that \( H^0(Z, \omega_Z) \to H^0(Z - \Delta, K_X + Z) \) is surjective, because \( H^1(\omega_\Delta) \to H^1(\omega_Z) \) is injective being the dual map of \( H^0(Z, \mathcal{O}_Z) \to H^0(\Delta, \mathcal{O}_\Delta) \). Take a non-zero section \( s \in H^0(Z - \Delta, K_X + Z) \simeq \mathbb{C} \). If \( s \) does not vanish identically on any components of \( Z - \Delta \), then \( s \) vanishes only at \( x \in \Delta \). Hence \( x \) is an isolated base point of \( |K_Z| \) in this case. So, we assume that \( s \) vanishes identically on some component and let \( \Gamma \) be the biggest curve on which \( s \) vanishes identically. Then \( \mathcal{O}_{Z - \Delta - \Gamma}(K_X + Z - \Gamma) \equiv \mathcal{O}_{Z - \Delta - \Gamma}(-Z - \Gamma) \) is nef. In particular, \( \deg \mathcal{O}_{Z - \Delta - \Gamma}(-Z) \geq (Z - \Delta - \Gamma)\Gamma \). Recall that \( \mathcal{O}_{Z - \Delta}(-Z) \) is nef of degree 1 and \( Z - \Delta \) is 1-connected. Therefore, we have \( \deg \mathcal{O}_{Z - \Delta - \Gamma}(-Z) = (Z - \Delta - \Gamma)\Gamma = 1 \). Then \( \Gamma \) is a 1-connected curve on which \( Z \) is numerically trivial. This shows that \( \Gamma \) is a \((-2)\)-cycle. We also know that \( Z - \Delta - \Gamma \) is 1-connected. We have \( \Delta \preceq Z - \Delta - \Gamma \), since \( \Delta \) must be the minimal model of the 1-connected curve \( Z - \Delta - \Gamma \) with \( p_a(Z - \Delta - \Gamma) = 1 \). Furthermore, since \( s \) induces a nowhere vanishing section of \( \mathcal{O}_{Z - \Delta - \Gamma}(K_X + Z - \Gamma) \), we see that \( \mathcal{O}_{Z - \Delta - \Gamma}(K_X + Z - \Gamma) \simeq \mathcal{O}_{Z - \Delta - \Gamma} \). Note that \( s \) is constant on every \( \Gamma_i \). Therefore, \( \Gamma \) is one of the maximal curves in \( \{ \Gamma_1, \Gamma_2, \Gamma_3 \} \) and \( x \in \Gamma \cap \Delta \).
Lemma 2.5. \( \omega_{Z-\Gamma} \simeq \mathcal{O}_{Z-\Gamma} \) and \( h^0(Z-\Gamma, \mathcal{O}_{Z-\Gamma}) = 2 \).

Proof. Consider the cohomology long exact sequence for

\[
0 \to \omega_{Z-\Gamma} \to \omega_Z \to \mathcal{O}_Z(-Z) \to 0.
\]

Since \( H^0(Z, \omega_Z) \to H^0(\Gamma, -Z) \) is the zero map, we get \( h^0(Z-\Gamma, \omega_{Z-\Gamma}) = h^0(Z, \omega_Z) = 2 \). We already know that \( \omega_{Z-\Gamma} \) is trivial on \( Z - \Delta - \Gamma \). Since \( \Delta \leq Z - \Delta - \Gamma \), we see that \( \omega_{Z-\Gamma} \) is numerically trivial. Then, by the Riemann-Roch theorem, we get \( h^1(Z-\Gamma, \omega_{Z-\Gamma}) = 2 \). By the Serre duality theorem, we get \( h^0(Z-\Gamma, \mathcal{O}_{Z-\Gamma}) = 2 \).

It remains to show that \( \omega_{Z-\Gamma} \simeq \mathcal{O}_{Z-\Gamma} \). For this purpose, we consider the cohomology long exact sequence for

\[
0 \to \omega_\Delta \to \omega_{Z-\Gamma} \to \mathcal{O}_{Z-\Delta-\Gamma} \to 0.
\]

We have \( h^0(\omega_\Delta) = h^0(Z-\Delta-\Gamma, \mathcal{O}) = 1 \). It follows that the restriction map \( H^0(Z-\Gamma, \omega_{Z-\Gamma}) \to H^0(Z-\Delta-\Gamma, \mathcal{O}_{Z-\Delta-\Gamma}) \) is surjective. Since \( \Delta \leq Z - \Delta - \Gamma \), we see that there is a nowhere vanishing section in \( H^0(Z-\Gamma, \omega_{Z-\Gamma}) \). Therefore, \( \omega_{Z-\Gamma} \simeq \mathcal{O}_{Z-\Gamma} \). \( \square \)

Lemma 2.6. With the above notation, \( Z - \Gamma \) is the canonical cycle of a maximally elliptic singularity of geometric genus two.

Proof. We shall show that \( Z-\Gamma = (Z-\Delta-\Gamma) + \Delta \) gives us the CCCD, i.e., the elliptic sequence. Recall that \( Z-\Delta-\Gamma \) and \( \Delta \) are 1-connected and \( \mathcal{O}_\Delta(-(Z-\Delta-\Gamma)) \simeq \mathcal{O}_\Delta(-(K_X+Z-\Gamma)) \simeq \mathcal{O}_\Delta \).
So it remains to show that \( Z-\Delta-\Gamma \) is the fundamental cycle on its own support. If \( C \leq Z - \Delta - \Gamma \) is a component such that \( C \not\simeq \Delta \), then \( C \Delta \geq 0 \) and, hence, \( C(Z - \Delta - \Gamma) = 2ZC - C\Delta \leq 2ZC \leq 0 \). This implies that \( \mathcal{O}_{Z-\Delta-\Gamma}(-(Z-\Delta-\Gamma)) \) is nef and, thus, \( Z-\Delta-\Gamma \) is the fundamental cycle on its support. Since the geometric genus \( h^0(Z-\Gamma, \mathcal{O}) = 2 \) coincides with the length of the CCCD, \( Z - \Gamma \) contracts to a maximally elliptic singularity of geometric genus 2. \( \square \)

Now, as a summary, we state a rough classification of the minimal even cycles according to the base locus of the canonical linear system and the numerical decomposition:

Proposition 2.7. Minimally even cycles \( Z \) of fundamental genus 2 fall into the following five classes (0), (i.a), (i.b), (ii.a) and (ii.b).

(0) \(|\omega_Z| \) is free from base points and \( Z \) is 2-connected.

(i) \(|\omega_Z| \) has an isolated base point \( x \).

  (i.a) \( Z = \Delta + \Delta_1, \Delta \cap \Delta_1 = \{x\} \).

  (i.b) \( Z = 2\Delta + \Gamma_1 + \Gamma_2 + \Gamma_3 \), the base point \( x \in \Delta \) is a non-singular point of \( Z_{\text{red}} \).

(ii) The fixed part \( \Gamma \) of \(|\omega_Z| \) is the fundamental cycle of a rational double point.

  (ii.a) \( Z = \Delta + \Delta_1 + \Gamma \), where the \( \Gamma \) is a simple chain of \((-2)\)-curves (of type (A)) with \( \Delta \cap \Gamma = \{x\} \) and \( \Delta_1 \cap \Gamma = \{y\} \).

  (ii.b) \( Z = 2\Delta + \Gamma_1 + \Gamma_2 + \Gamma_3 \), \( \Gamma \) is a maximal curve in \( \{\Gamma_1, \Gamma_2, \Gamma_3\} \).

We remark that, when \((V,o)\) is Gorenstein, the above substantially gives us the classification of the maximal ideal cycles.
2.2. General case. Let us consider the case \( m \geq 2 \).

Proposition 2.8. Let \((V, o)\) be an even singularity with \( p_f(V, o) = 2 \) and \( p_a(V, o) = m + 1 \geq 3 \). If \( Y = \sum_{i=1}^{m} D_i \) denotes the Yau cycle, then \( D_1 - D_m \) decomposes into a numerically disjoint sum of two fundamental cycles \( \Delta_1, \Delta_2 \) of rational double points: \( D_1 - D_m = \Delta_1 + \Delta_2, \ D_m \Delta_1 = 1, \mathcal{O}_{\Delta_2}(-\Delta_1) \cong \mathcal{O}_{\Delta_2} \). The possible types of \( \Delta_1, \Delta_2 \) are as follows.

\[
(2A_{m-1}) : \Delta_1 \text{ and } \Delta_2 \text{ are both of type } A_{m-1}, \text{Supp}(\Delta_1) \cap \text{Supp}(\Delta_2) = \emptyset. \\
(A_{2m-1}, A_{2m-3}) : \Delta_i \text{ is of type } A_{2m-2i-1} \ (i = 1, 2) \text{ and } \Delta_2 < \Delta_1. \\
(E_6, A_5) : m = 3, \Delta_1 \text{ is of type } E_6, \Delta_2 \text{ is of type } A_5 \text{ and } \Delta_2 < \Delta_1. \\
(D_l, A_1) : m = 2, \Delta_1 \text{ is of type } D_l \text{ for some } l \geq 4, \Delta_2 \text{ is of type } A_1 \text{ and } \Delta_2 < \Delta_1.
\]

Proof. Suppose that \( m \geq 2 \). We have \( K_X(D_1 - D_m) = 0 \), since \( p_a(D_1) = p_a(D_m) = -D_1^2 = -D_m^2 = 2 \) by Lemma 1.11 (1). Hence \( D_1 - D_m \) consists of \((-2)\)-curves. Let \( A \) be a component of \( D_1 - D_m \) and let \( i = i(A) \) be the biggest index such that \( A \leq D_i \). Then \( A(D_i + D_{i+1}) = 0 \) by (1.2) and \( 0 = K_X A = -2YA \).

We have \((D_1 - D_m)^2 = -4 \) by \( D_1 D_m = 0 \). It follows from Proposition 1.2 that \( D_1 - D_m \) decomposes as \( D_1 - D_m = \Delta_1 + \Delta_2, \ D_m \Delta_1 = 1, \Delta_2^2 = -2 \) and \( \mathcal{O}_{\Delta_2}(-\Delta_1) \cong \mathcal{O}_{\Delta_2} \). Since \( \mathcal{O}_{\Delta_2}(-\Delta_1) \) is nef of degree one, there exists a non-multiple component \( C \) of \( \Delta_2 \) with \(-CD_1 = 1\). Then \( CD_1 = 0 \) for \( i \geq 3 \) by (1.2) and \( CD_2 = 1 \).

We have \( 0 > C(D_1 - D_m) = C\Delta_1 + C\Delta_2 = C\Delta_2 \) by \( CD_m \geq 0 \) and \( \mathcal{O}_{\Delta_2}(-\Delta_1) \cong \mathcal{O}_{\Delta_2} \). From the A-D-E classification, we see that \( \Delta_2 \) is of Dynkin type \( A_r \) for some \( r \). Furthermore, we have \( r = 1 \) (i.e., \( C\Delta_2 = -2 \)) if and only if \( CD_m = 1 \), i.e., \( m = 2 \).

Assume that \( \Delta_1 \) and \( \Delta_2 \) are disjoint. Then the same argument as above shows that \( \Delta_1 \) is of Dynkin type \( A_s \) for some \( s \). Let \( C' \) be a component of \( \Delta_1 \) with \(-C'D_1 = 1 \). Then \( C \) and \( C' \) are \((-1)D_1\)-curves. It follows that \( D_1 - C - C' \) is a chain-connected curve with \( p_a(D_1 - C - C') = 2 \). Furthermore, \( \mathcal{O}_{D_1 - C - C'}(-\Delta_1) \) is numerically trivial by \( D_1^2 = -2 \). Hence, \( D_2 = D_1 - C - C' \). Now, we can repeat the argument inductively for \( D_i - D_m \) \((i = 1, \ldots, m - 1)\) to find \( r = s = m - 1 \). This gives \((2A_{m-1})\).

Assume that \( \Delta_2 \prec \Delta_1 \). Then it follows from Proposition 1.2 (3) that \( \Delta_1 \) is of type either \( A_{r+2} \) or \( D_l \) \((r = 1, l = 5, r = 3, l = 6) \) or \( E_6, A_5 \) \((r = 5) \). In this case, \( C \) is of multiplicity 2 in \( D_1 - D_2 \) and \( D_2 \leq D_1 - 2C \). It is easy to see that \(-D_1 - D_2 \) is nef on \( D_1 - D_2 \). Put \( \Gamma = D_1 - C - D_2 \). Then \( \Gamma^2 = -2 \) and \( CT = 0 \). Since \( \Gamma \) is the fundamental cycle on its support, as in Proposition 1.2 (3), we see that \( \Gamma \) is of type either \( A_3 \) or \( D_{l'} \) for some \( l' \geq 4 \). From this, we inductively see that \( r = 2m - 3 \) when \( \Delta_1 \) is of type \( A_{r+2}, m = 3 \) when \( \Delta_1 \) is of type \( E_6, \) and that \((D_5, A_3)\) is impossible.

The weighted dual graphs corresponding to the cases in the above Proposition are as in Figures 4–6. Note that, though \( D_m \) is treated as if it were a single curve and \( D_m \) has no common component with \( \Delta_1 + \Delta_2 \) in the figures, in reality \( D_m \) may well be reducible and, moreover, \( D_m \) and \( \Delta_1 + \Delta_2 \) may have a common component.

Example 2.9. Each can be realized by a double point with \( D_m \) non-singular. Sample equations are as follows.

\[
(2A_{m-1}) : \ z^2 = x^6 + y^{6m} \\
(A_{2m-1}, A_{2m-3}) : \ z^2 = x(y + x^{m-1})(y^5 - x^{5(m-1)}), \ m \geq 3 \\
(E_6, A_5) \ (m = 3) : \ z^2 = (x^2 - y^3)(x^5 - y^{15}) \\
(D_l, A_1) \ (m = 2) : \ z^2 = (x^{10} - y^5)(x^2 - y^{l-1}), \ l \geq 3
\]
Not all combinations are possible and we cannot attach an arbitrary type of 
\((\Delta_1, \Delta_2)\) to a given minimally even cycle \(D_m\). We call a curve \(D\) essentially irreducible, 
if there is a non-multiple irreducible component \(C\) of \(D\) such that \(D - C\) consists of 
\((-2)\)-curves at most.

**Lemma 2.10.** Assume that \(p_a(V, o) = m + 1 \geq 3\) and put \(D_1 - D_m = \Delta_1 + \Delta_2\) 
as above.

1. If \(A\) is an irreducible component of \(D_m\), then \(AD_m < 0\) if and only if \((D_{m-1} - D_m)A > 0\).

2. Every irreducible component \(B\) of \(D_{m-1}\) satisfying \(BD_{m-1} < 0\) meets \(D_m\) at 
one non-singular point of a non-multiple component \(C\) of \(D_m\). If \(CD_m = 0\), then \(C\) 
is a \((-2)\)-curve and \(C \leq D_{m-1} - D_m\).

3. If \((\Delta_1, \Delta_2)\) is of type \((2A_{m-1})\), then \(D_m\) is of type either \((0), (i.a)\) or \((ii.a)\). 
If it is of type \((E_6, A_5)\), then \(D_m\) is of type \((0)\) and is essentially irreducible. If it is 
of type either \((A_{2m-1}, A_{2m-3})\) or \((D_1, A_1)\), then \(D_m\) is of type \((0)\) and is essentially 
irreducible, unless \(D_m\) and \(D_{m-1} - D_m\) have a common component.

**Proof.** (1) is clear. In fact, we have \((D_{m-1} - D_m)A = -D_m A\).

(2), (3): We compare \(D_{m-1}\) and \(D_m\). We take an irreducible component \(B\) of 
\(D_{m-1}\) satisfying \(BD_{m-1} < 0\). Since \(B\) is not a component of \(D_m\), it is a \((-2)\)-curve. 
Then \(0 = K_B - 2YB = 0\), from which we get \((D_{m-1} + D_m)B = 0\). Furthermore, 
since \(D_{m-1}\) is a reducible numerically 1-connected curve, we get \(1 \leq (D_{m-1} - B)B = D_mB + 2\) and, thus, \(BD_{m-1} = -1\) and \(BD_m = 1\). Then, \(D_m\) has a unique non-
multiple component \(C\) meeting \(B\) at a point. If \(CD_m = 0\), then \((D_{m-1} - D_m)C = 0\) 
and \(C\) is a component of \(D_{m-1} - D_m\) (since \(BC > 0\) and \(B \leq D_{m-1} - D_m\)). In 
particular, \(C\) is a \((-2)\)-curve and \(B + C\) is the configuration of type \(A_2\) in this case.
We also remark that we have \(D_m \cap (D_1 - D_{m-1}) = \emptyset\) for \(m \geq 3\) by (1.2) and, thus, 
\(C\) cannot be a component of \(D_1 - D_{m-1}\). From these, one easily see that the case 
\(CD_m = 0\) cannot happen when \((\Delta_1, \Delta_2)\) is of type either \((2A_{m-1})\) or \((E_6, A_5)\).

Assume that \(CD_m < 0\). Then, \(D_m\) cannot be of types \((i.b), (ii.b)\) because \(D_m\) 
has a non-multiple component \(C\) which is not a \((-2)\)-curve. This is the case when
$D_1 - D_m$ is of type either $(2A_{m-1})$ or $(E_6, A_5)$ as we saw above. Except when $(\Delta_1, \Delta_2)$ is of type $(2A_{m-1})$, $B$ as above is unique and it is a component of multiplicity 2 of $D_{m-1}$. Then $(D_{m-1} - D_m)C \geq 2BC = 2$, i.e., $-D_mC \geq 2$. This implies that $D_m$ is essentially irreducible, since the other components, if any, are $(-2)$-curves by $4 = K_X D_m = -2D_mD_m \geq -2D_mC \geq 4$. In particular, when $(E_6, A_5)$, the only possible type of $D_m$ is $(0)$ which is essentially irreducible.

We assume that $CD_m = 0$. Assume that $D_{m-1} - D_m$ is of type $(A_3, A_1)$. Then $C$ is one of the end components of $A_3$. As in (1) does not meet $B$, but meets $C$ or another end component $C'$ of $A_3$. When $m \geq 3$, $C$ has to be the end component of $A_{2m-1}$ adjacent to $B$, and $A$ meets $C$. If $D_{m-1} - D_m$ is of type $(D_l, A_1)$, then $m = 2$ and $C$ have to be the component adjacent to $B$ in the $D_l$ configuration.

If $D_1 - D_2$ is of type $(D_l, A_1)$ and $CD_2 = 0$, then $C$ is a component of multiplicity 3 in $Z$. From this, one immediately sees that the only possible case is $l = 4$ and $D_2$ is of type (ii.a) as in Fig. 8 by the nefness of $-Z$ (examined on the $D_l$ configuration).

In Fig. 9, we give an example of the case where $D_m$ and $D_1 - D_m$ have a common component, where $D_m$ is of type (ii.b) (see, Fig. 2 (a)) and $D_1 - D_m$ is of type $(A_{2m-1}, A_{2m-2})$.

3. A geometric genus formula. The purpose of this section is to establish a formula computing the geometric genus of the even singular point $(V, o)$ of fundamental genus 2, modeled on Okuma's formula [11] for elliptic singularities.

Recall that $Z_K = 2Y = 2(D_1 + \cdots + D_m)$. By (1.2), $\mathcal{O}_{D_m}(Z) \simeq \mathcal{O}_{D_m}(D_l)$ for
1 \leq i \leq m - 1. We put \( Y_i = \sum_{j=1}^{m} D_j \) and

\[ F_{i-1} = Y - Y_i = \sum_{j=1}^{i-1} D_j \] (3.1)

for \( 1 \leq i \leq m \). Then \( Y_i \) is the Yau cycle on \( \text{Supp}(D_i) \) and, as shown in [6, Lemma 1.3], \(-F_{i-1}\) is \( \pi \)-nef for \( 1 \leq i \leq m \). Note that \( \mathcal{O}_{Y_i}(F_{i-1}) \) is numerically trivial.

We consider two subsets of \( \{1, 2, \ldots, m\} \) given by

\[
\mathcal{A} = \{ i \mid \mathcal{O}_{2Y_i-D_i}(F_{i-1}) \cong \mathcal{O}_{2Y_i-D_j}, 1 \leq i \leq m \}, \\
\mathcal{B} = \{ i \mid \omega_{2Y_i}(F_{i-1}) \cong \mathcal{O}_{2Y_i}, 1 \leq i \leq m \}. \tag{3.2}
\]

We always have \( 1 \in \mathcal{A} \), and we have \( 1 \in \mathcal{B} \) if and only if \((V, o)\) is Gorenstein.

**Lemma 3.1.** The geometric genus of an even singularity \((V, o)\) with \( p_f(V, o) = 2 \) and \( p_a(V, o) = m + 1 \) is given by \( p_g(V, o) = m + \# \mathcal{A} + \# \mathcal{B} \), where \( \# T \) denotes the cardinality of the set \( T \).

**Proof.** First, we consider the case \( m = 1 \). Then \( Y = D_1 = Z \) and \( Z_K = 2Z \).

It follows from [14] or [5] that \( p_g(V, o) = 2p_f(V, o) + 1 = 3 \) with equality holding only when \((V, o)\) is Gorenstein. If \((V, o)\) is Gorenstein, then \( 2 = p_a(V, o) < p_g(V, o) \) by Lemma 1.4. Hence we have \( p_g(V, o) = 3 \) if and only if \((V, o)\) is Gorenstein and, otherwise, we have \( p_g(V, o) = 2 \). Hence the formula for \( p_g(V, o) \) in the statement holds for \( m = 1 \).

Next, we consider the case \( m \geq 2 \). Put \( \omega_{Z_K} = \mathcal{O}_{Z_K}(\kappa) \). For \( i = 1, 2, \ldots, m \), consider the cohomology long exact sequence for

\[ 0 \to \omega_{2Y_i-D_i}(F_{i-1}) \to \omega_{2Y_i}(F_{i-1}) \to \mathcal{O}_{D_i}(\kappa - F_{i-1}) \to 0. \]

Note that \( \omega_{2Y_i}(F_{i-1}) \) is numerically trivial. Since \( \text{Supp}(Y_i) \) is connected and \( D_i \) is the chain-connected component of \( 2Y_i \), the restriction map \( H^0(2Y_i, \omega_{2Y_i}(F_{i-1})) \to H^0(D_i, \kappa - F_{i-1}) \) is not the zero map if and only if \( \omega_{2Y_i}(F_{i-1}) \cong \mathcal{O}_{2Y_i} \). Therefore,

\[
h^0(2Y_i, \omega_{2Y_i}(F_{i-1})) = h^0(2Y_i - D_i, \omega_{2Y_i-D_i}(F_{i-1})) + \begin{cases} 1, & \text{if } i \in \mathcal{B}, \\ 0, & \text{if } i \not\in \mathcal{B}. \end{cases} \tag{3.3}
\]

Next, for \( i = 1, 2, \ldots, m - 1 \), consider the exact sequence

\[ 0 \to \mathcal{O}_{2Y_{i+1}}(-F_i) \to \mathcal{O}_{2Y_i-D_i}(-F_{i-1}) \to \mathcal{O}_{D_i}(-F_{i-1}) \to 0. \]

By the same reasoning as above, the restriction map \( H^0(2Y_i - D_i, -F_{i-1}) \to H^0(D_i, -F_{i-1}) \) is not the zero map if and only if \( \mathcal{O}_{2Y_i-D_i}(F_{i-1}) \cong \mathcal{O}_{2Y_i-D_i} \) and we have

\[
h^1(2Y_i - D_i, -F_{i-1}) = h^1(2Y_{i+1}, -F_i) + \begin{cases} h^1(D_i, -F_{i-1}), & \text{if } i \in \mathcal{A}, \\ -\chi(D_i, -F_{i-1}), & \text{if } i \not\in \mathcal{A}. \end{cases}
\]
Since \( p_a(D) = 2 \), we get \( h^1(D_i, -F_{i-1}) = h^0(D_i, -F_{i-1}) + 1 \). From this and the duality theorem, we get

\[
h^0(2Y_i - D_i, \omega_{2Y_i - D_i}(F_{i-1})) = h^0(2Y_{i+1}, \omega_{2Y_{i+1}}(F_i)) = \begin{cases} 2, & \text{if } i \in A, \\ 1, & \text{if } i \not\in A. \end{cases}
\] (3.4)

Putting \( h_i = h^0(2Y_i, \omega_{2Y_i}(F_{i-1})) \), we get

\[
h_i = h_{i+1} + 1 + \begin{cases} 2, & \text{if } i \in A \cap B, \\ 1, & \text{if } i \in A \cup B \setminus (A \cap B), \\ 0, & \text{if } i \not\in A \cup B, \end{cases}
\]

by (3.3) and (3.4). This also holds for \( i = m \) with \( h_{m+1} = 0 \). Hence, summing up, we get the desired formula for \( p_g(V, o) = h_1 \).

A geometric interpretation of \( A \) is as follows. Note that \( \mathcal{O}_{2Y_i - D_i}(F_{i-1}) \simeq \mathcal{O}_{2Y_i - D_i, ((i-1)Z)} \) by (1.2). If \( i > 1 \) and \( i \in A \), then this means that \( \mathcal{O}_{2Y_i - D_i}(Z) \) is a torsion element and, putting \( \alpha = \text{ord}(\mathcal{O}_{2Y_i - D_i}(Z)) \), \( i - 1 \) is a multiple of \( \alpha \). Conversely, if \( j = 1 + k\alpha \) with a non-negative integer \( k \) and \( j \leq m \), then we have \( j \in A \). Hence,

\[
A = \left\{ 1 + k\alpha \mid 0 \leq k \leq \left[ \frac{m-1}{\alpha} \right] \right\},
\]

where \( [x] \) denotes the biggest integer not exceeding \( x \in \mathbb{R} \). We remark that, when \( m \geq 2 \), \( \alpha \) does not depend on \( i > 1 \) in the sense that we have \( \text{ord}(\mathcal{O}_{D_m}(Z)) = \alpha \) by [11, Lemma 3.7], since we have \( H^1(2Y_i - D_i, Z) \simeq H^1(D_m, Z) \).

A similar observation can be done also for \( B \). Assume that \( B \neq \emptyset \) and put \( \beta = \min B \). We have \( \omega_{2Y_i}(F_{i-1}) \simeq \mathcal{O}_{2Y_i, (\kappa - (i-1)Z)} \). If \( i \in B \) and \( i > \beta \), then restricting \( \mathcal{O}_{2Y_i}(\kappa - (\beta - 1)Z) \simeq \mathcal{O}_{2Y_i} \) to \( 2Y_i \), we get \( \mathcal{O}_{2Y_i}(\kappa - (\beta - 1)Z) \simeq \mathcal{O}_{2Y_i} \), and, therefore, \( \mathcal{O}_{2Y_i, ((i-\beta)Z)} \simeq \mathcal{O}_{2Y_i} \). Hence \( i - \beta \) can be divided by \( \alpha \) (which is also the order of \( \mathcal{O}_{2Y_i}(Z) \) by the same reasoning as above), and we get

\[
B = \left\{ \beta + k\alpha \mid 0 \leq k \leq \left[ \frac{m-\beta}{\alpha} \right] \right\}
\]
as far as \( B \neq \emptyset \). In particular, we have \( A = B \) if and only if \( (V, o) \) is Gorenstein.

We have shown the following:

**Theorem 3.2.** Let \( (V, o) \) be an even singularity with \( \nu_f(V, o) = 2 \) and \( p_a(V, o) = m + 1 \). Put

\[
\alpha = \begin{cases} 1, & \text{if } m = 1, \\ \text{ord}(\mathcal{O}_{D_m}(Z)), & \text{if } m \geq 2, \end{cases}
\] (3.5)

and \( \beta = \min B \) when \( B \neq \emptyset \). Then,

\[
p_g(V, o) = m + 2 + \left[ \frac{m-1}{\alpha} \right] + \left[ \frac{m-\beta}{\alpha} \right] \quad \text{if } B \neq \emptyset
\]

and \( p_g(V, o) = m + 1 + [(m-1)/\alpha] \) if \( B = \emptyset \).

**Remark 3.3.** (1) When \( B = \emptyset \), we have \( p_g(V, o) = h^1(Y, \mathcal{O}_Y) \) by [6, Theorem 1.5]. This means that the cohomological cycle is a subcurve of the Yau cycle.
(2) When \((V,o)\) is Gorenstein, \(\alpha\) may divide \(m - 1\).

Recall that \((V_{\nu},o_{\nu})\) is the singular point obtained by contracting \(D_{\nu}\) \((1 \leq \nu \leq m)\), \((V_1,o_1) = (V,o)\).

**Corollary 3.4.** Let \((V,o)\) be an even singularity with \(p_f(V,o) = 2\) and \(p_a(V,o) = m + 1\). Then \(p_g(V,o) \leq 3m = 3p_a(V,o) - 3\). If the equality sign holds, then \((V_{\nu},o_{\nu})\) is Gorenstein and \(p_g(V_{\nu},o_{\nu}) = 3(m - \nu + 1)\) for all \(\nu = 1, \ldots, m\).

**Definition 3.5.** Let \((V,o)\) be an even singularity with \(p_f(V,o) = 2\). It is called a maximally even singularity if the equality \(p_g(V,o) = 3p_a(V,o) - 3\) holds. Then, it is Gorenstein.

**4. Maximal ideal cycles.** In this section, we study the maximal ideal cycles of even Gorenstein singularities of fundamental genus 2 and show the following:

**Theorem 4.1.** Let \((V,o)\) be an even Gorenstein singular point with \(p_f(V,o) = 2\) and \(p_a(V,o) = m + 1\), \(m\) the ideal sheaf of \(o \in V\) and \(\pi : X \to V\) the minimal resolution. Put \(F_{\alpha} = \sum_{i=1}^{\alpha} D_i\) assuming that \(\alpha < m\) when \(m \geq 2\), where \(Y = \sum_{i=1}^{m} D_i\) is the Yau cycle and \(\alpha\) is the number defined in (3.5). Then, one of the following three holds.

1. \(F_{\alpha}\) is the maximal ideal cycle and \(mO_X \simeq O_X(-F_{\alpha})\).
2. \(F_{\alpha}\) is the maximal ideal cycle, but there exists a point \(p\) such that \(mO_X \simeq m_pO_X(-F_{\alpha})\).
3. There exists a \((-2)\)-cycle \(\Gamma \prec Z\) such that \(F_{\alpha} + \Gamma\) is the maximal ideal cycle and \(mO_X \simeq O_X(-F_{\alpha} - \Gamma)\). \(\Gamma\) is the fixed part of \(|\omega_Z|\) when \(m = 1\) and it is a chain-connected component of \(Z - D_{\alpha}\) when \(m > \alpha \geq 2\).

When \(m \geq 2\), (I) occurs only when either \(\alpha = 1\) or \(\alpha \nmid m - 1\); (II) occurs only when \(\alpha \nmid m - 1\).

Therefore, even Gorenstein singularities of fundamental genus 2 fall into 4 types (0), (I) and (II) in Theorem 4.1 and (III) a class consisting of those with \(\alpha \geq m \geq 2\).

In the course of the proof, we freely use the notation in the previous sections.

We first consider the case where \(m = 1\). Then, \(F_{\alpha} = Z\) and \(\omega_Z = O_Z(-Z)\) since \((V,o)\) is Gorenstein. Therefore, the assertion follows from what we saw in §2.1 (especially, Lemma 2.1) except the statement in (II) for \(Z\) of type (ii) in Proposition 2.7.

**Lemma 4.2.** Let \((V,o)\) be an even Gorenstein singularity with \(p_f(V,o) = p_a(V,o) = 2\). If it is of type (ii) and \(\Gamma\) is the fixed part of \(|\omega_Z|\) described in §2.1, then \(O_X(-Z - \Gamma)\) is \(\pi\)-free.

**Proof.** Since \((V,o)\) is Gorenstein, \(K_X\) is linearly equivalent to \(-2Z\). So, we have \(\omega_{-\Gamma} = O_{-\Gamma}\). As we have already seen in Case 1 in §2.1 and Lemma 2.5, \(-Z - \Gamma\) is trivial on \(Z - \Gamma\). We have \(O_{\Gamma}(-Z - \Gamma) \simeq O_{\Gamma}(-\Gamma)\). Hence \(-Z - \Gamma\) is nef also on \(\Gamma\), since \(\Gamma\) is the fundamental cycle of a rational double point. In sum, \(O_Z(-Z - \Gamma)\) is nef.

Using the exact sequences

\[
0 \to O_X(-Z_K - Z - \Gamma) \to O_X(-Z - \Gamma) \to O_{Z_K}(-Z - \Gamma) \to 0
\]

and

\[
0 \to O_Z(-2Z - \Gamma) \to O_{Z_K}(-Z - \Gamma) \to O_Z(-Z - \Gamma) \to 0,
\]
one can easily show that the restriction $H^0(X, -Z - \Gamma) \to H^0(Z, -Z - \Gamma)$ is surjective.

We know that $\mathcal{O}_Z(-Z - \Gamma)$ is nef of degree 2. Recall that $\mathcal{O}_{Z - \Gamma}(-Z - \Gamma) \simeq \mathcal{O}_{Z - \Gamma}$ and consider the cohomology long exact sequence for

$$0 \to \mathcal{O}_\Gamma(-2Z) \to \mathcal{O}_Z(-Z - \Gamma) \to \mathcal{O}_{Z - \Gamma} \to 0.$$ 

Since $\mathcal{O}_\Gamma(-2Z) \simeq \mathcal{O}_\Gamma$, we get $h^0(\Gamma, -2Z) = 1$ and $h^1(\Gamma, -2Z) = 0$. Hence, $H^0(Z, -Z - \Gamma) \to H^0(Z - \Gamma, \mathcal{O}_{Z - \Gamma})$ is surjective, and we get $h^0(Z, -Z - \Gamma) = h^0(\Gamma, -2Z) + h^0(Z - \Gamma, \mathcal{O}_{Z - \Gamma}) = 3$. Since $h^0(Z, -Z - \Gamma) = \deg \mathcal{O}_Z(-Z - \Gamma) + 1$ holds, $|\mathcal{O}_Z(-Z - \Gamma)|$ is free from base point by [5, Theorem 2.1]. This shows that $\mathcal{O}_X(-Z - \Gamma)$ is $\pi$-free. \[ \square \]

Now, we assume $m \geq 2$ and $\alpha < m$.

For $1 \leq i \leq m$, consider the exact sequence

$$0 \to \mathcal{O}_X(-Z_K) \to \mathcal{O}_X(-F_{i-1}) \to \mathcal{O}_{Y_{i+1}}(-F_{i-1}) \to 0.$$ 

Since $H^1(X, -Z_K) = 0$ by the vanishing theorem, the restriction map $H^0(X, -F_{i-1}) \to H^0(Y + Y_{i+1}, -F_{i-1})$ is surjective. Next, consider the exact sequence

$$0 \to \mathcal{O}_{Y_{i+1}}(-F_i) \to \mathcal{O}_{Y_{i+1}}(-F_{i-1}) \to \mathcal{O}_{D_i}(-F_{i-1}) \to 0$$

for $1 \leq i \leq m$ with the convention that $Y_{m+1} = 0$, $F_m = Y$. When $i = 1$, $H^0(Z_K, \mathcal{O}_{Z_K}) \to H^0(Z, \mathcal{O}_Z)$ is surjective and we have $h^0(Y + Y_2, -Z) = p_g(V, O) - 1$. When $i \geq 2$ and $i \notin A$, we have $H^0(D_i, -F_{i-1}) = 0$ and it follows $H^0(Y + Y_{i+1}, -F_i) \simeq H^0(Y + Y_i, -F_{i-1})$. Therefore, when $\alpha \leq m$, we have $H^0(Y + Y_{\alpha+1}, -F_\alpha) \simeq H^0(Y + Y_2, -Z)$. From the commutative diagram

$$\begin{array}{ccc}
H^0(X, -Z_K) & \to & H^0(X, -Z) \\
\| & & \uparrow \approx \\
H^0(X, -Z_K) & \to & H^0(Y + Y_{\alpha+1}, -F_\alpha)
\end{array}$$

we see that any holomorphic function on $X$ vanishing identically on $Z$ necessarily vanishes identically on $F_\alpha$.

We claim that the restriction map $H^0(Y + Y_{\alpha+1}, -F_\alpha) \to H^0(D_{\alpha+1}, \mathcal{O}_{D_{\alpha+1}})$ is surjective. To see this, it suffices to show that the restriction map $H^0(Y + Y_{\alpha+1}, -F_\alpha) \to H^0(Y_{\alpha+1}, -F_\alpha)$ is surjective, since we have $\mathcal{O}_{Y_{\alpha+1}}(-F_\alpha) \simeq \mathcal{O}_{Y_{\alpha+1}}$ and the restriction map $H^0(Y_{\alpha+1}, \mathcal{O}_{Y_{\alpha+1}}) \to H^0(D_{\alpha+1}, \mathcal{O}_{D_{\alpha+1}})$ is surjective. Consider the exact sequence

$$0 \to \mathcal{O}_Y(-Y) \to \mathcal{O}_{Y + Y_{\alpha+1}}(-F_\alpha) \to \mathcal{O}_{Y_{\alpha+1}}(-F_\alpha) \to 0.$$ 

Since $(V, O)$ is Gorenstein, we have $\mathcal{O}_Y(-Y) \simeq \omega_Y$ and $h^0(Y, -Y) = h^1(Y, \mathcal{O}_Y) = m + 1 + [(m - 1)/\alpha] = p_g(V, O) - 1 - [(m - 1)/\alpha]$ by [6, Theorem 1.5]. Similarly, we have $h^0(Y_{\alpha+1}, \mathcal{O}_{Y_{\alpha+1}}) = h^1(Y_{\alpha+1}, \mathcal{O}_{Y_{\alpha+1}}) - (m - \alpha) = 1 + [(m - \alpha - 1)/\alpha] = [(m - 1)/\alpha]$ by the Riemann-Roch theorem. Then, $h^0(Y + Y_{\alpha+1}, -F_\alpha) = p_g(V, O) - 1 = h^0(Y, -Y) + h^0(Y_{\alpha+1}, \mathcal{O}_{Y_{\alpha+1}})$. This implies that $H^0(Y + Y_{\alpha+1}, -F_\alpha) \to H^0(Y_{\alpha+1}, \mathcal{O}_{Y_{\alpha+1}})$ is surjective.

We have shown that $|\mathcal{O}_X(-F_\alpha)|$ has no base points on $D_{\alpha+1}$. So our next task is to consider the base points on $Z - D_{\alpha+1}$. Recall that $Z - D_{\alpha+1}$ consists of two fundamental cycles of rational double points. Since the surjective
map $H^0(Y + Y_{a+1}, -F_a) \to H^0(D_{a+1}, \mathcal{O}_{D_{a+1}})$ factors through $H^0(Z, -F_a)$, the restriction map $H^0(Y + Y_{a+1}, -F_a) \to H^0(\mathcal{O}_{Z-D_{a+1}})$ is non-zero and $H^0(Z, -F_a) \to H^0(D_{a+1}, \mathcal{O}_{D_{a+1}})$ is surjective. Note also that we have $\mathcal{O}_{Z-D_{a+1}}(-F_{a+1}) \simeq \mathcal{O}_{Z-D_{a+1}}$ being a numerically trivial invertible sheaf on the rational curve $Z - D_{a+1}$. Then, from the exact sequence

$$0 \to \mathcal{O}_{Z-D_{a+1}}(-F_{a+1}) \to \mathcal{O}_Z(-F_a) \to \mathcal{O}_{D_{a+1}} \to 0,$$  

(4.2)

we get $h^0(Z, -F_a) = 1 + h^0(Z - D_{a+1}, \mathcal{O}) = 3$. Since $\mathcal{O}_Z(-F_a)$ is nef of degree 2, it follows from [5, Theorem 2.1] that $|\mathcal{O}_Z(-F_a)|$ is free from base points. We next consider

$$0 \to H^0(Y_2 + Y_{a+1}, -Z - F_a) \to H^0(Y + Y_{a+1}, -F_a) \to H^0(Z, -F_a).$$

If $\alpha = 1$, then $H^0(Y_2 + Y_{a+1}, -Z - F_a) = H^0(2Y_2, -2Z) \simeq H^0(2Y_2, \omega_{2Y_2})$ which is of dimension $p_g(Y_2, o_2) = 3m - 3$. It follows that $H^0(Y + Y_2, -Z) \to H^0(Z, -Z)$ is of rank 2. In order to estimate $h^0(Y_2 + Y_{a+1}, -Z - F_a)$ for $\alpha \geq 2$, we consider

$$0 \to \mathcal{O}_{Y_2+Y_{a+1}}(-Z - F_i) \to \mathcal{O}_{Y_2+Y_i}(-Z - F_{i-1}) \to \mathcal{O}_{D_i}(-Z - F_{i-1}) \to 0$$

for $i \geq 2$ and proceed similarly as what we did with (4.1). Then, one sees $h^0(Y_2 + Y_{a+1}, -Z - F_1) = h^0(Y_2 + Y_1, -Z - F_{i-1})$ for $i < \alpha$, and $h^0(Y_2 + Y_{a+1}, -Z - F_a) = h^0(Y_2 + Y_{a+1}, -Z - F_{a-1}) - 1$ and it follows $h^0(Y_2 + Y_{a+1}, -Z - F_a) = h^0(2Y_2, -2Z) - 1$. On the other hand, since $\mathcal{O}_{2Y_2}(-2Z) \simeq \omega_{2Y_2}$ and the invariant $\beta$ for $(V_2, o_2)$ is given by $\alpha - 1$, we get $h^0(2Y_2, -2Z) = (m - 1) + 2 + [(m - 2)/\alpha] + [(m - 1 - \alpha)/\alpha] = m + [(m - 2)/\alpha] + [(m - 1)/\alpha]$. Hence, $h^0(Y + Y_{a+1}, -F_a) = h^0(Y_2 + Y_{a+1}, -Z - F_a) = 2 + \left\lceil \frac{m - 1}{\alpha} \right\rceil - \left\lfloor \frac{m - 2}{\alpha} \right\rfloor$ and we get the following:

**Lemma 4.3.** Assume that $m \geq 2$ and $\alpha < m$. Then the restriction map $H^0(Y + Y_{a+1}, -F_a) \to H^0(Z, -F_a)$ is surjective if $\alpha \geq 2$ and $\alpha$ divides $m - 1$; otherwise, its image is 2-dimensional.

If $\alpha \geq 2$ and $\alpha \mid (m - 1)$, then $\mathcal{O}_X(-F_a)$ is $\pi$-free, by what we have seen above. Hence $m\mathcal{O}_X = \mathcal{O}_X(-F_a)$ in this case.

We consider the case that $H^0(Y + Y_{a+1}, -F_a) \to H^0(Z, -F_a)$ is of rank 2. Take $s_0, s_1 \in H^0(Y + Y_{a+1}, -F_a)$ which span the image. We can assume that $s_0$ is a non-zero constant on $D_{a+1}$, whereas $s_1$ vanishes identically on $D_{a+1}$. Then $s_0$ cannot vanish identically on any components of $D_a - D_{a+1}$ and $s_1$ induces a non-zero element $t_1 \in H^0(Z - D_{a+1}, -F_{a+1})$ from the exact sequence (4.2) and the fact that $\mathcal{O}_{Z-D_{a+1}}(-F_{a+1}) \simeq \mathcal{O}_{Z-D_{a+1}}$.

Assume that $s_0$ has isolated zeros only. If $t_1$ does not vanish on any components, then $s_0$ and $s_1$ have no common zeros. In this case, $m\mathcal{O}_X \simeq \mathcal{O}_X(-F_a)$. If $t_1$ vanishes identically on a component, then it vanishes identically on a chain-connected component $\Gamma_{a+1}$ of $Z - D_{a+1}$ and is a non-zero constant on $Z - D_{a+1} - \Gamma_{a+1}$; it follows that $s_0$ and $s_1$ have common zeros only at a point $p \in \Gamma_{a+1}$, $p \not\in D_{a+1}$. That is, $Bs(\mathcal{O}_X(-F_a)) = \{p\}$.

Assume that $s_0$ vanishes identically on a component of $Z - D_a$. Then $\alpha \geq 2$ and $s_0$ vanishes identically on at least one maximal chain-connected component of $Z - D_a$. 
If $t_1$ does not vanish on any components of $Z - D_{α+1}$, then $s_0$ and $s_1$ have no common zeros and we have $mO_X \simeq O_X(-F_α)$. If $t_1$ vanishes on a component, then it vanishes on $Γ_{α+1}$ as above. Therefore, there are two possibilities: either $Z - D_{α+1}$ is of type $(2A_0)$ and $s_0, s_1$ has no common zeros, or $s_0, s_1$ vanishes identically on a maximal chain-connected component $Γ$ of $Z - D_α$ satisfying $Γ \subset Γ_{α+1}$. In the former case, we have $mO_X \simeq O_X(-F_α)$. As to the latter, we have the following.

**Lemma 4.4.** If $s_0$ and $s_1$ vanish identically on $Γ$, then $mO_X \simeq O_X(-F)$, where $F = F_α + Γ$.

**Proof.** Recall that we have $α \geq 2$ and $α \nmid m - 1$.

We first confirm that $O_X(-F_α - Γ)$ is $π$-nef. Put $Γ' = Z - D_α - Γ$. Then $Γ'$ is the fundamental cycle of a rational double point and $F_α + Γ = F_{α+1} + Z - Γ'$. Let $C$ be an irreducible component of $Z$ and assume that $C(F_α + Γ) = C(F_{α+1} + Z - Γ') > 0$. We know that $-F_α - F_{α+1}$ and $-Z$ are nef on $π^{-1}(o)$ by [6, Lemma 1.3]. From $C(F_α + Γ) > 0$, we get $CΓ > 0$. On the other hand, from $C(F_{α+1} + Z - Γ') > 0$, we get $CΓ' < 0$ and, thus, $C$ is a component of $Γ'$. This contradicts $CΓ > 0$, because $OΓ(-Γ)$ is trivial. Therefore, $-F$ is nef on $π^{-1}(o)$, where $F = F_α + Γ$.

It remains to show that $O_X(-F)$ is $π$-free. For this purpose, we consider the commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & O_X(-Z_K - Γ) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & O_X(-Z_K) \\
\end{array}
\begin{array}{ccc}
& & \longrightarrow \end{array}
\begin{array}{ccc}
O_X(-F) & \longrightarrow & O_{Y+Y_{α+1}}(-F) \\
\downarrow & & \downarrow \\
& \longrightarrow & O_{Y+Y_{α+1} - Γ}(-F) \\
\end{array}
\begin{array}{ccc}
0 & \longrightarrow & 0
\end{array}
$$

of sheaves with exact rows. Since there are no curves with support in $π^{-1}(o)$ whose self-intersection numbers are odd, we know that $ω_X$ is $π$-free by a result in [4]. Hence $H^0(X, -Z_K) \rightarrow H^0(Γ, O_Γ)$ is surjective, and we get $H^1(X, -Z_K - Γ) = 0$ by $H^1(X, -Z_K) = 0$. It follows that $H^0(X, -F) \rightarrow H^0(Y + Y_{α+1}, -F)$ and $H^0(Y + Y_{α+1} - Γ, -F)$ are both surjective.

Since $Γ$ is the fixed part of $|O_X(-F_α)|$, the natural inclusion $H^0(Y + Y_{α+1} - Γ, -F) \rightarrow H^0(Y + Y_{α+1} - F_α)$ is an isomorphism. Then, the exact sequence

$$
0 \rightarrow O_{Y_{2}+Y_{α+1}}(-F_α - Z) \rightarrow O_{Y+Y_{α+1} - Γ}(-F) \rightarrow O_{Z - Γ}(-F) \rightarrow 0
$$

and the computation done just before Lemma 4.3 show that the image of $H^0(Y + Y_{α+1} - Γ, -F) \rightarrow H^0(Z - Γ, -F)$ is 2-dimensional. We remark that $Z - Γ$ is numerically 1-connected, since so is $Z$ and $(Z - Γ)Γ = 1$. Since $O_{Z - Γ}(-F)$ is nef of degree one, we have $h^0(Z - Γ, -F) \leq 2$. Therefore, $H^0(Y + Y_{α+1} - Γ, -F) \rightarrow H^0(Z - Γ, -F)$ is surjective and $O_{Z - Γ}(-F)$ is generated by its global sections by [5, Theorem 2.1]. In particular, we know that $|O_X(-F)|$ has no base points on $Z - Γ$.

Now, we consider the commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & O_Γ(-Z_K) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & O_Γ(-F_α - Z) \\
\end{array}
\begin{array}{ccc}
& & \longrightarrow \end{array}
\begin{array}{ccc}
O_{Y+Y_{α+1}}(-F) & \longrightarrow & O_{Y+Y_{α+1} - Γ}(-F) \\
\downarrow & & \downarrow \\
& \longrightarrow & O_{Z - Γ}(-F) \\
\end{array}
\begin{array}{ccc}
0 & \longrightarrow & 0
\end{array}
$$

with exact rows, where the vertical maps are restriction maps. A non-zero element of $H^0(Γ, -Z_K) \simeq H^0(Γ, O_Γ)$ gives us an element of $H^0(Y + Y_{α+1}, -F)$ that does not vanish on $Γ \setminus (D_α \cap Γ)$. In sum, $O_X(-F)$ is $π$-free. □
Lemma 4.5. $h^1(Y + Y_{\alpha+1}, \mathcal{O}_{Y + Y_{\alpha+1}}) = p_g(V, o) - 1$, $h^1(Y + Y_{\alpha+1}, -F_{\alpha}) = p_g(V, o) - 1 - \alpha$ and $h^1(Y + Y_{\alpha+1}, -nF_{\alpha}) = p_g(V, o) - 2 - \alpha$ for any $n \geq 2$.

Proof. Firstly, we consider the case where $m = 1$.

$$0 \to \mathcal{O}_Z(-(n+1)Z) \to \mathcal{O}_{ZK}(-nZ) \to \mathcal{O}_Z(-nZ) \to 0.$$

Since $\mathcal{O}_Z(-Z) = \omega_Z$, by duality, we get $h^1(Z, -(n+1)Z) = h^0(Z, nZ) = 0$ for $n \geq 1$. Then, $H^1(ZK, -nZ) = H^1(Z, \omega_Z(-(n-1)Z))$. We get $h^1(ZK, -nZ) = 0$ for $n \geq 2$. Since $p_g(V, o) = 3$, we are done.

Next, we consider the case $m \geq 2$. The restriction map $H^0(X, -F_{\alpha}) \to H^0(Y + Y_{\alpha+1}, -F_{\alpha})$ is surjective and $H^1(X, -F_{\alpha}) \simeq H^1(Y + Y_{\alpha+1}, -F_{\alpha})$. Since $h^1(Y + Y_{\alpha+1}, -F_{\alpha}) = p_g(V, o) - 1$ as we already saw, we get $h^1(Y + Y_{\alpha+1}, -F_{\alpha}) = p_g(V, o) - 1 - \alpha$ by the Riemann-Roch theorem. It follows $h^1(ZK, -F_{\alpha}) = p_g(V, o) - 1 - \alpha$. For $n \geq 2$, we consider

$$0 \to \mathcal{O}_X(-Z_K - (n-2)F_{\alpha}) \to \mathcal{O}_X(-nF_{\alpha}) \to \mathcal{O}_{2Y_{\alpha+1}}(-nF_{\alpha}) \to 0.$$

We have $H^1(X, -Z_K - (n-2)F_{\alpha}) = 0$ when $n \geq 2$ and $H^1(X, -nF_{\alpha}) \simeq H^1(2Y_{\alpha+1}, -nF_{\alpha}) \simeq H^1(2Y_{\alpha+1}, \mathcal{O})$. Therefore, $h^1(ZK, -nF_{\alpha}) = p_g(V_{\alpha+1}, o_{\alpha+1}) = p_g(V, o) - 2 - \alpha$ for $n \geq 2$. We note that $\mathcal{O}_{Y + Y_{\alpha+1}}(-F_{\alpha}) \simeq \omega_{Y + Y_{\alpha+1}}$. Hence $h^1(Y + Y_{\alpha+1}, \mathcal{O}) = h^0(Y + Y_{\alpha+1}, -F_{\alpha}) = h^1(Y + Y_{\alpha+1}, -F_{\alpha}) + \alpha$ by the Riemann-Roch theorem. \(\Box\)

The following completes the proof of Theorem 4.1.

Lemma 4.6. Assume that $Bs|\mathcal{O}_X(-F_{\alpha})| = \{p\}$ and let $\rho : \widetilde{X} \to X$ be the blowing-up at $p$. Put $E = \rho^{-1}(p)$ and $F := \rho^*F_{\alpha} + E$. Then $m\mathcal{O}_X \simeq \mathcal{O}_{\widetilde{X}}(-F)$. In particular, $\mu t(V, o) = F^2 = 2\alpha + 1$.

Proof. Recall that $p \in D_\alpha - D_{\alpha+1}$ when $m \geq 2$. Since every irreducible component through $p$ is non-singular at $p$ and has negative intersection number with $F_{\alpha}$, one easily sees that $-F$ is nef on $(\pi \circ \rho)^{-1}(o)$. It remains to show that $\mathcal{O}_{\widetilde{X}}(-F)$ has no base points on $E$. This will be done along an analogous line to the proof of Lemma 4.4.

Since $p$ is the base point of $[\mathcal{O}_{Y + Y_{\alpha+1}}(-F_{\alpha})]$, the restriction map $H^0(\rho^*(Y + Y_{\alpha+1}), -\rho^*F_{\alpha}) \to H^0(\rho^*(E, \mathcal{O}_E))$ is the zero map and we get $h^0(\rho^*(Y + Y_{\alpha+1}) - E, -F) = h^0(\rho^*(Y + Y_{\alpha+1}), -\rho^*F_{\alpha}) = p_g(V, o) - 1$ and $h^1(\rho^*(Y + Y_{\alpha+1}) - E, -F) = h^1(\rho^*(Y + Y_{\alpha+1}), -\rho^*F_{\alpha}) + h^0(\rho^*(E, \mathcal{O}_E)) = p_g(V, o) - \alpha$. When $m = 1$, this gives us $h^0(\rho^*Z - E, -F) = 2$. When $m \geq 2$, we recall that $h^0(Y_2 + Y_{\alpha+1}, -Z - F_{\alpha}) = m - 1 + [(m - 2)/\alpha] + [(m - 1)/\alpha] = p_g(V, o) - 3$ and consider

$$0 \to H^0(\rho^*(Y_2 + Y_{\alpha+1}), -\rho^*(Z + F_{\alpha})) \to H^0(\rho^*(Y + Y_{\alpha+1}) - E, -F) \to H^0(\rho^*Z - E, -F)$$

to find $H^0(\rho^*(Y + Y_{\alpha+1}) - E, -F) \to H^0(\rho^*Z - E, -F)$ is of rank 2 and, hence, $h^0(\rho^*Z - E, -F) \geq 2$. Since $\mathcal{O}_{\rho^*Z - E}(-F)$ is nef of degree 1 and $\rho^*Z - E$ is chain-connected by Lemma 1.1, it follows from [5, Theorem 2.1] that $h^0(\rho^*Z - E, -F) = 2$ and $\mathcal{O}_{\rho^*Z - E}(-F)$ is generated by its global sections. From

$$0 \to \mathcal{O}_{\widetilde{X}}(-\rho^*Z_K) \to \mathcal{O}_{\widetilde{X}}(-F) \to \mathcal{O}_{\rho^*(Y + Y_{\alpha+1}) - E}(-F) \to 0,$$

we see that $H^0(\widetilde{X}, -F) \to H^0(\rho^*(Y + Y_{\alpha+1}) - E, -F)$ is surjective. As we already remarked in the proof of Lemma 4.4, $\omega_{\widetilde{X}}$ is $\pi$-free. Thus the restriction
map $H^0(\tilde{X}, -\rho^*Z_K) \to H^0(E, \mathcal{O}_E)$ is surjective and we have $H^1(\tilde{X}, -\rho^*Z_K - E) \simeq H^1(\tilde{X}, -\rho^*Z_K) = 0$. Then, it follows from the exact sequence

$$0 \to \mathcal{O}_{\tilde{X}}(-\rho^*Z_K - E) \to \mathcal{O}_{\tilde{X}}(-F) \to \mathcal{O}_{\rho^*(Y + Y_{\alpha+1})}(-F) \to 0$$

that $H^0(\tilde{X}, -F) \to H^0(\rho^*(Y + Y_{\alpha+1}), -F)$ is surjective and $H^1(\tilde{X}, -F) \simeq H^1(\rho^*(Y + Y_{\alpha+1}), -F)$. We now consider the commutative diagram

$$\begin{array}{ccc}
H^0(E, -\rho^*Z_K) & \xrightarrow{|\cdot|} & H^0(\rho^*(Y + Y_{\alpha+1}), -F) \\
\downarrow & & \downarrow \\
H^0(E, -\rho^*(F_\alpha + Z)) & \xrightarrow{|\cdot|} & H^0(\rho^*Z, -F) \\
\end{array}$$

where the vertical arrows are restriction maps, to conclude that $H^0(\rho^*(Y + Y_{\alpha+1}), -F)$ is surjective and $H^1(\rho^*(Y + Y_{\alpha+1}), -F)$ is surjective. Since $\mathcal{O}_{\rho^*Z}(-F)$ is nef of degree 2 and $h^0(\rho^*Z, -F) = 3$, we see that $\mathcal{O}_{\rho^*Z}(-F)$ is generated by its global sections. Since $H^0(\tilde{X}, -F) \to H^0(\rho^*Z, -F)$ is surjective, we see that $\mathcal{O}_{\tilde{X}}(-F)$ is $(\pi \circ \rho)$-free and we have $m\mathcal{O}_{\tilde{X}} \simeq \mathcal{O}_{\tilde{X}}(-F)$.

By Theorem 4.1 and its proof, we get the following:

**Corollary 4.7.** Let the situation be as in Theorem 4.1. Then, the multiplicity of $(V, o)$ is given by

$$\text{mult}(V, o) = \begin{cases} 
2\alpha, & \text{if } m\mathcal{O}_X \simeq \mathcal{O}_X(-F_\alpha), \\
2\alpha + 1, & \text{if } m\mathcal{O}_X \simeq m_p\mathcal{O}_X(-F_\alpha), \\
2\alpha + 2 & \text{if } m\mathcal{O}_X \simeq \mathcal{O}_X(-F_\alpha - \Gamma).
\end{cases}$$

**Proof.** See, [12, 4.6]. □

**Remark 4.8.** It will be interesting to compare the fundamental genus and the arithmetic genus of the maximal ideal cycle described in Theorem 4.1. We have $p_\alpha(F_\alpha) = \alpha + 1$ and $p_\alpha(F_\alpha + \Gamma) = \alpha$. Hence it is smaller than the fundamental genus only when it is of type (II) with $\alpha = 1$.

5. Embedding dimensions. In this section, we compute the embedding dimension of a maximally even singularity of fundamental genus 2. For this purpose, we sometimes need the following:

**Proposition 5.1.** Let $L$ be a line bundle of degree $d > 0$ on a curve $D$ such that $|L|$ is free from base points. Then $h^0(D, L) \leq d + h^0(D, \mathcal{O}_D)$. Furthermore, the graded ring $R(D, L) = \bigoplus_{n=0}^{\infty} H^0(D, nL)$ is generated in degree one in the following cases.

1. $h^0(D, L) = d + h^0(D, \mathcal{O}_D)$.
2. $h^0(D, L) = d + h^0(D, \mathcal{O}_D) - 1$, $d \geq 3$ and $\omega_D \simeq \mathcal{O}_D$.

**Proof.** Take a general section $s \in H^0(D, L)$. Since $|L|$ is free from base points, $\zeta = (s)$ is an effective Cartier divisor of degree $d$. By using the identifications $\mathcal{O}_D(L) \simeq \mathcal{O}_D(\zeta)$ and $\mathcal{O}_\zeta(\zeta) \simeq \mathcal{O}_\zeta$ by another general section of $L$, we get an exact sequence

$$0 \to \mathcal{O}_D \to \mathcal{O}_D(L) \to \mathcal{O}_\zeta \to 0.$$
From this, we get $h^0(D, L) \leq h^0(\mathcal{O}_\mathbb{C}) + h^0(D, \mathcal{O}_D)$. If (1) is the case, then $H^0(D, L) \to H^0(\mathcal{O}_\mathbb{C})$ is surjective. It follows that $H^0(D, nL) \to H^0(\mathcal{O}_\mathbb{C})$ is surjective for any $n \geq 1$, because so is the composite $\text{Sym}^n H^0(D, L) \to H^0(D, nL) \to H^0(\mathcal{O}_\mathbb{C})$. Then we have $h^0(D, nL) = h^0(D, \mathcal{O}_D) + nd$ for $n \geq 0$ in this case. If (2) is the case, then $H^0(D, L) \to H^0(\mathcal{O}_\mathbb{C})$ is not surjective, but one can prove that $H^0(D, L) \to H^0(\mathcal{O}_\mathbb{C})$ is surjective for any effective subscheme $\zeta' \subseteq \zeta$ of length $d - 1$ as in [10, §6]. This shows that $\zeta$ is “in general position” with respect to the quadrics (cf. [12]) and it follows that $\text{Sym}^n H^0(D, L) \to H^n(D, 2L)$ and the composite $\text{Sym}^n H^0(D, L) \to H^0(D, 2L) \to H^0(\mathcal{O}_\mathbb{C})$ are surjective. Then $h^0(D, nL) = h^0(D, \mathcal{O}_D) + nd - 1$ for $n > 0$.

In both of (1) and (2), the rest follows from Castelnuovo’s free pencil trick. Choose a general subspace $W \subset H^0(D, L)$ of dimension 2 and consider the Koszul exact sequence

$$0 \to 2 W \otimes \mathcal{O}_D((n - 1)L) \to W \otimes \mathcal{O}_D(nL) \to \mathcal{O}_D((n + 1)L) \to 0$$

for $n > 0$. Since $2h^0(D, nL) = h^0(D, (n + 1)L) + h^0(D, (n - 1)L)$, $W \otimes H^0(D, nL) \to H^0(D, (n + 1)L)$ is surjective for $n \geq 1$ when (1), and for $n \geq 2$ when (2).

We shall show the following with several lemmas:

**Theorem 5.2.** Let $(V, o)$ be a maximally even singular point with $p_f(V, o) = 2$. Then the embedding dimension is given by

$$\text{embdim}(V, o) = \begin{cases} 3, & \text{if } \text{mult}(V, o) = 2, 3, \\ 4, & \text{if } \text{mult}(V, o) = 4. \end{cases}$$

**Lemma 5.3.** Let $(V, o)$ be a maximally even singular point with $p_f(V, o) = 2$. If $\mathcal{B}_s|\mathcal{O}_X(-Z)| = \emptyset$, then $R(Y + Y_2, -Z) := \bigoplus_{n \geq 0} H^0(Y + Y_2, -nZ)$ is generated by $s_{0, 1} \in H^0(Y + Y_2, -Z)$ and $s_2 \in H^0(Y + Y_2, -3Z)$. In particular, $\text{embdim}(V, o) = 3$.

**Proof.** We first claim that the rank of the restriction map $H^0(Y + Y_2, -nZ) \to H^0(Z, -nZ)$ is $2n - 1$ when $n \geq 2$. This follows from the exact sequence

$$0 \to \mathcal{O}_{2Y_2}(-(n + 1)Z) \to \mathcal{O}_{Y + Y_2}(-nZ) \to \mathcal{O}_Z(-nZ) \to 0,$$

since we have $h^0(Y + Y_2, -nZ) - h^0(2Y_2, -(n + 1)Z) = h^1(Y + Y_2, -nZ) + 2n - 1 - p_9(V_2, o_2) = 2n - 1$ by Lemma 4.5. Recall that $s_{0, 1}$ generate a $(n + 1)$ dimensional subspace of $H^0(Z, -nZ)$ spanned by $s_{0, 1}^i s_{1}^{n - i}$ ($0 \leq i \leq n$). Hence, we find a new element $s_2 \in H^0(Y + Y_2, -3Z)$ linearly independent from $s_{0, 1}^i s_{1}^{n - i}$ ($0 \leq i \leq 3$).

Recall that the image of the restriction map $H^0(Y + Y_2, -Z) \to H^0(Z, -Z)$ is of dimension 2. We denote by $W = \langle s_{0, 1} \rangle \subset H^0(Y + Y_2, -Z)$ the subspace of dimension 2 which restricts to span the image of $H^0(Y + Y_2, -Z) \to H^0(Z, -Z)$. Then $W$ induces a pencil that is free from base points and we have a Koszul exact sequence

$$0 \to 2 W \otimes \mathcal{O}_{Y + Y_2}(-(n - 1)Z) \to W \otimes \mathcal{O}_{Y + Y_2}(-nZ) \to \mathcal{O}_{Y + Y_2}(-(n + 1)Z) \to 0$$

for any non-negative integer $n$. We have $2h^1(Y + Y_2, -nZ) = h^1(Y + Y_2, -(n - 1)Z) + h^1(Y + Y_2, -(n + 1)Z)$ when $n \geq 3$ by Lemma 4.5. It follows that the multiplication map $W \otimes H^0(Y + Y_2, -nZ) \to H^0(Y + Y_2, -(n + 1)Z)$ is surjective for $n \geq 3$. Hence, the graded ring $R(Y + Y_2, -Z)$ is generated by elements of degrees $\leq 3$. Similarly,
we can show that $W \otimes H^0(Y + Y_2, -Z) \to H^0(Y + Y_2, -2Z)$ is surjective and $W \otimes H^0(Y + Y_2, -2Z) \to H^0(Y + Y_2, -3Z)$ has 1-dimensional cokernel. Indeed, as we saw above, we have $s_2 \in H^0(Y + Y_2, -3Z)$. Clearly, there is a relation of the form $s_2^2 = 0$.

Let $x_0, x_1 \in H^0(X, -Z)$ and $x_2 \in H^0(X, -3Z)$ be preimages of $s_0, s_1$ and $s_2$. We show that $\mathfrak{m}/\mathfrak{m}^2$ is generated by $x_0, x_1, x_2$. Indeed, we have $H^0(X, \mathcal{O}_X) \simeq H^0(X, -Z)$ and $H^0(X, -2Z)/H^0(X, \mathcal{O}_X)$ is generated by $x_2$. Then,

$$\text{embdim}(V, o) = \dim H^0(X, -Z)/H^0(X, -2Z) + \dim H^0(X, -3Z)/H^0(X, \mathcal{O}_X) = \dim H^0(X, -Z)/H^0(X, -2Z) + 1.$$

As is already seen, the restriction map $H^0(X, -nZ) \to H^0(Y + Y_2, -nZ)$ is surjective for any positive integer $n$. Hence, $\dim H^0(X, -Z)/H^0(X, -2Z) = \dim \text{Im}\{H^0(X, -Z) \to H^0(Z, -Z)\} = \dim \text{Im}\{H^0(Y + Y_2, -Z) \to H^0(Z, -Z)\}$. We have $h^0(Y + Y_2, -Z) = p_g(V, o) - 1$ by Lemma 4.5 and the Riemann-Roch theorem. We consider the exact sequence

$$0 \to \mathcal{O}_{2Y_2}(-2Z) \to \mathcal{O}_{Y+Y_2}(-Z) \to \mathcal{O}_Z(-Z) \to 0.$$

Since $\mathcal{O}_{2Y_2}(-2Z) \simeq \omega_{2Y_2}$, we get $h^0(2Y_2, -2Z) = p_g(V, o) - 3$. Therefore, $\text{embdim}(V, o) = (p_g - 1) - (p_g - 3) + 1 = 3$. □

See, Example 2.9 for maximally even double points.

**Lemma 5.4.** Let $(V, o)$ be a maximally even singular point with $p_f(V, o) = 2$. If $\text{B}_s(\mathcal{O}_X(-Z)) = \{p\}$, then $\text{embdim}(V, o) = 3$.

**Proof.** Put $F = \rho^*Z - E$. We let $\tilde{Z}_K = \rho^*Z_K - E$ be the canonical cycle on $\tilde{X}$. We shall show $h^0(\tilde{Z}_K, -F) = \deg(\mathcal{O}_{\tilde{Z}_K}(-F)) + p_g(V, o) - 1$. We have $H^1(\tilde{X}, -F) \simeq H^1(\tilde{Z}_K, -F)$. Since $H^1(\tilde{X}, -F) \simeq H^1(\rho^*(Y + Y_2) - E, -F)$, we get $h^1(\tilde{Z}_K, -F) = p_g(V, o) - 1$. Hence, by the Riemann-Roch theorem, we get $h^0(\tilde{Z}_K, -F) = \deg(\mathcal{O}_{\tilde{Z}_K}(-F)) + p_g(V, o) - 1$ as wished. Since $\deg(\mathcal{O}_{\tilde{Z}_K}(-F)) = 3$, it follows from Proposition 5.1 (2) that $R(\tilde{Z}_K, -F)$ is generated in degree 1. From this, one has $\mathfrak{m}^n \simeq (\pi \circ \rho)_s \mathcal{O}_X(-nF)$ for any positive integer $n$ as in [12, §4]. Then we have $\text{embdim}(V, o) = \dim \text{Im}\{H^0(\tilde{X}, -F) \to H^0(F, -F)\}$.

Since the intersection number of any curve with support in $\pi^{-1}(o)$ is even, we see that $\mathcal{O}_X(K_X - Z)$ is $\pi$-free (cf. [4]). This implies that $H^0(\tilde{X}, -\rho^*(Z_K + Z) \to H^0(E, \mathcal{O}_E)$ is surjective and we get $H^1(\tilde{X}, -\rho^*Z_K - F) = 0$ from $H^1(\tilde{X}, -Z_K - Z) = 0$. It follows that $H^0(\tilde{X}, -F) \to H^0(\rho^*Z_K, -F)$ is surjective and $H^1(\tilde{X}, -F) \simeq H^1(\rho^*Z_K, -F)$. In particular, $h^1(\rho^*Z_K, -F) = p_g(V, o) - 1$. By the Riemann-Roch theorem, we get $h^0(\rho^*Z_K, -F) = p_g(V, o) + 3$. We consider

$$0 \to H^0(\rho^*(Y + Y_2) - E, -2F) \to H^0(\rho^*Z_K, -F) \to H^0(F, -F).$$

Then $\text{embdim}(V, o) = h^0(\rho^*Z_K, -F) - h^0(\rho^*(Y + Y_2) - E, -2F)$. We showed in the proof of Lemma 4.6 that $H^0(\rho^*(Y + Y_2) - E, -2F) \to H^0(\rho^*Z - E, -F)$ is surjective. Since $R(\rho^*Z - E, -F)$ is generated in degree 1 by Proposition 5.1 (1), $H^0(\rho^*(Y + Y_2) - E, -2F) \to H^0(\rho^*Z - E, -2F)$ is also surjective. From the exact sequence

$$0 \to H^0(2\rho^*Y_2, -3\rho^*Z - E) \to H^0(\rho^*(Y + Y_2) - E, -2F) \to H^0(\rho^*Z - E, -2F) \to 0,$$
we get $h^{0}(\rho^{*}(Y + Y_2) - E, -2F) = p_g(V_2, o_2) + 3 = p_g(V, o)$. In sum, $\text{embdim}(V, o) = 3$. □

**Example 5.5.** The minimal resolution of the hypersurface triple point \{\(z^3 = (x^2 - y^4)(x^4 + y^2)\)\} has two \((-2)\)-elliptic curves meeting normally at a point as the exceptional set. Hence it is of type (i.a). The exceptional set for \(z^3 = x^4 + y^6\) consists of one \((-2)\)-elliptic curve $E_0$ and three \((-2)\)-curves $E_1, E_2, E_3$ such that $E_0E_i = 1$ (\(i = 1, 2, 3\)) and $E_1E_j = 0$ (\(i, j \in \{1, 2, 3\}, i \neq j\)). Hence it is of type (i.b) (see, Fig. 2 (a)). \(z^3 = x(x^3 + y^5)\) is another example of type (i.b) (see, Fig. 3 (c)).

If $\alpha = 1$, then Type (II) occurs only when $m = 1$.

**Lemma 5.6.** Let $(V, o)$ be of type (II), $m = 1$. Then the ring $R(Z_K, -F)$ is generated in degree one. Furthermore, $m^n\mathcal{O}_X \simeq H^0(X, -nF)$ for any positive integer $n$ and $\text{embdim}(V, o) = 4$.

**Proof.** We only have to show that $h^0(Z_K, -F) = 6 = \deg(\mathcal{O}_{Z_K}(-F)) - 1 + h^0(Z_K, \mathcal{O}_{Z_K})$.

$$0 \to \mathcal{O}_Z(-2Z - \Gamma) \to \mathcal{O}_{Z_K}(-F) \to \mathcal{O}_Z(-F) \to 0.$$  

We have $h^1(Z, -2Z - \Gamma) = 0$ and $h^0(Z, -2Z - \Gamma) = 3$, since $\omega_Z = \mathcal{O}_Z(-Z)$. As we already saw in the proof of Lemma 4.2, we have $h^0(Z, -F) = 3$. Hence $h^0(Z_K, -F) = 3 + 3 = 6$. By Proposition 5.1, $R(Z_K, -F)$ is generated in degree 1. This shows that $m^n \simeq \pi_*\mathcal{O}_X(-nF)$ as in [12]. In particular, $\text{embdim}(V, o) = \varepsilon(V, o)$. Since $H^0(X, -F) \to H^0(Z_K, -F)$ is surjective, by using

$$0 \to \mathcal{O}_{Z-\Gamma}(-2F) \to \mathcal{O}_{Z_K}(-F) \to \mathcal{O}_F(-F) \to 0$$

and $h^0(Z - \Gamma, \mathcal{O}_{Z-\Gamma}) = 2$, we get $\text{embdim}(V, o) = 6 - 2 = 4$. □

**Remark 5.7.** For a 1-connected curve $D$ with $p_a(D) = 2$ and $K_D$ nef, the structure of the canonical algebra $R(D, K_D)$ was extensively studied in [8]. When $D$ is not 2-connected, her deep analysis shows:

$$R(D, K_D) \simeq \mathbb{C}[x_0, x_1, y, z]/(\varphi_2, \varphi_6),$$

where $\deg x_0 = \deg x_1 = 1$, $\deg y = 2$ and $\deg z = 3$, and the $\varphi_i$’s are homogeneous forms of degree $i$. A normal form is like:

$$\varphi_2 = x_0(x_0 - \lambda x_1), \quad \varphi_6 = z^2 - y^3 + \cdots \quad (\lambda \in \mathbb{C}).$$

In other words, the canonical model of $D$ is a weighted complete intersection of type $(2, 6)$ in $\mathbb{P}(1, 1, 2, 3)$.

**References**


