

THE NUMBER OF EQUISINGULAR MODULI OF A RATIONAL SURFACE SINGULARITY*

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To Henry Laufer on his 70th birthday

Abstract. We consider a conjectured topological inequality for the number of equisingular moduli of a rational surface singularity, and prove it in some natural special cases. When the resolution dual graph is “sufficiently negative” (in a precise sense), we verify the inequality via an easy cohomological vanishing theorem, which implies that this number is computed simply from the graph (Theorem 3.10). To consider an important and less restrictive meaning of “sufficiently negative” requires a much more difficult “hard vanishing theorem” (Theorem 4.5), which is false in characteristic p . Theorem 7.9 verifies the conjectured inequality in this more general situation. As a corollary, we classify in characteristic p all taut singularities with reduced fundamental cycle (Theorem 9.2).

Key words. Rational singularity, equisingular deformation, tautness, characteristic p singularities, quasi-homogeneity.

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1. Introduction. The following is a special case of a conjectured inequality in [14] for complex normal surface singularities:

RATIONAL CONJECTURE. *Let $(V, 0)$ be (the germ of) a complex rational surface singularity, $(X, E) \rightarrow (V, 0)$ the minimal good resolution (or MGR). Define $S_X = (\Omega_X^1(\log E))^*$, the bundle on X of derivations logarithmic along E . Then*

$$h^1(X, S_X) \leq h^1(X, -(K_X + E)),$$

with equality if and only if $(V, 0)$ is quasi-homogeneous.

The right-hand term is $h^1(X, \wedge^2 S_X)$, the *second plurigenus* of the singularity, which can be computed from the resolution graph Γ , hence is a topological invariant. On the other hand, $h^1(S_X)$ is the dimension of the smooth space of *equisingular deformations* of the singularity ([11], (5.16)), i.e., those deformations with the same graph; it is difficult to compute, and can vary in equisingular families. In [14] the Conjecture was verified when the graph Γ is star-shaped, or when $(V, 0)$ admits a smoothing whose total space is $(\mathbb{C}^3/G, 0)$.

In this paper we prove the Conjecture for graphs Γ which are “sufficiently negative at the nodes” by computing $h^1(S_X)$ in all those cases. Let r be the number of ends of the graph Γ , and for an exceptional curve E_i let t_i be its valence and $d_i = -E_i \cdot E_i$. The easiest version is the following:

THEOREM (3.10). *Let $(X, E) \rightarrow (V, 0)$ be the MGR of a rational surface singularity. Suppose that for all i , one has*

$$d_i \geq 2t_i - 2.$$

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- (1) If Γ is star-shaped, then $(V, 0)$ is weighted homogeneous, and $h^1(S_X) = h^1(-(K_X + E)) = r - 3$.
- (2) If Γ is not star-shaped, then $h^1(S_X) = h^1(-(K_X + E)) - 1 = r - 4$.

For instance, if Γ is any trivalent tree (not star-shaped) with $d \geq 4$ at each node, then the “number of equisingular moduli” is exactly the number of ends minus 4. Note however that the reduced curve E is itself rigid.

The base space of a semi-universal deformation of $(V, 0)$ contains a unique irreducible Artin component, parametrizing deformations which resolve simultaneously after base change (see e.g. [9], p. 115). It is smooth of dimension $h^1(\Theta_X) = h^1(S_X) + \Sigma(d_i - 1)$. Combining with the results of [2] yields

COROLLARY (3.11). *For a rational singularity $(V, 0)$ with $d_i \geq 2t_i - 2$, all i , one has*

- (1) if Γ is star-shaped, then $\dim T_V^1 = \sum_i (2d_i - 3) + r - 4$.
- (2) if Γ is not star-shaped, then $\dim T_V^1 = \sum_i (2d_i - 3) + r - 5$.

The origin of the Rational Conjecture is the

MAIN CONJECTURE OF [14]. *Let $(X, E) \rightarrow (V, 0)$ be the MGR of a complex normal surface singularity. Define $S_X = (\Omega_X^1(\log E))^*$. If $(V, 0)$ is not Gorenstein, then*

$$h^1(\mathcal{O}_X) - h^1(S_X) + h^1(-(K_X + E)) \geq 0,$$

with equality if and only if $(V, 0)$ is quasihomogeneous.

For $(V, 0)$ Gorenstein, it followed from [13] that the cohomological expression above is always greater than or equal to 1, with equality exactly in the quasihomogeneous case. That result was a generalization of an inequality for isolated hypersurface singularities: the Milnor number μ is greater than or equal to the Tjurina number τ , with equality exactly in the quasihomogeneous case [7]. The connection is that for a two-dimensional hypersurface, on the minimal good resolution one has

$$1 + \mu - \tau = h^1(\mathcal{O}_X) - h^1(S_X) + h^1(-(K_X + E)).$$

The reader may consult [14] to see the derivation of the expression in the Main Conjecture, the verification for certain quotients of hypersurface singularities, and the proof that one does have equality in the quasihomogeneous case. (Cf. also [6].)

In this more general setting, $h^1(S_X)$ counts the first-order deformations of X to which each exceptional curve E_i lifts. It is the tangent space to a smooth family of equisingular deformations of the resolution; but only sometimes (e.g., when $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_E)$) do these deformations “blow down” to actual deformations of the singularity ([11], (2.7)).

Our initial approach to the Rational Conjecture applies as well to the Main Conjecture. By Propositions 2.6 and 2.10, one can compute h^1 of the restrictions of the 3 bundles to E solely from the graph Γ . The following completely general result is instructive.

PROPOSITION (2.11). *Let $(X, E) \rightarrow (V, 0)$ be the MGR of a normal surface singularity, not a simple elliptic or cusp singularity. Then*

$$h^1(\mathcal{O}_E) - h^1(S_X \otimes \mathcal{O}_E) + h^1(-(K_X + E) \otimes \mathcal{O}_E) = 1 - \delta,$$

where δ is 1 if the resolution dual graph Γ is star-shaped, 0 otherwise.

From this result follows the key observation: the inequality in the Main Conjecture holds in those cases for which

$$h^1(S_X) = h^1(S_X \otimes \mathcal{O}_E),$$

e.g. if $h^1(S_X(-E)) = 0$. If the d_i are big enough relative to $g_i = \text{genus } E_i$ and t_i , then such a cohomological vanishing result can be proved via so-called “easy vanishing theorems” of [12], as recalled in Section 3. Deducing quasihomogeneity is often possible via a result in [13] (Proposition 2.2 below).

THEOREM (3.8). *Let $(X, E) \rightarrow (V, 0)$ be the MGR of a normal surface singularity. Suppose that for all i one has*

$$d_i \geq 4g_i - 4 + 3t_i,$$

with strict inequality for at least one i . Then h^1 of each of the three sheaves \mathcal{O}_X, S_X , and $-(K_X + E)$ is equal to h^1 of its restriction to E , and the Main Conjecture holds.

This should be compared with H. Grauert’s old result: if E is a single smooth curve and $d > 4g - 4$, then $(V, 0)$ is the cone over a curve, determined by E and its conormal bundle. Here, $h^1(S_X) = h^1(S_X \otimes \mathcal{O}_E) = (3g - 3) + g$.

Returning to the Rational Conjecture, it is desirable (and necessary for applications) to prove it in a slightly more general situation than Theorem 3.10, with a weaker inequality for the d_i . However, this requires a much more delicate argument and a “hard vanishing theorem” (as in [13]), which is false in prime characteristic. The main work of the paper is to prove vanishing under certain conditions.

Let E' denote the sum of the curves which are not end-curves; and for each curve, let t'_i denote the number of intersections with curves in E' . The condition

$$(**) \ d_i \geq t_i + t'_i - 2, \text{ all } i$$

is exactly what simplifies the computation of $h^1(-(K_X + E))$; by Proposition 4.2, it equals $r - 3$ (except for cyclic quotients). As $h^1(S_X \otimes \mathcal{O}_E)$ is $r - 4$ in the non-star-shaped case (Proposition 2.6) and $h^1(S_X) \geq h^1(S_X \otimes \mathcal{O}_E)$, we deduce:

PROPOSITION (4.4). *Let $(V, 0)$ be a rational singularity whose graph satisfies (**). Then the Rational Conjecture for $(V, 0)$ is equivalent to*

$$H^1(S_X(-E')) = 0.$$

A long and technical argument over several sections of the paper eventually yields the following, which is somewhat weaker than desired:

THEOREM (4.5). *Let $(V, 0)$ be a rational singularity whose graph satisfies (**). Then $H^1(S_X(-E') \otimes \mathcal{O}_E) = 0$.*

This result does yield the Rational Conjecture in an important case.

COROLLARY (2.12). *Suppose a rational singularity with reduced fundamental cycle satisfies (**). Then $H^1(S_X(-E')) = 0$ and $h^1(S_X) = r - 4 + \delta$.*

The set-up used above can be stretched to prove a more precise result.

THEOREM (7.9). *Suppose a rational singularity, with reduced fundamental cycle, has all curves satisfying $d_i \geq t_i + t'_i - 2$ for all i , except that one also allows that either*

- (1) *exactly one curve satisfies $d = t + t' - 3$, or*
- (2) *exactly two curves, separated by a (possibly empty) string of rational curves, satisfy $d = t + t' - 3$.*

Then $h^1(S_X(-E')) = 0$, $h^1(S_X) = h^1(S_X \otimes \mathcal{O}_E)$, and the Rational Conjecture holds.

(If some $d \leq t + t' - 4$, then $h^1(S_X(-E') \otimes \mathcal{O}_E) \neq 0$ (Lemma 5.1), and $h^1(S_X) > h^1(S_X \otimes \mathcal{O}_E)$.)

REMARK. Some of the issues in this paper originate with the work of Henry Laufer [3], [4]. In [4], he made a complete list of graphs Γ for which there corresponds a unique analytic type; these are the *taut* singularities, characterized by the vanishing of $h^1(S_X)$ for *every* singularity with graph Γ . He also listed those Γ for which the singularity is uniquely determined by the analytic type of the reduced curve E (i.e., in the rational case, by cross-ratios at the nodes). Theorem 7.9 allows one to recover easily these classifications for rational singularities with reduced fundamental cycle. (For instance, Laufer's condition on line 5 of p.162 of [4] is equivalent to $d \geq t + t' - 3$.) Of course, these form a very small portion of Laufer's lists!

More importantly, the methods of this paper allow one to extend this partial classification to characteristic p . In [1], M. Artin listed all the rational double points in characteristic p ; he showed that for a graph of type D or E , there were certain primes for which the graph was not taut. Lee-Nakayama [5] showed that all cyclic quotients are taut. Work of F. Schüller [8] extends some work of Laufer to characteristic p , so that a graph Γ is taut if and only if $h^1(S_X) = 0$ for every singularity with that graph.

THEOREM (9.2). *In characteristic p , there is a complete list of the taut singularities with reduced fundamental cycle.*

The paper is organized as follows: In Section 2 we compute explicitly h^0 and h^1 of the restrictions of the three relevant sheaves to E , finding (Proposition 2.11) an equality close to the Main Conjecture. The divisor E' (which is E minus the rational end-curves) becomes important. In Section 3, "easy vanishing theorems" (in the sense of [12]) give explicit conditions, in both the general and rational cases, for h^1 's of the relevant bundles on X to be computable from restriction to E . From Section 4 on, only rational singularities are considered, and one attempts to get weaker restrictions on the d_i to imply the vanishing of $H^1(S_X(-E'))$. This is the technical heart of the paper: one uses an inductive procedure on subgraphs, starting at the end-curves and growing towards the interior. Section 7 generalizes the preceding arguments and gives the strongest results of the paper. Finally, in Section 8 previous proofs are examined and modified to get analogous results valid in characteristic p . The taut singularities with reduced fundamental cycle are listed in Section 9.

2. Restriction to E and E' . Consider the minimal good resolution $(X, E) \rightarrow (V, 0)$ of a normal surface singularity, with weighted dual graph Γ . For each exceptional curve E_i , let g_i be the genus, $d_i = -E_i \cdot E_i$ the degree of the conormal bundle, and t_i the number of intersections with other curves (or, valency of the vertex in Γ). A curve (or vertex) is called a *node* if $t_i \geq 3$, and a *star* if $2g_i + t_i > 2$.

Let $S = S_X$ be the rank 2 bundle of derivations logarithmic along E , defined by the short exact sequence ([12], 1.7.1)

$$0 \rightarrow S \rightarrow \Theta_X \rightarrow \oplus N_{E_i} \rightarrow 0.$$

In local coordinates x, y , if E is given by $y = 0$, then S is generated by $\partial/\partial x$ and $y\partial/\partial y$; if E is given by $xy = 0$, S is generated by $x\partial/\partial x$ and $y\partial/\partial y$. For each i there is an exact sequence ([12], 1.10.2)

$$0 \rightarrow \mathcal{O}_{E_i} \rightarrow S \otimes \mathcal{O}_{E_i} \rightarrow \Theta_{E_i}(-t_i) \rightarrow 0.$$

(We abuse notation slightly, as $-t_i$ represents the negative of an effective divisor of degree t_i .) This sequence splits unless E consists of a single smooth curve. The global section of $S \otimes \mathcal{O}_{E_i}$ from the left hand injection sends 1 to $y\partial/\partial y$, where $y = 0$ is any local equation for E_i on X ; for, if y' is another local equation for E , then $y'\partial/\partial y' = y\partial/\partial y$ modulo $y = 0$. We record a useful

LEMMA 2.1. *If $E_i \cap E_j = P_{ij}$, then $H^0(S \otimes \mathbb{C}_{P_{ij}})$ has a natural ordered basis $\{x\partial/\partial x, y\partial/\partial y\}$, where $y = 0$ (respectively $x = 0$) is any local analytic equation defining E_i (resp. E_j).*

Proof. If x' (resp. y') are other equations, write $x' = ux, y' = vy$, where u, v are units in the local ring at P_{ij} ; then, compare $x'\partial/\partial x'$ and $y'\partial/\partial y'$ with the previous choices, and reduce the coefficients modulo the maximal ideal. \square

From now on, we restrict attention to graphs which are *not* one of the following types:

- (1) A chain of smooth rational curves (= cyclic quotient singularity)
- (2) A cycle of smooth rational curves (= cusp singularity)
- (3) A smooth elliptic curve (= simple elliptic singularity)

In every other case, E contains at least one “star” curve E_0 , with $2g_0 + t_0 > 2$. For such a curve, $h^0(\Theta_{E_0}(-t_0)) = 0$, hence $h^0(S \otimes \mathcal{O}_{E_0}) = 1$. We recall a useful result:

PROPOSITION 2.2. ([13]) *Assume that $H^0(S) \rightarrow H^0(S \otimes \mathcal{O}_{E_0})$ is surjective, where E_0 is a star curve. Then $(V, 0)$ is weighted homogeneous.*

Proof. While this result is not stated explicitly in [13], a complete proof is found there from (3.12) through (3.16). One lifts a non-0 element of $H^0(S \otimes \mathcal{O}_{E_0})$ to a $D \in H^0(S)$, a derivation of the local ring of V . A local argument along E_0 shows it is a non-nilpotent derivation, whence by a theorem of Scheja-Wiebe one has quasi-homogeneity. \square

Let R be the union of the rational end curves (with $g_i = 0, t_i = 1$), and $E' = E - R$ the union of the other curves. E' is connected and is the union of the stars plus rational curves with $t = 2$.

LEMMA 2.3. *Suppose F is a connected and reduced cycle in E' containing a star E_0 . Then one has an inclusion into a one-dimensional space:*

$$H^0(S \otimes \mathcal{O}_F) \subset H^0(S \otimes \mathcal{O}_{E_0}).$$

Proof. Induct on the number of components of a connected F' between E_0 and F . Given such an $F' < F$, choose an E_i in $F - F'$ which intersects F' . Then $E_i \cdot F' > 0$, and an easy check shows one has vanishing of H^0 of the first term in

$$0 \rightarrow S \otimes \mathcal{O}_{E_i}(-F') \rightarrow S \otimes \mathcal{O}_{F'+E_i} \rightarrow S \otimes \mathcal{O}_{F'} \rightarrow 0.$$

\square

A graph is called *star-shaped* if it consists of one star out of which emanate strings of rational curves; it is the graph of a weighted homogeneous singularity.

LEMMA 2.4. *Suppose the graph of X is star-shaped. Then $h^0(S \otimes \mathcal{O}_{E'}) = 1$.*

Proof. As in 2.3, the relevant connected cycles F are constructed from the central star E_0 by adding smooth rational curves E_i with $t_i = 2$ with $F' \cdot E_i = 1$. Then

$$S \otimes \mathcal{O}_{E_i}(-F') = \mathcal{O}_{E_i}(-1) \oplus \mathcal{O}_{E_i}(-1),$$

so $h^0(S \otimes \mathcal{O}_{F'+E_i}) = h^0(S \otimes \mathcal{O}_{F'})$. \square

LEMMA 2.5. *Suppose the graph of X is not star-shaped. Then $h^0(S \otimes \mathcal{O}_{E'}) = 0$.*

Proof. Since the graph is not star-shaped, one concludes that either

- (1) E' contains two stars E_0 and E'_0 connected by a (possibly empty) chain of rational curves with $t_i = 2$
- (2) E' contains one star E_0 and a chain of rational curves with $t_i = 2$ and intersecting E_0 at least twice.

In either case, we claim that the union F of the star and the rational curves in the chain satisfies $h^0(S \otimes \mathcal{O}_F) = 0$. The assertion then follows using the method of proof of Lemma 2.3.

The result is clearest in the case where two stars E_0 and E'_0 meet at a point P . In an appropriate affine open neighborhood U of P , choose local functions x and y whose vanishing defines the two curves E'_0 and E_0 , respectively, and so that dx and dy form a basis for the local 1-forms. Then S is locally a free $\mathcal{O}(U)$ -module with basis $x\partial/\partial x$ and $y\partial/\partial y$. The one-dimensional spaces of global sections of $S \otimes \mathcal{O}_{E_0}$ and $S \otimes \mathcal{O}_{E'_0}$ are of the form $ay\partial/\partial y$ and $bx\partial/\partial x$, respectively, where a, b are constants. These agree at P and hence extend to a global section of $S \otimes \mathcal{O}_F$ only when $a = b = 0$.

In the general case, we need to choose compatible “coordinates” on the components of F , as done in [12] (itself modeled closely on [4]). Denote the rational curves in the chain in order by E_1, \dots, E_r , and let $E_{r+1} = E'_0$ be the star at the end. Let $P_i = E_i \cap E_{i+1}$, $0 \leq i \leq r$. For $1 \leq i \leq r$, the embedding of E_i in X is locally analytically the embedding in the normal line bundle of degree $-d_i$. We choose coordinates so that P_{i-1} corresponds to $\{0\}$ and P_i to $\{\infty\}$. Cover the scheme $2E_i$ by 2 affines

$$U_{i,1} = \text{Spec } k[x_i, y_i]/y_i^2$$

$$U_{i,2} = \text{Spec } k[x'_i, y'_i]/y'^2_i,$$

on whose intersection one has

$$x'_i = 1/x_i, \quad y'_i = x_i^{d_i} y_i.$$

We may also assume that y_i (and y'_i) are local equations for $E_i \subset 2E_i$, and (possibly replacing x_i by $x_i + y_i g(x_i)$, and similarly for x'_i) that the divisor of the predecessor $E_{i-1} \cap 2E_i \subset 2E_i$ has local equation given by $x_i = 0$ (and similarly $x'_i = 0$ at ∞). In particular, we can assume that at each intersection point, the functions x_i, y_i are restrictions (modulo higher order terms) of local equations for the intersecting curves. Furthermore, starting at P_1 and adjusting constants, we can assume that in the tangent space of P_i , we have $x'_i = y_{i+1}$, $y'_i = x_{i+1}$ (for $i < r$). The standard exact sequence on E_i

$$0 \rightarrow \mathcal{O}_{E_i} \rightarrow S \otimes \mathcal{O}_{E_i} \rightarrow \Theta_{E_i}(-t_i) \rightarrow 0$$

may be expressed on $U_{i,1}$ (since $t_i = 2$) as

$$0 \rightarrow \{y_i \partial / \partial y_i\} \rightarrow \{y_i \partial / \partial y_i, x_i \partial / \partial x_i\} \rightarrow \{x_i \partial / \partial x_i\} \rightarrow 0,$$

and similarly on $U_{i,2}$. The patching condition is

$$x_i \partial / \partial x_i = -x'_i \partial / \partial x'_i + d_i y'_i \partial / \partial y'_i$$

$$y_i \partial / \partial y_i = y'_i \partial / \partial y'_i.$$

For $1 \leq i \leq r$, a global section of $S \otimes \mathcal{O}_{E_i}$ is of the form

$$A_i y_i \partial / \partial y_i + B_i x_i \partial / \partial x_i = (A_i + d_i B_i) y'_i \partial / \partial y'_i - B_i x'_i \partial / \partial x'_i,$$

for some constants A_i, B_i .

For $1 \leq i \leq r - 1$, the two-dimensional space $S \otimes \mathbb{C}_{P_i}$ has the natural ordered basis

$$x'_i \partial / \partial x'_i = y_{i+1} \partial / \partial y_{i+1}, \quad y'_i \partial / \partial y'_i = x_{i+1} \partial / \partial x_{i+1}.$$

One similarly has an ordered basis at both P_0 and P_r . Via patching, we have that

$$H^0(S \otimes \mathcal{O}_F) = \text{Ker} \left(\bigoplus_{i=0}^{r+1} H^0(S \otimes \mathcal{O}_{E_i}) \rightarrow \bigoplus_{i=0}^r H^0(S \otimes \mathbb{C}_{P_i}) \right).$$

Compatibility of the global sections above of the $H^0(S \otimes \mathcal{O}_{E_i})$ at P_1, \dots, P_{r-1} means

$$-B_i = A_{i+1}, \quad A_i + d_i B_i = B_{i+1}, \quad i = 1, \dots, r - 1.$$

A global section of the one-dimensional space $H^0(S \otimes \mathcal{O}_{E_0})$ is of the form $Bx \partial / \partial x$, where x is a local equation of E_0 near P_0 . Therefore, its image in the space $S \otimes \mathbb{C}_{P_0}$ is $Bx_1 \partial / \partial x_1$. A section of $S \otimes \mathcal{O}_{E_1}$ patches compatibly if $A_1 = 0$. Similarly, a global section of $H^0(S \otimes \mathcal{O}_{E_r})$ patches compatibly with a section of $H^0(S \otimes \mathcal{O}_{E'_0})$ if $A_r + d_r B_r = 0$. These $2r$ equations in the A_i, B_i become r equations in the B_i , namely

$$-d_1 B_1 + B_2 = 0$$

$$B_1 - d_2 B_2 + B_3 = 0$$

.....

$$B_{r-1} - d_r B_r = 0.$$

One recognizes the matrix of these equations as the intersection matrix of the cyclic quotient singularity whose resolution dual graph is that of the r curves between E_0 and E'_0 . In particular, the determinant is $\pm n$, where one has an n/q cyclic quotient. Thus, the B_i are all 0.

Note that the same proof applies in case $E_0 = E'_0$, except that one then has an additional condition that $B_1 = -B_r$. \square

PROPOSITION 2.6. *Consider the minimal good resolution $(X, E) \rightarrow (V, 0)$ of a normal surface singularity, with graph Γ , excluding the 3 cases above. Let $E' = E - R$, where R is the union of the rational end curves, r in number. Then*

(1) if Γ is star shaped, then

$$h^1(S \otimes \mathcal{O}_{E'}) = h^1(S \otimes \mathcal{O}_E) = r + 4h^1(\mathcal{O}_E) - 3.$$

(2) if Γ is not star-shaped, then

$$h^1(S \otimes \mathcal{O}_{E'}) = h^1(S \otimes \mathcal{O}_E) = r + 4h^1(\mathcal{O}_E) - 4.$$

Proof. We claim that

$$\chi(S \otimes \mathcal{O}_{E'}) = 4 - 4h^1(\mathcal{O}_E) - r.$$

For, Riemann-Roch for a rank 2 vector bundle G on X restricted to a cycle Z supported on E states

$$\chi(G \otimes \mathcal{O}_Z) = -Z \cdot (Z + K) + Z \cdot \det G.$$

(This may be easily deduced from the formula when G is a line bundle.) We let $Z = E' = E - R$ and $G = S$, and note that $\det(S) = -(K + E)$ and $E \cdot (E + K) = 2h^1(\mathcal{O}_E) - 2$. Now a small calculation establishes the claim. We conclude

$$h^1(S \otimes \mathcal{O}_{E'}) = r + 4h^1(\mathcal{O}_E) - 4 + h^0(S \otimes \mathcal{O}_{E'}).$$

Now apply Lemmas 2.4 and 2.5.

Finally, the short exact sequence

$$0 \rightarrow \mathcal{O}_R(-E') \rightarrow \mathcal{O}_E \rightarrow \mathcal{O}_{E'} \rightarrow 0$$

has as first term the direct sum of $\mathcal{O}(-1)$ for the rational end curves. Tensoring with S , H^1 of the first term is 0, whence the equality of H^1 of $S \otimes \mathcal{O}_E$ and $S \otimes \mathcal{O}_{E'}$. \square

REMARK 2.7. The last short exact sequence also implies that in the star-shaped case, $H^0(S \otimes \mathcal{O}_E) \rightarrow H^0(S \otimes \mathcal{O}_{E'})$ is a surjection onto a one-dimensional space, hence either space surjects onto $H^0(S \otimes \mathcal{O}_{E_0})$, where E_0 is the central curve.

REMARK 2.8. In case E consists of one smooth curve of genus $g > 1$, one understands $h^1(S \otimes \mathcal{O}_E) = 4g - 3$ as corresponding to $3g - 3$ deformations of the curve and g deformations of the conormal bundle.

We conclude with the useful

PROPOSITION 2.9. *Notation as above,*

- (1) $h^1(S) = h^1(S \otimes \mathcal{O}_E) + h^1(S(-E'))$ if Γ is not star-shaped or $(V, 0)$ is weighted-homogeneous
- (2) $h^1(S) = h^1(S \otimes \mathcal{O}_E) + h^1(S(-E')) - 1$ if Γ is star-shaped but $(V, 0)$ is not weighted homogeneous.

Proof. Using Proposition 2.2 and the previous lemmas, one concludes that $H^0(S) \rightarrow H^0(S \otimes \mathcal{O}_{E'})$ is the zero-map except when $(V, 0)$ is not weighted homogeneous. \square

It is much easier to determine the cohomology of the restriction to E of the determinant of S , namely $-(K_X + E)$.

PROPOSITION 2.10. *Consider the minimal good resolution $(X, E) \rightarrow (V, 0)$ of a normal surface singularity, with graph Γ , excluding the 3 cases above. Then*

$$h^1(-(K_X + E) \otimes \mathcal{O}_E) = r + 3h^1(\mathcal{O}_E) - 3.$$

Proof. Since $-(K_X + E) \cdot E_i = -(2g_i - 2 + t_i) \leq 0$ except at rational end curves, and is strictly negative for at least one E_i , one easily concludes that $h^0(-(K_X + E) \otimes \mathcal{O}_{E'}) = 0$. As at the end of the proof of the Proposition 2.6, we have

$$h^0(-(K_X + E) \otimes \mathcal{O}_E) = h^0(-(K_X + E) \otimes \mathcal{O}_R(-E')),$$

which equals r . The assertion now follows from Riemann-Roch, as

$$\chi(-(K_X + E) \otimes \mathcal{O}_E) = (-3/2)E \cdot (E + K_X) = 3(1 - h^1(\mathcal{O}_E)).$$

□

Recall that the Main Conjecture of [14] is an inequality about

$$h^1(X, \mathcal{O}_X) - h^1(X, S_X) + h^1(X, -(K_X + E)).$$

For the corresponding expression for the cohomology of the restriction of these bundles to E , the previous results give a precise formula.

PROPOSITION 2.11. *Let $(X, E) \rightarrow (V, 0)$ be the minimal good resolution of a normal surface singularity, excluding the 3 cases above. Then*

$$h^1(\mathcal{O}_E) - h^1(S \otimes \mathcal{O}_E) + h^1(-(K_X + E) \otimes \mathcal{O}_E) = 1 - \delta,$$

where δ is 1 if the graph is star-shaped, 0 otherwise.

Thus, the Main Conjecture can be verified by explicit calculation in those cases for which h^1 of the bundles on X agrees with h^1 of their restriction to E . Since all one needs is an inequality, we have

COROLLARY 2.12. *Let $(X, E) \rightarrow (V, 0)$ be the minimal good resolution of a normal surface singularity, excluding the 3 cases above. If $H^1(S(-E')) = 0$, then the Main Conjecture holds.*

Proof. By Proposition 2.6, $h^1(S) = h^1(S \otimes \mathcal{O}_{E'}) = h^1(S \otimes \mathcal{O}_E)$. By the last Proposition,

$$h^1(S) = h^1(\mathcal{O}_E) + h^1(-(K_X + E) \otimes \mathcal{O}_E) - 1 + \delta \leq h^1(\mathcal{O}_X) + h^1(-(K_X + E)) - 1 + \delta,$$

so that

$$h^1(\mathcal{O}_X) - h^1(S) + h^1(-(K_X + E)) \geq 1 - \delta \geq 0.$$

The Main Conjecture asserts that in the non-Gorenstein case, the left hand side is non-negative, and equals 0 if and only the singularity is weighted homogeneous. The inequality is now clear. If the expression equals 0, then $\delta = 1$, so the graph is star-shaped. For E_0 the star, consider the maps

$$H^0(S) \rightarrow H^0(S \otimes \mathcal{O}_{E'}) \rightarrow H^0(S \otimes \mathcal{O}_{E_0}).$$

The first map is surjective via the vanishing cohomology assumption, while the second is an isomorphism via Lemmas 2.3 and 2.4. Proposition 2.2 then yields quasihomogeneity. Finally, it is proved in ([14], Theorem 3.3) that for a non-Gorenstein quasihomogeneous singularity, the expression is 0. □

REMARK 2.13. It is easy to see that $H^1(S(-E)) = 0$ implies $H^1(S(-E')) = 0$, though the converse is false in general, even for rational singularities.

3. Vanishing theorems. Let $(X, E) \rightarrow (V, 0)$ be a good resolution of a normal surface singularity, \mathcal{F} a vector bundle on X . We recall “easy vanishing theorems” (as in [12]) for the local cohomology

$$H_E^1(\mathcal{F}) = \varinjlim H^0(\mathcal{F} \otimes \mathcal{O}_Z(Z)).$$

PROPOSITION 3.1. (*Vanishing Theorem*) *Suppose that for every i , $Z \cdot E_i < 0$ implies $H^0(\mathcal{F} \otimes \mathcal{O}_{E_i}(Z)) = 0$. Then $H_E^1(\mathcal{F}) = 0$.*

Proof. Use a “downward induction” on Z ; if E_i is in the support of Z , use the exact sequence

$$0 \rightarrow \mathcal{F} \otimes \mathcal{O}_{Z-E_i}(Z - E_i) \rightarrow \mathcal{F} \otimes \mathcal{O}_Z(Z) \rightarrow \mathcal{F} \otimes \mathcal{O}_{E_i}(Z) \rightarrow 0.$$

□

COROLLARY 3.2. *If L is a line bundle on X with $L \cdot E_i \leq 0$, all i , then $H_E^1(L) = 0$.*

There is a slight refinement of the last Corollary which is occasionally useful.

PROPOSITION 3.3. *If L is a line bundle with $L \cdot E_i \leq 0$, all i , then either*

- (1) $H_E^1(L(-E)) = 0$, or
- (2) $L \otimes \mathcal{O}_E \simeq \mathcal{O}_E$ and $\dim H_E^1(L(-E)) = 1$.

Proof. One has the standard exact sequence

$$0 = H_E^0(L) \rightarrow H^0(L \otimes \mathcal{O}_E) \rightarrow H_E^1(L(-E)) \rightarrow H_E^1(L) = 0.$$

Suppose that $H^0(L \otimes \mathcal{O}_{E_i}) = 0$, some i . By induction, we show that for a connected F between E_i and E , one has $H^0(L \otimes \mathcal{O}_F) = 0$. For, given an $F' < F$ with $H^0(L \otimes \mathcal{O}_{F'}) = 0$, choose an E_j in $F - F'$ which intersects F' , and consider

$$0 \rightarrow H^0(L \otimes \mathcal{O}_{E_j}(-F')) \rightarrow H^0(L \otimes \mathcal{O}_{F'+E_j}) \rightarrow H^0(L \otimes \mathcal{O}_{F'}) = 0.$$

By assumption, the degree of $L(-F') \otimes \mathcal{O}_{E_j}$ is negative, whence the assertion.

The only other possibility is that $L \otimes \mathcal{O}_{E_i} \simeq \mathcal{O}_{E_i}$, all i . The same argument as above shows $\dim H^0(L \otimes \mathcal{O}_E) = 1$. This global section of $L \otimes \mathcal{O}_E$ induces an isomorphism on each E_i , so is an isomorphism itself. □

If $(V, 0)$ has a rational singularity, then $Z \cdot (Z + K) < 0$, so every Z contains an E_i with $(Z + K) \cdot E_i < 0$, i.e. $Z \cdot E_i < 2 - d_i$. This yields the following refinements.

PROPOSITION 3.4. *Assume $(V, 0)$ is a rational singularity. Suppose that for every i , $Z \cdot E_i < 2 - d_i$ implies $H^0(\mathcal{F} \otimes \mathcal{O}_{E_i}(Z)) = 0$. Then $H_E^1(\mathcal{F}) = 0$.*

COROLLARY 3.5. *Assume $(V, 0)$ is a rational singularity and L is a line bundle on X . If $L \cdot E_i \leq d_i - 2$, all i , then $H_E^1(L) = 0$.*

We wish to study $H^1(S)$ and $H^1(-(K_X + E))$ in both the general and rational cases. By duality, $h^1(\mathcal{F}) = h_E^1(\mathcal{F}^* \otimes K_X)$, so one can use the easy vanishing theorems, using the duals of the short exact sequences

$$0 \rightarrow \mathcal{O}_{E_i} \rightarrow S \otimes \mathcal{O}_{E_i} \rightarrow \Theta_{E_i}(-t_i) \rightarrow 0.$$

PROPOSITION 3.6. *Let $(X, E) \rightarrow (V, 0)$ be the minimal good resolution of a normal surface singularity. Then for any exceptional divisor D ,*

(1) $H^1(S(-D)) = 0$ if for all i ,

$$D \cdot E_i \leq \min \{2(2 - 2g_i) - t_i - d_i, 2 - 2g_i - d_i\}.$$

(2) $H^1(-(K_X + E)(-D)) = 0$ if for all i ,

$$D \cdot E_i \leq 2(2 - 2g_i) - t_i - d_i.$$

Proof. For the first statement, note $h^1(S(-D)) = h^1_E(S^*(K_X + D))$. From the standard sequence for $S \otimes \mathcal{O}_{E_i}$, one has

$$0 \rightarrow K_{E_i}(t_i) \rightarrow S^* \otimes \mathcal{O}_{E_i} \rightarrow \mathcal{O}_{E_i} \rightarrow 0,$$

hence for any effective cycle Z ,

$$0 \rightarrow K_{E_i}^{\otimes 2}(t_i + d_i)(D + Z) \rightarrow S^*(K_X + D) \otimes \mathcal{O}_{E_i}(Z) \rightarrow K_{E_i}(d_i)(D + Z) \rightarrow 0,$$

Via Proposition 3.1, one has only to check needed inequalities for $D \cdot E_i$ to guarantee that the first and third line bundles have negative degree whenever $Z \cdot E_i < 0$. One must handle separately the case of a rational end-curve ($g_i = 0, t_i = 1$), for which the second bound in the minimum is needed.

The second statement is proved similarly. \square

PROPOSITION 3.7. *Let $(X, E) \rightarrow (V, 0)$ be the minimal good resolution of a rational surface singularity. Then*

(1) $H^1(S(-D)) = 0$ if for all i ,

$$D \cdot E_i \leq \min \{2 - t_i, 0\}.$$

(2) $H^1(-(K_X + E)(-D)) = 0$ if for all i ,

$$D \cdot E_i \leq 2 - t_i.$$

Proof. Same as the preceding proposition, but now using Proposition 3.4. \square

As indicated by Proposition 3.3 above, if one of the vanishing theorems holds for $D = nE$, one can sometimes conclude vanishing for $D = (n - 1)E$, with only a mildly more restrictive condition.

THEOREM 3.8. *Let $(X, E) \rightarrow (V, 0)$ be the MGR of a normal surface singularity. Suppose that for all i , one has*

$$(*) \quad d_i \geq 2(2g_i - 2) + 3t_i,$$

with strict inequality for at least one i . Then

- (1) $h^1(S(-E)) = h^1(-(K_X + E)(-E)) = h^1(\mathcal{O}_X(-E)) = 0$.
- (2) If Γ is star-shaped, then $(V, 0)$ is weighted homogeneous, and $h^1(S) = r + 4h^1(\mathcal{O}_E) - 3$.
- (3) If Γ is not star-shaped, then $h^1(S) = r + 4h^1(\mathcal{O}_E) - 4$.
- (4) $h^1(\mathcal{O}_X) - h^1(S_X) + h^1(-(K_X + E)) = 1 - \delta$, where δ is 1 if $(V, 0)$ is weighted homogeneous, 0 otherwise.

Proof. By hypothesis, $(2K_X + 3E) \cdot E_i \leq 0$, all i . By Proposition 3.3, this implies that $h^1_E(2K_X + 2E) = 0$, as long as the inequality is strict for some i . Dually, $h^1(-(K_X + E)(-E)) = 0$. Similarly, one can show $h^1(\mathcal{O}_X(-E)) = 0$ as long as $d_i \geq 2g_i - 2 + 2t_i$, all i , with at least one strict inequality.

Applying Proposition 3.6(1) with $D = 2E$, one has $h^1(S(-2E)) = 0$ as long as one has the inequalities $(*)$ and $d_i \geq 2t_i - 2$. This second inequality is a consequence of $(*)$ unless $g_i = 0, t_i = 1$; but in that case the extra inequality is $d_i \geq 0$, which is automatic. Thus $h^1(S(-E)) = h^1(S(-E) \otimes \mathcal{O}_E)$. The dualizing sheaf of E is the restriction of $K_X + E$, so by duality on E , the last space has dimension $h^0(S^*(K_X + 2E) \otimes \mathcal{O}_E)$. For each i we have the exact sequence

$$0 \rightarrow K_{E_i}^{\otimes 2}(3t_i - d_i) \rightarrow S^*(K_X + 2E) \otimes \mathcal{O}_{E_i} \rightarrow K_{E_i}(2t_i - d_i) \rightarrow 0.$$

The two line bundles are easily checked to have degree ≤ 0 given $(*)$. If for some i we have $d_i > 2(2g_i - 2) + 3t_i$, both line bundles have strictly negative degree (one deals separately with the case $t_i = 1, g_i = 0$). Therefore, $h^0(S^*(K_X + 2E) \otimes \mathcal{O}_{E_i}) = 0$. One can now apply the same induction trick on connected subcycles as in the proof of Proposition 3.3 to conclude that $h^0(S^*(K_X + 2E) \otimes \mathcal{O}_E) = 0$.

By the cohomology vanishing of (1), the expressions in (2) and (3) come from Propositions 2.6 and 2.9; also, (4) is given by Proposition 2.11. It remains to show that the graph is star-shaped if and only if the singularity is weighted homogeneous. One direction is obvious; the other is proved as in Corollary 2.12, replacing E' there by E . \square

REMARK 3.9. The condition $(*)$ as well as the elimination of certain simple graphs means that none of the singularities in the Theorem could be Gorenstein. The condition $(*)$ in case E is one smooth curve is Grauert's well-known theorem that such a singularity is a cone.

THEOREM 3.10. *Let $(X, E) \rightarrow (V, 0)$ be the MGR of a rational surface singularity. Suppose that for all i , one has*

$$d_i \geq 2t_i - 2.$$

- (1) $h^1(S(-E)) = h^1(-(K_X + E)(-E)) = 0$.
- (2) If Γ is star-shaped, then $(V, 0)$ is weighted homogeneous, and $h^1(S) = h^1(-(K_X + E)) = r - 3$.
- (3) If Γ is not star-shaped, then $h^1(S) = h^1(-(K_X + E)) - 1 = r - 4$.

Proof. (1) follows directly from Proposition 3.7 with $D = E$; the other statements follow as in the preceding proof. \square

COROLLARY 3.11. *For a rational singularity $(V, 0)$ with $d_i \geq 2t_i - 2$, all i , one has*

- (1) if Γ is star-shaped, then $\dim T^1_V = \sum_i (2d_i - 3) + r - 4$.
- (2) if Γ is not star-shaped, then $\dim T^1_V = \sum_i (2d_i - 3) + r - 5$.

Proof. That $d_i \geq 2t_i - 2$ implies that $d_i \geq t_i$, hence the fundamental cycle is reduced. Further, $d_i > t_i$ unless $d_i = t_i = 2$. Therefore, blowing up the maximal ideal yields only rational double point singularities. According to the main formula of [2], the dimension of T^1 minus the dimension of the Artin component equals the multiplicity of the singularity minus 3; there is no contribution from the other infinitely near points. The formulas now follow. \square

Of course, in general higher-order infinitesimal neighborhoods of E can contribute to the three invariants in the Main Conjecture.

PROPOSITION 3.12. *Suppose the MGR $(X, E) \rightarrow (V, 0)$ has E a smooth curve of genus $g \geq 2$, whose conormal bundle $\mathcal{O}_E(-E) \cong L$ has degree $d > 2g - 2$. Then*

- (1) $h^1(\mathcal{O}_X) = g$
- (2) $h^1(-(K_X + E)) = 3g - 3 + h^0(E, 2K_E - L)$
- (3) $h^1(S) \leq h^1(\mathcal{O}_X) + h^1(-(K_X + E))$, and equality is equivalent to quasi-homogeneity.

Proof. One computes directly that $h^1(S(-2E)) = 0$, using that

$$H^0(S^*(2E + K_X) \otimes \mathcal{O}_E(nE)) = 0, n \geq 1.$$

Similarly, $h^1(-(K_X + E)(-2E)) = h^1(\mathcal{O}_X(-E)) = 0$. Thus, $H^0(S) \rightarrow H^0(S \otimes \mathcal{O}_{2E})$ is surjective, and $h^1(S) = h^1(S \otimes \mathcal{O}_{2E})$. Now use the long exact sequence

$$H^0(S \otimes \mathcal{O}_{2E}) \rightarrow H^0(S \otimes \mathcal{O}_E) \rightarrow H^1(S \otimes \mathcal{O}_E(-E)) \rightarrow H^1(S \otimes \mathcal{O}_{2E}) \rightarrow H^1(S \otimes \mathcal{O}_E) \rightarrow 0.$$

The first map, into a one-dimensional space, is surjective if and only if $H^0(S) \rightarrow H^0(S \otimes \mathcal{O}_E)$ is surjective, which as mentioned earlier is equivalent to quasi-homogeneity. One examines the same sequence with S replaced by $-(K_X + E)$. Everything now follows easily using Propositions 2.6 and 2.10. \square

REMARK 3.13. A singularity of this type is an equisingular deformation of the cone over (E, L) . For, consider in the local ring the filtration given by order of vanishing along E ; then there is an equisingular degeneration to the associated graded ring, corresponding to the cone (cf. [14], 4.8). For the cone, equisingular deformations are counted by the graded $T^1(i)$, where $i \geq 0$. As computed in ([11], 3.3), the weight 0 part has dimension $4g - 3$ (E varies in $3g - 3$ ways and L in g); in weight 1 one has $h^1(E, \Theta \otimes L^{-1})$ non-conical equisingular deformations leaving E and the conormal bundle L unchanged.

4. Rational singularities. For rational singularities, there is an explicit topological formula for the second plurigenus $h^1(-(K_X + E))$.

THEOREM 4.1. [14],(4.4) *On the MGR of a rational surface singularity (not an RDP), let Y be the smallest effective cycle satisfying $Y \cdot E_i \leq 2 - d_i$, all i . If $Z = Y - E$, then*

$$h^1(-(K_X + E)) = Z \cdot (Z + 3K)/2 + Z \cdot E.$$

At a node, $E \cdot E_i = t_i - d_i > 2 - d_i$, so Y has multiplicity at least 2 there; the same then applies for any neighbors with $t = 2$. Therefore, unless $Y = 0$ (the RDP case) or $Y = E$ (cyclic quotient), one has $Y \geq E + E'$.

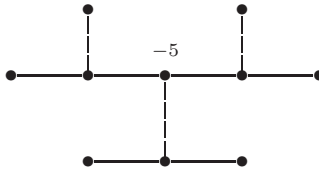
PROPOSITION 4.2. *Exclude RDP's and cyclic quotients. Then $Y = E + E'$ iff*

$$(**) d_i \geq t_i + t'_i - 2,$$

all i . In this case, $h^1(-(K_X + E)) = r - 3$, where r is the number of ends of the graph.

Proof. Examine $(E + E') \cdot E_i \leq 2 - d_i$ and Theorem 4.1 (cf. also Proposition 2.6). \square

EXAMPLE 4.3. The rational graph below satisfies (**) and has multiplicity 8; as usual, the unmarked bullets are -2 's.



Thus, for graphs satisfying (**) (a generalization of the already considered case $d_i \geq 2t_i - 2$), the Rational Conjecture states that $h^1(S) \leq r - 3$, with equality exactly in the quasihomogeneous case. But $h^1(S) \geq h^1(S \otimes \mathcal{O}_E)$, and the second term is $r - 3$ or $r - 4$, depending upon whether the graph is star-shaped or not. So, in this case the Rational Conjecture is equivalent to the assertion $h^1(S) = h^1(S \otimes \mathcal{O}_E) = h^1(S \otimes \mathcal{O}_{E'})$. In fact, there is a converse to Corollary 2.12:

PROPOSITION 4.4. *Let $(V, 0)$ be a rational singularity whose graph satisfies*

$$(**) \quad d_i \geq t_i + t'_i - 2, \text{ all } i.$$

Then the Rational Conjecture for $(V, 0)$ is equivalent to $H^1(S(-E')) = 0$.

Proof. The first map in the exact sequence

$$H^0(S) \rightarrow H^0(S \otimes \mathcal{O}_{E'}) \rightarrow H^1(S(-E')) \rightarrow H^1(S) \rightarrow H^1(S \otimes \mathcal{O}_{E'}) \rightarrow 0$$

is the zero-map unless the singularity is weighted homogeneous, in which case it is surjective onto a one-dimensional space. The missed cases of RDP's and cyclic quotients are easily verified separately. \square

The next few sections will be devoted to proving the following result.

THEOREM 4.5. *Let $(V, 0)$ be a rational singularity whose graph satisfies*

$$(**) \quad d_i \geq t_i + t'_i - 2, \text{ all } i.$$

Then $H^1(S(-E') \otimes \mathcal{O}_E) = 0$.

Unfortunately, at present we cannot conclude that $H^1(S(-E')) = 0$ without a further hypothesis.

COROLLARY 4.6. *Suppose a rational singularity satisfying (**) has a reduced fundamental cycle, i.e. $d_i \geq t_i$, all i . Then $H^1(S(-E')) = 0$, $h^1(S) = h^1(S \otimes \mathcal{O}_E)$, and the Rational Conjecture is true.*

Proof. $h^1(S(-(E+E'))) = 0$ by Proposition 3.7(1) and the hypotheses, as a simple calculation checks. Thus, $h^1(S(-E')) = h^1(S(-E') \otimes \mathcal{O}_E) = 0$, by the Theorem. \square

Note Example 4.3 is not covered by the Corollary; but see Example 7.10 below. Theorem 4.5 does not follow from an "easy vanishing theorem," and is false in characteristic p (Example 8.7).

5. Computing $H^1(S(-E') \otimes \mathcal{O}_E)$. There is an exact sequence

$$0 \rightarrow \mathcal{O}_E \rightarrow \bigoplus \mathcal{O}_{E_i} \rightarrow \bigoplus \mathbb{C}_{P_{ij}} \rightarrow 0,$$

where the key map compares functions on adjacent curves E_i and E_j with their values at the intersection point P_{ij} . (That is, on each E_i , at an intersection point P_{ij} one sends a function f to $\pm f(P_{ij})$, doing the opposite for E_j at that point.) Tensoring with $S(-E')$ gives the important map

$$\Phi_E : \bigoplus_{E_i \subset E} H^0(S(-E') \otimes \mathcal{O}_{E_i}) \rightarrow \bigoplus_{P_{ij}=E_i \cap E_j} H^0(S(-E') \otimes \mathbb{C}_{P_{ij}}).$$

LEMMA 5.1.

- (1) $H^1(S(-E') \otimes \mathcal{O}_{E_i}) = 0$ unless $d_i \leq t_i + t'_i - 4$.
- (2) If $d_i \geq t_i + t'_i - 3$ for all i , then

$$\text{Coker } \Phi_E = H^1(S(-E') \otimes \mathcal{O}_E).$$

Proof. The first assertion is easily verified. For the second, tensor the short exact sequence with $S(-E')$ and take cohomology. \square

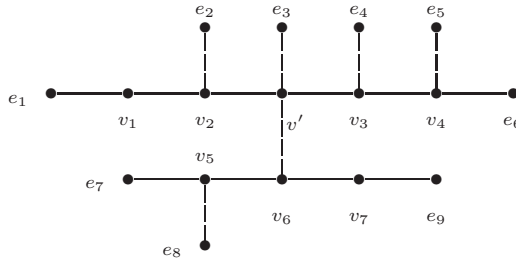
The task is therefore to prove surjectivity of Φ_E . For $P_{ij} = E_i \cap E_j$, one gets contributions to the two-dimensional space $S(-E') \otimes \mathbb{C}_{P_{ij}}$ from $H^0(S(-E') \otimes \mathcal{O}_{E_i})$ and $H^0(S(-E') \otimes \mathcal{O}_{E_j})$. We speak of the contribution of E_i to its t_i “slots”. The goal is to account systematically for contributions at the intersection points from the curves intersecting there. We achieve this via an induction process using an increasing sequence of certain reduced exceptional divisors, called *cones*, where one new curve (and hence one new intersection point) is added at each step.

This process is best explained dually by considering a class of subtrees of the graph Γ . Let v be a vertex, p an adjacent edge. Define the *cone* $\mathcal{C}(v, p)$ to be the connected component of v in the graph $\Gamma - \{p\}$, plus the edge p sticking out of it. In other words, $\mathcal{C}(v, p)$ consists of all vertices and edges on the “other side” of v , away from p ; but we keep the edge p as well. Thus, $\mathcal{C}(v, p)$ arises from adding v and p to $t_v - 1$ other cones $\mathcal{C}(v_i, p_i)$, where the p_i are the other edges emanating from v , with v_i the other vertex of p_i ; here t_v is the valency of v . We say that the $\{\mathcal{C}(v_i, p_i)\}$ are *completed* by adding v and p , forming $\mathcal{C}(v, p)$.

In Example 5.2 below, let p' be the edge connecting v' and v_6 . Then $\mathcal{C}(v', p')$ consists of all vertices and edges “not below p' ”, and is the completion of the three cones centered at v_2, v_3 , and e_3 , with edges pointing towards v' .

We describe several sequences of cones as follows: in Round 1, consider cones consisting of an end vertex and its one edge. In Round 2, consider those v all but one of whose neighbors are ends, with p the edge leading to the other neighbor; form the cones $\mathcal{C}(v, p)$. The new vertices are the ends of the graph you get by removing the ends and their edges from Γ . In Round 3, remove all the vertices and edges from previous cones, and for each end vertex v of the corresponding graph, form a new cone $\mathcal{C}(v, p)$, where p is the edge of v not in a previous cone. That is, we *complete* several previously considered cones. Continue this process, until removing from Γ all the vertices and edges in previous cones, the remaining graph has 1 or 2 ends. If two such end vertices remain, make a choice of one, use it to form a cone, and continue. In this way, one is eventually left with a graph with one vertex, all of whose neighbors occur in previously chosen cones.

EXAMPLE 5.2.



We describe the Rounds for this example, without mentioning the edges p , which will always be “towards the center of the graph v' ”, in an obvious sense. In Round 1, one has the nine cones corresponding to the e_i . In Round 2, one has the cones leading out from v_1, v_4, v_5 , and v_7 . In Round 3, we have the cones leading out from v_2, v_3 , and v_6 . This leaves the vertices v' and v_6 and the edge p' connecting them, and we choose one of them (say, v_6) to form the corresponding cone. Thus, after Round 4, one has 4 cones (forming the graph $\Gamma - \{v'\}$). There remains only v' , all of whose neighbors have been accounted for in previously chosen cones.

Alternatively, one may start with any interior vertex v , whose edges p_1, \dots, p_t lead to vertices v_1, \dots, v_t , and consider it the terminal stage of t cones $\mathcal{C}(v_i, p_i)$. Then, take apart each $\mathcal{C}(v_i, p_i)$ by reversing the completion process. But for our induction, one starts at the ends, and works one’s way up to a “terminal” vertex v .

Return to the point of view of exceptional divisors. The pair (v, p) corresponds to a curve E_0 and an intersection point P_0 , and the cone $\mathcal{C}(E_0, P_0)$ is the union of all curves leading away from E_0 via the $t_0 - 1$ intersection points other than P_0 (but with P_0 a distinguished point.) We have described above how an increasing sequence of cones eventually exhausts the graph.

Each pair (v, p) corresponds to a curve E_0 and an intersection point P_0 , and the cone $\mathcal{C}(E_0, P_0)$ is the union of all curves leading away from E_0 via the $t_0 - 1$ intersection points other than P_0 (but with P_0 a distinguished point.) From now on, we shall use curve (rather than node) notation.

We outline the induction process using increasing sequences of cones. Suppose E_0 is an end-curve, intersecting at P_0 with another curve E_1 . We will show below that

$$H^0(S(-E') \otimes \mathcal{O}_{E_0}) \rightarrow H^0(S(-E') \otimes \mathbb{C}_{P_0})$$

is an inclusion of a one-dimensional space; so in showing surjectivity of Φ_E we will have to account for the missing dimension at P_0 by using a contribution from E_1 . At the next step, change notation so that E_0 has valency t and intersects $t - 1$ end-curves at P_1, \dots, P_{t-1} ; if P_0 is the remaining intersection point, we are considering the cone $\mathcal{C}(E_0, P_0)$. We need $H^0(S(-E') \otimes \mathcal{O}_{E_0})$ to have enough sections to account for the missing dimensions at P_1, \dots, P_{t-1} ; that is, for each of these P_i we need a section that vanishes at the other $t - 2$ points and contributes the needed dimension at $S(-E') \otimes \mathbb{C}_{P_i}$. That would take care of the desired surjectivity at $t - 1$ intersection points. Optimally, we would also like the space of sections of $H^0(S(-E') \otimes \mathcal{O}_{E_0})$ which vanish at these $t - 1$ points to map onto $S(-E') \otimes \mathbb{C}_{P_0}$. Then we would not have to worry about these t points for the rest of the induction. We’ll call this the Type I case. But we will be satisfied if we can find a section vanishing on the $t - 1$ points and giving a “useful” element of $S(-E') \otimes \mathbb{C}_{P_0}$; call this case of Type II. The

general case will consist of completing cones for which appropriate surjectivity results have been established. At the last step, one has an E_0 all of whose t intersection points arise from previously considered cones.

Recall that if E_i is locally defined by $y = 0$, E_j by $x = 0$, then $S \otimes \mathbb{C}_{P_{ij}}$ has a natural ordered basis $x\partial/\partial x, y\partial/\partial y$ which is independent of the choice of x and y . Multiplying by a local equation of E' (either x , y , or xy) gives an ordered basis for $S(-E') \otimes \mathbb{C}_{P_{ij}}$, so that elements are given by an ordered pair of numbers (a, b) ; this is equal (up to a scalar multiplication) to the element (b, a) viewed from considering $E_j \cap E_i$.

Returning to cohomological considerations, for a cone $\mathcal{C}(E_0, P_0)$, we consider two natural “evaluation” maps:

$$\Phi_{E_0, P_0} : \bigoplus_{E_i \subset \mathcal{C}} H^0(S(-E') \otimes \mathcal{O}_{E_i}) \rightarrow \bigoplus_{P_{ij}=E_i \cap E_j} H^0(S(-E') \otimes \mathbb{C}_{P_{ij}})$$

$$\Psi_{E_0, P_0} : \bigoplus_{E_i \subset \mathcal{C}} H^0(S(-E') \otimes \mathcal{O}_{E_i}) \rightarrow \bigoplus_{P_{ij}=E_i \cap E_j} H^0(S(-E') \otimes \mathbb{C}_{P_{ij}}) \oplus H^0(S(-E') \otimes \mathbb{C}_{P_0})$$

In other words, for Φ we consider all points of intersection of curves in \mathcal{C} , $t_0 - 1$ of which are on E_0 ; for Ψ , we also consider evaluation at the additional point P_0 . Clearly,

$$\Phi_{E_0, P_0} = \pi \cdot \Psi_{E_0, P_0},$$

where π is projection off the two-dimensional direct summand $H^0(S(-E') \otimes \mathbb{C}_{P_0})$. We will study the cokernel of these maps for our judiciously chosen increasing sequence of cones, ultimately concluding the desired vanishing result.

It will turn out that if $d_i \geq t_i + t'_i - 2$ for all i , then all cones (except from end-curves) will have one of two properties:

DEFINITION 5.3. $\mathcal{C}(E_0, P_0)$ is of *Type I* if Ψ_{E_0, P_0} is surjective.

DEFINITION 5.4. $\mathcal{C}(E_0, P_0)$ is of *Type II* if

- (1) Ψ_{E_0, P_0} has image of codimension 1
- (2) Φ_{E_0, P_0} is surjective
- (3) $\text{Im } \Psi \cap \text{Ker } \pi$ contains an element in $S(-E') \otimes \mathbb{C}_{P_0}$ with coordinates $(1, -b)$, where $b \geq 1$.

This will be done by a variant of the method in Section 2; local coordinates on $2E$ will be chosen in the same way, but now one allows that t_i may be greater than 2.

6. The cokernels of Φ and Ψ for cones. In considering a cone $\mathcal{C}(E_0, P_0)$, we choose coordinates for the curves as in Section 2. To simplify notation, we write $d = d_0$, $t = t_0$, and $t' = t'_0$. An E_0 will be defined in a first chart by $y = 0$, with P_0 given by $x = 0$, and the other intersection points (if any) P_1, \dots, P_{t-1} given by $x = a_1, \dots, a_{t-1}$ (no intersection points at ∞). Then $S \otimes \mathcal{O}_{E_0}$ will be generated by the images of $x \prod_{j=1}^{t-1} (x - a_j) \partial/\partial x$ and $y\partial/\partial y$. The second chart, defined by x', y' , will be as in Section 2. We write out the elements of $H^0(S(-E') \otimes \mathcal{O}_{E_0})$ and compute their images in the various “slots”, i.e., the two-dimensional spaces $S(-E') \otimes \mathbb{C}_{P_j}$. Note while $S \otimes \mathbb{C}_P$ has a natural ordered basis (Lemma 2.1), the ordered basis for $S(-E') \otimes \mathbb{C}_P$ is unique up to a scalar multiple.

Note that the bundles we consider satisfy

- (1) $t > 1$: $S(-E') \otimes \mathcal{O}_{E_0} \cong \mathcal{O}_{E_0}(d-t') \oplus \mathcal{O}_{E_0}(d-t-t'+2)$
(2) $t = 1$: $S(-E') \otimes \mathcal{O}_{E_0} \cong \mathcal{O}_{E_0}(-1) \oplus \mathcal{O}_{E_0}$.

In the “easy case,” $H^1(S(-E'))(-\sum_{i=0}^{t-1} P_i) \otimes \mathcal{O}_{E_0}$ vanishes, hence

$$H^0(S(-E') \otimes \mathcal{O}_{E_0}) \rightarrow \bigoplus_{i=0}^{t-1} H^0(S(-E') \otimes \mathbb{C}_{P_i})$$

is surjective, guaranteeing that E_0 fills all the slot entries at its t intersection points. We start with

LEMMA 6.1. *For a curve E_0 with $t = 2$, suppose $t' = 1$ or $d \geq 3$. Then*

$$H^0(S(-E') \otimes \mathcal{O}_{E_0}) \rightarrow \bigoplus_{i=0}^1 H^0(S(-E') \otimes \mathbb{C}_{P_i})$$

is surjective.

LEMMA 6.2. *Consider a cone $\mathcal{C}(E_0, P_0)$ consisting of a string of curves starting with an end-curve, for which $t_0 = 2$. Then the cone has Type I.*

Proof. Starting from the end-curve, name the curves in the string as E_r, \dots, E_1, E_0 , with intersection points $P_i = E_i \cap E_{i-1}, i = 1, \dots, r$. Then by Lemma 6.1, E_{r-1} fills up both of its slots, so $\mathcal{C}(E_{r-1}, P_{r-1})$ has Type I. Moving along the chain, a later E_i need only fill the slot at P_i , which is automatic because $H^1(S(-E') \otimes \mathcal{O}_{E_i}(-P_i)) = 0$. \square

The general case requires more delicate argument.

LEMMA 6.3. *Suppose E_0 is an end-curve, with self-intersection $-d$ and intersection point P_0 . Then $H^0(S(-E') \otimes \mathcal{O}_{E_0})$ is one-dimensional, with basis $x \cdot x\partial/\partial x - dx \cdot y\partial/\partial y$, whose image in $S(-E') \otimes \mathbb{C}_{P_0}$ is $(1, -d)$ in the ordered basis $x \cdot x\partial/\partial x, x \cdot y\partial/\partial y$.*

Proof. In the first chart, E' is defined by $x = 0$, so a section of $S(-E') \otimes \mathcal{O}_{E_0}$ is of the form

$$A(x)x \cdot x\partial/\partial x + B(x)x \cdot y\partial/\partial y.$$

In the other chart, where E' is empty, this becomes

$$-A(1/x')\partial/\partial x' + \{(d/x')A(1/x') + 1/x'B(1/x')\}y'\partial/\partial y'.$$

To be a global section requires that A is a constant, say 1, in which case B must be the constant $-d$. \square

Thus, a Round 1 cone $\mathcal{C}(E_0, P_0)$ could be considered of Type II, but we view these separately.

Now consider a cone $\mathcal{C}(E_0, P_0)$, where P_0 does not intersect an end-curve. We write out the global sections of $S(-E') \otimes \mathcal{O}_{E_0}$. Use coordinates for which P_0 is given by $x = 0$, and the other intersection points P_i are at $x = a_i, i = 1, \dots, t-1$. Assume further that the last $t-t'$ of these are the points intersection with end-curves. $S \otimes \mathcal{O}_{E_0}$ is generated on the first chart by $x \prod (x - a_i)\partial/\partial x$ and $y\partial/\partial y$, and on the second by $\prod (x' - (1/a_i))\partial/\partial x'$ and $y'\partial/\partial y'$. An equation for E' is given by $z = y \cdot x \prod_{i=1}^{t'-1} (x - a_i)$

in the first chart, and $z' = y' \prod_{i=1}^{t'-1} (x' - (1/a_i))$ on the second. For convenience, let $\alpha = \prod_{i=1}^{t'-1} (-a_i)$, $\beta = \prod_{j=1}^{t-1} (-a_j)$. A calculation yields the

LEMMA 6.4. *In the coordinates above, the sections of $S(-E') \otimes \mathcal{O}_{E_0}$ are written in the two charts as*

$$\begin{aligned} & A(x)zx \prod_{i=1}^{t-1} (x - a_i) \partial/\partial x + B(x)zy \partial/\partial y \\ &= z' \prod (x' - (1/a_i)) \partial/\partial x' \{ \alpha \beta (-x'^{d-t-t'+2} A(1/x')) \} \\ & \quad + z' y' \partial/\partial y' \{ \alpha \beta \prod (x' - (1/a_i)) dx'^{d-t-t'+1} A(1/x') + \alpha x'^{d-t'} B(1/x') \}. \end{aligned}$$

These are global sections exactly when

- (1) $A(x)$ is a polynomial of degree $d - t - t' + 2$, with coefficient of $x^{d-t-t'+2}$ denoted C
- (2) $B(x)$ is a polynomial of degree $d - t' + 1$, with coefficient of $x^{d-t'+1}$ denoted C'
- (3) $C' = -dC$.

Proof. Convert to coordinates in the second chart, carefully. \square

We record that for a global section as above, the element induced in the slot at P_j for $j > 0$ has coordinates

$$(A(a_j)a_j \prod_{i \neq j} (a_j - a_i), B(a_j)),$$

while the coordinates at P_0 are

$$(A(0)\beta, B(0)).$$

LEMMA 6.5. *Suppose $d = t = t' = 2$, with P_1 given by $a_1 = 1$. Then the general element of $H^0(S(-E') \otimes \mathcal{O}_{E_0})$ is of the form $Azx(x - 1)\partial/\partial x + (B - dAx)zy\partial/\partial y$, with A, B arbitrary. Its image in $S(-E') \otimes \mathbb{C}_{P_i}$ is $(-A, B)$ for $i = 0$, $(A, B - dA)$ for $i = 1$.*

Excluding for the moment the case that Γ is star-shaped with length one arms, one can form Round 2 cones.

LEMMA 6.6. *Suppose $\mathcal{C}(E_0, P_0)$ is a Round 2 cone; thus $t' = 1$. Then either*

- (1) $d \geq t$ and \mathcal{C} is of Type I, or
- (2) $d = t - 1$ (so $t \geq 3$) and \mathcal{C} is of Type II.

Proof. Assume first that $d \geq t$. Then in the notation of Lemma 6.4, we certainly have global sections with $A(x) = 0$ and $B(x)$ a polynomial of degree $t - 1$. In particular, for every $P_i, i \geq 0$, we can choose $B(x)$ to be a polynomial taking value 1 at P_i and 0 at $P_j, j \neq i$. For $i \geq 1$, this contributes $(0, 1)$ in the coordinates of $S(-E') \otimes \mathbb{C}_{P_i}$, and $(0, 0)$ in the $t - 1$ other slots. As the contribution from the corresponding end-cone in this slot is $(-d_i, 1)$, we conclude that $\text{Im } \Psi_{E_0, P_0}$ contains the direct summand $\oplus_{i=1}^{t-1} S(-E') \otimes \mathbb{C}_{P_i}$. On the other hand, we have $H^0(S(-E') \otimes \mathcal{O}_{E_0}) \rightarrow S(-E') \otimes \mathbb{C}_{P_0}$ is surjective since $H^1(S(-E') \otimes \mathcal{O}_{E_0}(-P_0)) = 0$. Thus, Ψ_{E_0, P_0} is surjective, and the cone is of Type I.

Next suppose $d = t - 1$. There is a global section with $A(x) = 0$ and $B(x)$ a polynomial of degree $t - 2$. Thus for every $1 \leq i \leq t - 1$, we can find a polynomial $B_i(x)$ which vanishes at all the P_j except for P_i and P_0 . Combining with the contribution from the end-curves, we conclude that Φ_{E_0, P_0} is surjective. To find an element in $\text{Im } \Psi \cap \text{Ker } \pi$, we consider the global section of $S(-E') \otimes \mathcal{O}_{E_0}$ with $A(x) = 1$ and

$$B(x) = -d \prod_{i=1}^{t-1} (x - a_i) - \sum_{i=1}^{t-1} (a_i/d_i) \prod_{k \neq i} (x - a_k).$$

For $j > 0$, $B(a_j) = -(a_j/d_j) \prod_{k \neq j} (a_j - a_k)$, so by the results above the contribution of the global section at this slot has coordinates

$$(a_j \prod_{k \neq j} (a_j - a_k), -(a_j/d_j) \prod_{k \neq j} (a_j - a_k)).$$

This is a non-0 multiple of the section $(-d_j, 1)$, which can be matched by a contribution from the corresponding end-curve. Furthermore, $B(0) = \beta(-d + \sum(1/d_i))$, hence the contribution of the section at P_0 has coordinates

$$(\beta, \beta(-d + \sum(1/d_i))).$$

Therefore, subtracting off contributions from the $t - 1$ end-curves gives an element in the image of Ψ whose only non-0 entries are at P_0 , and it is a multiple of $(1, -(d - \sum(1/d_i)))$. But as $d_i \geq 2$, we conclude that

$$d - \sum(1/d_i) \geq (t - 1) - (t - 1)/2 = (t - 1)/2 \geq 1$$

(as $t \geq 3$). Therefore, the corresponding cone is of Type II. \square

We can now consider a cone $\mathcal{C}(E_0, P_0)$ in an arbitrary round, postponing the terminal situation for which the previously considered cones involve all but one curve.

LEMMA 6.7. *Consider a cone $\mathcal{C}(E_0, P_0)$, with intersection points at P_1, \dots, P_{t-1} coming from previously considered Rounds. Assume as before that $d \geq t + t' - 2$. Then this cone has Type I unless $d = t + t' - 2$ and P_1, \dots, P_{t-1} come from end-curves or Type II cones. In that case, the cone has Type II.*

Proof. Recall that there is a global section of $S(-E') \otimes \mathcal{O}_{E_0}$ with $A(x) = 0$ and $B(x)$ a polynomial of degree $d - t'$. Suppose that some number t^* of the $t - 1$ points come from Type II cones or end-curves. Then for any of these points, there is a polynomial $B_i(x)$ of degree t^* which is non-0 at that point but vanishes at the $t^* - 1$ other points and also at P_0 . If $d - t' \geq t^*$, such polynomials can be used to construct global sections, and so we can conclude surjectivity of Ψ exactly as in the proof of the previous Lemma. In other words, the cone is of Type I unless $d - t' < t^*$. But by hypothesis $d - t' \geq t - 2$; so the only case not covered is that $t^* = t - 1$ and $d = t + t' - 2$.

Considering this remaining case, one can as in the above Lemma choose for every P_i ($i > 0$) a polynomial of degree $d - t' = t - 2$ which vanishes exactly at all $P_k, k \neq 0, i$. As in the last Lemma, one concludes surjectivity of Φ . It remains to produce an element in $\text{Im } \Psi \cap \text{Ker } \pi$ whose coordinates in the P_0 slot are of the form $(1, -u)$, where u is a rational number ≥ 1 . For this, we choose the global section as in the proof of the last Lemma, except that we must match the contribution $(1, -u_i)$ at P_i , where

$u_i \geq 1$. This choice will produce a contribution in the slot at P_0 of $(1, -d + \sum(1/u_i))$. Thus, it remains only to verify that

$$d - \sum_{i=1}^{t-1} (1/u_i) \geq 1.$$

As $u_i \geq 1$ and $d = t + t' - 2$, the inequality follows easily unless $t' = 1$. But that means all P_i ($i > 0$) come from end-curves, and that case was handled in the preceding Lemma. \square

We are ready to consider the terminal situation. Suppose E_0 is a curve with intersection points P_1, \dots, P_t intersecting with other curves E_1, \dots, E_t , and the corresponding cones $\mathcal{C}(E_i, P_i)$ have already been shown to be end-curves or of Type I or Type II as above.

LEMMA 6.8. *In the situation above, the map Φ_E is surjective, hence*

$$H^1(S(-E') \otimes \mathcal{O}_E) = 0.$$

Proof. We need to produce global sections of $S(-E') \otimes \mathcal{O}_{E_0}$ which account for missing entries in slots coming from Type II cones and end-curves. Assume there are $t^* \leq t$ of these. As above, if $d - t' \geq t^* - 1$ we can find at each of these points a suitable global section with slot entry $(0, 1)$ there, but vanishing at the other $t^* - 1$ points. This suffices to prove the surjectivity of Φ_E in that case.

But $d - t' < t^* - 1$ only when $t^* = t$ and $d = t + t' - 2$. However, we claim that this cannot happen, because the graph is that of a rational singularity. Consider the cycle $Z = E + E'$. For an end-curve E_i , we have $Z \cdot E_i = 2 - d_i$, hence $(Z + K) \cdot E_i = 0$. The condition $t^* = t$ means that every cone along the way has been of Type II; for any other curve E_i not E_0 , we have $d_i = t_i + t'_i - 2$. As $Z \cdot E_i = t_i + t'_i - 2d_i$, we conclude that also in this case $(Z + K) \cdot E_i = 0$. By rationality, $Z \cdot (Z + K) \leq -2$, so we must have $(Z + K) \cdot E_0 < 0$. But this says $t + t' - d - 2 < 0$, contradicting the hypothesis. \square

REMARK 6.9. Note that if the fundamental cycle is reduced, i.e., $d_i \geq t_i$ for all i , then there are no curves of Type II; one has only end-curves and Type I curves. By Lemma 6.6, this is clear at Round 2. In a later Round, by Lemma 6.7 the only new Type II case would occur if there were $t - 1$ end-curves; but that case was handled in the previous Round.

7. Some sharpened results. We have shown that if $d_i \geq t_i + t'_i - 2$, all i , then $H^1(S(-E') \otimes \mathcal{O}_E) = 0$. As is clear from Lemma 5.1, vanishing is not possible if some $d_i \leq t_i + t'_i - 4$. In this Section we discuss some vanishing for graphs with one or more vertices satisfying

$$d = t + t' - 3.$$

As $d \geq t - 1$, we have $t' \geq 2$. According to Lemma 6.2, on such a curve a global section of $S(-E') \otimes \mathcal{O}_{E_0}$ has $A(x) = 0$ and $B(x)$ a polynomial of degree $t - 3$. So, for any set of the $t - 3$ intersection points, one may chose a section vanishing at all of them, and giving a non-zero contribution of the form $(0, \cdot)$ at each of the other 3 points. We easily conclude the following two results.

LEMMA 7.1. Consider a cone $\mathcal{C}(E_0, P_0)$ so that E_0 satisfies $d = t+t' - 3$. Suppose that at least 2 of the vertices P_1, \dots, P_{t-1} correspond to Type I cones. Then $\mathcal{C}(E_0, P_0)$ has Type II, except that the contribution at P_0 is $(0, 1)$.

LEMMA 7.2. At the terminal stage, suppose E_0 satisfies $d = t + t' - 3$, and at least two of the t intersection points come from Type I cones. Then Φ_E is surjective, hence $H^1(S(-E') \otimes \mathcal{O}_E) = 0$.

COROLLARY 7.3. For a star-shaped rational graph whose central curve satisfies $d \geq t + t' - 3$, one has $H^1(S(-E') \otimes \mathcal{O}_E) = 0$.

Proof. Since $d \geq t - 1$, one has in the exceptional case $d = t + t' - 3$ that $t' \geq 2$, so there are at least 2 Type I strings. \square

LEMMA 7.4. Suppose the graph contains two curves E_0 and E'_0 satisfying $d = t + t' - 3$, connected by a (possibly empty) string of rational curves, while all other curves satisfy $d \geq t+t'-2$. For the $t_0 - 1$ intersection points of E_0 not pointing towards E'_0 , assume as in Lemma 7.1 that at least two correspond to Type I cones; make the same assumption for E'_0 . Then Φ_E is surjective, hence $H^1(S(-E') \otimes \mathcal{O}_E) = 0$.

Proof. By assumption, E_0 is connected at some P_0 by a string of curves with $t = t' = 2$ to some $P'_0 \in E'_0$. If $P_0 = P'_0$, then via Lemma 7.1 the contributions from the two curves to $S(-E') \otimes \mathbb{C}_{P_0}$ span the whole space, so Φ_E is surjective.

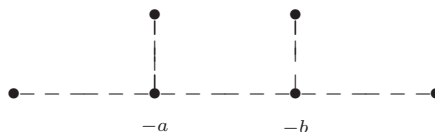
So, suppose E_0 is joined to E'_0 by a chain of r rational curves E_1, \dots, E_r , with $P_i = E_i \cap E_{i+1}$ ($i < r$). Assume first that all intermediary curves E_i have $d_i = 2$. Moving from E_0 towards E'_0 , we show that all intermediate cones $\mathcal{C}(E_i, P_i)$ are of Type II, except that the extra contribution at P_i is $(0, 1)$. By Lemma 6.5, the global sections of $S(-E') \otimes \mathcal{O}_{E_i}$ give a contribution of $(-A_i, B_i)$ at P_{i-1} , and $(A_i, B_i - 2A_i)$ at P_i . So, use $A_i = -1, B_i = 0$ to fill in the slot at P_{i-1} , then use $A_i = 0$ and $B_i = 1$ to make the contribution $(0, 1)$ at P_i . At the last stage, E'_0 , which already had a $(1, 0)$ at P'_0 , now receives a $(0, 1)$ from the last curve in the string.

If some E_i satisfies $d_i \geq 3$, then by Lemma 6.1 $H^0(S(-E') \otimes \mathcal{O}_{E_i})$ maps onto the sum of the two spaces $S(-E') \otimes \mathbb{C}_P$, P an intersection point of E_i . By Lemma 6.5, then each curve adjacent to E_i maps onto the space $S(-E') \otimes \mathbb{C}_P$ for P the outer point in the direction away from E_i . Continuing in this way gives the desired surjectivity, without even using the contributions of E_0 and E'_0 at P_0 and P'_0 . \square

REMARK 7.5. In the previous Lemma, if one assumes that the fundamental cycle is reduced, then as already mentioned there are no Type II curves except end-curves. But among the $t_0 - 1$ neighbors of E_0 are at least 2 non-end-curves, as $d - t_0 = t'_0 - 3 \geq 0$. So, the conditions on intersection points are automatically satisfied.

We illustrate this case with several examples. The first is known by [4] to be a taut singularity, which fact may be deduced in two steps.

EXAMPLE 7.6. For $a, b \geq 3$, a singularity with resolution graph

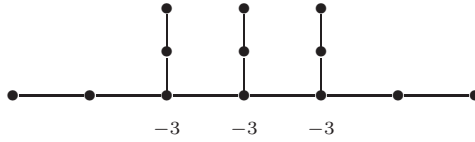


satisfies $H^1(S(-E') \otimes \mathcal{O}_E) = 0$.

Proof. Assume that all end-strings emanating from the two nodes E_1 and E_2 contain at least 2 curves; the other cases are similar or easier. Then Lemma 7.4 applies. \square

REMARK 7.7. Note that in the above example, the plurigenus $h^1(-(K_X + E))$ may have dimension much bigger than $h^1(S) = 0$. For instance, if all non-nodal curves are $-2's$, and the outward end-strings each have length n , then the plurigenus equals n .

EXAMPLE 7.8. Singularities with the graph below have $H^1(S(-E') \otimes \mathcal{O}_E) \neq 0$, hence $h^1(S) > h^1(S \otimes \mathcal{O}_E) = 1$.



Proof. The 3 curves E_i (corresponding to the 3 nodes) each satisfy $d_i = t_i + t'_i - 3$, so the sum of dimensions of the spaces of sections of $S(-E') \otimes \mathcal{O}_{E_i}$ equals 3. But only they can contribute to the two two-dimensional spaces $S(-E') \otimes \mathbb{C}_P$ at the two intersection points (the edges joining the nodes). So Φ_E cannot be surjective.

This example is still consistent with the Rational Conjecture, as the plurigenus equals 4. \square

THEOREM 7.9. Suppose a rational singularity, with reduced fundamental cycle, has all curves satisfying $d_i \geq t_i + t'_i - 2$ for all i , except that one allows that either

- (1) one curve satisfies $d = t + t' - 3$, or
- (2) two curves, separated by a (possibly empty) string of rational curves, satisfy $d = t + t' - 3$.

Then $h^1(S(-E')) = 0$, $h^1(S) = h^1(S \otimes \mathcal{O}_E)$, and the Rational Conjecture is satisfied.

Proof. Combining Lemmas 7.2 and 7.4 and Remark 7.5, we conclude that $H^1(S(-E') \otimes \mathcal{O}_E) = 0$. It suffices to show that $H^1(S(-E') \otimes \mathcal{O}_{nE}) = 0$ for all $n \geq 2$. For each $n \geq 1$, one proceeds inductively from the divisor nE to $(n + 1)E$ via $nE + F$, for a judiciously chosen $F \geq 0$ which is effective and reduced. Consider for a curve E_i not contained in F the sequence

$$0 \rightarrow S(-E' - nE - F) \otimes \mathcal{O}_{E_i} \rightarrow S(-E') \otimes \mathcal{O}_{nE + F + E_i} \rightarrow S(-E') \otimes \mathcal{O}_{nE + F} \rightarrow 0.$$

The requirement for the induction is that H^1 of the first term is 0. One has that $S(-E' - nE) \otimes \mathcal{O}_{E_i}$ equals

- (1) $\mathcal{O}_{E_i}(d_i - t'_i + n(d_i - t_i)) \oplus \mathcal{O}_{E_i}(2 - t_i + d_i - t'_i + n(d_i - t_i))$, if $t_i > 1$
- (2) $\mathcal{O}_{E_i}(-1 + n(d_i - 1)) \oplus \mathcal{O}_{E_i}(n(d_i - 1))$, if $t_i = 1$.

So, H^1 of the twist with $\mathcal{O}_{E_i}(-F)$ equals 0 as long as

- (1) $F \cdot E_i \leq 3 - t_i - t'_i + d_i + n(d_i - t_i)$, if $t_i \geq 2$
- (2) $F \cdot E_i \leq n(d_i - 1)$, if $t_i = 1$.

One way to proceed is to first choose some E_0 and go from nE to $nE + E_0$, and then step by step add a curve adjacent to what has already been chosen (i.e., go from

$nE + F$ to $nE + F + E_i$ if $F \cdot E_i = 1$). Given that $d_i \geq t_i$ and $d_i \geq t_i + t'_i - 3$, then this procedure will work starting with any E_0 unless there is a curve with

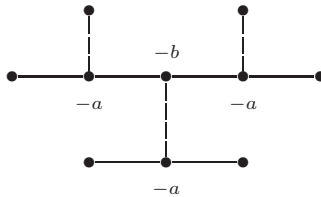
$$3 - t_i - t'_i + d_i + n(d_i - t_i) = 0,$$

i.e. $d_i = t_i$ and $t'_i = 3$. If there is only one such curve, then let E_0 be that curve; then the above procedure will get one from nE to $nE + E_0$ and then on to $(n + 1)E$, and the desired vanishing of $H^1(S(-E'))$ holds.

Now suppose there are two curves E_0 and E'_0 with $d = t + t' - 3$. If the path between them contains a curve E_1 with $d_1 > t_1 + t'_1 - 2$, then start with $F = E_0 + E'_0$, and successively adds the curves between E_0 and E'_0 , up to E_1 . At this point, the new F will satisfy $F \cdot E_1 = 2$, but now the inequality of (1) is satisfied, and the induction may proceed as before.

We are left with the case that the only curves (if any) in between E_0 and E'_0 satisfy $d = t + t' - 2$, i.e. $d = t = t' = 2$. In that case, let F be the sum of E_0 , E'_0 and all the curves in between. To proceed in the induction from nE to $nE + F$, it suffices to show that $H^1(S(-E' - nE) \otimes \mathcal{O}_F) = 0$. But by assumption, every curve E_i in F satisfies $E \cdot E_i = 0$, hence $\mathcal{O}_F(-nE) \cong \mathcal{O}_F$. But $H^1(S(-E') \otimes \mathcal{O}_F)$ is a quotient of $H^1(S(-E') \otimes \mathcal{O}_E)$, which is 0 as already mentioned. \square

EXAMPLE 7.10. Assuming $b \geq 3$, the graph below is rational if either $a \geq 3$, or $a = 2$ and $b \geq 5$:



We ask whether $H^1(S(-E')) = 0$, i.e. $h^1(S) = 2$. This is true in the following cases:

- (1) For $a, b \geq 4$, by the “Easy Vanishing Theorem” 3.10.
- (2) For $a \geq 3$ and $b \geq 4$, by Corollary 4.6.
- (3) For $a \geq 3$ and $b \geq 3$ by Theorem 7.9.
- (4) For $a = 2$ and $b \geq 7$ by Proposition 3.7 applied to D equals E plus the 3 outer nodes, concluding $H^1(S(-E')) \cong H^1(S(-E') \otimes \mathcal{O}_{D-E'})$, and noting the last term is a quotient of $H^1(S(-E') \otimes \mathcal{O}_E)$, which is 0 by Theorem 4.5.

Our methods can not handle the case $a = 2$ and $b = 5$ or 6, though $d_i \geq t_i + t'_i - 2$, all i .

8. Results in characteristic p . Throughout this section we consider rational surface singularities in characteristic $p > 0$. We analyze earlier proofs to find sufficient conditions for the same calculations of $h^1(S)$ to hold. Arguments that use Riemann-Roch (e.g., easy vanishing theorems, the Euler characteristic of $S \otimes \mathcal{O}_{E'}$) remain valid. So we restate Theorem 3.10:

THEOREM 8.1. If $d_i \geq 2t_i - 2$ for all i , then in all characteristics $h^1(S) = h^1(S \otimes \mathcal{O}_E)$.

One needs to revisit the calculation of $h^0(S \otimes \mathcal{O}_{E'})$.

LEMMA 8.2. Exclude cyclic quotients, cusps, and simple elliptic singularities.

- (1) If the graph Γ is star-shaped, then $h^0(S \otimes \mathcal{O}_{E'}) = 1$.
- (2) Suppose Γ contains two stars connected by a chain of rational curves, the determinant n of whose intersection matrix is not divisible by p . Then $h^0(S \otimes \mathcal{O}_{E'}) = 0$.

Proof. The previous proofs of Lemmas 2.4 and 2.5 are valid in characteristic p , given that the hypothesis in (2) implies that the equations for the B_i 's admit only the 0 solution. \square

Examining the proof of Theorem 4.5, the arguments involving Type II cones involved some inequalities; we avoid that case by restricting to the case of reduced fundamental cycle (cf. Remark 7.5). The Riemann-Roch argument of Lemma 6.2 is valid in characteristic p , so a string of length at least two starting from an end-curve is still of Type I.

On the other hand, according to Lemma 6.3 an end-curve's contribution $(1, -d)$ becomes $(1, 0)$ when p divides d . In this case, its neighbor receives $(0, 1)$ in the corresponding slot. For each curve E_i , define \bar{t}_i to be the number of adjacent end-curves whose degrees are divisible by p . Note $\bar{t}_i \leq t_i - t'_i$. The situation is clarified by the following analogue of Lemma 6.6.

LEMMA 8.3. *Suppose $\mathcal{C}(E_0, P_0)$ is a Round 2 cone (so $t' = 1$), and assume $d \geq t$. Then*

- (1) $d < t + \bar{t} - 2$ implies Φ_{E_0, P_0} is not surjective, hence Φ_E is not surjective
- (2) $d = t + \bar{t} - 2$ implies \mathcal{C} is of Type II, except that the contribution at P_0 is of the form $(0, 1)$.
- (3) $d > t + \bar{t} - 2$ implies \mathcal{C} is of Type I.

Proof. Since $d \geq t$, as in the proof of Lemma 6.6 we can always produce contributions of $(0, 1)$ at each of the t points of E_0 . But for \bar{t} of the points, we need a contribution of the type $(1, *)$. This requires having an $A(x)$ which vanishes at all but one of these points; this means that

$$d - t + 1 \geq \bar{t} - 1,$$

or $d \geq t + \bar{t} - 2$. The proof should now be clear. \square

One avoids the new type II condition in the last result via the inequality

$$d \geq t + t' + \bar{t} - 2.$$

These are exactly the conditions one needs to generalize all previous results from characteristic 0, since the induction involves only end-curves and Type I curves.

THEOREM 8.4. *Suppose a rational singularity, with reduced fundamental cycle, has all curves satisfying $d_i \geq t_i + t'_i + \bar{t}_i - 2$ for all i , except that one allows that either*

- (1) one curve satisfies $d = t + t' + \bar{t} - 3$, or
- (2) two curves, separated by a (possibly empty) string of rational curves, satisfy $d = t + t' + \bar{t} - 3$.

Then $h^1(S(-E')) = 0$, so $h^1(S) = h^1(S \otimes \mathcal{O}_E)$.

Proof. To prove first that $H^1(S(-E') \otimes \mathcal{O}_E) = 0$, consider the inductive step of Lemma 6.7 for a curve E_0 . There are no Type I curves, so the only slots that need filling are from the $t - t'$ end-curves plus the curve at P_0 . Contributions of the type $(0, 1)$ are handled by choosing $A(x) = 0$ and various $B(x)$ to vanish at all but one of

these points. This can happen because $d - t' \geq t - t'$. We also need contributions of type $(1, 0)$ at P_0 and the \bar{t} points. This requires choosing various $A(x)$ to vanish at all but one of these points; but $d - t - t' + 2 \geq \bar{t}$ by hypothesis, so this can happen. A similar argument handles the terminal situation, except that choosing $A(x)$ to vanish at all but one of the \bar{t} points requires only the weaker condition that $d - t - t' + 2 \geq \bar{t} - 1$ (the situation in (1) above.) Moving to the situation of (2), assume that the induction has led to a cone \mathcal{C} with the weaker condition $d = t + t' + \bar{t} - 3$. Then the argument above shows the cone has Type II, except that the contribution at P_0 is $(0, 1)$. Now consider the case that one has two such cones, separated by a string of rational curves. Then the argument in Lemma 7.4 works exactly as before. Therefore, in all cases of the Theorem one has $H^1(S(-E') \otimes \mathcal{O}_E) = 0$. To conclude that $H^1(S(-E')) = 0$, the proof of Theorem 7.9 is valid in all characteristics. \square

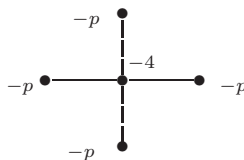
COROLLARY 8.5. *Suppose the graph is star-shaped, and the central curve satisfies $d \geq t$. Then $h^1(S(-E') \otimes \mathcal{O}_E) = 0$ if and only if*

$$d \geq t + t' + \bar{t} - 3.$$

Proof. At the node, there are t' Type I curves. Letting $A(x) = 0$, one can choose various $B(x)$ of degree $d - t'$ to vanish at all but one of the $t - t'$ end-curve intersection points. If $\bar{t} = 0$, then automatically Φ_E is surjective, so for vanishing of H^1 one only needs that $d \geq t + t' - 3$ (obvious converse to Lemma 5.1). If $\bar{t} > 0$, then to separate out those points requires finding various $A(x)$ of degree at least $\bar{t} - 1$; this means $d - t - t' + 2 \geq \bar{t} - 1$. \square

REMARK 8.6. The inequality in the Corollary is automatic if $d \geq 2t - 3$, so the case $d = t = 3$ is covered. However, if $d = t = \bar{t} = 4$ (a degree 4 central curve plus 4 end-curves whose degrees are divisible by p), then the cohomology group does not vanish, and there are “extra” equisingular deformation beyond those arising from the cross-ratio of the 4 intersection points on the central curve.

EXAMPLE 8.7. Consider a star-shaped graph whose central curve has $d = 4$, and each of whose 4 branches consists of a single $-p$ curve:



The argument in the proof of Lemma 8.3 shows that in characteristic p , one has $h^1(S(-E') \otimes \mathcal{O}_E) \neq 0$, hence $h^1(S) \geq 2$.

REMARK 8.8. Recall that in characteristic 0, the “hard” vanishing theorem $H^1_E(S) = 0$ implies that the space of equisingular deformations of a rational resolution inject into a smooth subspace of the base space of the semi-universal deformation of the singularity [12]. That vanishing result need no longer be true in characteristic p , although it is not known whether the equisingular deformations still inject into the base-space; the non-vanishing may simply reflect the failure to lift vector fields from the singularity to the MGR. We note, however, that if the fundamental cycle is reduced and at least one non-end curve has $d_i > t_i$, then the vanishing theorem still holds (by [12], (2.16)).

9. Taut singularities in characteristic p with reduced fundamental cycle. We shall use the following criterion to determine whether a graph Γ is taut.

THEOREM 9.1. (*[3], (3.9); [8]*) *A graph Γ is taut if and only if for every singularity with graph Γ , on the MGR one has $H^1(S) = 0$.*

Proof. Note that if $H^1(S) \neq 0$, then there would be a non-trivial smooth equisingular family of resolutions ([11], 5.16); but tautness implies one has a unique singularity, hence a unique resolution.

Next suppose $H^1(S) = 0$ for every resolution. For an effective cycle Z on a resolution, there is an easily verified surjection $S \rightarrow \Theta_Z$. Thus, $H^1(\Theta_Z) = 0$. Laufer takes a graph Γ which is “potentially taut” (i.e., all $t_i \leq 3$) and a formal sum $Z = \sum n_i E_i$, converting it into a “plumbing scheme” $P = P_Z$; this is an actual exceptional divisor on a resolution of a specific singularity with graph Γ . The requisite characteristic p construction is similar, done in Section 3 of [8]. The authors show that if Z is sufficiently big, then $H^1(P, \Theta_P) = 0$ implies tautness ([8], Proposition 3.16). The point is that a combinatorially equivalent divisor on another resolution can be connected to P by a connected family (actually a more general result is proved by Laufer in [3], Theorem 3.2). \square

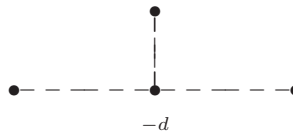
If $h^1(S) = 0$, then so is $h^1(S \otimes \mathcal{O}_E)$. By Proposition 2.6, except for the excluded cases, taut singularities are star-shaped with 3 ends or are not star-shaped and have 4 ends. One thus considers also Γ a chain of rational curves (a cyclic quotient) or a cycle of rational curves (a “cusp” singularity). In the following, every vertex is allowed to have any degree ≥ 2 unless otherwise specified.

THEOREM 9.2. *The following are the taut singularities in characteristic p with reduced fundamental cycle:*

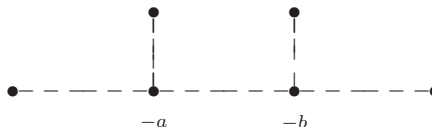
- (1) For all p ,



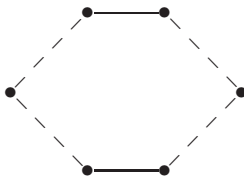
- (2) For $d \geq 3$ and all p ,



- (3) For $a, b \geq 3$ and p not dividing the determinant of the string of curves between the nodes,



(4) For p not dividing the determinant of the intersection matrix of the cusp,



Proof. Except for restrictions on the prime p , the above list includes all taut singularities in characteristic 0 with reduced fundamental cycle.

Theorem 8.1 says that $h^1(S) = h^1(S \otimes \mathcal{O}_E)$ in all cases above except (2) for $d = 3$ and (3) for a or b equal to 3. But those cases are covered by Theorem 8.4.

In cases (2) and (3), Proposition 2.6 gives the value of $h^1(S \otimes \mathcal{O}_E)$ as long as $h^0(S \otimes \mathcal{O}_E)$ is as it was in characteristic 0. By Lemma 8.2, we conclude that for (2) and (3), one has $h^1(S) = 0$; these are indeed taut.

For (3), we show that if p does divide the determinant, then $h^1(S \otimes \mathcal{O}_E) \neq 0$. For, there then exist non-trivial solutions A_i and B_i (in the notation of the proof of Lemma 2.4), hence $h^0(S \otimes \mathcal{O}_F) = 1$. Now go from F to E' arguing as in the proof to show $h^0(S \otimes \mathcal{O}_F) = h^0(S \otimes \mathcal{O}_{E'}) = 1$. The first line of the proof of Proposition 2.6 gives that $\chi(S \otimes \mathcal{O}_{E'}) = 0$, hence $h^1(S \otimes \mathcal{O}_{E'}) = 1$, and so $h^1(S) = 1$.

For (1), start with any curve in the string and proceed as in Lemmas 2.3 and 2.4 to conclude that $h^0(S \otimes \mathcal{O}_E) = 4$; the Euler characteristic is also 4, so $h^1(S) = 0$.

Finally, for the cusps of (4), the Euler characteristic of $S \otimes \mathcal{O}_E$ is 0, so $h^1(S \otimes \mathcal{O}_E) = h^0(S \otimes \mathcal{O}_E)$. A calculation similar to that in the proof of Lemma 2.5 gives equations for the coefficients A_i, B_i of sections of $H^0(S \otimes \mathcal{O}_{E_i})$; they reduce to homogeneous equations in the B_i whose determinant is that of the intersection matrix of the graph. Thus, there is a non-trivial solution if and only if p divides this determinant. \square

REMARK 9.3. The graphs pictured for the cusps of (4) show more than one curve, but the same argument applies when the minimal resolution is a nodal curve whose degree is not divisible by p .

REMARK 9.4. Lee and Nakayama have already proved [5] that the cyclic quotient singularities in (1) are taut in all characteristics. Other cases of tautness in characteristic p have been proved by Y. Tanaka [10] and F. Schüller [8].

REMARK 9.5. Of course, there are many star-shaped rational graphs (e.g., E_8) which are taut in characteristic 0, but not in certain positive characteristic (e.g., [1]). Necessarily, those must have d equal 2.

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