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006

SEXTIC CURVES WITH SIX DOUBLE POINTS ON A CONIC*

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Abstract. Let C_6 be a plane sextic curve with 6 double points that are not nodes. It is shown that if they are on a conic C_2, then the unique possible case is that all of them are ordinary cusps. From this it follows that C_6 is irreducible. Moreover, there is a plane cubic curve C_3 such that $C_6 = C_2^3 + C_3^2$. Such curves are closely related to both the branch curve of the projection to a plane of the general cubic surface from a point outside it and canonical surfaces in $\mathbb{P}^3$ or $\mathbb{P}^4$ whose desingularizations have birational invariants $q > 0$, $p_g = 4$ or $p_g = 5$, $P_2 \leq 23$.

Key words. Plane curves, ordinary cusps, tacnodes, surfaces of general type.

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Introduction. In this paper, we study, a priori not necessarily irreducible, singular plane curves, paying special attentions to double points that are not nodes on them. Here, a node is a double point with two distinct singular tangent lines, e.g., $y^2 + x^2 = 0$. Therefore, a double point that is not a node has two coincident singular tangent lines, e.g., $y^2 + x^n = 0, n \geq 3$. In this case, the line $y = 0$ is called the *singular tangent line*. In the case $y^2 + x^3 = 0$ the double point is called an *ordinary cusp* and in the case $y^2 + x^4 = 0$ it is called a *tacnode*.

Let C_6 be a plane sextic curve that has 6 singular points each of which is either an ordinary cusp or a tacnode. If these 6 singular points are not on a single conic, then every possible combination actually occurs. Namely, for every integer i with $0 \leq i \leq 6$, there exists a sextic curve with $6 - i$ cusps and i tacnodes such that these 6 singular points are not on a conic (cf. [K]). However, if there is a conic passing through all the 6 singular points, then the situation changes drastically as the following theorem, our main result in this paper, shows.

Theorem. *Let C_6 be a (not necessarily irreducible) plane sextic curve that has 6 distinct double points $K_1, \dots, K_6$, none of which is a node, lying on a conic C_2. Then, there exists a cubic curve C_3 meeting C_2 transversally at the 6 points K_i and the tangent line to C_3 at K_i coincides with the singular tangent line to C_6, $i \in \{1, 2, \dots, 6\}$. Therefore, the only one case is possible: the 6 singular points are all ordinary cusps, C_6 is irreducible and the equation of C_6 is given by $C_3^2 + C_2^3 = 0$.*

Indeed, although the authors were familiar enough with sextic curves with ordinary cusps or tacnodes as singularities (cf. [Z1], [Z2], [CPS, Section 1.1], [K], [St1], [St2] and [KS]), they did not imagine that such a surprising statement holds true. What convinced them of its validity were the explicit examples constructed in [K]:

Examples. Affine equations of sextic curves with only ordinary cusps and tacnodes. ($i^2 = -1$).

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(1) Seven cusps:

$$C_6(x,y) = 4x^6 + y^6 - 4x^3y^2 - 2y^4 - 4x^3 + 5y^2$$
$$= (y^2-1)^3 + (-2x^3 + y^2 + 1)^2 = 0.$$

(2) Six cusps and one tacnode:

$$C_6(x,y) = x^6 - 18x^4y^2 - 27x^2y^4 - 12x^4 + 36x^2y^2 + 48x^2 + 36y^2 - 64$$
$$= (x^2 + 3y^2 - 4)^3 + [i\sqrt{27}(x^2y + y^3 - 2y)]^2 = 0.$$

(3) Eight cusps:

$$C_6(x,y) = x^6 + y^6 + 2i\sqrt{3}xy(x^4 - y^4) - 5x^2y^2(x^2+y^2)$$
$$- 4i\sqrt{3}xy(x^2-y^2) + 12x^2y^2 - 4$$
$$= [\sqrt[3]{4}(x^2-1)]^3 + [y^3 - \sqrt{3}ixy^2 + x^2y - \sqrt{3}ix(x^2-2)]^2 = 0.$$

(4) Nine cusps:

$$C_6(x,y) = x^6 + y^6 + 1 - 2x^3y^3 - 2x^3 - 2y^3$$
$$= (-\sqrt[3]{4}xy)^3 + (x^3 + y^3 - 1)^2 = 0.$$

In these examples, the sextic curve C_6 has more than 6 singular points, but any of them is either a cusp or a tacnode. Since we imposed particular symmetries on the curves, we could easily realize that 6 ordinary cusps are on the conic C_2 in the expression $C_6 = C_2^3 + C_3^2$ given above, where and in the sequel we denote by $C_i = 0$ the equation of the curve C_i of degree i.

As far as known examples concern, any sextic curve with only cusps or tacnodes has 6 cusps on a conic, when the number of the singular points is at least 7. Then, a natural question arises: *Is this true in general? That is, does any sextic C_6, with at least 7 singular points each of which is either a cusp or a tacnode, have 6 cusps on a conic C_2?* If the answer is affirmative, as we think so, then the equation of C_6 is of the form $C_3^2 + C_2^3 = 0$ by the above theorem.

When the sextic C_6 has exactly 6 cusps on a conic (and no other singularities), our result can be applied to the branch curve of the projection to a plane of the general cubic surface from a point outside it, as discussed in [Se, Ch. IV, p. 19] and [M]. In these papers, curves C_{6m}, C_{3m} and C_{2m} of respective degrees $6m, 3m$ and $2m$ were considered, whereas we consider here the case $m = 1$ only. Hence, Theorem is well known in the case of 6 cusps if we assume the generality of curves. Nevertheless, their methods seem not extend to more general situations; for example, when C_6 has singularities other than the six cusps, and so $\notin C_2$, or when C_2 is reducible. Thus, our result is more general than theirs when $m = 1$. Furthermore, our proof is new and quite elementary at the cost of considerable length. When C_6 is irreducible, we have another argument to prove Theorem suggested by M. Oka: show that the Alexander polynomial is non-trivial and apply the result of Degtyarev [D] to see that C_6 is a $(2,3)$ torus curve, i.e., $C_6 = C_2^3 + C_3^2$. Again, ours has an advantage that C_6 can be reducible.

Returning to the case that C_6 has 6 cusps on a conic (and no other singularities), Theorem has an interesting consequence which is closely related to irregular canonical surfaces in $\mathbb{P}^4$. To explain it, let us recall the work of O. Zariski, cf. [Z1], [Z2]. Indeed,

C_6 gives us a normal surface S in $\mathbb{P}^3$ whose affine equation is $z^6 - C_6(x, y) = 0$. A desingularization X of S is an irregular surface with $q = 1$ and $p_g = 5$, where q and p_g respectively denote the irregularity and the geometric genus of X. The reason why $q > 0$ is because the 6 cusps are on a conic, as shown in [Z1]. Furthermore, it is not so hard to see that the canonical transformation $\varphi_{|K_X|}$ of X is birational. So, the image under $\varphi_{|K_X|}$, which is called a *canonical surface*, is a surface of general type in $\mathbb{P}^4$. Our result affirms that we cannot modify Zariski's example by replacing some cusps by tacnodes on the sextic. This is one of the important consequences of ours. On the other hand, such a replacement is possible if the 6 cusps are not on a conic.

In fact, we constructed in [K] normal surfaces $S \subset \mathbb{P}^3$ whose affine equation is $z^6 - C_6(x, y) = 0$ and the desingularizations have the birational invariants: $q = 0, p_g = 4$ and $P_2 = 23 - i$, by giving C_6 with $6 - i$ cusps and i tacnodes not lying on a conic ($i = 0, \ldots, 6$). The case of $P_2 = 23$, i.e., C_6 with 6 cusps, is again due to Zariski [Z1], [Z2]. The importance of these surfaces S is in the facts that they themselves are canonical (image of the canonical transformation) and that they are the only known examples of regular, normal and canonical surfaces in $\mathbb{P}^{\mathbf{3}}$ of degree > 5, whereas the case of non-normal, regular and canonical surfaces in $\mathbb{P}^3$ is a classical subject dates back to F. Enriques (cf. [E, Ch. VIII]). In other words, Zariski's example can be extended, by "replacing" cusps with tacnodes, to the normal canonical surfaces in $\mathbb{P}^3$, whose desingularizations have $q = 0, p_g = 4$ and $P_2 = 22, 21, 20, 19, 18, 17$. This makes it more strange that the replacement is impossible when the six singular points are on a conic.

Although we assume that the ground field k is the algebraically closed field of characteristic zero, what we actually need is only a "weak" version of Bézout's theorem, i.e., if a curve of degree n and a curve of degree m have at least $mn + 1$ points in common, then they have a common component. Such a weak version of Bézout theorem needs less hypotheses on the ground field.

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1. Proof. Let the notation and the assumption be as in Theorem in the Introduction.

LEMMA 1.1. *If the conic C_2 passing through $K_1, \ldots, K_6$ is reducible and splits in two distinct lines as $C_2 = \ell_1\ell_2$, then none of the K_i's are the point $\ell_1 \cap \ell_2$.*

Proof. If a singular point of C_6 is at $\ell_1 \cap \ell_2$, then one of the two lines, say ℓ_1, contains 4 of the 6 singular points. By Bézout's theorem, C_6 splits into ℓ_1 and a quintic C_5. Recall that none of the 4 singular points can be a node. It follows that C_5 and ℓ_1 have to be tangent to each other at the 4 points. By Bézout's theorem again, C_5 has ℓ_1 as a component and, thus, C_6 can be divided by ℓ_1^2. Then, it would follow that on ℓ_1 there are no double points of C_6. Therefore, none of the 6 singular points of C_6 are at $\ell_1 \cap \ell_2$. □

COROLLARY 1.2. *If C_2 splits in two lines $C_2 = \ell_1\ell_2$, then on each line ℓ_i there are* 3 *distinct singular points of C_6.*

LEMMA 1.3. *C_6 and C_2 cannot have a common component.*

Proof. We first assume that C_2 is irreducible. If C_6 has at least one cusp, then C_6 cannot have C_2 as a component, because C_2 passes through the cusp (which has only one local analytic branch). If all the 6 singular points on C_2 are different from cusps, then $C_6 = D_2D_2'D_2''$ with D_2, D_2', D_2'' being two by two bitangent conics by [C1], [C2]. In this case, however, C_2 can be none of the three conics D_2, D_2', D_2'', because each of them contains only 4 tacnodes. Next, we assume that $C_2 = \ell_1\ell_2$ and that ℓ_1 is a component of C_6. Put $C_6 = C_5\ell_1$. In this case none of the 3 singular points on ℓ_1 (Corollary 1.2) is a cusp. This means that C_5 and ℓ_1 have at least 6 points in common (counting multiplicities). By Bézout's theorem, C_5 has ℓ_1 as a component and, thus, C_6 can be divided by ℓ_1^2, which is absurd. □

Corollary 1.4. *C_6 cuts out on C_2 the divisor $2\sum_{i=1}^{6} K_i$.*

Proof. The conic C_2 cannot be tangent at a double point to the singular tangent line of C_6, otherwise, by Bézout's theorem C_6 and C_2 have a common component, which Lemma 1.3 forbids. □

Now, since the space of homogeneous polynomials of degree 3 in three variables is of dimension 10, we can find a cubic C_3 passing through $K_1, K_2, \dots, K_6$ which is tangent at K_1, K_2, K_3 to the singular tangent line of C_6. This tangency can be understood also improperly, in the sense that "improperly tangent" means that C_3 has K_1, or K_2, or K_3 as a singular point.

Furthermore, we can assume that K_1, K_2, K_3 are not collinear, by a suitable relabeling if necessary.

Lemma 1.5. *Let the choice of K_1, K_2, K_3 and the cubic C_3 be as above. Then C_3 and C_2 have no common components.*

Proof. We first assume that the conic C_2 is irreducible and that $C_3 = C_2E_1$, where E_1 is a line. From Corollary 1.4, the conic C_2 has no tangents coincident with the singular tangent line of C_6 at the three points K_1, K_2, K_3. Thus E_1 must pass through K_1, K_2 and K_3, which is impossible because K_1, K_2 and K_3 are not collinear. Next, let us assume that C_2 is reducible, $C_2 = \ell_1\ell_2$, and that $C_3 = D_2\ell_1$ with a conic D_2. From Corollary 1.2, the conic D_2 meets ℓ_2 at three points and, thus, D_2 can be divided by ℓ_2. Then, $C_3 = \ell_1\ell_2F_1$ with the third line F_1. By the choice of C_3, the line F_1 must pass through K_1, K_2 and K_3, which is again impossible since they are not collinear. □

Corollary 1.6. *C_3 cuts out on C_2 the divisor $\sum_{i=1}^{6} K_i$.*

Proof. Indeed C_3 and C_2 cannot be tangent at any point K_i, otherwise they have a common component, which Lemma 1.5 forbids. □

Let C_6 and C_3 be the two curves considered above. In particular C_3 at K_1, K_2, K_3 is tangent to the singular tangent line of C_6. Let us consider the following pencil of plane sextic curves, given by the equation

$$\lambda C_6 + \mu C_3^2 = 0, \quad \forall \lambda, \mu \in k. \tag{1.1}$$

It may happen that C_6 and C_3 have some common components (even, C_3 may be a component of C_6), nevertheless the equation (1.1) defines a pencil of curves because C_6 and C_3^2 are different.

Lemma 1.7. *In the pencil* (1.1)*, there exists a sextic curve of the form C_2C_4 with a suitable quartic C_4. That is, there are two values $\overline{\lambda}$ and $\overline{\mu}$ of λ and μ such that the following identity of polynomials holds true*

$$\overline{\lambda} C_6 + \overline{\mu} C_3^2 = C_2C_4. \tag{1.2}$$

Proof. From Corollaries 1.4 and 1.6, the generic curve of the pencil cuts out on C_2 the divisor $2\sum_{i=1}^{6} K_i$. So, if we consider a curve of the pencil passing through a point $P \in C_2, P \neq K_i, \forall i$, and in the case $C_2 = \ell_1\ell_2$ is reducible we choose $P = \ell_1 \cap \ell_2$, then such a curve intersects C_2 in at least 13 points (counting multiplicities) if C_2 is irreducible and in at least 14 points if C_2 is reducible (cf. Lemma 1.1). In any case, by Bézout theorem, it can be divided by C_2 and takes the form C_2C_4 with a quartic C_4, i.e., the identity (1.2) holds true. □

LEMMA 1.8. *The quartic C_4 appearing in* (1.2) *is equal to C_2^2. Thus, there exists $\overline{\nu} \in k$ such that $\overline{\lambda}C_6 = -\overline{\mu}C_3^2 + \overline{\nu}C_2^3$, where the cubic C_3 passes through $K_1, K_2, \dots, K_6$ and its tangent line at each K_i is the singular tangent line at K_i of C_6.*

Proof. Let us consider the singular point K_i. The K_i's are singular points of both of the two generators of the pencil (1.1). It follows that C_4 passes through every K_i $(1 \leq i \leq 6)$.

Now, we consider K_1. We can assume that $K_1 = (0,0)$ in the affine plane (x,y) and that the singular tangent line at K_1 to C_6 is $y = 0$. With the above considerations and assumptions on C_3 at K_1, the identity (1.2) can be written in the following way

$$\overline{\lambda}[ay^2 + (\geq 3)] + \overline{\mu}[by + (\geq 2)]^2 = [cx + dy + (\geq 2)][\alpha x + \beta y + (\geq 2)], \tag{1.3}$$

where $a, b, c, d, \alpha, \beta \in k$ and the symbol $(\geq m)$ indicates that it is a sum of monomials in x and y of degree $\geq m$.

From (1.3), considering the terms of degree 2, we obtain

$$(\overline{\lambda}a + \overline{\mu}b^2)y^2 = (cx + dy)(\alpha x + \beta y). \tag{1.4}$$

Then, comparing the both sides of (1.4), we deduce

$$\begin{cases} c\alpha = 0, \\ c\beta + d\alpha = 0. \end{cases}$$

Since at K_1 the tangent line to C_2 is different from $y = 0$ (Corollary 1.2), we have $c \neq 0$. This implies $\alpha = 0$ and from this $\beta = 0$. This means that C_4 has at K_1 a singular point of multiplicity ≥ 2. Repeating for K_2 and K_3 what we did for K_1, we see that C_4 has at each of the three points K_1, K_2 and K_3 a singular point of multiplicity ≥ 2.

In addition, C_4 also passes through K_4, K_5 and K_6. Thus the common points of C_4 and C_2, counting multiplicities, are at least 9. Now, if C_2 is irreducible, then we have $C_4 = C_2E_2$ by Bézout's theorem, where E_2 is a suitable conic. If C_2 is reducible and $C_2 = \ell_1\ell_2$, then two of the three non-collinear points K_1, K_2, K_3 are on a line. Assume that this line is ℓ_1 and $K_1, K_2, K_4 \in \ell_1$. The quartic C_4 and ℓ_1 have at least 5 points in common and therefore ℓ_1 is a component of C_4. Put $C_4 = \ell_1 D_3$. The cubic D_3 cuts out on ℓ_2 the divisor $2K_3 + K_5 + K_6$. Therefore, D_3 contains ℓ_2. In conclusion, we have again $C_4 = C_2E_2$.

Next, substituting $C_4 = C_2E_2$ in (1.2) and considering the terms of degree 2, we obtain an identity similar to (1.4). More precisely,

$$(\overline{\lambda}a + \overline{\mu}b^2)y^2 = (cx + dy)^2\gamma, \tag{1.5}$$

where γ is the constant appearing in the equation $\gamma + (\geq 1) = 0$ of E_2.

From (1.5), we obtain $c^2\gamma = 0$. As we already remarked, $c \neq 0$. Hence we have $\gamma = 0$. This means that the conic E_2 is passing through K_1. Similarly, we see that E_2 is passing through K_2 and K_3.

Let us consider the singularity K_4. Substituting $C_4 = C_2E_2$ in (1.2), we obtain $\overline{\lambda}C_6 + \overline{\mu}C_3^2 = C_2^2E_2$. Assuming also here $K_4 = (0,0)$ with singular tangent line $y = 0$, we can rewrite it in the following way, similarly as in (1.3)

$$\overline{\lambda}[a''y^2 + (\geq 3)] + \overline{\mu}[a'x + b'y + (\geq 2)]^2 = [c'x + d'y + (\geq 2)]^2[\gamma' + (\geq 1)], \tag{1.6}$$

where $\gamma' + (\geq 1) = 0$, $\gamma' \in k$, is the equation of the conic E_2. The terms of degree 2 in (1.6) give us

$$\overline{\lambda}a''y^2 + \overline{\mu}(a'x + b'y)^2 = (c'x + d'y)^2\gamma'$$

and we obtain

$$\begin{cases} \overline{\mu}{a'}^2 = c'^2\gamma', \\ \overline{\mu}a'b' = c'd'\gamma'. \end{cases}$$

We note that $\overline{\mu} \neq 0$, otherwise in (1.2) C_2 is a component of C_6, contradicting Lemma 1.3. Since at K_4 the tangent line to C_2 is different from $y = 0$ by Corollary 1.4, we have $c' \neq 0$.

If $\gamma' \neq 0$, then $a' \neq 0$ and it follows from the two equalities $\frac{b'}{a'} = \frac{d'}{c'}$. This means that $a'x + b'y = a'(x + \frac{d'}{c'}y) = \frac{a'}{c'}(c'x + d'y)$. In other words, the two curves C_3 and C_2 have the same tangent at K_4, contradicting Corollary 1.6. This implies that $\gamma' = 0$ and, thus, $a' = 0$. Now, $a' = 0$ tells us that the cubic C_3 is tangent at K_4 to the singular tangent line at K_4 of the sextic C_6. Moreover, $\gamma' = 0$ tells us that the conic E_2 passes through K_4.

We can say the same things for K_5, K_6. Therefore, we conclude that the tangent line at any K_i of the cubic C_3 coincides with the singular tangent line at K_i to C_6, for any i. The conics E_2 and C_2 have the six points K_i in common and therefore they coincide, in either case that C_2 is irreducible or not. □

Proof of Theorem in the Introduction. Using Lemma 1.8, we can complete the proof of Theorem easily. If at least one of the six singular points K_i, $1 \leq i \leq 6$, is not an ordinary cusp, then the sextic C_6 and the cubic C_3 have at least 19 points in common (counting multiplicities). By Bézout's theorem, C_6 and C_3 have a common component. Then, from the identity $\overline{\lambda}C_6 = -\overline{\mu}C_3^2 + \overline{\nu}C_2^3$ in Lemma 1.8, we deduce that such a component is a component of C_2^3. However, this is impossible, since C_3 and C_2 have no common components by Lemma 1.5. Therefore, the only possible case is that the six singularities $K_1, K_2, \dots, K_6$ on the conic are all ordinary cusps. In particular C_6 is irreducible, because no reducible curve of degree 6 can have 6 cusps. □

The existence of six ordinary cusps of C_6 on the conic C_2 is clear from the equation $-\overline{\mu}C_3^2 + \overline{\nu}C_2^3 = [\sqrt{-\overline{\mu}}C_3]^2 + [\sqrt[3]{\overline{\nu}}C_2]^3 = 0$, where the cubic C_3 and the conic C_2 have only to satisfy the condition that they intersect at distinct 6 points.

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