

THE GRAUERT–GROTHENDIECK COMPLEX ON DIFFERENTIABLE SPACES AND A SHEAF COMPLEX OF BRYLINSKI*

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Dedicated to Henri Laufer on the occasion of his 70th birthday

Abstract. We use the Grauert–Grothendieck complex on differentiable spaces to study basic relative forms on the inertia space of a compact Lie group action on a manifold. We prove that the sheaf complex of basic relative forms on the inertia space is a fine resolution of Brylinski’s sheaf of functions on the inertia space.

Key words. Grauert–Grothendieck complex, relative basic form, differentiable space.

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1. Introduction. In his paper [Bry87b], JEAN-LUC BRYLINSKI studies the cyclic homology theory of the smooth crossed-product algebra $\mathcal{A} := \mathcal{C}_c^\infty(G \times M)$ associated to a manifold M which carries a smooth action of a Lie group G . The crossed-product algebra carries the convolution product $*$ defined by

$$(u * v)(g, x) := \int_M u(gh, h^{-1}x) \cdot v(h^{-1}, x) d\mu(h) \quad \text{for } u, v \in \mathcal{A}, (g, x) \in G \times M. \quad (1.1)$$

The symbol μ hereby denotes a fixed left-invariant Haar measure on G . BRYLINSKI asserts in his article that the Hochschild homology $HH_k(\mathcal{A})$ of the (topological) algebra \mathcal{A} coincides naturally with the space $\Omega_{\text{b.r.}}^k(M/G)$ of so-called *basic relative k -forms*. The sheaf complex of basic relative forms will be constructed in Section 3. Despite it being a sheaf complex over the orbit space M/G , basic relative forms are defined as forms on the so-called *loop space*

$$\Lambda_0(G \ltimes M) := \bigcup_{g \in G} \{g\} \times M^g \subset G \times M \quad (1.2)$$

which essentially consists of a disjoint union of the fixed point manifolds M^g , $g \in G$ of the smooth action $G \times M \rightarrow M$ together with an appropriate topology on it. In Section 3 we will also see that the loop space carries even the structure of a differentiable stratified space. BRYLINSKI has also claimed in [Bry87b] that the sheaf complex of basic relative forms is a resolution of a certain sheaf \mathcal{B} on the orbit space M/G . The section spaces of that sheaf over $O \subset M/G$ open are given by

$$\begin{aligned} \mathcal{B}(O) := \{f \in \mathcal{C}_{\Lambda_0}^\infty(s_{|\Lambda_0}^{-1}\pi^{-1}(O)) \mid f \text{ is } G\text{-invariant} \\ \text{and } f(g, -) \text{ locally constant for all } g \in G\}. \end{aligned} \quad (1.3)$$

Hereby, the map $s_{|\Lambda_0} : \Lambda_0(G \ltimes M) \rightarrow M$ is given by $s_{|\Lambda_0}(g, p) = p$, $\pi : M \rightarrow G/M$ is the orbit projection, the sheaf $\mathcal{C}_{\Lambda_0}^\infty$ is defined as the sheaf of continuous functions

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on the loop space which are locally restrictions of smooth functions on $G \times M$, the G -action on $\Lambda_0(G \times M)$ is given by the diagonal action with conjugation in the first coordinate, and the function $f(g, -)$ for $f \in \mathcal{C}^\infty(\Lambda_0(G \times M))$ and $g \in G$ is the map $M^g \rightarrow \mathbb{R}$, $p \rightarrow f(g, p)$. Since the sheaf \mathcal{B} has been defined first in [Bry87b], we call it *Brylinski's sheaf*.

For the proof that $HH_k(\mathcal{A})$ is isomorphic to $\Omega_{\text{b.r.}}^k(M/G)$, BRYLINSKI refers to the unpublished paper [Bry87a], a proof of the second claim is missing.

The purpose of these notes is to shed some light onto Brylinski's sheaf \mathcal{B} and the sheaf complex of basic relative forms. We interpret the latter as a certain subcomplex of the Grauert-Grothendieck complex of differential forms on the loop space of the G -manifold M . Then, in Theorem 3.1, we show that it is what it is claimed to be: an acyclic complex of fine sheaves whose cohomology in degree zero coincides with Brylinski's sheaf \mathcal{B} .

Let us mention that the cyclic homology theory of crossed product algebras has been studied also by BLOCK-GETZLER [BG94] and NISTOR [Nis93]. Moreover, FARSI-PFLAUM-SEATON have described in [FPS15b, FPS15a] the stratification theory of the loop space and proved a de Rham Theorem for it.

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2. Differential forms. In this section, we will define the Grauert-Grothendieck complex of differential forms on a differentiable space (X, \mathcal{O}) which we allow to be non-reduced. To this end consider first an affine open subset of X and let $\iota : U \hookrightarrow \tilde{U} \subset \mathbb{R}^n$ be a singular chart. Denote by \mathcal{I}_ι the ideal sheaf in $\mathcal{C}_{\tilde{U}}^\infty$ such that $\mathcal{O}_U \cong \iota^{-1}(\mathcal{O}_\iota)$, where \mathcal{O}_ι is the restriction of the sheaf $\mathcal{C}_{\tilde{U}}^\infty/\mathcal{I}_\iota$ to $\iota(U)$. We then define Ω_ι^0 as the sheaf \mathcal{O}_ι and, for $k \in \mathbb{N}^*$, the sheaf Ω_ι^k as the inverse image sheaf

$$\Omega_\iota^k := \iota^{-1}(\Omega_{\tilde{U}}^k/\mathcal{I}_\iota\Omega_{\tilde{U}}^k + d\mathcal{I}_\iota \wedge \Omega_{\tilde{U}}^{k-1}) = \iota^*(\Omega_{\tilde{U}}^k/\mathcal{I}_\iota\Omega_{\tilde{U}}^k + d\mathcal{I}_\iota \wedge \Omega_{\tilde{U}}^{k-1}). \tag{2.1}$$

The latter equality holds true because $\mathcal{O}_U \cong \iota^{-1}(\mathcal{O}_\iota)$, and because the sheaf

$$\Omega_{\tilde{U}}^k/\mathcal{I}_\iota\Omega_{\tilde{U}}^k + d\mathcal{I}_\iota \wedge \Omega_{\tilde{U}}^{k-1}$$

is an \mathcal{O}_ι -module. By construction, the exterior differential d descends to sheaf morphisms $d : \Omega_\iota^k \rightarrow \Omega_\iota^{k-1}$ so that we obtain a complex of sheaves $(\Omega_\iota^\bullet, d)$ over U . In the following we will show that the sheaf complexes $(\Omega_\iota^\bullet, d)$ glue to a sheaf complex (Ω^\bullet, d) on X , when ι runs through a singular atlas of (X, \mathcal{O}) . To construct the gluing maps we will proceed by proving a sequence of lemmata.

LEMMA 2.1. *Let $\iota : U \hookrightarrow \tilde{U} \subset \mathbb{R}^n$ and $\kappa : U \hookrightarrow \hat{U} \subset \mathbb{R}^m$ be two singular charts of (X, \mathcal{O}) for which there exists a smooth embedding $H : \hat{U} \hookrightarrow \tilde{U}$ such that $H(\hat{U})$ is closed in \tilde{U} and such that the pullback $H^* : \mathcal{C}_{\tilde{U}}^\infty \rightarrow \mathcal{C}_{\hat{U}}^\infty$ induces an isomorphism of locally ringed spaces $(H|_{\kappa(U)}, H^*) : (\kappa(U), \mathcal{O}_\kappa) \rightarrow (\iota(U), \mathcal{O}_\iota)$. Then there exists a*

unique isomorphism of sheaf complexes $\eta_{\kappa,\iota} : \Omega_\iota^\bullet \rightarrow \Omega_\kappa^\bullet$ such that $\eta_{\kappa,\iota}(f) = H^*f$ for all $f \in \mathcal{O}_\iota(V)$ with $V \subset U$ open.

Proof. By construction we have the following canonical identifications

$$\Omega_\iota^k(U) \cong \Omega^k(\tilde{U}) / (J_\iota \Omega^k(\tilde{U}) + dJ_\iota \wedge \Omega^{k-1}(\tilde{U})), \quad \text{where } J_\iota := \mathcal{J}_\iota(\tilde{U}) \tag{2.2}$$

and

$$\Omega_\kappa^k(U) \cong \Omega^k(\hat{U}) / (J_\kappa \Omega^k(\hat{U}) + dJ_\kappa \wedge \Omega^{k-1}(\hat{U})), \quad \text{where } J_\kappa := \mathcal{J}_\kappa(\hat{U}). \tag{2.3}$$

Denote by $I \subset \mathcal{C}^\infty(\tilde{U})$ the vanishing ideal of $H(\hat{U})$. The pull-back morphism $H^* : \Omega^k(\tilde{U}) \rightarrow \Omega^k(\hat{U})$ then is surjective with kernel $I\Omega^k(\tilde{U}) + dI \wedge \Omega^{k-1}(\tilde{U})$, since $H(\hat{U})$ is a closed submanifold of \tilde{U} . Since $H^*J_\iota \subset J_\kappa$ and since d commutes with H^* , pull-back by H induces a surjective map denoted by the same symbol

$$H^* : \Omega^k(\tilde{U}) / (J_\iota \Omega^k(\tilde{U}) + dJ_\iota \wedge \Omega^{k-1}(\tilde{U})) \longrightarrow \Omega^k(\hat{U}) / (J_\kappa \Omega^k(\hat{U}) + dJ_\kappa \wedge \Omega^{k-1}(\hat{U})).$$

This map is injective since $I\Omega^k(\tilde{U}) + dI \wedge \Omega^{k-1}(\tilde{U})$ is contained in $J_\iota \Omega^k(\tilde{U}) + dJ_\iota \wedge \Omega^{k-1}(\tilde{U})$ and since $H^*J_\iota = J_\kappa$. After choosing for each open set $V \subset U$ an open $\hat{V} \subset \hat{U}$ such that $V = \kappa^{-1}(\hat{V})$, pull-back via $H|_{\hat{V}}$ induces in the same way isomorphisms $H_V^* : \Omega_\iota^k(V) \rightarrow \Omega_\kappa^k(V)$ for each $k \in \mathbb{N}$ and $V \subset U$ open. The family of isomorphisms H_V^* then defines the desired sheaf isomorphism $\eta_{\kappa,\iota}$. Since each H_V^* commutes with the differentials, $\eta_{\kappa,\iota}$ is a morphism of sheaf complexes, indeed. Moreover, since Ω_ι^\bullet is generated as a sheaf of differential graded algebras by Ω_ι^0 , $\eta_{\kappa,\iota}$ is uniquely determined by its action on Ω_ι^0 . This finishes the proof. \square

LEMMA 2.2. *Under the assumption of the preceding lemma let $G : \check{U} \hookrightarrow \tilde{U}$ be a second smooth embedding defined on an open neighborhood $\check{U} \subset \mathbb{R}^n$ of $\kappa(U)$ such that $(G|_{\kappa(U)}, G^*) : (\kappa(U), \mathcal{O}_\kappa) \rightarrow (\iota(U), \mathcal{O}_\iota)$ is an isomorphism of locally ringed spaces. Then $H_V^* : \Omega_\iota^k(V) \rightarrow \Omega_\kappa^k(V)$ and $G_V^* : \Omega_\iota^k(V) \rightarrow \Omega_\kappa^k(V)$ coincide for all open $V \subset U$ which means that the sheaf isomorphism $\eta_{\kappa,\iota}$ does not depend on the particular embedding inducing an isomorphism between $(\kappa(U), \mathcal{O}_\kappa)$ and $(\iota(U), \mathcal{O}_\iota)$.*

Proof. After possibly shrinking \tilde{U} we can assume that $G(\check{U})$ is closed in \tilde{U} as well. For every smooth function $f \in \mathcal{C}^\infty(\tilde{U})$ we then have

$$(H^*f)|_{\check{U} \cap \tilde{U}} - (G^*f)|_{\check{U} \cap \tilde{U}} \in \mathcal{J}_\kappa(\tilde{U} \cap \check{U}), \quad \text{and} \quad (H^*df)|_{\check{U} \cap \tilde{U}} - (G^*df)|_{\check{U} \cap \tilde{U}} \in d\mathcal{J}_\kappa(\tilde{U} \cap \check{U}).$$

That implies that the actions of H^* and G^* on $\Omega_\iota^k(U)$ coincide. Likewise $H_V^* = G_V^*$ for all open $V \subset U$, hence $\eta_{\kappa,\iota}$ is independent of the particularly chosen embedding H . \square

Next let $\iota : U \hookrightarrow \mathbb{R}^n$ and $\kappa : V \hookrightarrow \mathbb{R}^m$ be two singular charts of X defined on open $U, V \subset X$. We will construct a sheaf morphism $\eta_{\kappa,\iota} : \Omega_\iota^k|_{U \cap V} \rightarrow \Omega_\kappa^k|_{U \cap V}$. Let $x \in U \cap V$. By the embedding theorem A.3 there exists a singular chart $\lambda : W_x \hookrightarrow \mathbb{R}^{\text{rk } x}$ defined over an open neighborhood $W_x \subset U \cap V$ of x . Moreover, after possibly shrinking W_x , there exist embeddings $H : \tilde{W}_x \hookrightarrow \mathbb{R}^n$ and $G : \tilde{W}_x \hookrightarrow \mathbb{R}^m$ of an open neighborhood \tilde{W}_x of $\lambda(x)$ such that $\iota|_{W_x} = H \circ \lambda$ and $\kappa|_{W_x} = G \circ \lambda$, and such that H^* induces an isomorphism from $(\iota(U), \mathcal{O}_\iota)$ to $(\lambda(U), \mathcal{O}_\lambda)$ and G^* one from $(\kappa(U), \mathcal{O}_\kappa)$ to $(\lambda(U), \mathcal{O}_\lambda)$. By Lemma 2.1 we obtain isomorphisms of sheaf complexes

$\eta_{\lambda,\iota} : \Omega_{\iota|W_x}^\bullet \rightarrow \Omega_\lambda^\bullet$ and $\eta_{\lambda,\kappa} : \Omega_{\kappa|W_x}^\bullet \rightarrow \Omega_\lambda^\bullet$. Put $\eta_{\kappa,\iota}^{W_x} := (\eta_{\lambda,\kappa})^{-1} \circ \eta_{\lambda,\iota}$. Then $\eta_{\kappa,\iota}^{W_x}$ is a sheaf isomorphism from $\Omega_{\iota|W_x}^\bullet$ to $\Omega_{\kappa|W_x}^\bullet$ which by Lemma 2.2 does not depend on the particular choice of λ and the embeddings H and G . Moreover, if y is another point of $U \cap V$, an argument using Lemma 2.2 and the embedding theorem A.3 again shows that the sheaf isomorphisms $\eta_{\kappa,\iota}^{W_x}$ and $\eta_{\kappa,\iota}^{W_y}$ coincide on the overlap $W_x \cap W_y$. This proves the next lemma.

LEMMA 2.3. *Given two singular charts $\iota : U \hookrightarrow \mathbb{R}^n$ and $\kappa : V \hookrightarrow \mathbb{R}^m$ of (X, \mathcal{O}) there exists a unique sheaf morphism $\eta_{\kappa,\iota} : \Omega_{\kappa|U \cap V}^\bullet \rightarrow \Omega_{\iota|U \cap V}^\bullet$ such that*

$$\eta_{\kappa,\iota|W_x} = (\eta_{\lambda,\kappa})^{-1} \circ \eta_{\lambda,\iota}$$

for each $x \in U \cap V$ and each singular chart $\lambda : W_x \hookrightarrow \mathbb{R}^{k_x}$ defined on a sufficiently small open neighborhood $W_x \subset U \cap V$ of x .

Application of Lemma 2.2 and the embedding theorem A.3 a last time entails the final lemma.

LEMMA 2.4. *Assume that $\iota : U \hookrightarrow \mathbb{R}^n$, $\kappa : V \hookrightarrow \mathbb{R}^m$, and $\lambda : W \hookrightarrow \mathbb{R}^l$ are three singular charts of X . Then the following cocycle condition holds true over the intersection $U \cap V \cap W$:*

$$\eta_{\kappa,\iota} = \eta_{\kappa,\lambda} \circ \eta_{\lambda,\iota}. \tag{2.4}$$

The cocycle condition holding true for any triple of singular charts entails that the sheaf complexes Ω_ι^k glue together to a global sheaf Ω_X^k on X . This sheaf is the *sheaf of abstract k -forms on X* . We sometimes denote it briefly by Ω^k . By construction, each of the $\eta_{\kappa,\iota}$ commutes with the exterior differential d , hence the operators $d : \Omega_\kappa^k \rightarrow \Omega_\kappa^{k+1}$ glue together to a sheaf morphism $d : \Omega_X^k \rightarrow \Omega_X^{k+1}$. So finally we obtain a sheaf complex (Ω_X^\bullet, d) of commutative differential graded algebras. The complex of global sections $(\Omega^\bullet(X), d)$ will be called the *Grauert–Grothendieck complex of X* .

REMARK 2.5. For $X \subset \mathbb{C}^n$ a complex space or algebraic variety, the complex $\Omega^\bullet(X)$ of abstract forms on X has been first constructed by Grauert [GK64] and Grothendieck [Gro66] using sheaves of holomorphic respectively regular functions. In [Spa71, §4.], Spallek has given a construction of the sheaf of k -forms of class \mathcal{C}^∞ on an affine differentiable space (X, \mathcal{O}) which essentially corresponds to the one given here in Eq. (2.1). It is claimed in [Spa71] that given a (singular) atlas for (X, \mathcal{O}) , the sheaves of k -forms over affine domains induce a global sheaf of k -forms on X . Several details of the corresponding construction, in particular a verification of the cocycle condition, are missing.

REMARK 2.6. Recall that a morphism of differentiable spaces $(f, \varphi) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ consists of a continuous map $f : X \rightarrow Y$ and a morphism of sheaves $\varphi : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$. Using the construction of the sheaves of abstract forms on X and Y one extends the morphism φ in a unique way to a morphism of sheaves of commutative differential algebras $\varphi : \Omega_Y^\bullet \rightarrow f_*\Omega_X^\bullet$. One concludes that forming the sheaf complex of abstract forms is a functor defined on the category of differentiable spaces.

3. Basic relative forms. Let us first recall the notion of *relative forms* associated to a smooth map $p : M \rightarrow N$ between manifolds M and N . By relative forms one understands sections of the sheaf complex $\Omega_{M \xrightarrow{p} N}^\bullet$ defined as the quotient sheaf

$$\Omega_M^\bullet / d(p^{-1}\mathcal{C}_N^\infty) \wedge \Omega_M^\bullet$$

together with the differential induced by the differential on Ω_M^\bullet . If $p : M \rightarrow N$ is a surjective submersion, the space of global sections $\Omega_{M \xrightarrow{p} N}^k(M)$ can be identified with the space of smooth families $(\omega_y)_{y \in N}$ of forms $\omega_y \in \Omega^k(p^{-1}(y))$. The differential acts fiberwise on $(\omega_y)_{y \in N}$ which means that

$$d((\omega_y)_{y \in N}) = (d\omega_y)_{y \in N} .$$

If the underlying map p is projection onto the first factor of a product $N \times M$, one can identify $\Omega_{M \xrightarrow{p} N}^k$ with the sheaf of smooth sections $\Gamma^\infty(-, \wedge^k s^* T^* M)$, where $s : N \times M \rightarrow M$ is projection onto the second factor. More precisely, one has in this case a sequence of sheaf morphisms, whose composition is an isomorphism:

$$\Gamma^\infty(-, \wedge^k s^* T^* M) \hookrightarrow \Omega_{N \times M}^k \longrightarrow \Omega_{N \times M \xrightarrow{p} N}^k = \Omega_{N \times M}^k / d(p^{-1}\mathcal{C}_N^\infty) \wedge \Omega_{N \times M}^{k-1} . \quad (3.1)$$

Note that even though $\Gamma^\infty(-, \wedge^k s^* T^* M)$ is a subsheaf of Ω_M^k for each k , one does not obtain that way a subsheaf complex of Ω_M^\bullet since the exterior derivative on Ω_M^\bullet does in general not map $\Gamma^\infty(-, \wedge^k s^* T^* M)$ to $\Gamma^\infty(-, \wedge^{k-1} s^* T^* M)$. The correct differential on $\Gamma^\infty(-, \wedge^k s^* T^* M)$ acts fiberwise as explained above.

After these preliminary remarks we now assume that G is a compact Lie group acting on a smooth manifold M . By $G \times M$ we denote the corresponding action groupoid. More precisely, this is the Lie groupoid with arrow space $G \times M$, object space M , source map $s : G \times M \rightarrow M$, $(g, p) \mapsto p$, target map $t : G \times M \rightarrow M$, $(g, p) \mapsto gp$ and multiplication $m : (G \times M)^{(2)} \rightarrow G \times M$, $((h, gp), (g, p)) \mapsto (hg, p)$ defined on the fibered product $(G \times M)^{(2)} := (G \times M)_{s \times t} (G \times M)$. The unit map of the action groupoid is $u : M \rightarrow G \times M$, $p \mapsto (e, p)$ with e denoting the identity element of G , and the inversion map is $G \times M \rightarrow G \times M$, $(g, p) \mapsto (g^{-1}, gp)$. We will denote the orbit space of the action groupoid $G \times M$ by M/G , and the orbit map by $\pi : M \rightarrow M/G$. For more details on Lie groupoids see [MM03].

The space $\Lambda_0(G \times M)$ defined in Eq. (1.2) corresponds to the loop space of the action groupoid $G \times M$ that means to the space of all $(g, p) \in G \times M$ such that $s(g, p) = t(g, p)$. Since $\Lambda_0(G \times M)$ is a closed subspace of the manifold $G \times M$, the loop space inherits from the ambient manifold the structure of a differentiable space. We denote the canonical embedding by $\iota : \Lambda_0(G \times M) \hookrightarrow G \times M$. Since by its definition the loop space is locally semialgebraic, it carries a minimal Whitney B stratification, so $\Lambda_0(G \times M)$ becomes a differentiable stratified space. For simplicity, we denote the loop space shortly by Λ_0 . Moreover, we denote the structure sheaf of smooth functions on Λ_0 by $\mathcal{C}_{\Lambda_0}^\infty$, and the sheaf of smooth functions on $G \times M$ vanishing on Λ_0 by \mathcal{J} . Now consider the Grauert-Grothendieck complex $\Omega_{\Lambda_0}^\bullet$ of differential forms on the loop space. Following BRYLINSKI, we define the sheaf $\Omega_{\Lambda_0 \rightarrow G}^k$ of *relative forms on the loop space* as the quotient sheaf

$$\Omega_{\Lambda_0 \rightarrow G}^k := \iota^{-1} \left(\Omega_{G \times M \rightarrow G}^k / (\mathcal{J} \Omega_{G \times M \rightarrow G}^k + d\mathcal{J} \wedge \Omega_{G \times M \rightarrow G}^{k-1}) \right) . \quad (3.2)$$

The graded sheaf $\Omega_{\Lambda_0 \rightarrow G}^\bullet$ inherits from $\Omega_{G \times M \rightarrow G}^\bullet$ a differential turning it into a sheaf complex. Let us give another representation of the sheaf or relative forms on Λ_0 .

To this end observe that the pull-back bundle s^*T^*M induces a monomorphism of bundles $\wedge^k s^*T^*M \hookrightarrow \wedge^k T^*(G \times M)$, hence the following morphism of sheaves:

$$\Gamma_{\Lambda_0}^\infty(-, \wedge^k s^*T^*M) \longrightarrow \Omega_{\Lambda_0}^k = \iota^{-1}\left(\Omega_{G \times M}^k / (\mathcal{J}\Omega_{G \times M}^k + d\mathcal{J} \wedge \Omega_{G \times M}^{k-1})\right). \quad (3.3)$$

Here, $\Gamma_{\Lambda_0}^\infty(-, \wedge^k s^*T^*M)$ stands for the sheaf of smooth sections of the vector bundle $\wedge^k s^*T^*M \rightarrow G \times M$ over the subspace Λ_0 . In other words, the section space $\Gamma_{\Lambda_0}^\infty(U, E)$ for $U \subset \Lambda_0$ open can be identified with the quotient space $\Gamma^\infty(\tilde{U}, E) / \mathcal{J}(\tilde{U})\Gamma^\infty(\tilde{U}, E)$, where $\tilde{U} \subset G \times M$ open is chosen so that $\tilde{U} \cap \Lambda_0 = U$. Since the composition of sheaf morphisms in (3.1) is an isomorphism, the sheaf morphism (3.3) induces a canonical identification

$$\begin{aligned} \Omega_{\Lambda_0 \rightarrow G}^k(U) \cong \Gamma^\infty(\tilde{U}, \wedge^k s^*T^*M) / (\mathcal{J}(\tilde{U})\Gamma^\infty(\tilde{U}, \wedge^k s^*T^*M) \\ + d\mathcal{J}(\tilde{U}) \wedge \Gamma^\infty(\tilde{U}, \wedge^{k-1} s^*T^*M)), \end{aligned}$$

where \tilde{U} is chosen as before. For a section $\omega \in \Gamma^\infty(\tilde{U}, \wedge^k s^*T^*M)$ we denote its image in $\Omega_{\Lambda_0 \rightarrow G}^k(U)$ by $[\omega]$. Since Λ_0 is the union of the fibers $\{g\} \times M^g$, $g \in G$, the relative form $[\omega]$ can be identified with the smooth family $(\omega_g)_{g \in G}$ of restrictions $\omega_g := \omega|_{M^g}$, and any smooth family $(\omega_g)_{g \in G}$ of forms $\omega_g \in \Omega^k(M^g)$ gives rise to a unique relative form on the loop space. Under this identification the differential of $[\omega] = (\omega_g)_{g \in G}$ is given by the smooth family $(d\omega_g)_{g \in G}$.

Next recall that the G -action on M gives rise for each $p \in M$ to the normal space $N_p M := T_p M / T_p \mathcal{O}_p$, where \mathcal{O}_p denotes the G -orbit through p . The family N^* associating to each $p \in M$ the conormal space $N_p^* M \subset T^*M$ is a smooth generalized vector subbundle of the cotangent bundle T^*M in the sense of DRAGER–LEE–PARK–RICHARDSON [DLPR12]. Note that under the isomorphism between the tangent and cotangent bundle induced by a riemannian metric on M the generalized vector bundle N^* becomes a generalized distribution in the sense of STEFAN–SUESSMANN, cf. [Ste74, Sus73]. The restriction of N^* to an orbit or a stratum of fixed orbit type is a vector bundle; see [PPT14]. After these preparatory remarks it is clear that $\wedge^k N^*$, $k \in \mathbb{N}^*$, is a smooth generalized vector subbundle of $\wedge^k T^*M$. Likewise, if one puts $N_p M^g := T_p M^g / (T_p \mathcal{O}_p \cap T_p M^g)$ for $g \in G$, the alternating power $\wedge^k (N M^g)^*$ is a smooth generalized vector subbundle of $\wedge^k T^*M^g$ for every $k \in \mathbb{N}^*$. The space $\Omega_{\text{h.r.}}^k(U)$ of *horizontal relative k -forms* over $U \subset \Lambda_0$ open is now defined as

$$\begin{aligned} \Omega_{\text{h.r.}}^k(U) &:= \{[\omega] = (\omega_g)_{g \in G} \in \Omega_{\Lambda_0 \rightarrow G}^k(U) \mid \omega_{(g,p)} \in \wedge^k N_p^* \text{ for all } (g,p) \in \Lambda_0\} = \\ &= \{[\omega] = (\omega_g)_{g \in G} \in \Omega_{\Lambda_0 \rightarrow G}^k(U) \mid \omega_g \in \Gamma^\infty(U \cap M^g, \wedge^k (N M^g)^*)\}. \end{aligned}$$

Obviously, we thus obtain a subsheaf $\Omega_{\text{h.r.}}^k \subset \Omega_{\Lambda_0 \rightarrow G}^k$ having these spaces as its section spaces. Now observe that the G -action on the cotangent bundle T^*M coming from the G -action on M induces a G -action on relative forms, hence we can speak of *invariant relative k -forms*. These are exactly those $[\omega] \in \Omega_{\text{h.r.}}^k(U)$ which satisfy

$$\sigma_{(hgh^{-1}, hx)}(hv_1, \dots, hv_k) = \omega_{(g,x)}(v_1, \dots, v_k) \quad (3.4)$$

for all $(g,x) \in U$ and $h \in G$ such that $(hgh^{-1}, hx) \in U$ and $v_1, \dots, v_k \in N_x$. Note that by definition invariance of $[\omega]$ does not depend on the particular choice of a representative. If one writes $[\omega]$ as a smooth family $(\omega_g)_{g \in G}$ of forms ω_g on $U \cap M^g$ and if U is G -invariant, invariance of $[\omega]$ can be equivalently expressed by

$$h^* \omega_{hgh^{-1}} = \omega_g \quad \text{for all } g, h \in G. \quad (3.5)$$

This relation implies in particular that the differential on $\Omega_{\Lambda_0 \rightarrow G}^\bullet$ maps an invariant family $(\omega_g)_{g \in G}$ over a G -invariant open U to the invariant family $(d\omega_g)_{g \in G}$. Since the G -action on T^*M leaves the conormal bundle N^* invariant, we can even speak of *invariant horizontal relative k -forms*. Now we are ready to put for $O \subset M/G$ open

$$\Omega_{\text{b.r.}}^k(O) := \{[\omega] \in \Omega_{\text{h.r.}}^k(s_{|\Lambda_0}^{-1}\pi^{-1}(O)) \mid [\omega] \text{ is invariant}\}. \tag{3.6}$$

These spaces are the section spaces of a sheaf $\Omega_{\text{b.r.}}^k$. Observe that the sheaf $\Omega_{\text{b.r.}}^k$ is defined over the orbit space M/G , not the loop space. Following BRYLINSKI again, we call sections of $\Omega_{\text{b.r.}}^k$ *basic relative k -forms*. In case we need to clarify the action groupoid underlying a sheaf of basic relative k -forms we will denote that sheaf more clearly by $\Omega_{G \times M\text{-b.r.}}^k$. The differential d maps $\Omega_{\text{b.r.}}^k$ to $\Omega_{\text{b.r.}}^{k+1}$. This follows from Cartan’s magic formula since it entails for $[\omega] = (\omega_g)_{g \in G} \in \Omega_{\text{b.r.}}^k(O)$, $g \in G$, and every element ξ of the Lie algebra \mathfrak{g}_g of the centralizer $G_g := Z_G(g)$ of g the equality

$$i_{\xi_{M^g}} d\omega_g = \mathcal{L}_{\xi_{M^g}} \omega_g - di_{\xi_{M^g}} \omega_g = 0,$$

where ξ_{M^g} denotes the fundamental vector field of ξ on M^g . So we finally obtain a complex of sheaves $(\Omega_{\text{b.r.}}^\bullet, d)$ over the orbit space M/G . Since each of the $\Omega_{\text{b.r.}}^k$ is in a natural way a $\mathcal{C}_{M/G}^\infty$ -module, $(\Omega_{\text{b.r.}}^\bullet, d)$ is even a complex of fine sheaves. To formulate our main result let us remind the reader that we denote by \mathcal{B} Brylinski’s sheaf over the orbit space M/G and that this sheaf has section spaces given by (1.3).

EXAMPLE 3.1. As an example let us consider the S^1 -action on \mathbb{R}^2 by rotation. We parametrize S^1 by $e^{2\pi i\theta}$, where $\theta \in \mathbb{R}$. In coordinates, the action is expressed as

$$\mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2, (\theta, (x, y)) \mapsto (\cos(\theta)x - \sin(\theta)y, \sin(\theta)x + \cos(\theta)y) .$$

The loop space is given by

$$\Lambda_0(S^1 \times \mathbb{R}^2) = \{(e^{2\pi i\theta}, (x, y)) \in S^1 \times \mathbb{R}^2 \mid \theta = 0 \text{ or } x = y = 0\}.$$

The minimal stratification of $\Lambda_0(S^1 \times \mathbb{R}^2)$ is given by the decomposition into the strata $S_0 = \{1\} \times (\mathbb{R}^2 \setminus \{(0, 0)\})$, $S_1 = (S^1 \setminus \{1\}) \times \{(0, 0)\}$, and $S_2 = \{(1, (0, 0))\}$. Using the given parametrization of S^1 it is clear that in a neighborhood of the subset $\{1\} \times \mathbb{R}^2 \subset \Lambda_0(S^1 \times \mathbb{R}^2)$ the loop space looks like a neighborhood of $\{0\} \times \mathbb{R}^2$ in the space

$$\Lambda'_0 := (\mathbb{R} \times \{(0, 0)\}) \cup (\{0\} \times \mathbb{R}^2) .$$

The loop space is smooth around each point of the stratum S_1 , hence to describe the sheaf of smooth functions on $\Lambda_0(S^1 \times \mathbb{R}^2)$ we need to only understand how smooth functions on Λ'_0 look around a neighborhood of the origin. To this end let $I \subset \mathcal{C}^\infty(\mathbb{R}^3)$ denote the ideal of smooth functions vanishing on Λ'_0 . If $f \in I$, then $f = \theta g$ for some $g \in \mathcal{C}^\infty(\mathbb{R}^3)$ and with $\theta : \mathbb{R}^3 \rightarrow \mathbb{R}$ denoting here the projection onto the first coordinate. Since f vanishes on the θ -axis, g does so, too, hence $g = xf_1 + yf_2$ for some $f_1, f_2 \in \mathcal{C}^\infty(\mathbb{R}^3)$. We obtain the representation $f = \theta xf_1 + \theta yf_2$. Therefore, the differential graded ideal $I\Omega^\bullet(\mathbb{R}^3) + dI \wedge \Omega^\bullet(\mathbb{R}^3) \subset \Omega^\bullet(\mathbb{R}^3)$ consists of all sums of forms of the form

$$\theta x\omega_1 + \theta y\omega_2 + x d\theta \wedge \omega_3 + y d\theta \wedge \omega_4 + \theta dx \wedge \omega_3 + \theta dy \wedge \omega_4,$$

where $\omega_1, \omega_2 \in \Omega^k(\mathbb{R}^3)$ and $\omega_3, \omega_4 \in \Omega^{k-1}(\mathbb{R}^3)$. Now observe that in the space of relative forms $\Omega_{\mathbb{R}^3 \xrightarrow{\theta} \mathbb{R}}^k(\mathbb{R}^3)$ the form $d\theta$ vanishes. Moreover, the pullback of a relative

form of degree $k \geq 1$ to the stratum $\mathbb{R}^* \times \{(0, 0)\}$ vanishes as well. One concludes that, for $k \geq 1$ the space

$$\Omega_{\Lambda'_0 \xrightarrow{\theta} \mathbb{R}}^k(\Lambda'_0) := \Omega_{\mathbb{R}^3 \xrightarrow{\theta} \mathbb{R}}^k(\mathbb{R}^3) / I\Omega_{\mathbb{R}^3 \xrightarrow{\theta} \mathbb{R}}^k(\mathbb{R}^3) + dI \wedge \Omega_{\mathbb{R}^3 \xrightarrow{\theta} \mathbb{R}}^{k-1}(\mathbb{R}^3)$$

of relative k -forms on Λ'_0 can be identified with the space of smooth families $(\omega_\theta)_{\theta \in \mathbb{R}}$, where

$$\omega_\theta \in \begin{cases} \Omega^k(\mathbb{R}^2), & \text{if } \theta = 0, \\ \{0\}, & \text{else.} \end{cases}$$

So one concludes that $\Omega_{\Lambda_0(S^1 \times \mathbb{R}^2) \rightarrow S^1}^k(\Lambda_0(S^1 \times \mathbb{R}^2)) \cong \Omega^k(\mathbb{R}^2)$ for $k \geq 1$ and that, under this isomorphism, $\Omega_{\text{b.r.}}^k(\mathbb{R}^2/S^1) \cong \Omega_{\text{bas}}^k(\mathbb{R}^2/S^1)$, where the latter denotes the space of basic k -forms on \mathbb{R}^2 . In case $k = 0$, one has $\Omega_{\text{b.r.}}^0(\mathbb{R}^2/S^1) \cong \mathcal{C}^\infty(\mathbb{R}^2/S^1)$. Finally in this example, Brylinski's sheaf can be identified with the sheaf (over the orbit space) of G -invariant locally constant functions on \mathbb{R}^2 .

THEOREM 3.1. *Let G be a compact Lie group acting on a manifold M . The sheaf complex $(\Omega_{\text{b.r.}}^\bullet, d)$ of basic relative forms together with the natural monomorphism of sheaves*

$$d_{-1} : \mathcal{B} \hookrightarrow \Omega_{\text{b.r.}}^0 = \pi_* (\mathcal{C}_{\Lambda_0}^\infty)^G$$

then forms a fine resolution of Brylinski's sheaf \mathcal{B} .

Proof. Let $O \subset M/G$ be open, and $f \in \mathcal{B}(O)$. By definition through Eq. (1.3) f then is a smooth G -invariant function on $s_{|\Lambda_0}^{-1} \pi^{-1}(O)$, and $f(g, -) : M^g \rightarrow \mathbb{R}$ is locally constant for all $g \in G$. Hence $df(g, -) \in \Omega^1(M^g)$ vanishes for every g . This entails that $\mathcal{B} \hookrightarrow \Omega_{\text{b.r.}}^\bullet$ is a cochain complex of sheaves over the orbit space M/G .

It remains to show that that cochain complex of sheaves is exact, meaning that for each orbit $\mathcal{O} \in M/G$ the complex of stalks $\mathcal{B}_{\mathcal{O}} \hookrightarrow \Omega_{\text{b.r.,}\mathcal{O}}^\bullet$ is exact. To this end we proceed in several steps. In the first step we consider the case, where M is a finite dimensional vector space V carrying a linear G -action, and where $\mathcal{O} = \{0\}$, the orbit through the origin. Choose a G -invariant scalar product on V . Assume that $B \subset V$ is an open ball around the origin, and consider the homothety $h : [0, 1] \times V \rightarrow V$, $(t, v) \rightarrow tv$. The homothety h leaves each of the subsets $B^g \subset B$ invariant, and commutes with the G -action. Hence

$$K : \Omega_{\text{b.r.}}^k(B/G) \rightarrow \Omega_{\text{b.r.}}^{k-1}(B/G),$$

$$\omega = (\omega_g)_{g \in G} \mapsto K\omega := \begin{cases} \Lambda_0(G \times M) \ni (g, p) \mapsto \omega_g(0), & \text{for } k = 0, \\ \left(\int_0^1 h_t^*(\xi_t \lrcorner \omega_g) \right)_{g \in G}, & \text{for } k \geq 1, \end{cases}$$

is a well-defined operator, where $\Omega_{\text{b.r.}}^{-1}$ is Brylinski's sheaf \mathcal{B} , h_t equals $h(t, -) : B \rightarrow B$, $v \mapsto tv$, and $\xi_t : B \rightarrow TB$ is the vector field given by $\xi_t := \partial_t h_t$. Cartan's magic formula implies that

$$(\omega_g)_{g \in G} = dK(\omega_g)_{g \in G} + Kd(\omega_g)_{g \in G} \quad \text{for all } (\omega_g)_{g \in G} \in \Omega_{\text{b.r.}}^k(B) \text{ and } k \in \mathbb{N},$$

since h_1^* acts by identity on $(\omega_g)_{g \in G}$, $(h_0^* \omega_g)_{g \in G} = 0$ for $k \geq 1$, and $h_0^* \omega = \omega(-, 0)$ for $k = 0$. Hence $\mathcal{B}_{\mathcal{O}} \hookrightarrow \Omega_{\text{b.r.,}\mathcal{O}}^\bullet$ is exact when $\mathcal{O} = \{0\}$.

In the second step we come back to the general case of a G -action on an arbitrary manifold M . Choose a G -invariant riemannian metric η on M . Let $p \in M$ be a point, \mathcal{O} the orbit through p , and $N_p := T_p M / T_p \mathcal{O}$ the normal space to \mathcal{O} at p . Via the riemannian metric we can identify N_p with the orthogonal complement of $T_p \mathcal{O}$ in $T_p M$. The isotropy group G_p acts in a natural way on N_p , cf. [Pfl01, Sec. 4.2.5]. Choose an open ball B around the origin of N_p with radius smaller than the injectivity radius at p . The slice theorem [Pfl01, Thm. 4.2.6] entails that the G -action on M induces an action of the isotropy group G_p on the slice $S_p := \exp(B)$ and that the exponential map intertwines the G_p -actions on N_p and S_p . Moreover, the exponential map provides an equivariant diffeomorphism between $G \times_{G_p} B$ and the G -saturation $U := G \cdot S_p$ of the slice. It therefore suffices to verify the claim for the case where M has the form $G \times_H B$ with $H \subset G$ being a compact subgroup and B an open H -invariant ball around the origin of a finite dimensional H -representation space V , and where \mathcal{O} is the orbit through the point $[e, 0] \in G \times_H B$. From now on we will consider only this setting. Note that here and in the following we will denote by $[g, v]$ the equivalence class of a point $(g, v) \in G \times B$ in $G \times_H B$.

In the third step we provide a description of the tangent bundle $T(G \times_H V)$. To this end choose a bi-invariant riemannian metric on G . Let \mathcal{Q} be the foliation of G given by the orbits of the canonical right action of H on G . For each $g \in G$ let E_g be the orthogonal complement of the tangent space $T_g \mathcal{O}$ to the leaf of \mathcal{Q} through g . One thus obtains a vector bundle $E \rightarrow G$ which is invariant under the left action of G on TG and invariant under the right action of H . The latter follows from the fact that the right action of H on G maps leaves of \mathcal{Q} to leaves, since $g \exp(t\xi)h = gh \exp(t \operatorname{Ad}_{h^{-1}}(\xi))$ for all $g \in G$, $h \in H$, $\xi \in \mathfrak{h}$, and $t \in \mathbb{R}$. One concludes that E can be identified with the trivial bundle $G \times \mathfrak{m}$, where \mathfrak{m} is the orthogonal complement of \mathfrak{h} in \mathfrak{g} . Now call two elements $(\Xi, (v, X)), (Z, (w, Y)) \in E \times TV = E \times V \times V$ equivalent, if there is an $h \in H$ with $\Xi = Zh$ and $h(v, X) = (w, Y)$. We denote by $[\Xi, (v, X)]$ the equivalence class of $(\Xi, (v, X)) \in E \times TV$. One checks immediately that the quotient space of $E \times TV$ by this equivalence relation can be canonically identified with the tangent bundle $T(G \times_H V)$. Under this identification, an element of the tangent space $T_{[g,v]}(G \times_H V) \cong \mathfrak{m} \times V$ over the footpoint $[g, v] \in G \times_H V$ has a unique representation of the form $[(g, \xi), (v, X)]$ with $\xi \in \mathfrak{m}$ and $X \in V$. For later purposes let us remark that if $[g, v] = [g', v']$, i.e. if $(g', v') = (gh^{-1}, hv)$ for some $h \in H$, then $[(g, \xi), (v, X)] = [(g', \xi'), (v', X')]$ with $\xi' = \xi h^{-1}$ and $v' = hv$. In the following step we will denote the equivalence class $[(g, \xi), (v, X)]$ shortly by $[\xi, X]_{(g,v)}$.

The fourth step consists in verifying that the embedding map $\iota : B \hookrightarrow M := G \times_H B$, $v \mapsto [e, v]$ induces an isomorphism between the sheaves $\Omega_{G \times M\text{-b.r.}}^k$ and $\Omega_{H \times B\text{-b.r.}}^k$. Observe that both sheaves live on the same topological space, since the orbit spaces M/G and B/H are canonically isomorphic since ι is a Morita equivalence. The isomorphism is given by pullback via ι . More precisely, for $O \subset M/G$ and a basic relative form $\omega = (\omega_g)_{g \in G} \in \Omega_{G \times M\text{-b.r.}}^k(O)$ let $\iota^* \omega$ be the family $(\iota_{O,h}^* \omega_h)_{h \in H}$, where $\iota_{O,h}$ denotes the restriction of the embedding ι to $(\pi_{H \times B}^{-1}(O))^h$ with $\pi_{H \times B} : B \rightarrow B/H$ being the orbit map of the groupoid $H \times B$. Obviously, $\iota_{O,h}^* \omega_h$ is a $Z_H(h)$ -invariant horizontal form on $(\pi_{H \times B}^{-1}(O))^h$, and the family $(\iota_{O,h}^* \omega_h)_{h \in H}$ is H -invariant. Hence $\iota^* \omega \in \Omega_{H \times B\text{-b.r.}}^k(O)$, so we obtain a morphism of sheaf complexes $\iota^* : \Omega_{G \times M\text{-b.r.}}^\bullet \rightarrow \Omega_{H \times B\text{-b.r.}}^\bullet$. Let us show that it is an isomorphism. To this end note first that for every $g \in G$ the invariant space M^g is given by $M^g = \{[f, v] \in G \times_H B \mid g \in fH_v f^{-1}\}$. Now

let $(\varrho_h)_{h \in H} \in \Omega_{H \times B\text{-b.r.}}^k(O)$, and define $\omega \in \Omega_{G \times M\text{-b.r.}}^k(O)$ by

$$\omega_{g,[f,v]}([\xi_1, X_1]_{(f,v)}, \dots, [\xi_k, X_k]_{(f,v)}) := \varrho_{f^{-1}gf,v}(f^{-1}gfX_1, \dots, f^{-1}gfX_k),$$

where $g \in G$, $[f, v] \in M^g \cap s_{\Lambda_0(G \times M)}^{-1} \pi_{|G \times M}^{-1}(O)$, and $[\xi_1, X_1]_{(f,v)}, \dots, [\xi_k, X_k]_{(f,v)} \in T_{(f,v)}M$. Since $\varrho = (\varrho_h)_{h \in H}$ is H -invariant, one obtains for every $h \in H$ the equality

$$\begin{aligned} \omega_{g,[fh^{-1},hv]}([\xi_1 h^{-1}, hX_1]_{(fh^{-1},hv)}, \dots, [\xi_k h^{-1}, hX_k]_{(fh^{-1},hv)}) &= \\ &= \varrho_{hf^{-1}gf, hv}(hf^{-1}gfX_1, \dots, hf^{-1}gfX_k) = \\ &= \varrho_{f^{-1}gf,v}(f^{-1}gfX_1, \dots, f^{-1}gfX_k). \end{aligned}$$

This shows that ω_g is independent of the choices made, and an element of $\Omega^k(M^g \cap s_{\Lambda_0(G \times M)}^{-1} \pi_{|G \times M}^{-1}(O))$. Moreover, ω_g is a horizontal form by construction. The family $\omega = (\omega_g)_{g \in G}$ is also G -invariant. To verify this let $h \in G$, and observe that by definition

$$\omega_{hg, h^{-1}, [hf, v]}([h\xi_1, X_1]_{(hf, v)}, \dots, [h\xi_k, X_k]_{(hf, v)}) = \varrho_{f^{-1}gf, v}(f^{-1}gfX_1, \dots, f^{-1}gfX_k),$$

where $g \in G$, $[f, v]$, and $[\xi_1, X_1]_{(f,v)}, \dots, [\xi_k, X_k]_{(f,v)}$ are as above. This proves G -invariance of the family ω , hence $\omega \in \Omega_{G \times M\text{-b.r.}}^\bullet(O)$ indeed. By construction it is clear that $\iota^* \omega = \varrho$. By G -invariance, ω is uniquely determined by $\iota^* \omega$. So ι is a sheaf isomorphism as claimed.

In the fifth and final step we show that $\mathcal{B}_\mathcal{O} \hookrightarrow \Omega_{G \times M\text{-b.r.}, \mathcal{O}}^\bullet$ is an exact sheaf complex in the case where $M = G \times_H B$ and \mathcal{O} is the orbit through the point $[e, 0]$. By the second step, it suffices to consider this case. By the fourth step, the embedding $\iota : B \hookrightarrow M$ induces an isomorphism of sheaf complexes $\iota^* : \Omega_{G \times M\text{-b.r.}} \rightarrow \Omega_{H \times B\text{-b.r.}}^\bullet$. By the first step, the cochain complex $\mathcal{B}_{\{0\}} \hookrightarrow \Omega_{H \times B\text{-b.r.}, \{0\}}^\bullet$ is exact, hence $\mathcal{B}_\mathcal{O} \hookrightarrow \Omega_{G \times M\text{-b.r.}, \mathcal{O}}^\bullet$ is so, too, since the orbit through $\iota(0)$ coincides with \mathcal{O} . The proof is finished. \square

Appendix A. Differentiable stratified spaces. For the convenience of the reader we briefly recall here the notion of a differentiable space, mainly following [NGSdS03], and then describe what it means that a stratification is compatible with the differentiable structure.

DEFINITION A.1. An algebra over \mathbb{R} of the form $A = \mathcal{C}^\infty(\mathbb{R}^n)/J$, where $J \subset \mathcal{C}^\infty(\mathbb{R}^n)$ is a closed ideal, is called a *differentiable algebra*. Denote by $\text{spec } A$ the maximal spectrum of a differentiable algebra A , and by \mathcal{O}_A its structure sheaf, which is the sheafification of the presheaf $U \mapsto A_U$, where $U \subset \text{spec } A$ is open and A_U is the localization of A over the subset of elements which do not vanish over U ; cf. [NGSdS03, Sec. 3.1].

A differentiable algebra carries in a natural way the structure of a Fréchet algebra. Moreover, given a differentiable algebra A , the pair $(\text{spec } A, \mathcal{O}_A)$ is a commutative locally ringed space.

DEFINITION A.2. A commutative locally \mathbb{R} -ringed space (X, \mathcal{O}) is called an *affine differentiable space*, if it is isomorphic as a commutative locally ringed space to $(\text{spec } A, \mathcal{O}_A)$ for some differentiable algebra A . The ringed space (X, \mathcal{O}) is called a *differentiable space*, if for every $x \in X$ there exists an open neighborhood $U \subset X$ such that $(U, \mathcal{O}|_U)$ is an affine differentiable space. By a *morphism of differentiable spaces*

we understand a morphism of locally \mathbb{R} -ringed spaces $(f, \varphi) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ between differentiable spaces (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) .

Locally, a differentiable space (X, \mathcal{O}) can be embedded into euclidean space. We call an embedding $\iota : U \hookrightarrow \mathbb{R}^n$ over an open $U \subset X$ together with a morphism of differentiable algebras $\iota^* : \mathcal{C}^\infty(\iota(U)) \rightarrow \mathcal{O}(U)$ such that $(\iota^*(x_1)(x), \dots, \iota^*(x_n)(x)) = \iota(x)$ for all $x \in U$ a *singular chart* for (X, \mathcal{O}) if these data induce an isomorphism of locally \mathbb{R} -ringed spaces

$$(\iota, \iota^*) : (U, \mathcal{O}|_U) \rightarrow (\iota(U), \mathcal{C}_{\iota(U)}^\infty / \mathcal{I}_\iota),$$

where \mathcal{I}_ι is the kernel of the morphism of sheaves $\iota^* : \mathcal{C}_{\iota(U)}^\infty \rightarrow \mathcal{O}|_U$. By a *singular atlas* of (X, \mathcal{O}) we understand a family \mathfrak{A} of singular charts such that the family of domains $\{\text{dom}(\iota)\}_{(\iota, \iota^*) \in \mathfrak{A}}$ is an open cover of X . If (X, \mathcal{O}) is reduced, the embedding $\iota : U \hookrightarrow \mathbb{R}^n$ completely determines the singular chart (ι, ι^*) , because ι^* then is just pullback by the embedding. By abuse of language we denote a singular chart even in the non-reduced case just by the embedding $\iota : U \hookrightarrow \mathbb{R}^n$.

The following result shows that around a point of a differentiable space the minimal embedding dimension is given by the dimension of the Zariski tangent space.

THEOREM A.3 ([Pff01, Prop. 1.3.10 & Corollaries]). *Let (X, \mathcal{O}) be differentiable space, and $x \in X$ a point. Then there exists an open affine neighborhood W of x together with a singular chart $\lambda : W \hookrightarrow \mathbb{R}^{\text{rk } x}$, where $\text{rk } x$ is the dimension of the Zariski tangent space $T_x X$ at x . Moreover, if $\iota : U \hookrightarrow \mathbb{R}^n$ is another singular chart defined on an open affine neighborhood of x , then $\text{rk } x \leq n$, and there exists an open affine neighborhood $V \subset W \cap U$ of x , an open neighborhood $\tilde{V} \subset \mathbb{R}^{\text{rk } x}$ of $\lambda(x)$ and a smooth embedding $H : \tilde{V} \hookrightarrow \mathbb{R}^n$ such that $\iota(V)$ is closed in \tilde{V} and such that $H \circ \iota|_V = \kappa|_V$.*

DEFINITION A.4. A stratification \mathcal{S} of the topological space X underlying a differentiable space (X, \mathcal{O}) is said to be *compatible with (X, \mathcal{O})* or just *with \mathcal{O}* if for each stratum $S \in \mathcal{S}$ and singular chart $\iota : U \hookrightarrow \mathbb{R}^n$ the image $\iota(S \cap U)$ of the stratum under ι is a submanifold of \mathbb{R}^n . We call a differentiable space (X, \mathcal{O}) together with a compatible stratification \mathcal{S} of X a *differentiable stratified space*.

EXAMPLE A.1. Typical examples of differentiable stratified spaces are real or complex algebraic varieties and orbit spaces of compact Lie group actions on manifolds. In [PPT14] it has been shown that the orbit space of a proper Lie groupoid carries the structure of a differentiable stratified space in a natural way.

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