

CONSTRAINED SYSTEMS OF CONSERVATION LAWS: A GEOMETRIC THEORY*

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Abstract. We address the Riemann and Cauchy problems for systems of n conservation laws in m unknowns which are subject to $m - n$ constraints ($m \geq n$). Such constrained systems generalize systems of conservation laws in standard form to include various examples of conservation laws in Physics and Engineering beyond gas dynamics, e.g., multi-phase flow in porous media. We prove local well-posedness of the Riemann problem and global existence of the Cauchy problem for initial data with sufficiently small total variation, in one spatial dimension. The key to our existence theory is to generalize the $m \times n$ systems of constrained conservation laws to $n \times n$ systems of conservation laws with states taking values in an n -dimensional manifold and to extend Lax’s theory for local existence as well as Glimm’s random choice method to our geometric framework. Our resulting existence theory allows for the accumulation function to be *non-invertible* across hypersurfaces.

Key words. Shock waves, hyperbolic conservation laws, Glimm scheme, Riemann problem, relaxation systems, curved state space.

Mathematics Subject Classification. 35L65.

1. Introduction. In this paper we develop a geometric framework for proving well-posedness of the Riemann and Cauchy problems for systems of conservation laws by extending the (flat) space of states to a *manifold of states*. Our motivation comes from constrained conservation laws that appear in Physics and Engineering, systems of the form

$$G(u)_t + F(u)_x = 0, \tag{1.1}$$

$$C(u) = 0, \tag{1.2}$$

with states $u(x, t) \in \hat{U} \subset \mathbb{R}^m$, $t \geq 0$, $x \in \mathbb{R}$, and where $G, F : \hat{U} \rightarrow \mathbb{R}^n$ and $C : \hat{U} \rightarrow \mathbb{R}^{m-n}$ are smooth functions and $m \geq n$. The function C gives the constraints on the states and is determined by physical and chemical principles in applications. Often C is not given explicitly but must be derived from the equations. In this case, until the constraints are identified, the system appears to have fewer eigenvalues than equations. The Euler equations of gas dynamics are a constrained system for the gas equation of state being a (trivial) constraint. Non-trivial examples of conservation laws in Physics or Engineering of form (1.1) - (1.2) often reflect transport of components that may coexist in liquid or gaseous form under thermodynamic equilibrium, see for instance [11, 12, 16, 17].¹ More generally, hyperbolic conservation laws with relaxation terms give rise to constraints, determining the equilibrium states of the relaxation system [15] (see also [2, 9]).

For nonlinear systems of conservation laws shock waves form generically and it is most natural to study the Riemann problem. That is, piece-wise constant initial data containing one discontinuity. In [13], Lax proved local existence for the Riemann

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¹The discussion leading to equation (2.3) in [11] nicely illustrates how constraints arise.

problem for hyperbolic systems of conservation laws in *standard form*,

$$u_t + f(u)_x = 0. \quad (1.3)$$

Building on Lax's work, Glimm proved global existence of solutions of the Cauchy problem for (1.3).

From a theoretical perspective, assuming G and C are invertible, we can easily prove existence of solutions for (1.1) - (1.2). Namely, we first solve the constraint locally, say, using the implicit function theorem which gives us a local parameterization of the solution space $C^{-1}(0)$. Secondly, we introduce the change of variable $U \equiv G(u) \in \mathbb{R}^n$ for all states u lying in the image of the parameterization, which then allows us to write (1.1) in standard form (1.3) for

$$f(U) \equiv F \circ G^{-1}(U). \quad (1.4)$$

We can now apply Lax's and Glimm's method to conclude with the existence of solutions for the Riemann and the Cauchy problem.

However, this approach has three major drawbacks. First, there exists balance laws in Petroleum Engineering for which G fails to be invertible on points or hypersurfaces in state space [10, Chapter 5.1]. Secondly, in practice, inverting the accumulation function G *explicitly* is often an obstacle too hard to overcome, regardless whether G^{-1} exists theoretically.² Thirdly, a single parameterization does generally not cover the entire solution space $C^{-1}(0)$ which could restrict the set of states connectable by shock and rarefaction curves artificially. To cover $C^{-1}(0)$ entirely, a suitable collection of such parameterizations is needed, which then defines a manifold.

Motivated to overcome these drawbacks, our approach here is to extend Lax's and Glimm's existence theories beyond standard form (1.3), instead of reducing (1.1) - (1.2) to standard form. We thus avoid inverting the accumulation function G entirely. Without introducing the variable $U \equiv G(u)$, one is automatically forced to consider the manifold of constraints as state space, unless one is willing to accept artificial restrictions on state space and dependence on the choice of a local parameterization. We therefore extend Lax's and Glimm's methods to the geometric framework of Riemannian manifolds of states, so that the resulting existence theory is independent of local parameterizations. Remarkably, it is possible to extend Lax's and Glimm's methods even further and allow for the accumulation function G to be *non-invertible* on a finite union of hypersurfaces in state space. In Proposition (1.4), we show that inverting C explicitly can be circumvented in practice.

Let us finally remark that the fruitful geometric treatment of constraints on state space in Classical Mechanics has been further motivation for us to extend Lax's and Glimm's methods to a similar geometric setting. To give a broader context, Differential Geometry plays a central role in the theory of shock waves in General Relativity where (in contrast to this paper) the conservation laws of fluid dynamics are defined *on* a manifold [1, 8], for instance, the question whether shock waves create essential spacetime singularities [20, 21, 19] has a deeper geometric structure [22].

We now present our main results. In order to obtain an existence theory for (1.1) - (1.2), we consider the following $n \times n$ systems of conservation laws with states

²The Euler equations can be written in standard form, after changing variables, however, this change of variables is non-trivial and one cannot expect to explicitly write all conservation laws of Physics or Engineering in standard form.

assuming values in some *abstract* differentiable manifold \mathcal{M} :

$$g(w)_t + f(w)_x = 0, \tag{1.5}$$

where $w(x, t) \in \mathcal{M}$, and g and f are C^3 functions from \mathcal{M} to \mathbb{R}^n , (c.f. [5, 24] for an introduction into Riemannian Geometry). The system of constrained conservation laws (1.1) - (1.2) assumes the above form by setting $u = w$ whenever u lies in the manifold $C^{-1}(0)$, and by defining g and f to be the restrictions of G and F to $C^{-1}(0)$. The initial data for the Riemann problem for (1.5) is

$$w(x, 0) = \begin{cases} w_l, & x < 0 \\ w_r, & x > 0 \end{cases}, \tag{1.6}$$

for w_l and w_r points in \mathcal{M} , so-called constant states. Our first theorem generalizes Lax’s existence theory to our framework of a manifold of states.

THEOREM 1.1. *Assume \mathcal{M} is a C^3 manifold and let $w_l \in \mathcal{M}$. Assume (1.5) is strictly hyperbolic in the sense of Definition 2.1 and assume that each characteristic field is either genuinely nonlinear or linearly degenerate. Then there exists a neighborhood \mathcal{U} of w_l in \mathcal{M} such that if $w_r \in \mathcal{U}$, the Riemann problem (1.5) with initial data (1.6) has a solution which consists of (at most) $n + 1$ constant states in \mathcal{M} separated by shocks, rarefaction waves or contact discontinuities, such that the shocks satisfy the Lax-admissibility condition (3.23) in \mathcal{U} . This solution is unique within the class of contact discontinuities, admissible shocks and rarefaction waves separated by (at most) $n + 1$ constant states.*

Our definition of strict hyperbolicity is general enough to allow for dg to have a non-trivial null-space on set of measure zero, that is, invertibility of dg is allowed to fail at lower dimensional surfaces, as explained in detail below Definition 2.1. The extended Lax method for proving Theorem 1.1 is independent of local parameterizations and the construction of shock and rarefaction curves is *not* restricted to coordinate neighborhoods. Remarkably, parameterizing the wave curves by arc-length, the most natural parameterization for curves in Riemannian Geometry, implies the C^2 contact between shock and rarefaction curves, c.f. Lemma 3.4.

Building on Theorem 1.1, we address in Section 4 the *Cauchy problem* for (1.5) with initial data of sufficiently small total variation and prove existence of weak solutions by extending Glimm’s famous random choice method to our framework, as recorded in the next theorem. We define the total variation ($T.V.(.)$) and our analog of the L^∞ -norm ($d_\infty(\cdot, \cdot)$) in terms of the canonical distance function on \mathcal{M} , c.f. (4.1) - (4.4).

THEOREM 1.2. *Assume the system (1.5) is strictly hyperbolic in the sense of Definition 2.1 and each characteristic field is genuinely non-linear or linearly degenerate in some neighborhood of some point $\bar{w} \in \mathcal{M}$. Given some curve*

$$w_0 : \mathbb{R} \rightarrow \mathcal{M}$$

such that $T.V.(w_0)$ and $d_\infty(w_0, \bar{w})$ are sufficiently small, then there exists a weak solution $w(x, t) \in \mathcal{M}$ of (1.5) for all $x \in \mathbb{R}$ and all $t \geq 0$ with initial data w_0 , and there exists a constant $C > 0$ such that

$$T.V.(w(\cdot, t)) + d_\infty(w(\cdot, t), \bar{w}) \leq C (d_\infty(w_0, \bar{w}) + T.V.(w_0)), \quad \forall t \geq 0,$$

$$\int_{-\infty}^{\infty} d_{\mathcal{M}}(w(x, t_2), w(x, t_1)) dx \leq C |t_2 - t_1| T.V.(w_0).$$

Glimm’s estimates for wave interaction extend quite naturally to our framework, since wave strength is measured by the parameters of wave curves. We thus obtain a uniform bound on the Glimm functionals over the approximate solutions generated by the random choice method, by using Glimm’s original argument modified to our framework. However, to conclude the existence of a convergent subsequence of the approximate solutions constructed with Glimm’s scheme, one faces the difficulty of defining appropriate norms. In particular, a supremums norm on the states themselves does not seem to make sense and one has to abandon convergence in L^1 of the approximate solutions. We overcome these difficulties by using the canonical distance function of a Riemannian metric on \mathcal{M} to measure the distance of states, which does not define a norm, but which allows us to define an adapted expression of total variation for curves in \mathcal{M} and for the supremums norm on states. After extending Helly’s Theorem to our framework, we obtain point-wise convergence of the approximate solutions and convergence in L^1 of the flux $f(w)$ and the accumulation $g(w)$, (abandoning L^1 convergence of states all together), which suffices to prove the existence of a weak solution to (1.5).

From Theorem 1.1 we obtain the following theorem regarding the constrained system (1.1) - (1.2).

THEOREM 1.3. *Let $G, F \in C^3(\hat{\mathcal{U}}, \mathbb{R}^n)$ and $C \in C^3(\hat{\mathcal{U}}, \mathbb{R}^{m-n})$ for some open set $\hat{\mathcal{U}} \subset \mathbb{R}^m$. Assume that for all $u \in \hat{\mathcal{U}}$ with $C(u) = 0$*

$$\det \begin{bmatrix} DG(u) \\ DC(u) \end{bmatrix} \neq 0, \tag{1.7}$$

(D denotes differentiation in $\hat{\mathcal{U}} \subset \mathbb{R}^m$), and such that there exists $r_k(u) \in \mathbb{R}^m$ and $\lambda_k(u) \in \mathbb{R}$, for all $k \in \{1, \dots, n\}$, solving

$$(\lambda_k DG - DF) r_k = 0, \tag{1.8}$$

$$DC r_k = 0, \tag{1.9}$$

with $\lambda_1(u) < \dots < \lambda_n(u)$. Assume each field is genuinely non-linear or linearly degenerate. Let $u_l \in \hat{\mathcal{U}}$ with $C(u_l) = 0$. Then, there exists a neighborhood $\mathcal{U} \subset \hat{\mathcal{U}}$ of u_l such that if $u_r \in \mathcal{U}$ and $C(u_r) = 0$, then the Riemann problem for (u_l, u_r) of (1.2) has the unique solution stated in Theorem 1.1 for the manifold $\mathcal{M} \equiv C^{-1}(0)$. Moreover, this existence result still holds when the assumption (1.7) is weakened such that DG is not invertible, as specified in Definition 2.1.

Theorem 1.3 follows from Theorem 1.1, since (1.7) implies that $C^{-1}(0)$ defines a manifold whenever DC has full rank, c.f. [24, Theorem 5.1]. Equations (1.8) and (1.9) are the generalized eigenvalue problem and (1.9) is a consistency condition which ensures that the wave curves of the system remain in the solution space of the constraint. One can weaken condition (1.7) for DG to be non-invertible on sets of measure zero in $\mathcal{M} \equiv C^{-1}(0)$ as specified by our notion of hyperbolicity in Definition 2.1.

The extended Glimm’s method of Theorem 1.2 yields existence of weak solutions to the Cauchy problem for (1.1) - (1.2), under the assumption (1.7) - (1.9) and corresponding small data. Since this is rather straightforward, we are content not to summarize the application of Theorem 1.2 as a separate theorem.

One can avoid inverting the constraint C to obtain the solution space $C^{-1}(0)$, explicitly, since the integral curves of the r_k 's describe $C^{-1}(0)$ locally, as recorded in the next proposition. This follows from the consistency condition (1.9) together with the construction underlying Theorem 1.3. The result of the next proposition is very useful for applications, since it reduces the inversion problem for C to methods from Linear Algebra.

PROPOSITION 1.4. *Assume (1.7) - (1.9) hold for all u in some open subset of \mathbb{R}^m . Then, locally, $C^{-1}(0)$ equals the image of the integral curves of the eigenvectors r_k . Moreover, the integral curves of r_k define a local parameterization of $C^{-1}(0)$ if and only if there exists a coordinate system of Riemann invariants.*

We develop the geometric framework underlying Theorem 1.1 in Section 2 and prove Theorem 1.1 in Section 3. In Section 5, we prove Theorem 1.3 and Proposition 1.4. In Section 4, we generalize Glimm's random choice method to our framework and prove global existence of solutions for the Cauchy problem with initial data of small total variation, recorded in Theorem 4.1.

2. The geometric framework.

2.1. Basic notions. We now introduce the geometric framework underlying Theorem 1.1, first extending the notion of hyperbolicity, genuine nonlinearity and linear degeneracy to states taking values in a manifold \mathcal{M} . (See [5] or [24] for an introduction to Differential Geometry.) To begin, write (1.5) in the equivalent form

$$dg\Big|_w(w_t) + df\Big|_w(w_x) = 0, \tag{2.1}$$

where dg and df denote the differential of g and f respectively. That is, dg and df are point-wise linear mappings from $T_w\mathcal{M}$, the space of tangent vectors at w , to \mathbb{R}^n . Note that $w_t, w_x \in T_w\mathcal{M}$, since $t \mapsto w(x, t)$ for x fixed and $x \mapsto w(x, t)$ for t fixed define curves on \mathcal{M} , and since the derivative of a curve with respect to its parameter is a tangent vector. System (2.1) now gives rise to the eigenvalue problem

$$\lambda_k(w) dg\Big|_w(r_k(w)) = df\Big|_w(r_k(w)), \tag{2.2}$$

where $\lambda_k(w)$ is a real number and $r_k(w) \in T_w\mathcal{M}$ is a vector, the so-called right-eigenvector. Thus, $\lambda_k : \mathcal{M} \rightarrow \mathbb{R}$ is a real-valued scalar and r_k is a vector field on \mathcal{M} . Note that a solution (λ_k, r_k) is independent of the choice of coordinate, which is fundamental to our framework. We mostly refer to the pair (λ_k, r_k) , but sometimes to r_k alone, as the k -th characteristic field.

We now define the notion of hyperbolicity in our framework and thereby specify in what sense $dg|_w$ is allowed to have a non-trivial null-space. If $dg(r_k) = 0$ at w , then, for the eigenvalue problem (2.2) to hold, r_k must also lie in the null-space of $df|_w$, so that $\lambda_k(w)$ would be left undetermined at the point w in \mathcal{M} . Nevertheless, if Σ , the set of points $w \in \mathcal{M}$ at which $dg|_w$ fails to be invertible, is a hypersurface in \mathcal{M} and if λ_k is continuous across Σ , then we define λ_k on Σ uniquely as the continuous extension of λ_k to \mathcal{M} . This motivates the next definition.

DEFINITION 2.1. *We call (1.5) a hyperbolic system of conservation laws in \mathcal{M} , if the following holds:*

- (i) $dg|_w$ is invertible for all $w \in \mathcal{M} \setminus \Sigma$, where $\Sigma \subset \mathcal{M}$ denote the union of a finite family of co-dimension one surfaces;

- (ii) for each $w \in \mathcal{M} \setminus \Sigma$ and $k \in \{1, \dots, n\}$ there exists $r_k(w) \in T_w \mathcal{M}$ and $\lambda_k(w) \in \mathbb{R}$ that solve the eigenvalue problem (2.2), such that the eigenvectors $r_k(w)$ are linearly independent;
- (iii) λ_k and r_k can be extended as C^2 functions to \mathcal{M} , such that the resulting r_k are linearly independent on \mathcal{M} .

We call (1.5) strictly hyperbolic in \mathcal{M} if in addition, for all $w \in \mathcal{M}$,

$$\lambda_1(w) < \dots < \lambda_n(w). \tag{2.3}$$

The essential point, regarding our assumption for Σ in Definition (2.1), is Σ being of measure zero so that the extension of λ_k is unique.³ Defining λ_k as the continuous extension then suffices to construct solutions of (1.5) following Lax’s method. However, to prove the existence of shock curves we need to assume that Σ consists of only finitely many hypersurfaces for the characteristic polynomial associated with the Hugoniot locus to have the required stability property. (Our methods also work for surfaces of higher co-dimension.) Below we give some examples of hyperbolic systems of conservation laws, in the sense of Definition 2.1, for which dg fails to be invertible everywhere.

Whether the k -th characteristic field corresponds to a solution that is either a shock or a rarefaction wave, or whether it corresponds to a contact discontinuity depends on the fields being genuinely nonlinear or linearly degenerate, respectively. We now define these notions.

DEFINITION 2.2. *We say the k -th characteristic field is genuinely nonlinear in \mathcal{M} , if for all $w \in \mathcal{M}$*

$$r_k \Big|_w (\lambda_k) > 0. \tag{2.4}$$

In Definition 2.2 we use the interpretation of vectors as directional derivatives of scalar functions. That is, expressed in coordinates y^μ on a subset of \mathcal{M} , we have $r_k|_w = r_k^\mu(y(w)) \frac{\partial}{\partial y^\mu}$ and (2.4) thus becomes

$$r_k \Big|_w (\lambda_k) = r_k^\mu(y(w)) \frac{\partial (\lambda \circ y^{-1})(y(w))}{\partial y^\mu},$$

which is indeed a real number independent of the choice of coordinates y^μ . In the special case $\mathcal{M} = \mathbb{R}^n$, (2.4) reduces to the standard definition of genuine nonlinearity in \mathbb{R}^n , namely,

$$\langle \nabla \lambda_k, r_k \rangle_{\mathbb{R}^n} > 0.$$

Note that (2.4) leaves the freedom to scale the length of r_k and this is often used to set $r_k \Big|_w (\lambda_k) = 1$. However, we are *not* using this normalization, but work with the normalization condition

$$\langle r_k, r_k \rangle_{\mathcal{M}} = 1,$$

³If Σ were an open set, then the eigenvalue problem (2.2) should rather be considered as additional constraints of the system and not as part of the evolution.

in terms of the metric tensor $\langle \cdot, \cdot \rangle_{\mathcal{M}}$, which allows us to parameterize the shock and rarefaction curves with respect to arc-length. Analogously to (2.4), the following definition generalizes the notion of linear degeneracy to states in a manifold.

DEFINITION 2.3. *We say the k -th characteristic field is linearly degenerate in \mathcal{M} , if for all $w \in \mathcal{M}$*

$$r_k \Big|_w (\lambda_k) = 0. \tag{2.5}$$

We assume in this paper that the characteristic fields are either linearly degenerate or genuinely non-linear to prove well-posedness of the Riemann problem and develop the framework in a simplest setting. We expect however that Liu’s construction [14] extends to our geometric framework, in particular to our notion of hyperbolicity in Definition 2.1, so that genuine non-linearity is allowed to fail across hypersurfaces in state space.

To close this section, we give two examples of systems of conservation laws hyperbolic in the sense of Definition 2.1, for which dg is not invertible.

EXAMPLE 2.4. Take as accumulation and flux function in (1.5)

$$g(u, v) \equiv \left(\begin{array}{c} \frac{1}{2}u^2 \\ \frac{1}{3}v^3 \end{array} \right) \quad \text{and} \quad f(u, v) \equiv \left(\begin{array}{c} \frac{1}{3}u^3 \\ v^3 \end{array} \right),$$

for $u \in (-\infty, 3)$ and $v \in \mathbb{R}$. Their derivatives are

$$dg(u, v) \equiv \left(\begin{array}{cc} u & 0 \\ 0 & v^2 \end{array} \right) \quad \text{and} \quad df(u, v) \equiv \left(\begin{array}{cc} u^2 & 0 \\ 0 & 3v^2 \end{array} \right),$$

and the eigenvalue problem (2.2) has the solutions $r_1 = (1, 0)$ with $\lambda_1 = u$ and $r_2 = (0, 1)$ with $\lambda_2 = 3$. Thus, this system is strictly hyperbolic on its domain $(u, v) \in (-\infty, 3) \times \mathbb{R}$ and dg has a non-trivial null space on the lines $\{u = 0\}$ and $\{v = 0\}$. Moreover, the first field is genuinely non-linear and the second linearly degenerate.

EXAMPLE 2.5. We modify the p -system in [23], by considering

$$g(u, v) \equiv \left(\begin{array}{c} uv + v \\ \frac{1}{2}u^2v \end{array} \right) \quad \text{and} \quad f(u, v) \equiv \left(\begin{array}{c} u \\ u^2p(v) \end{array} \right), \tag{2.6}$$

as accumulation and flux function in (1.5). We assume $p(v) > 0$ is a given smooth function and we restrict to $v > 0$. In matrix form, (2.6) is given by

$$\left(\begin{array}{cc} 1+u & v \\ \frac{1}{2}u^2 & uv \end{array} \right) \left(\begin{array}{c} v_t \\ u_t \end{array} \right) + \left(\begin{array}{cc} 0 & 1 \\ u^2p' & 2up \end{array} \right) \left(\begin{array}{c} v_x \\ u_x \end{array} \right) = 0, \tag{2.7}$$

where p' is the derivative of $p = p(v)$. Clearly, $dg(v, u)$ has a non-trivial null-space at $u = 0$ for all $v > 0$. The resulting eigenvalue problem (2.2) is well-posed with eigenvalues

$$\lambda_{\pm} = \frac{1}{2v(2+u)} \left(\alpha(v, u) \pm \sqrt{8uv(2+u)p' + \alpha(v, u)^2} \right), \tag{2.8}$$

for $\alpha(v, u) \equiv 4p(v)(1 + u) - u(1 + 2vp')$. Since at $u = 0$

$$\lambda_+ = \frac{2p(v)}{v} > 0 \quad \text{and} \quad \lambda_- = 0,$$

it follows that λ_+ and λ_- are distinct for all u sufficiently close to zero. Moreover, for u sufficiently small, λ_{\pm} are smooth functions for all $v > 0$. Thus, since the flux and accumulation functions in (2.6) are smooth, the eigenvectors of λ_{\pm} are smooth. To conclude that (2.7) is strictly hyperbolic in the sense of Definition 2.1, we need to choose $p(v)$ such that the resulting eigenvectors are linearly independent.

For this, we here consider the simple (but non-trivial) special case

$$p(v) = v.$$

The eigenvalues simplify to $\lambda_+ = 2$ and $\lambda_- = -\frac{u}{(2+u)v}$ and the corresponding normalized eigenvectors are given by

$$r_+ = \frac{1}{\sqrt{(1 - 2v)^2 + 4(1 + u)^2}} \begin{pmatrix} 1 - 2v \\ 2(1 + u) \end{pmatrix}$$

$$r_- = \frac{1}{\sqrt{4v^2 + u^2}} \begin{pmatrix} -2v \\ u \end{pmatrix}.$$

Clearly r_- and r_+ remain linearly independent in a neighbourhood of $u = 0$, for suitable values of $v > 0$. A computation shows that (λ_-, r_-) is genuinely non-linear for all $u \neq 0$ and all $v > 0$, while (λ_+, r_+) is linearly degenerate.

2.2. Weak solutions and Rankine Hugoniot conditions. We now define weak solutions and derive the Rankine Hugoniot jump conditions for system (1.5). As in [23], we define $C_0^1(\Omega)$ to be the set of real valued continuously differentiable functions on $\Omega \equiv \{(x, t) : t \geq 0\}$ with compact support (which can be non-vanishing on compact subsets of the line $\{t = 0\}$). Now, multiplying (1.5) by some $\psi \in C_0^1(\Omega)$, integrating over the resulting equation and applying Gauss’s Divergence Theorem to shift derivatives, leads to our definition of weak solutions.

DEFINITION 2.6. *Let $w(x, t) \in \mathcal{M}$ be a bounded function such that $f(w(x, t))$ and $g(w(x, t))$ are measurable in (x, t) and let $w_0(x) \in \mathcal{M}$ be a bounded function such that $g(w_0(x))$ is measurable in t . We say that $w(x, t) \in \mathcal{M}$ is a weak solution of (1.5) with initial data w_0 if*

$$\iint_{t \geq 0} (g(w) \psi_t + f(w) \psi_x) dxdt + \int_{\mathbb{R}} \psi(\cdot, 0) g(w_0) dx = 0, \quad \forall \psi \in C_0^1(\Omega).$$

To clarify, the boundedness of $w(x, t)$ assumed in Definition 2.6 shall be understood with respect to the canonical distance function $d_{\mathcal{M}}$, defined in (4.1), which turns \mathcal{M} into a metric-space.

We next show that the discontinuity of a weak solution across a surface $\Gamma(t) \equiv (x(t), t) \in \Omega$ satisfies the following generalized Rankine Hugoniot jump conditions:

$$s[g(w)] = [f(w)], \tag{2.9}$$

where $s(t) = \dot{x}(t)$ is the shock speed, and $[\cdot]$ denotes the jump across Γ , i.e., $[w] = w_l - w_r$, $[g(w)] = g(w_l) - g(w_r)$ and $[f(w)] = f(w_l) - f(w_r)$, with w_l and w_r denoting the left- and right-limit of $w(x, t)$ at Γ .

PROPOSITION 2.7. *Let $\Gamma(t) \equiv (x(t), t)$ be a differentiable curve across which $w(x, t)$ is discontinuous and such that $w(x, t)$ is in C^1 on the complement of Γ and $w(x, t)$ has well-defined limits on both sides of Γ . Suppose $w(x, t)$ is a strong solution of (1.5) in the complement of Γ . Then, $w(x, t)$ is a weak solution of (1.5) if and only if (2.9) hold across Γ .*

Proof. The proof closely follows the reasoning in [23] and is only included for completeness. Consider some point p on Γ and let $B \subset \{(x, t) : t \geq 0\}$ be some small open ball centered at p . Denote with B_1 and B_2 the two open regions in B to the right and left of Γ , respectively. Now, assuming $w(x, t)$ is a weak solution and that Ψ has compact support in B , we find from (2.6) that

$$\begin{aligned} 0 &= \iint_B (g(w)\psi_t + f(w)\psi_x) dxdt \\ &= \iint_{B_1} (g(w)\psi_t + f(w)\psi_x) dxdt + \iint_{B_2} (g(w)\psi_t + f(w)\psi_x) dxdt. \end{aligned} \tag{2.10}$$

Using the fact that $w(x, t)$ is a strong solution of (1.5) on each B_i , $i = 1, 2$, since $w \in C^1(B_i)$, we get

$$\iint_{B_i} (g(w)\psi_t + f(w)\psi_x) dxdt = \iint_{B_i} (g(w)\psi)_t + (f(w)\psi)_x dxdt.$$

Now, using the divergence theorem together with the fact that the outward pointing unit normal vector of $\partial B_1 = \Gamma$ and of $\partial B_2 = \Gamma$, N_1 and N_2 , both point along the direction of $(-s, 1)$, but with opposite sign, we write (2.10) as

$$\begin{aligned} \iint_{B_1} (g(w)\psi_t + f(w)\psi_x) dxdt &= \int_{\Gamma} \psi \begin{pmatrix} g(w_r) \\ f(w_r) \end{pmatrix} \cdot N_1 dt, \\ \iint_{B_2} (g(w)\psi_t + f(w)\psi_x) dxdt &= - \int_{\Gamma} \psi \begin{pmatrix} g(w_l) \\ f(w_l) \end{pmatrix} \cdot N_1 dt. \end{aligned}$$

Inserting this into (2.10) and rescaling N_1 to coincide with $(-s, 1)$, gives

$$0 = \iint_B (g(w)\psi_t + f(w)\psi_x) = \int \psi (s[g(w)] - [f(w)]) dt,$$

for all $\psi \in C_0^1$, from which we conclude that (2.9) holds if and only if $w(x, t)$ is a weak solution of (1.5). \square

2.3. Left eigen-one-forms. To prove Glimm’s Theorem within our framework and to show the admissibility of shock waves, we use the following eigenvalue problem for eigen-one-forms l_k ,

$$\lambda_k l_k(dg) = l_k(df), \tag{2.11}$$

by which we mean that l_k is a smooth mapping from \mathcal{M} to the space of linear maps from \mathbb{R}^n to \mathbb{R} , $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$. That is, $l_k|_w : \mathbb{R}^n \rightarrow \mathbb{R}$ is linear and smooth in $w \in \mathcal{M}$, and we have

$$\lambda_k(w) l_k \Big|_w \left(dg \Big|_w(v) \right) = l_k \Big|_w \left(df \Big|_w(v) \right), \quad \forall v \in T_w \mathcal{M}.$$

The above equation closes, since $dg|_w(v) \in T_{g(w)}\mathbb{R}^n = \mathbb{R}^n$ and $df|_w(v) \in T_{f(w)}\mathbb{R}^n = \mathbb{R}^n$ for all $v \in T_w\mathcal{M}$, where we use the canonical identification between $T_p\mathbb{R}^n$ and \mathbb{R}^n , for all $p \in \mathbb{R}^n$, through the standard basis on \mathbb{R}^n . By a slight abuse of language we refer to the l_k 's as one-forms.

Let us remark here that one can write the eigenvalue problem for eigen-one-forms (2.11) equivalently as the left-eigenvector problem used in [6, 23]. Namely, by the Riesz representation Theorem, there exists a unique vector $\vec{l}_k \in \mathbb{R}^n$ such that $l_k(v) = \langle \vec{l}_k, v \rangle_{\mathbb{R}^n}$ for all $v \in \mathbb{R}^n$, where $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ denotes the Euclidean inner product on \mathbb{R}^n .

The following lemma shows that the existence of left eigen-one-forms follows from the existence of right eigenvectors, under the assumption that dg is point-wise invertible in \mathcal{M} . We apply Lemma 2.8 in Section 3 to prove C^2 contact of the shock and rarefaction curves at w_l and to prove the admissibility of the lower branch of shock curves.

LEMMA 2.8. *Assume $\det(dg|_w) \neq 0$ and that there exists n linearly independent right eigenvectors $r_k(w) \in T_w\mathcal{M}$ which solve (2.2) with eigenvalues $\lambda_k(w) \in \mathbb{R}$, then there exists n linearly independent one-forms, l_k , which solve (2.11). In particular, there exist a normalization of the l_k 's such that the l_k 's and r_k 's are mutually orthonormal in the sense that*

$$l_j(dg(r_k)) = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} . \tag{2.12}$$

Proof. It suffices to prove the lemma at a fixed point $w \in \mathcal{M}$. Using that $dg|_w$ has the inverse

$$dg|_{g(w)}^{-1} : \mathbb{R}^n \longrightarrow T_w\mathcal{M},$$

where we identify $T_{g(w)}g(\mathcal{M})$ with \mathbb{R}^n , we define the linear map

$$A(w) = dg|_{g(w)}^{-1}df|_w,$$

which maps $T_w\mathcal{M}$ into itself. Now, subsequently suppressing the dependence on w , we write (2.2) as

$$A r_k = \lambda_k r_k. \tag{2.13}$$

Let e_k be the k -th standard basis vector of \mathbb{R}^n , that is, all entries of e_k are 0 except the k -th entry, which equals 1. Define the linear mapping $U : \mathbb{R}^n \longrightarrow T_w\mathcal{M}$ by setting $U(e_k) \equiv r_k$. Then U is invertible and, in light of (2.13), we obtain

$$U^{-1}AU = \text{diag}(\lambda_1, \dots, \lambda_n),$$

where we represent the linear mapping on the left hand by its matrix in the standard basis on \mathbb{R}^n . Multiplying the above equation from the left by e_k^* , where star denotes the transpose of a matrix, yields,

$$e_k^* \cdot U^{-1}AU = \lambda_k e_k^*,$$

which is equivalent to

$$e_k^* \cdot U^{-1}A = \lambda_k e_k^* \cdot U^{-1}.$$

Substituting now the definition of A , that is, $A = dg^{-1}df$, into the above equation we obtain

$$e_k^* \cdot U^{-1}dg^{-1}df = \lambda_k \cdot e_k^* \cdot U^{-1}dg^{-1}dg.$$

The left- and right-hand side of the above equation are both linear mappings from $T_w\mathcal{M}$ to \mathbb{R} , and applying them to some $v \in T_w\mathcal{M}$, yields

$$\left\langle (U^{-1}dg^{-1})^* e_k, df(v) \right\rangle_{\mathbb{R}^n} = \lambda_k \cdot \left\langle (U^{-1}dg^{-1})^* e_k, dg(v) \right\rangle_{\mathbb{R}^n}.$$

We conclude that the sought after one-form, satisfying (2.11), is given by

$$l_k \Big|_w (\cdot) = \left\langle (U^{-1}dg^{-1})^* e_k, \cdot \right\rangle_{\mathbb{R}^n}. \tag{2.14}$$

To verify (2.12), we use (2.14) and compute

$$\begin{aligned} l_k (dg(r_j)) &= \left\langle (U^{-1}dg^{-1})^* e_k, dg(r_j) \right\rangle_{\mathbb{R}^n} \\ &= \left\langle e_k, U^{-1}(r_j) \right\rangle_{\mathbb{R}^n} \\ &= \left\langle e_k, e_j \right\rangle_{\mathbb{R}^n}, \end{aligned}$$

which completes the proof. \square

A useful application of left eigen-one-forms, though not further used in this article, is the following Lemma (which is also proven in [17] for \mathbb{R}^n as state space).

LEMMA 2.9. *Assume that (1.5) is hyperbolic, that the flux f and the accumulation g are both C^2 regular and that $dg|_w$ is invertible for all $w \in \mathcal{M}$, then*

$$r_k(\lambda_k) \cdot l_k(dg(r_k)) = l_k((\nabla_{r_k} df)(r_k)) - \lambda_k \cdot l_k((\nabla_{r_k} dg)(r_k)), \tag{2.15}$$

where $\nabla_{r_k} df$ and $\nabla_{r_k} dg$ are the covariant derivatives of the one forms df^i and dg^i for $i = 1, \dots, n$ fixed.

Proof. This proof generalizes the one in [23]. To begin, note that the expressions on the left and right hand side of the k -th characteristic eigenvalue problem, (2.2), are both differentiable maps from \mathcal{M} to \mathbb{R}^n . Fixing Cartesian coordinates x^i on \mathbb{R}^n , we differentiate the i -th component of (2.2) in the direction of r_k , which yields

$$r_k \Big|_w (\lambda_k dg^i \Big|_w (r_k)) = r_k \Big|_w (df^i \Big|_w (r_k)),$$

where $g^i \equiv x^i \circ g$ and $f^i \equiv x^i \circ f$, for $i = 1, \dots, n$ fixed. By the product rule of partial differentiation, applied for instance in some coordinates on \mathcal{M} , we write the above equation as

$$r_k \Big|_w (\lambda_k dg^i \Big|_w (r_k) + \lambda_k(w) r_k \Big|_w (dg^i \Big|_w (r_k))) = r_k \Big|_w (df^i \Big|_w (r_k)). \tag{2.16}$$

Analogous to the computation in (3.17), we find by the Leibnitz rule of covariant differentiation [5, 24] that

$$r_k \Big|_w (df^i \Big|_w (r_k)) = (\nabla_{r_k} df^i) \Big|_w (r_k) + df^i \Big|_w (\nabla_{r_k} r_k),$$

where $\nabla_{r_k} df^i$ denotes the covariant derivative of the one-form df^i and where we used as in (3.17) that the Christoffel symbols enter covariant differentiation of vector fields and of one-forms with opposite sign. Now, since an analogous result holds for dg^i , we write (2.16) as

$$\begin{aligned} & r_k \Big|_w (\lambda_k) dg^i \Big|_w (r_k) + \lambda_k(w) (\nabla_{r_k} dg^i) \Big|_w (r_k) + \lambda_k(w) dg^i \Big|_w (\nabla_{r_k} r_k) \\ &= (\nabla_{r_k} df^i) \Big|_w (r_k) + df^i \Big|_w (\nabla_{r_k} r_k). \end{aligned}$$

Writing the k -th characteristic one-form in coordinates x^i as $l_k \equiv (l_k)_i dx^i$, we multiply the previous equation by $(l_k)_i$ and sum over the resulting expression with respect to i , which yields

$$\begin{aligned} & r_k \Big|_w (\lambda_k) l_k \left(dg \Big|_w (r_k) \right) + \lambda_k(w) l_k \left((\nabla_{r_k} dg) \Big|_w (r_k) \right) + \lambda_k(w) l_k \left(dg \Big|_w (\nabla_{r_k} r_k) \right) \\ &= l_k \left((\nabla_{r_k} df) \Big|_w (r_k) \right) + l_k \left(df \Big|_w (\nabla_{r_k} r_k) \right). \end{aligned} \tag{2.17}$$

Now, from the eigen-value problem (2.11), we find

$$\lambda_k(w) l_k \left(dg \Big|_w (\nabla_{r_k} r_k) \right) = l_k \left(df \Big|_w (\nabla_{r_k} r_k) \right),$$

so that (2.17) simplifies to

$$r_k \Big|_w (\lambda_k) l_k \left(dg \Big|_w (r_k) \right) + \lambda_k(w) l_k \left((\nabla_{r_k} dg) \Big|_w (r_k) \right) = l_k \left((\nabla_{r_k} df) \Big|_w (r_k) \right),$$

which is the sought after equation (2.15), independent of any choice of coordinates. This completes the proof of Lemma 2.9. \square

3. The Riemann problem in the manifold of states. In this section, we extend Lax’s existence theory [13] to the geometric framework introduced in Section 2, for which we use mostly the exposition in [23] as a starting point. As in [13], the solutions of Riemann problems constructed are compositions of contact discontinuities, shock and rarefaction waves.

3.1. Rarefaction waves. We begin by introducing the k -th characteristic curve of (1.5) as the (real valued) solution of the ODE

$$\frac{d}{dt} \chi(x, t) = \lambda_k(w(\chi(x, t), t)), \quad \text{with} \quad \chi(x, 0) = x. \tag{3.1}$$

Following [4], we now define rarefaction waves as C^1 solutions of (1.5) constant along a family of characteristic curve and such that the corresponding wave speed increase monotonously with respect to x , so that the family of characteristic curves diverge to form a fan-like region.

DEFINITION 3.1. *We say that a C^1 solution of (1.5), $(x, t) \mapsto w(x, t) \in \mathcal{M}$, is a k -simple wave, if it is constant along the k -th characteristic curve, that is, if for fixed x the tangent vector of the curve $t \mapsto w(\chi(x, t), t)$ satisfies*

$$\frac{d}{dt} w(\chi(x, t), t) = 0.$$

A k -rarefaction wave is a k -simple wave with $\frac{\partial}{\partial x} \lambda_k(w(x, t)) > 0$ and a k -compression wave is a k -simple wave with $\frac{\partial}{\partial x} \lambda_k(w(x, t)) < 0$.

It follows immediately that characteristic curves are straight lines for simple waves.

Under the assumption that the k -th characteristic field is genuinely nonlinear, the following theorem proves that for each $w_l \in \mathcal{M}$ there exists a one-parameter family of states w_r for which the Riemann problem with w_l on the left and w_r on the right is solved by a rarefaction wave.

THEOREM 3.2. *Assume (1.5) is strictly hyperbolic, let w_l be some point in \mathcal{M} and assume the k -th characteristic field of (1.5) is genuinely nonlinear in some neighborhood of w_l with $r_k \Big|_w (\lambda_k) > 0$. Then there exists a one-parameter family of states $w(\epsilon)$, with $\epsilon \in [0, a)$ for some $a > 0$, such that $w(0) = w_l$ and $w(\epsilon)$ can be connected to w_l on the right by a k -rarefaction wave in \mathcal{M} for each $\epsilon \in [0, a)$. We call the curve $\epsilon \mapsto w(\epsilon)$ the k -th rarefaction curve. This curve is unique modulo its parameterization and it can be parameterized so that its tangent vector is given by $\dot{w}(\epsilon) = r_k(w(\epsilon))$.*

Proof. Consider the ODE

$$\frac{dv}{d\epsilon} = r_k(v(\epsilon)) \quad \text{with} \quad v(\lambda_k(w_l)) = w_l. \tag{3.2}$$

Since df and dg are assumed C^1 regular, we conclude that r_k and λ_k are C^1 as well. Now the Picard Lindelöf theorem yields the existence of a constant $a > 0$ and a function $v(\epsilon)$, for $\epsilon \in [\lambda_k(w_l), \lambda_k(w_l) + a)$, which solves (3.2). Consider now the following Riemann problem for the scalar conservation law

$$\epsilon_t + \lambda_k(v(\epsilon)) \epsilon_x = 0, \quad \epsilon(x, 0) = \begin{cases} \lambda_k(w_l), & x < 0 \\ \lambda_k(w_l) + \hat{a}, & x > 0, \end{cases} \tag{3.3}$$

for some $0 < \hat{a} < a$. By genuine non-linearity of the k -th characteristic field, we find that

$$\frac{d}{d\epsilon} \lambda_k(v(\epsilon)) = d\lambda_k \Big|_{v(\epsilon)} \left(\frac{dv}{d\epsilon} \right) = r_k \Big|_{v(\epsilon)} (\lambda_k) > 0,$$

which implies $\lambda_k \circ v$ to be monotone in ϵ . Thus, following the basic theory of scalar conservation laws [23], there exist a rarefaction wave solving (3.3), given by

$$\epsilon(x, t) = (\lambda_k \circ v)^{-1} \left(\frac{x}{t} \right), \tag{3.4}$$

for (x, t) lying in the wave fan region

$$\lambda_k(w_l) \leq \frac{x}{t} \leq \lambda_k \left(v(\lambda_k(w_l) + \hat{a}) \right), \tag{3.5}$$

where we used the initial data in (3.2) to simplify the lower bound in (3.5). Notice that $\epsilon(x, t)$ is constant along the characteristic lines of (3.3), (the straight lines inside the wave fan (3.5)), and its range is the domain of definition of v , that is, $[\lambda_k(w_l), \lambda_k(w_l) + \hat{a}]$. This allows us to define $(x, t) \mapsto w(x, t)$ as

$$w(x, t) \equiv v(\epsilon(x, t)), \tag{3.6}$$

for (x, t) lying in the wave fan (3.5). Since $\epsilon(x, t)$ is constant along the k -characteristic curves and C^1 regular inside the wave fan (3.5) and Lipschitz continuous across its boundary lines, we conclude that w is a k -simple wave provided it solves (1.5)-(1.6).

We now show that w is a solution of the Riemann problem (1.6) for the system (1.5). For this, we compute

$$\begin{aligned} g(w)_t + f(w)_x &= dg \Big|_w (w_t) + df \Big|_w (w_x) \\ &= dg \Big|_w \left(\frac{dv}{d\epsilon} \right) \epsilon_t + df \Big|_w \left(\frac{dv}{d\epsilon} \right) \epsilon_x, \end{aligned}$$

and using (3.2) together with the eigenvalue problem (2.2), we find

$$\begin{aligned} g(w)_t + f(w)_x &= dg \Big|_w (r_k(v)) \epsilon_t + df \Big|_w (r_k(v)) \epsilon_x \\ &= dg \Big|_w (r_k(v)) (\epsilon_t + \lambda_k(v) \epsilon_x). \end{aligned} \tag{3.7}$$

From (3.3), it follows that the right hand side in the previous equation vanishes, from which we conclude that w is in fact a solution of (1.5) and that w is a simple wave.

Finally, use $\lambda_k(w) = \lambda_k(v(\epsilon))$ and (3.4) to compute

$$\frac{\partial}{\partial x} \lambda_k(w) = \frac{1}{t} > 0,$$

which implies that $w(x, t)$ is indeed a rarefaction wave.

Varying now $\hat{a} \in (0, a)$ yields the sought after one-parameter family of states connected to w_l by a rarefaction wave. Uniqueness of the rarefaction curve follows from the uniqueness of r_k (modulo its length) under the assumption that (1.5) is strictly hyperbolic. \square

Let us remark that if the set of points Σ , for which the null-space of dg is non-trivial, were not a set of measure zero, then the eigenvalues λ_k would not be specified over this set and (3.3) could not be introduced. Let us remark further, that one could avoid the inversion problem $(\lambda_k \circ v)^{-1}$ in (3.4) by normalizing r_k such that $r_k \Big|_w (\lambda_k) = 1$ holds, however, we avoid this normalization here since it would prevent us from introducing an arc-length parameterization for the wave curves, which is convenient for extending Glimm's Theorem to our framework.

3.2. Shock waves. We now prove the existence of n one-parameter families of states that can be connected to w_l on the right by a shock wave.

THEOREM 3.3. *Let $w_l \in \mathcal{M}$, and assume that (1.5) is strictly hyperbolic in some neighborhood of w_l . Then there exists n one-parameter families of states, $\epsilon \mapsto w_k(\epsilon) \in C^2(I, \mathcal{M})$ with $I \equiv (-a, a)$ for some $a > 0$, such that $w_k(0) = w_l$ and $w_k(\epsilon)$ satisfies the RH jump conditions (2.9) for some unique shock speed $s_k \in C^1(I, \mathbb{R})$, for all $\epsilon \in I$. The shock curves $\epsilon \mapsto w_k(\epsilon)$ are linearly independent at w_l , unique up to parameterization and have a C^2 -dependence on w_l .*

Proof. Given $w_l \in \mathcal{M}$, we choose coordinates y^μ defined in a neighborhood \mathcal{U} of w_l , such that any straight line in \mathbb{R}^n through $y(w_l)$ intersects $y(\Sigma \cap \mathcal{U})$ at most in finitely many points.⁴ Writing $y = y(w) \in \mathbb{R}^n$ and $y_l = y(w_l) \in \mathbb{R}^n$, we compute

$$f(w) - f(w_l) = f \circ y^{-1}(y) - f \circ y^{-1}(y_l)$$

⁴That is, we arrange for each hypersurface in $y(\Sigma \cap \mathcal{U})$ to be sufficiently curved.

$$\begin{aligned}
 &= \int_0^1 \frac{d}{d\sigma} f \circ y^{-1}(y_l + \sigma(y - y_l)) d\sigma \\
 &= \int_0^1 d(f \circ y^{-1})|_{y_l + \sigma(y - y_l)} d\sigma \cdot (y - y_l) \\
 &\equiv \mathcal{F}(y) \cdot (y - y_l)
 \end{aligned}$$

and similarly

$$\begin{aligned}
 g(w) - g(w_l) &= \int_0^1 d(g \circ y^{-1})|_{y_l + \sigma(y - y_l)} d\sigma \cdot (y - y_l) \\
 &\equiv \mathcal{G}(y) \cdot (y - y_l).
 \end{aligned}$$

We now write the RH condition (2.9) as the equivalent eigenvalue problem

$$(\mathcal{F}(y) - s \mathcal{G}(y))(y - y_l) = 0. \tag{3.8}$$

Assume for the moment that $w_l \notin \Sigma$, i.e., $dg|_{w_l}$ is invertible. At y_l , the above matrices simplify to $\mathcal{F}(y_l) = d(f \circ y^{-1})|_{y_l}$ and $\mathcal{G}(y_l) = d(g \circ y^{-1})|_{y_l}$ so that (3.8) reduces to the right eigenvalue problem (2.2) at y_l . Thus, for y sufficiently close to y_l , the characteristic polynomial \mathcal{P} of (3.8) is a small perturbation of the characteristic polynomial of (2.2) at y_l . By continuous dependence of the (complex) zeros of a polynomial on its coefficients, we conclude that \mathcal{P} has n mutually distinct zeros μ_i , $i = 1, \dots, n$, with their real parts satisfying

$$\operatorname{Re}(\mu_1) > \dots > \operatorname{Re}(\mu_n). \tag{3.9}$$

Now, since \mathcal{P} has real coefficients, the genuine complex roots of \mathcal{P} appear in complex conjugate pairs, so that (3.9) requires \mathcal{P} to have n real valued zeros. Therefore, (3.8) has n real valued eigenvalues $\mu_1 > \dots > \mu_n$, that is, there exists $\zeta_k(y) \in \mathbb{R}^n$ and $\mu_k(y) \in \mathbb{R}$ such that

$$(\mathcal{F}(y) - \mu_k(y)\mathcal{G}(y))\zeta_k(y) = 0, \tag{3.10}$$

for $k = 1, \dots, n$, and where

$$\zeta_k(y_l) = dy \Big|_{w_l} \begin{pmatrix} r_k \end{pmatrix} \quad \text{and} \quad \mu_k(y_l) = \lambda_k(y_l).$$

Now, assuming $dg|_w$ has a non-trivial null-space for all $w \in \Sigma$ and $\Sigma \neq \emptyset$, then the characteristic polynomial of (2.2) vanishes identically at each $w \in \Sigma$. However, since by our choice of coordinates y , we arranged for $y(\Sigma \cap \mathcal{U})$ to be intersected by the straight line $\sigma \mapsto y_l + \sigma(y - y_l)$ at most in finitely many points, it follows that \mathcal{G} is invertible, so that the characteristic polynomial of (3.8) is non-vanishing and again gives rise to n distinct real eigenvalues, that is, to (3.10).

We conclude that (3.8) is satisfied if and only if

$$y - y_l = \epsilon \zeta_k(y) \quad \text{and} \quad s = \mu_k(y), \tag{3.11}$$

for some non-zero $\epsilon \in \mathbb{R}$ and some $k = 1, \dots, n$. It remains to prove that for each k there exists a one-parameter family of states $y(\epsilon)$ which solves

$$H(y, \epsilon) \equiv y - y_l - \epsilon \zeta_k(y) = 0.$$

H is a C^2 function since g and f are assumed to be in C^3 . It is straightforward that $H(y_l, 0) = 0$ and

$$\frac{\partial}{\partial y^\nu} H^\mu \Big|_{(y_l, 0)} = I_\nu^\mu,$$

for I denoting the identity matrix on \mathbb{R}^n and $\mu, \nu \in \{1, \dots, n\}$ indices. Thus, the Implicit Function Theorem implies the existence of a C^2 curve $y_k(\epsilon)$, such that

$$H(y_k(\epsilon), \epsilon) = 0$$

for all ϵ sufficiently close to 0.

Defining $w_k(\epsilon) \equiv y^{-1}(y_k(\epsilon))$, it is immediate by construction that w_k is the sought after shock curve of the k -th characteristic field through w_l with $s_k(\epsilon) \equiv \mu_k \circ y(w_k(\epsilon))$ being the corresponding shock speed. Since \mathcal{M} is assumed to be a C^3 manifold, we have $w_k \in C^2(I_k, \mathcal{M})$. The above construction yields n unique shock curves with the property that $w_k(0) = w_l$ and $\dot{w}(0) = r_k(w_l)$, so that linear independence follows as well. The shock curves have a C^2 -dependence on y_l , since H has this dependence on y_l .

Finally, the above construction of shock curves and speeds is independent of the choice of coordinates, since $s(w)$, $[g^i(w)]$ and $[f^i(w)]$ are scalar functions over \mathcal{M} . \square

The next lemma shows that the shock and rarefaction curves have C^2 contact at w_l , most remarkably, when parameterized by their *arc-length*. This lemma is the first step in the construction which requires Riemannian Geometry over basic Differential Topology.

LEMMA 3.4. *Assume the hypotheses of Theorem 3.3. Assume further that \mathcal{M} is endowed with a Riemannian metric $\langle \cdot, \cdot \rangle_{\mathcal{M}}$ and that the characteristic fields are normalized such that*

$$\langle r_k, r_k \rangle_{\mathcal{M}}(w) = 1, \quad \forall w \in \mathcal{M}. \tag{3.12}$$

Then the shock curves defined in Theorem 3.3 can be parameterized by arc-length and, denoting the k -th shock curve in its arc-length parameterization by $\epsilon \mapsto w_k(\epsilon)$, they satisfy

$$\dot{w}_k(0) = r_k(w_l) \quad \text{and} \quad \ddot{w}_k(0) = \nabla_{r_k} r_k(w_l), \tag{3.13}$$

while the shock speeds satisfies

$$s_k(0) = \lambda_k(w_l) \quad \text{and} \quad \dot{s}_k(0) = \frac{1}{2} r_k \Big|_{w_l}(\lambda). \tag{3.14}$$

Here “ \cdot ” denotes $\frac{d}{d\epsilon}$, ∇ is the Levi Civita connection of $\langle \cdot, \cdot \rangle_{\mathcal{M}}$ and \ddot{w}_k denotes covariant differentiation of \dot{w}_k along the shock curve with respect to ∇ .

Proof. It suffices to prove the theorem for $w_l \notin \Sigma$, since the λ_k ’s and r_k ’s are assumed to be extendable to C^2 functions over \mathcal{M} (by Definition 2.1), so that the expressions on the right hand side of (3.13) and (3.14) all extend continuously to Σ as w_l varies, and since the shock curves have a C^2 dependence on w_l .

By construction of w_k and s_k we already have $w_k(0) = w_l$ and $s_k(0) = \lambda_k(w_l)$. Let y^μ be the coordinate system used in the proof of Theorem 3.3. Writing $w_k(\epsilon') \equiv y^{-1}(y_k(\epsilon'))$, for some parameter ϵ' , we find

$$\dot{w}_k(0) = dy^{-1} \Big|_{y_l} (\dot{y}_k(0)) \in T_{w_l} \mathcal{M}.$$

Differentiating the first equation in (3.11) with respect to ϵ' yields

$$\dot{y}_k(0) = \zeta_k(y_l) = dy \Big|_{w_l} (r_k).$$

Combining the above two equations, we obtain

$$\dot{w}_k(0) = r_k(w_l), \tag{3.15}$$

which holds independent of the choice of coordinates, as both sides are objects in $T_{w_l}\mathcal{M}$.

Now, since $\dot{w}_k(0) = r_k(w_l) \neq 0$ we can use the change of variables

$$\frac{d\epsilon}{d\epsilon'} = \sqrt{\left| \frac{dw_k}{d\epsilon'} \right|_{\mathcal{M}}} \quad \text{with} \quad \epsilon(0) = 0,$$

to obtain the arc-length parameter ϵ of the shock curves. By (3.15) and since r_k is normalized by $\langle r_k, r_k \rangle_{\mathcal{M}} = 1$, we have $\frac{d\epsilon}{d\epsilon'} = |\dot{w}_k(0)|_{\mathcal{M}} = 1$. Thus the shock curves parameterized by arc-length ϵ still satisfies $w_k(0) = w_l$ and (3.15) with “.” now denoting $\frac{d}{d\epsilon}$.

To derive the remaining identities, first differentiate the RH conditions (2.9) with respect to ϵ , which gives

$$\dot{s}(g(w_k) - g(w_l)) + sdg(\dot{w}_k) = df(\dot{w}_k), \tag{3.16}$$

where dg and df are evaluated at $w_k(\epsilon)$. We next differentiate (3.16) by ϵ , for this observe that $df^i(\cdot)$ and $dg^i(\cdot)$ for fixed $i = 1, \dots, n$ are one-forms on \mathcal{M} , so that, setting $a_\mu \equiv \frac{\partial f^i \circ x^{-1}}{\partial x^\mu}$ and $x^\mu(\epsilon) \equiv x^\mu \circ w_k(\epsilon)$ for some coordinate system x^μ , we find

$$\begin{aligned} \frac{d}{d\epsilon} df^i \Big|_{w_k} (\dot{w}_k) &= \frac{d}{d\epsilon} \left(\frac{\partial f^i \circ x^{-1}}{\partial x^\mu} \frac{dx^\mu}{d\epsilon} \right) \\ &= \frac{\partial a_\mu}{\partial x^\nu} \frac{dx^\nu}{d\epsilon} \frac{dx^\mu}{d\epsilon} + a_\mu \frac{d^2 x^\mu}{d\epsilon^2} \\ &= \frac{\partial a_\mu}{\partial x^\nu} \dot{w}_k^\nu \dot{w}_k^\mu - \Gamma_{\mu\nu}^\sigma a_\sigma \dot{w}_k^\nu \dot{w}_k^\mu + a_\mu \frac{d\dot{w}_k^\mu}{d\epsilon} + a_\sigma \Gamma_{\mu\nu}^\sigma \dot{w}_k^\nu \dot{w}_k^\mu \\ &= (\nabla_{\frac{\partial}{\partial \epsilon}} a_\mu) \dot{w}_k^\mu + a_\mu (\nabla_{\frac{\partial}{\partial \epsilon}} \dot{w}_k^\mu) \\ &= (\nabla_{\frac{\partial}{\partial \epsilon}} df^i) (\dot{w}_k) + df^i(\ddot{w}_k), \end{aligned} \tag{3.17}$$

where \ddot{w}_k denotes covariant differentiation along the shock curves, $\nabla_{\frac{\partial}{\partial \epsilon}} \dot{w}_k$. (Regarding the above cancellation, note that $\Gamma_{\sigma\rho}^\mu$ enters covariant differentiation of a vector field with a plus sign, but covariant differentiation of a one-form with a minus sign.) An analogous result holds for dg^i . Using now (3.17) for differentiating (3.16) by $\frac{d}{d\epsilon}$ component-wise then results in

$$\begin{aligned} &\ddot{s}(g^i(w_k) - g^i(w_l)) + 2\dot{s}dg^i(\dot{w}_k) + sdg^i(\ddot{w}_k) + s\nabla_{\frac{\partial}{\partial \epsilon}} dg^i(\dot{w}_k) \\ &= \nabla_{\frac{\partial}{\partial \epsilon}} df^i(\dot{w}_k) + df^i(\ddot{w}_k), \end{aligned}$$

Evaluating this equation at $\epsilon = 0$ and using that $w_k(0) = w_l$ and $\dot{w}_k(0) = r_k(w_l)$, yields

$$\begin{aligned} &2\dot{s}(0)dg^i \Big|_{w_l} (r_k) + \lambda_k(w_l)dg^i \Big|_{w_l} (\ddot{w}_k(0)) + \lambda_k(w_l)(\nabla_{r_k} dg^i) \Big|_{w_l} (r_k) \\ &= (\nabla_{r_k} df^i) \Big|_{w_l} (r_k) + df^i \Big|_{w_l} (\ddot{w}_k(0)). \end{aligned} \tag{3.18}$$

Differentiate now the eigenvalue problem (2.2) along w_k , which gives us

$$\dot{\lambda}_k dg^i(r_k) + \lambda_k dg^i(\nabla_{\frac{\partial}{\partial \epsilon}} r_k) + \lambda_k \left(\nabla_{\frac{\partial}{\partial \epsilon}} dg^i \right) (r_k) = df^i(\nabla_{\frac{\partial}{\partial \epsilon}} r_k) + \left(\nabla_{\frac{\partial}{\partial \epsilon}} df^i \right) (r_k).$$

Evaluating the previous equation at $\epsilon = 0$, using $\dot{w}_k(0) = r_k(w_l)$, and subtracting the resulting expression from (3.18), yields

$$(2\dot{s}(0) - \dot{\lambda}_k(0)) dg \Big|_{w_l}(r_k) = (df - \lambda_k dg) \Big|_{w_l}(\ddot{w}_k(0) - (\nabla_{r_k} r_k)(w_l)), \tag{3.19}$$

where we used that $\dot{w}_k(0) = r_k(w_l)$ implies $(\nabla_{\frac{\partial}{\partial \epsilon}} r_k)(w_k(0)) = \nabla_{r_k} r_k(w_l)$.

Now, in light of the eigenvalue problem (2.11) and since the left and right hand sides in (3.19) are both elements in $T_{w_l} \mathcal{M}$, applying the eigen-one-form l_k to both sides of (3.19) yields

$$(2\dot{s}(0) - \dot{\lambda}_k(0)) l_k \left(dg \Big|_{w_l}(r_k) \right) = 0.$$

Using that $l_k \left(dg \Big|_{w_l}(r_k) \right) = 1$ by (2.12) of Lemma 2.8 and that

$$\dot{\lambda}_k(0) = \frac{d}{d\epsilon} \Big|_{\epsilon=0} \lambda_k(w_k(\epsilon)) = d\lambda_k \Big|_{w_l}(\dot{w}_k(0)) = d\lambda_k \Big|_{w_l}(r_k) = r_k \Big|_{w_l}(\lambda_k), \tag{3.20}$$

it follows that

$$\dot{s}(0) = \frac{1}{2} r_k \Big|_{w_l}(\lambda_k), \tag{3.21}$$

which proves the second equation in (3.14).

To complete the proof, substitute (3.21) into (3.19), which gives us

$$(df - \lambda_k dg) \Big|_{w_l}(\ddot{w}_k(0) - \nabla_{r_k} r_k(w_l)) = 0,$$

from which we conclude in light of the eigenvalue problem (2.2) that

$$\ddot{w}_k(0) - \nabla_{r_k} r_k(w_l) = c r_k(0). \tag{3.22}$$

Now, since r_k is assumed to satisfy the normalization (3.12), using the Leibnitz rule for ∇ , we find that

$$0 = r_k \Big|_w \left(\langle r_k, r_k \rangle_{\mathcal{M}} \right) = 2 \langle \nabla_{r_k} r_k, r_k \rangle_{\mathcal{M}}(w).$$

Likewise, since $\epsilon \mapsto w_k(\epsilon)$ is parameterized by arc-length, it follows that the acceleration \ddot{w}_k is orthogonal to its velocity \dot{w}_k , $\langle \ddot{w}_k, \dot{w}_k \rangle_{\mathcal{M}} = 0$. Thus, applying the one-form $\langle \cdot, \dot{w}_k \rangle_{\mathcal{M}}$ to both sides of (3.22) and using that $\dot{w}_k(0) = r_k(w_l)$, we conclude that $c = 0$ and therefore (3.22) implies the sought after equation (3.13). \square

As for a criteria guaranteeing uniqueness of solutions of the Riemann problem for the system (1.5), we impose the so-called *Lax admissibility* conditions. That is, we say that the k -th shock curve is (Lax) admissible if the following inequalities hold

$$\lambda_{k-1}(0) < s_k(\epsilon) < \lambda_k(0),$$

$$\lambda_k(\epsilon) < s_k(\epsilon) < \lambda_{k+1}(\epsilon). \tag{3.23}$$

This condition is identical to the flat case and implies that the k -th characteristic lines impinge on the k -th shock, c.f. [13, 23].

The next lemma shows that the shock curves in the arc-length parameterization of Lemma 3.4 are admissible for $\epsilon < 0$. Its proof is taken from [23] and is presented here for completeness.

LEMMA 3.5. *Assume the hypotheses of Lemma (3.4) and let the k -th shock curve be parameterized by arc-length. Assume further that the k -th characteristic field is genuinely nonlinear with $r_k|_w(\lambda_k) > 0$. Then (3.23) holds for $s_k(\epsilon)$ and $\lambda_k(w_k(\epsilon))$ if and only if $\epsilon < 0$.*

Proof. Define the function $\Phi(\epsilon) \equiv \lambda_k(\epsilon) - s_k(\epsilon)$, then Lemma 3.4 implies that $\Phi(0) = 0$ and

$$\dot{\Phi}(0) = \dot{\lambda}_k(0) - \dot{s}_k(0) = \frac{1}{2}r_k|_{w_l}(\lambda) > 0,$$

where we used (3.20) for the substitution $\dot{\lambda}_k(0) = r_k|_{w_l}(\lambda_k)$. Assume now that the Lax admissibility conditions hold. This implies that $\Phi(\epsilon) < 0$ and thus that $\epsilon < 0$, since Φ is monotonically increasing by the previous computation.

Conversely, assuming that $\epsilon < 0$, the monotonicity of Φ implies that $\Phi(\epsilon) < 0$ and thus $\lambda_k(\epsilon) < s_k(\epsilon)$. Moreover, since $\lambda_k(0) = s_k(0)$ and $\dot{s}_k(0) = \frac{1}{2}r_k|_{w_l}(\lambda) > 0$, we conclude that $\lambda_k(0) > s_k(\epsilon)$ for ϵ sufficiently close to 0. Furthermore, as ϵ approaches 0 from below, $s_k(\epsilon)$ converges to $\lambda_k(0)$, which implies that $s_k(\epsilon) > \lambda_{k-1}(0)$, since $\lambda_k(0) > \lambda_{k-1}(0)$ and $\lambda_k(0) > s_k(\epsilon)$. Finally, since $\lambda_{k+1}(0) > \lambda_k(0) = s_k(0)$, we conclude that $\lambda_{k+1}(\epsilon) > s_k(\epsilon)$ for $\epsilon < 0$ sufficiently close to 0. In summary, we proved that (3.23) holds. \square

Following [13], we now define the *wave curve* for the genuinely non-linear characteristic fields as rarefaction curves whenever $\epsilon > 0$ and shock curves for $\epsilon < 0$.

DEFINITION 3.6. *Assume the k -th characteristic field is genuinely nonlinear with $r_k|_w(\lambda_k) > 0$ and assume r_k is normalized by (3.12). Let \bar{w}_k denote the k -th shock curve parameterized by arc-length and let \hat{w}_k denote the k -th rarefaction curve constructed in Theorem 3.2. We now define the “wave” curve $\epsilon \mapsto w_k(\epsilon)$ in \mathcal{M} as*

$$w_k(\epsilon) \equiv \begin{cases} \bar{w}_k(\epsilon), & \epsilon \leq 0 \\ \hat{w}_k(\epsilon), & \epsilon \geq 0. \end{cases} \tag{3.24}$$

Let us finally record the following basic properties of the wave curves being C^2 regular and parameterized by arc-length and of the shock speed being the arithmetic average of the left and right characteristic speeds.

PROPOSITION 3.7. *Assume the hypotheses of Lemma (3.4) and assume the k -th characteristic field is genuinely nonlinear with $r_k|_{w_l}(\lambda_k) > 0$. Then the curve $\epsilon \mapsto w_k(\epsilon)$ defined in (3.24), is C^2 regular and is parameterized by arc-length. Moreover, the shock speed satisfies*

$$s_k(\epsilon) = \frac{\lambda_k(w_l) - \lambda_k(\epsilon)}{2} + O(\epsilon^2). \tag{3.25}$$

Proof. Equation (3.25) follows by the proof of Theorem 17.16 in [23]. Note that $r_k|_{w_l}(\lambda_k)$ cancels out in (3.25), so that the normalization $r_k|_{w_l}(\lambda_k) = 1$ in [23] is irrelevant for the result here.

The rest of the corollary is immediate from Theorem 3.2 and Lemma 3.4. In particular, arc-length parameterization follows since $\langle r_k, r_k \rangle_{\mathcal{M}}(w) = 1$ for each $w \in \mathcal{M}$ and since the rarefaction curves satisfy $\frac{d}{d\epsilon} \hat{w}_k(\epsilon) = r_k(w_k(\epsilon))$, by Theorem 3.2. \square

3.3. Contact discontinuities. We now solve the Riemann problem (1.5) and (1.6) when the k -th characteristic field is linearly degenerate, c.f. Definition 2.5, within the class of contact discontinuities. We call a function $w(x, t)$ a *contact discontinuity* if it is a weak solution of the Riemann problem (1.5) - (1.6), discontinuous across the straight line $\{(st, t) : t \geq 0\}$, such that s coincides with the characteristic speed on both sides of that line.

THEOREM 3.8. *Let the k -th characteristic field be linearly degenerate in the sense of Definition 2.5. Then, given some $w_l \in \mathcal{M}$, there exists a one-parameter family of states, $w(\epsilon)$, for $\epsilon \in I$, for some interval I containing 0, such that for each $\epsilon \in I$, there exists a contact discontinuity solving the Riemann problem (1.5) - (1.6) for $w_l = w(0)$ and $w_r = w(\epsilon)$. If (1.5) is strictly hyperbolic in the sense of Definition 2.1, then this one-parameter family is unique.*

Proof. Given some $w_l \in \mathcal{M}$, define the curve $\epsilon \mapsto w(\epsilon)$, for ϵ sufficiently close to 0, as the solution of the ODE

$$\frac{dw}{d\epsilon} = r_k(w(\epsilon)), \tag{3.26}$$

for initial data $w(0) = w_l$. The assumption of linear degeneracy, that is, $r_k|_w(\lambda_k) = 0$ for all $w \in \mathcal{M}$, implies that λ_k is constant along the curve $\epsilon \mapsto w(\epsilon)$. That is,

$$\lambda_k(w(\epsilon)) = \lambda_k(w_l),$$

for all $\epsilon \in I$, where I denotes some open interval containing 0.

Now, given some $\epsilon \in I$, we define the function

$$v(x, t) \equiv \begin{cases} w_l, & x < t \lambda_k(w_l), \\ w(\epsilon), & x > t \lambda_k(w_l). \end{cases} \tag{3.27}$$

Clearly, v satisfies the initial data (1.6) for $w_r = w(\epsilon)$, and setting $s = \lambda_k(w_l)$, we find that

$$\frac{d}{d\epsilon} (f(w(\epsilon)) - sg(w(\epsilon))) = df|_{w} (r_k) - s dg|_{w} (r_k),$$

which vanishes by the eigenvalue problem (2.2). We conclude that

$$f(w(\epsilon)) - sg(w(\epsilon)) = f(w_l) - sg(w_l),$$

for all $\epsilon \in I$, which are in fact the RH conditions, (2.9). Therefore, v satisfies the RH conditions and is a weak solution of (1.5), according to Proposition 2.7. We conclude that w_l and $w(\epsilon)$ are connected by a contact discontinuity. Uniqueness of the family of states follows from (3.26) together with the uniqueness of the vectorfield r_k by strict hyperbolicity. \square

3.4. Proof of Theorem 1.1. Choose coordinates y^μ in some neighborhood \mathcal{U} of w_l in \mathcal{M} . We now follow the arguments in the proof of Theorem 17.18 in [23]. By the results from sections 3.1 - 3.3, there exists a neighborhood $\mathcal{U} \subset \mathcal{M}$ of w_l and for each $k = 1, \dots, n$ there exists a one-parameter family of maps

$$T_{\epsilon_k}^k : \mathcal{U} \rightarrow \mathcal{M},$$

with the defining property that any $w \in \mathcal{U}$ can be joined to $T_{\epsilon_k}^k w$ by either a k -shock, a k -rarefaction wave or a contact discontinuity, depending on whether the k -th characteristic field is genuinely non-linear or linearly degenerate. Set $\epsilon \equiv (\epsilon_1, \dots, \epsilon_n)$ and define the composition $T(\epsilon) \equiv T_{\epsilon_n}^n \dots T_{\epsilon_1}^1$. To complete the prove it remains to show that there exists exactly one ϵ such that $w_r = T(\epsilon)w_l$. This follows by the Inverse Function Theorem. In more detail, defining

$$F(\epsilon_1, \dots, \epsilon_n) \equiv T_{\epsilon_n}^n \dots T_{\epsilon_1}^1 w_l - w_l,$$

it follows that $F(0, \dots, 0) = 0$ and, by construction of the wave curves (see (3.26) and Lemma 3.4), that

$$\frac{\partial F}{\partial \epsilon_i}(0, \dots, 0) = r_i(w_l).$$

Now, since the r_k 's are linearly independent by the assumption of hyperbolicity, the Inverse Function Theorem implies the existence of a unique ϵ such that $w_r = T(\epsilon)w_l$. The result is independent of the choice of coordinates, since the intermediate states as well as the wave curves connecting them are independent of coordinates. This completes the proof of Theorem 1.1. \square

4. The Cauchy problem and Glimm's scheme. In this section we extend Glimm's method for proving global existence of the Cauchy problem with initial data of bounded total variation to our framework of a manifold of states and prove Theorem 1.2.

To begin, we introduce the canonical distance function $d_{\mathcal{M}}$, which turns \mathcal{M} into a metric space. That is, $d_{\mathcal{M}} : \mathcal{M} \times \mathcal{M} \rightarrow [0, \infty)$ is defined as

$$d_{\mathcal{M}}(p, q) = \inf_{\gamma} \int_a^b \sqrt{\langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle_{\mathcal{M}}} ds, \tag{4.1}$$

where $\langle \cdot, \cdot \rangle_{\mathcal{M}}$ denotes the metric tensor on \mathcal{M} and the infimum is taken over all continuous piecewise differentiable curves γ with $\gamma(a) = p$ and $\gamma(b) = q$.

Given a curve $w : [a, b] \rightarrow \mathcal{M}$, we now define the total variation of w as

$$T.V.|_a^b(w) \equiv \sup_{\mathcal{P}} \sum_{i=0}^{n_p-1} d_{\mathcal{M}}(w(x_{i+1}), w(x_i)) \tag{4.2}$$

where \mathcal{P} denotes the set of partitions of the interval $[a, b]$ and $\{x_0, \dots, x_{n_p}\} \in \mathcal{P}$ is a partition. For a curve $w : \mathbb{R} \rightarrow \mathcal{M}$, we define its *total variation* as

$$T.V.(w) \equiv \lim_{k \rightarrow \infty} T.V.|_{-k}^k(w|_{[-k, k]}). \tag{4.3}$$

Given a constant state $\bar{w} \in \mathcal{M}$, we define

$$d_{\infty}(\bar{w}, w) \equiv \sup_{x \in \mathbb{R}} d_{\mathcal{M}}(w(x), \bar{w}), \tag{4.4}$$

which generalizes the sup-norm on states in \mathbb{R}^n to our framework and which gives us a convenient notion of distance, sufficient to prove convergence of the Glimm scheme below. The main result of this section is the generalization of Glimm’s famous Theorem [6], recorded in Theorem 1.2, which we now recall.

THEOREM 4.1. *Assume the system (1.5) is strictly hyperbolic and each characteristic field is genuinely non-linear or linearly degenerate in some neighborhood of some point $\bar{w} \in \mathcal{M}$. Given some curve*

$$w_0 : \mathbb{R} \rightarrow \mathcal{M}$$

such that $T.V.(w_0)$ and $d_\infty(w_0, \bar{w})$ are sufficiently small, then there exists a weak solution $w(x, t) \in \mathcal{M}$ of (1.5) for all $x \in \mathbb{R}$ and all $t \geq 0$ with initial data w_0 , and there exists a constant $C > 0$ such that

$$\begin{aligned} T.V.(w(\cdot, t)) + d_\infty(w(\cdot, t), \bar{w}) &\leq C (d_\infty(w_0, \bar{w}) + T.V.(w_0)), \quad \forall t \geq 0, \\ \int_{-\infty}^{\infty} d_{\mathcal{M}}(w(x, t_2), w(x, t_1)) dx &\leq C |t_2 - t_1| T.V.(w_0). \end{aligned}$$

The remainder of this section is devoted to the proof of Theorem 4.1. Throughout this section, we assume (1.5) to be strictly hyperbolic and each characteristic field to be either genuinely non-linear or linearly degenerate.

4.1. Interaction estimates. We begin the proof of Theorem 4.1 by deriving interaction estimates. For this, assume we are given three constant states, w_l, w_m and w_r , sufficiently close to \bar{w} such that their mutual Riemann problems are well-posed, c.f. Theorem 1.1. Assume further that

$$\begin{aligned} w_r &= T(\epsilon)w_l \equiv T_{\epsilon_n}^n \dots T_{\epsilon_1}^1 w_l, \\ w_m &= T(\gamma)w_l \equiv T_{\gamma_n}^n \dots T_{\gamma_1}^1 w_l, \\ w_r &= T(\delta)w_m \equiv T_{\delta_n}^n \dots T_{\delta_1}^1 w_m, \end{aligned} \tag{4.5}$$

with T denoting the operator introduced in section 3.4. We call the curve on \mathcal{M} connecting w_l with $w_r = T(\epsilon)w_l$ the ϵ -wave, setting $\epsilon = (\epsilon_1, \dots, \epsilon_n)$. The curve connecting the state $T_{\epsilon_j}^j \dots T_{\epsilon_1}^1 w_l$ with $T_{\epsilon_{j-1}}^{j-1} \dots T_{\epsilon_1}^1 w_l$ is referred to as the ϵ_j -wave, or simply as the j -wave, and we call $|\epsilon_j|$ the strength of the j -wave.

PROPOSITION 4.2. *If w_l, w_m and w_r are sufficiently close to a constant state $\bar{w} \in \mathcal{M}$, then*

$$\epsilon_i = \gamma_i + \delta_i + O(|\gamma| |\delta|), \tag{4.6}$$

where $|\gamma| \equiv \max\{|\gamma_i| : i = 1, \dots, n\}$. Moreover, if there exists a coordinate system of Riemann invariants ω_j near \bar{w} , then

$$\epsilon_i = \gamma_i + \delta_i + O\left((|\gamma| + |\delta|)^3\right). \tag{4.7}$$

Proof. Choose local coordinates y^μ around \bar{w} such that w_l, w_m and w_r are contained in the coordinate neighborhood. By (4.5), $y_r \equiv y(w_r)$ is a C^2 function with respect to $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \mathbb{R}^n$. Taylor expanding y_r around $\epsilon = 0$, (keeping in mind that second order derivatives do in general not commute), gives

$$y_r^\mu - y_l^\mu = \sum_{j=1}^n \frac{\partial y_r^\mu}{\partial \epsilon_j} \Big|_0 \epsilon_j + \sum_{j \leq k} \epsilon_j \epsilon_k \left(1 - \frac{1}{2} \delta_{jk}\right) \frac{\partial^2 y_r^\mu}{\partial \epsilon_j \partial \epsilon_k} \Big|_0 + O(|\epsilon|^3) \tag{4.8}$$

where δ_{jk} denotes the Kronecker symbol. For genuinely non-linear fields, from (3.13) of Lemma 3.4, we have $\frac{\partial y_r^\mu}{\partial \epsilon_j} \Big|_0 = r_j^\mu(w_l)$ and thus, for $j < k$,

$$\frac{\partial^2 y_r^\mu}{\partial \epsilon_j \partial \epsilon_k} \Big|_0 = \frac{\partial r_k^\mu(T_{\epsilon_{k-1}}^{k-1} \dots T_{\epsilon_1}^1 y_l)}{\partial \epsilon_j} \Big|_0 = r_j^\nu \partial_\nu (r_k^\mu) \Big|_{y_l} \equiv r_j(r_k^\mu) \Big|_{y_l},$$

where ∂_ν refers to partial differentiation in coordinates y on \mathcal{M} , not to differentiation in parameter space $\epsilon = (\epsilon_1, \dots, \epsilon_n)$. For $j = k$, (3.13) implies that

$$\begin{aligned} \frac{\partial^2 y_r^\mu}{\partial \epsilon_k^2} \Big|_0 &= \nabla_{\frac{\partial}{\partial \epsilon_k}} \frac{\partial y_r^\mu}{\partial \epsilon_k} \Big|_0 - \Gamma_{\nu\sigma}^\mu \frac{\partial y_r^\nu}{\partial \epsilon_k} \frac{\partial y_r^\sigma}{\partial \epsilon_k} \Big|_0 \\ &= \nabla_{r_k} r_k^\mu \Big|_{y_l} - \Gamma_{\nu\sigma}^\mu r_k^\nu r_k^\sigma \Big|_{y_l} \\ &= r_k^\nu \partial_\nu (r_k^\mu) \Big|_{y_l} \equiv r_k(r_k^\mu) \Big|_{y_l}, \end{aligned}$$

where $\Gamma_{\nu\sigma}^\mu$ is the Christoffel symbol of the covariant derivative ∇ . By (3.26), the above equations also hold true for a contact discontinuity of a linearly degenerate field. Substituting the above identities into (4.8) yields

$$y_r^\mu - y_l^\mu = \sum_{j=1}^n r_j^\mu \Big|_{y_l} \epsilon_j + \sum_{j \leq k} \left(1 - \frac{1}{2} \delta_{jk}\right) r_k(r_j^\mu) \Big|_{y_l} \epsilon_j \epsilon_k + O(|\epsilon|^3). \tag{4.9}$$

Likewise, we find for $y_m = T(\gamma)y_l$ the expression

$$y_m^\mu - y_l^\mu = \sum_{j=1}^n r_j^\mu \Big|_{y_l} \gamma_j + \sum_{j \leq k} \left(1 - \frac{1}{2} \delta_{jk}\right) r_k(r_j^\mu) \Big|_{y_l} \gamma_j \gamma_k + O(|\gamma|^3), \tag{4.10}$$

while Taylor expanding $y_r = T(\delta)y_m$ at y_m gives

$$y_r^\mu - y_m^\mu = \sum_{j=1}^n r_j^\mu \Big|_{y_m} \delta_j + \sum_{j \leq k} \left(1 - \frac{1}{2} \delta_{jk}\right) r_k(r_j^\mu) \Big|_{y_m} \delta_j \delta_k + O(|\delta|^3). \tag{4.11}$$

To compare (4.9) - (4.11), we now derive an expression for (4.11) which is centered at y_l . For this, we Taylor expand $r_j \Big|_{y_m}$ and $r_k(r_j^\mu) \Big|_{y_m}$ at y_l and apply (3.13), which yields

$$\begin{aligned} r_j^\mu \Big|_{y_m} &= r_j^\mu \Big|_{y_l} + \sum_{i=1}^n r_i(r_j^\mu) \Big|_{y_l} \gamma_i + O(|\gamma|^2), \\ r_k(r_j^\mu) \Big|_{y_m} &= r_k(r_j^\mu) \Big|_{y_l} + O(|\gamma|). \end{aligned}$$

Substituting the previous two identities into (4.11) yields

$$\begin{aligned} y_r^\mu - y_m^\mu &= \sum_{j=1}^n r_j^\mu \Big|_{y_l} \delta_j + \sum_{i,j=1}^n r_i(r_j^\mu) \Big|_{y_l} \gamma_i \delta_j \\ &\quad + \sum_{j \leq k} \left(1 - \frac{1}{2} \delta_{jk}\right) r_k(r_j^\mu) \Big|_{y_l} \delta_j \delta_k + O(|\delta|^3 + |\gamma| |\delta|^2) \end{aligned} \tag{4.12}$$

To complete the proof, we add (4.12) to (4.10) and equate the resulting expression to the right hand side of (4.9), which gives

$$\begin{aligned} \sum_{i=1}^n (\epsilon_i - \gamma_i - \delta_i) r_i^\mu &= \sum_{j \leq k} \left(1 - \frac{1}{2} \delta_{jk}\right) r_k(r_j^\mu) \Big|_{y_i} (\gamma_j \gamma_k + \gamma_j \gamma_k - \epsilon_j \epsilon_k) \\ &\quad + \sum_{i,j=1}^n r_i(r_j^\mu) \Big|_{y_i} \gamma_i \delta_j + O\left((|\gamma| + |\delta|)^3 + |\epsilon|^3\right). \end{aligned} \tag{4.13}$$

Using now that $\epsilon_i = O(|\gamma| + |\delta|)$, since ϵ vanishes when γ and δ are zero, we conclude from (4.13) that

$$\epsilon_j = \gamma_j + \delta_j + O\left((|\gamma| + |\delta|)^2\right). \tag{4.14}$$

Substituting (4.14) into (4.13), a straightforward computation finally yields

$$\sum_{i=1}^n (\epsilon_i - \gamma_i - \delta_i) r_i^\mu = \sum_{j < i} \gamma_i \delta_j (r_i(r_j^\mu) - r_j(r_i^\mu)) + O\left((|\gamma| + |\delta|)^3\right),$$

from which (4.6) is immediate. Moreover, since the existence of a coordinate system of Riemann invariants implies the Lie brackets of the r_k 's in the previous equation to vanish, (4.7) follows. \square

From Proposition 4.2 one obtains the main interaction estimate, recorded in the following theorem. Before we state the theorem, consider the interacting waves in (4.5). We say the j -wave in γ and the k -wave in δ *approach* if either $j > k$ or if $j = k$ and at least one of the waves is a shock. (We often say γ in place of γ wave and so on.) That is, two waves approach if the wave on the left is faster than the wave on the right. Note that neither rarefaction waves nor the contact discontinuities constructed in Theorem (3.8) interact, if they belong to the same characteristic family, since the head of the left wave travels with the speed of the tail of the right wave. We now define

$$D(\gamma, \delta) \equiv \sum |\gamma_i| |\delta_j|,$$

where the sum is over all pairs for which the i wave in γ and the j -wave in δ approach.

THEOREM 4.3. *If w_l, w_m and w_r are sufficiently close to \bar{w} , then* ⁵

$$\epsilon_i = \gamma_i + \delta_i + D(\gamma, \delta)O(1) \quad \text{as } |\gamma| + |\delta| \rightarrow 0, \tag{4.15}$$

and if (4.7) hold, then

$$\epsilon_i = \gamma_i + \delta_i + D(\gamma, \delta)O(|\gamma| + |\delta|) \quad \text{as } |\gamma| + |\delta| \rightarrow 0.$$

Proof. The proof of Theorem 19.2 in [23] can be taken word by word. \square

⁵The $O(1)$ in (4.15) should be understood as a constant weight to each term in D and not as a constant.

4.2. The random choice method. We now introduce Glimm’s random choice method for constructing approximate solutions, c.f. [6, 23]. Adapting the scheme to our framework is straightforward and we include it only for completeness. To begin, divide the line $\{t = k\Delta t\}$ into the segments lying between the points $m\Delta x$, $m \in \mathbb{Z}$ with $k + m$ being an even number and $\Delta x, \Delta t > 0$. For $k \in \mathbb{N}$, this defines a grid over $\mathbb{R} \times [0, \infty)$. We let the ratio $\frac{\Delta x}{\Delta t} \equiv c$ be fixed and assume the stability condition

$$\frac{\Delta x}{\Delta t} > \sup\{|\lambda_j(w)| : w \in \hat{\mathcal{U}}, 1 \leq j \leq n\}, \tag{4.16}$$

to ensure that waves from different grid cells do not interact within a single time step. Here $\hat{\mathcal{U}}$ denotes the set containing \mathcal{U} such that the intermediate states of the solution to the Riemann problem (w_l, w_r) lie in $\hat{\mathcal{U}}$ for all $w_l, w_r \in \mathcal{U}$. Note that, by continuous dependence of the wave curves on their emanating states $w_l, \hat{\mathcal{U}}$ exist for \mathcal{U} sufficiently small.

Now, choosing for each $k \in \mathbb{N}$ some $\theta_k \in [-1, 1]$ randomly, the *mesh points* are defined as

$$a_{m,k} \equiv (m\Delta x + \theta_k\Delta x, k\Delta t),$$

for all $m \in \mathbb{Z}$ and $k \in \mathbb{N}$ with $k + m$ even. Assuming now some constant values $w_{m-1}^{k-1} \in \mathcal{M}$ and $w_{m+1}^{k-1} \in \mathcal{M}$ are given, and let $v_{m,k}(x, t) \in \mathcal{M}$ denote the solution of the Riemann problem

$$g(v)_t + f(v)_x = 0$$

with initial data at $t = (k - 1)\Delta t$ given by

$$v(x, (k - 1)\Delta t) = \begin{cases} w_{m-1}^{k-1}, & (m - 1)\Delta x \leq x \leq m\Delta x, \\ w_{m+1}^{k-1}, & m\Delta x \leq x \leq (m + 1)\Delta x. \end{cases}$$

We then introduce the constant state

$$w_m^k \equiv v_{m,k}(a_{m,k}),$$

for $m \in \mathbb{Z}$ with $m + k$ even.

This defines Glimm’s random choice scheme. Given constant states $w_m^0 \in \mathcal{M}$, each state w_m^0 being associated to the interval $m\Delta x \leq x \leq (m + 2)\Delta x$ at $t = 0$, the above scheme defines the values $w_m^k \in \mathcal{M}$ for all $k \in \mathbb{N}$ and $m \in \mathbb{Z}$, provided all such values stay in the neighborhood \mathcal{U} where the above Riemann problem is well-posed. We denote the approximate solution obtain by the above scheme with

$$w_{\theta, \Delta x}(x, t) \equiv v_{m,k}(x, t), \tag{4.17}$$

for $x \in [(m - 1)\Delta x, (m + 1)\Delta x]$ and $t \in [(k - 1)\Delta t, k\Delta t]$, where $k + m$ is assumed even. In Theorem 4.5, we prove that $w_{\theta, \Delta x}$ is indeed defined for all $t \geq 0$, by showing that $w_m^k \in \mathcal{U}$ for all $k \in \mathbb{N}$ and $m \in \mathbb{Z}$ with $k + m$ even.

4.3. The Glimm functional and total variation bounds. As in [6, 23] we define *mesh curves* as (unbounded) piecewise linear curves connecting mesh points by moving from “west” to either “north” or “south”, that is, a curve I connecting mesh points is a mesh curve if it is linear in between mesh points, for each $k \in \mathbb{Z}$

there exists exactly one $n \in \mathbb{N}$ such that $a_{k,n} \in I$ and, given some $a_{k,n} \in I$, either $a_{k+1,n+1} \in I$ or $a_{k+1,n-1} \in I$.

Given a mesh curve I , we define I^- as the set of mesh points in the $t \geq 0$ half-plane below I and I^+ as the set of mesh points above I . Given two mesh curves I and J , we say that $I > J$ if every point on I is either on J or lies in J^+ . This gives a partial ordering on the set of mesh curves. We say I is an immediate successor of J if $I > J$ and each point but one of I lies also on J .

Following [6], we define the following functionals on the mesh curve, J ,

$$L(J) \equiv \sum \{|\alpha| : \alpha \text{ crosses } J\}, \tag{4.18}$$

$$Q(J) \equiv \sum \{|\alpha||\beta| : \alpha, \beta \text{ cross } J \text{ and approach}\}, \tag{4.19}$$

where we refer to the wave $w_r = T(\alpha)w_l$, solving the Riemann problem for initial data w_l and w_r , as the α wave. Moreover, assuming α is a j -wave and β is a k -wave, both crossing the mesh curve J , we say that α and β approach if either $j > k$ and α lies to the left of β , or $j < k$ and β lies to the left of α , or $j = k$ and α and β are distinct waves (not connecting the same states) and at least one of them is a shock wave. The next lemma is crucial for proving bounds on the above functionals.

LEMMA 4.4. *For the set of approximates $w_{\theta, \Delta x}$, defined in (4.17), $T.V.(\cdot)$ and $L(\cdot)$ are equivalent in the sense that there exists a constant $C > 0$, independent of θ and Δx , such that for any mesh curve I we have*

$$\frac{1}{C} L(I) \leq T.V.(w_{\theta, \Delta x}|_I) \leq C L(I). \tag{4.20}$$

Proof. To begin, consider two states $w_l, w_r \in \mathcal{M}$ such that w_r is connected to w_l on the right by either a j -shock, j -rarefaction wave or a j contact discontinuity. We denote the corresponding curve by α and assume arc-length parameterization $\epsilon \mapsto \alpha(\epsilon)$ as in Lemma 3.4. Now, it follows from the definition of $d_{\mathcal{M}}$, (4.1), that

$$d_{\mathcal{M}}(w_l, w_r) \leq \int_a^b \sqrt{\langle \dot{\alpha}, \dot{\alpha} \rangle_{\mathcal{M}}} d\epsilon,$$

where by (3.24) a and b are such that $a = 0$ and $\alpha(b) = w_r$ for α being a rarefaction curve, while $b = 0$ and $\alpha(a) = w_r$ if α is a shock curve. In fact, $|b - a|$ is the strength of the wave. Now, since arc-length parameterization implies $\langle \dot{\alpha}, \dot{\alpha} \rangle_{\mathcal{M}} = 1$, and since $w_r = T_{\alpha_j}^j w_l$, the above inequality reduces to

$$d_{\mathcal{M}}(w_l, w_r) \leq |b - a| = |\alpha_j|. \tag{4.21}$$

Now, the total variation of $w \equiv w_{\theta, \Delta x}$ on some mesh curve I is the sum over the mutual distances (with respect to $d_{\mathcal{M}}$) of all intermediate states corresponding to the waves crossing I . Thus, applying (4.21) for all these intermediate states, with w_l and w_r playing the role of the individual intermediate states, immediately gives

$$T.V.(w|_I) \leq L(I). \tag{4.22}$$

To derive a lower bound on $T.V.(w)$, assume again two states $w_l, w_r \in \mathcal{M}$ such that w_r is connected to w_l on the right by a j -wave α_j , that is, $w_r = T_{\alpha_j}^j w_l$. We again

assume the j -wave to be parameterized by arc-length and, for simplicity, we assume $a = 0$ and $b = \alpha_j$. We now write

$$|\alpha_j| = \int_a^b d\epsilon = \frac{1}{|\dot{\gamma}_0(0)|_{\mathcal{M}}} \int_a^b |\dot{\gamma}_0(0)|_{\mathcal{M}} d\epsilon,$$

where γ_0 denotes the unique geodesic curve⁶ with $\gamma_0(a) = w_l$ and $\gamma_0(b) = w_r$ and where $|\cdot|_{\mathcal{M}}^2 \equiv \langle \cdot, \cdot \rangle_{\mathcal{M}}$. Using that geodesic curves have a constant parameterization, i.e., $|\dot{\gamma}_0(0)|_{\mathcal{M}} = |\dot{\gamma}_0(\epsilon)|_{\mathcal{M}}$ for all $\epsilon \in [a, b]$, it follows that

$$|\alpha_j| = \frac{1}{|\dot{\gamma}_0(0)|_{\mathcal{M}}} \int_a^b |\dot{\gamma}_0(\epsilon)|_{\mathcal{M}} d\epsilon = \frac{1}{|\dot{\gamma}_0(0)|_{\mathcal{M}}} d_{\mathcal{M}}(w_l, w_r). \tag{4.23}$$

To show that (4.23) is indeed valid, it remains to prove that $|\dot{\gamma}_0(0)|_{\mathcal{M}} > 0$ for $b > 0$ sufficiently small. For this, we denote α_j by α and Taylor expand $\alpha(b) - \gamma_0(b)$ at 0, using that $\alpha(b) = w_r = \gamma_0(b)$ and using that, by definition, second order covariant derivatives of geodesic curves vanish ($\nabla_{\frac{\partial}{\partial \epsilon}} \dot{\gamma}_0 = 0$), which leads to

$$\begin{aligned} \dot{\gamma}_0^\mu(0) - r_k^\mu(0) &= b \frac{d^2 \alpha^\mu}{d\epsilon^2}(0) + \frac{d^2 \gamma_0^\mu}{d\epsilon^2}(0) + O(b^2) \\ &= b \nabla_{\frac{\partial}{\partial \epsilon}} \dot{\alpha}^\mu(0) - b \Gamma_{\rho\sigma}^\mu (\dot{\alpha}^\rho \dot{\alpha}^\sigma - \dot{\gamma}_0^\rho \dot{\gamma}_0^\sigma) \Big|_0 + O(b^2) \\ &= b \nabla_{r_k} r_k^\mu \Big|_{w_l} - b \Gamma_{\rho\sigma}^\mu (r_k^\rho r_k^\sigma - \dot{\gamma}_0^\rho \dot{\gamma}_0^\sigma) \Big|_{w_l} + O(b^2), \end{aligned}$$

where we applied (3.13) of Lemma 3.4 for the last equality. Thus, taking the absolute value with respect to the metric yields

$$|\dot{\gamma}_0|_{\mathcal{M}} \geq |r_k|_{\mathcal{M}} - b (|\nabla_{r_k} r_k|_{\mathcal{M}} + |\Gamma_{\rho\sigma}^\mu (r_k^\rho r_k^\sigma - \dot{\gamma}_0^\rho \dot{\gamma}_0^\sigma)|_{\mathcal{M}} + O(b)), \tag{4.24}$$

(4.24) implies that $|\dot{\gamma}_0|_{\mathcal{M}} > 0$ for $b > 0$ sufficiently small, since $|\nabla_{r_k} r_k|_{\mathcal{M}}$ is uniformly bounded in \mathcal{U} (by the assumed C^2 regularity of f and g) and since $|\dot{\gamma}_0(0)|_{\mathcal{M}}$ is bounded by the maximum of $|r_k(\epsilon)|_{\mathcal{M}}$ along α , for geodesic curves minimizing arc-length.

Now, to obtain a bound on $L(I)$ in terms of the total variation of $w|_I$, we define

$$C^{-1} \equiv \inf \{ |\dot{\gamma}(0)|_{\mathcal{M}} \mid \gamma \text{ geodesic curve connecting two states in } \hat{\mathcal{U}} \},$$

where $\hat{\mathcal{U}}$ denotes the set containing \mathcal{U} such that the intermediate states of the solution to the Riemann problem (w_l, w_r) lie in $\hat{\mathcal{U}}$ for all $w_l, w_r \in \mathcal{U}$. By (4.24) and continuous dependence, one can restrict \mathcal{U} enough such that for any geodesic curve γ we have a uniform upper and lower bound on $|\dot{\gamma}(0)|_{\mathcal{M}}$, (on the order of $|r_k|_{\mathcal{M}}$) which then implies that $\infty > C > 0$. Now, by (4.23), for the C defined above, the lower bound for the total variation in (4.20) is immediate. \square

The following theorem shows that the Glimm functional, $L(I) + kQ(I)$, is non-increasing for some constant $k > 0$. This was the key insight in [6] to prove convergence of the scheme later on.

⁶When the infimum of $d_{\mathcal{M}}$ in (4.1) is attained, then the corresponding curve is a so-called geodesic curve, which is determined through the metric tensor alone. For p and q sufficiently close, there always exist a unique geodesic curve connecting both points and minimizing (4.1). Moreover, geodesic curves are one degree more regular than the metric, thus $d_{\mathcal{M}}$ has the same regularity as the metric.

THEOREM 4.5. *Consider two mesh curves, I and J , with $J > I$, and assume I is in the domain of definition of $w_{\theta, \Delta x}$. If $L(I)$ is sufficiently small, then J is in the domain of definition of $w_{\theta, \Delta x}$ and*

$$\begin{aligned} Q(J) &\leq Q(I), \\ L(J) + kQ(J) &\leq L(I) + kQ(I), \end{aligned} \tag{4.25}$$

for a constant $k > 0$ independent of I and J . In particular, if $T.V.(w_0)$ is sufficiently small, then $w_{\theta, \Delta x}$ is defined in $\mathbb{R} \times [0, \infty)$ and across each mesh curve \hat{I}

$$T.V.(w_{\theta, \Delta x}|_{\hat{I}}) \leq C T.V.(w_0) \tag{4.26}$$

for a constant $C > 0$ independent of J , θ and Δx .

Proof. The proof of the estimates (4.25) for J being an immediate successor of I follows by the exact same words as in the proof of Theorem 19.5 in [23]. A straightforward computation then shows that

$$L(J) + kQ(J) \leq L(I) + kQ(I) \leq L(I) + kL(I)^2 \leq 2L(I),$$

provided we assume $L(I) < \frac{1}{k}$. By (4.22), of the proof of Lemma 4.4, we then obtain the inequality

$$T.V.(w_{\theta, \Delta x}|_J) \leq L(J) + kQ(J) \leq 2L(I), \tag{4.27}$$

from which we conclude that $w_{\theta, \Delta x}$ is defined for the immediate successor J of I , as long that $L(I)$ is sufficiently small.

The inequalities (4.25) and (4.27) now extend to general mesh curves $J > I$ by induction, since the constant k comes from the $O(1)$ weight in Theorem 4.3 and thus only depends on the r_k 's and the Riemannian metric. Using then that (4.27) implies $w_{\theta, \Delta x}|_J$ to stay inside $\hat{\mathcal{U}}$, so that

$$\sup\{|\lambda_j(w_{\theta, \Delta x}|_J)|, 1 \leq j \leq n\} < \frac{\Delta x}{\Delta t}, \tag{4.28}$$

it follows that $w_{\theta, \Delta x}$ is defined for any J of I , as long that $T.V.(I)$ is sufficiently small.

To extend the domain of definition of $w_{\theta, \Delta x}$ to $t \geq 0$, we first introduce 0 as the unique mesh curve closest to $\{t = 0\}$, (connecting all points $a_{m,0}$ with $a_{m+1,1}$ for $m \in 2\mathbb{Z}$). We then obtain

$$L(J) + kQ(J) \leq L(0) + kQ(0) \leq L(0) + kL(0)^2 \leq 2L(0),$$

provided we assume $L(0) < \frac{1}{k}$. Now Lemma 4.4 yields again

$$T.V.(w_{\theta, \Delta x}|_J) \leq L(J) + kQ(J) \leq C T.V.(0), \tag{4.29}$$

which implies again that the wave speeds of $w_{\theta, \Delta x}$ are bounded by $\frac{\Delta x}{\Delta t}$, c.f. (4.28), so that $w_{\theta, \Delta x}$ is defined for all $t \geq 0$.

By (4.27), to prove (4.26), it remains only to bound $T.V.(0)$ by $T.V.(w_0)$. For this, consider a single Riemann problem (w_l, w_r) which is resolved through the wave $w_r = T(\epsilon)w_l$ and denote the intermediate states by $w^i \equiv T_{\epsilon_i}^i \dots T_{\epsilon_1}^1 w_l$, $i = 0, \dots, n$. With this notation we now define the constant

$$c \equiv \sup \left\{ \frac{d_{\mathcal{M}}(w^{i+1}, w^i)}{d_{\mathcal{M}}(w_l, w_r)} \mid i \in \{0, \dots, n-1\}, w_l, w_r \in \mathcal{U} \right\},$$

which is finite for \mathcal{U} sufficiently small, since in the critical case when $d_{\mathcal{M}}(w_l, w_r)$ is smaller than $d_{\mathcal{M}}(w^{i+1}, w^i)$, we have that $d_{\mathcal{M}}(w^{i+1}, w^i)$ converges to 0 faster than $d_{\mathcal{M}}(w_l, w_r)$, as $w_l \rightarrow w_r$. It follows that

$$T.V.(0) \leq cn T.V.(w_0),$$

where we used that $T.V.(w_0)$ is larger than the total variation of the piece-wise constant approximate of w_0 which are used as the initial states in the Glimm scheme. Now (4.26) immediately follows from (4.29). \square

The convergence of the $w_{\theta, \Delta x}$ as $\Delta x \rightarrow 0$, is in fact a consequence of the estimates of the next corollary.

COROLLARY 4.6. *If $T.V.(w_0)$ and $d_{\mathcal{M}}(w_0, \bar{w})$ are both sufficiently small, where $\bar{w} \in \mathcal{M}$ is some constant state, then, for all $t \geq 0$,*

$$T.V.(w_{\theta, \Delta x}(\cdot, t)) + d_{\infty}(w_{\theta, \Delta x}(\cdot, t), \bar{w}) \leq C (T.V.(w_0) + d_{\infty}(w_0, \bar{w})), \tag{4.30}$$

$$\int_{-\infty}^{\infty} d_{\mathcal{M}}(w_{\theta, \Delta x}(x, t_2), w_{\theta, \Delta x}(x, t_1)) dx \leq C (|t_2 - t_1| + \Delta t) T.V.(w_0), \tag{4.31}$$

for $C > 0$ denoting some constant independent of $\theta, \Delta x, k, \Delta t, t, t_1$ and t_2 .

Proof. For ease of notation we subsequently write w instead of $w_{\theta, \Delta x}$ and use C as a universal constant. Since the waves in $w_{\theta, \Delta x}$ from different grid cells do not interact due to (4.16), it follows that

$$\begin{aligned} d_{\infty}(w(\cdot, t), \bar{w}) &= d_{\infty}(w(\cdot, k\Delta t), \bar{w}), \\ T.V.(w(\cdot, t)) &= T.V.(w(\cdot, k\Delta t)) \end{aligned}$$

for all $t \in ((k - 1)\Delta t, k\Delta t]$. It thus suffices to prove (4.30) on the lines $t = k\Delta t$ only. Moreover, by the previous equation, the estimate

$$T.V.(w(\cdot, k\Delta t)) \leq C T.V.(w_0) \tag{4.32}$$

is an immediate consequence of (4.26) in Theorem 4.5. To derive the remaining bound on d_{∞} in (4.30), observe that

$$d_{\infty}(w(\cdot, k\Delta t), \bar{w}) \leq d_{\infty}(w(\cdot, k\Delta t), w(x_0, k\Delta t)) + d_{\mathcal{M}}(w(x_0, k\Delta t), \bar{w}) \tag{4.33}$$

for any $x_0 \in \mathbb{R}$. For the first term on the right hand side we find that

$$\begin{aligned} d_{\infty}(w(\cdot, k\Delta t), w(x_0, k\Delta t)) &\leq \limsup_{l \rightarrow \infty} \sup_{x \in [-l, l]} d_{\mathcal{M}}(w(\cdot, k\Delta t), w(x_0, k\Delta t)) \\ &\leq \lim_{l \rightarrow \infty} T.V. |_{-l}^l (w(\cdot, k\Delta t)) \\ &\equiv T.V.(w(\cdot, k\Delta t)) \\ &\leq C T.V.(w_0), \end{aligned}$$

where $T.V. |_{-l}^l$ is defined in (4.2). Thus (4.33) is bounded by

$$d_{\infty}(w(\cdot, k\Delta t), \bar{w}) \leq C T.V.(w_0) + d_{\mathcal{M}}(w(x_0, k\Delta t), \bar{w})$$

for any $x_0 \in \mathbb{R}$. Thus, as in [23], taking the limit of the previous inequality as $x_0 \rightarrow \infty$ and using that $T.V.(w(\cdot, k\Delta t)) \leq C T.V.(w_0) < \infty$ implies that $w(x_0, k\Delta t)$ converges to some point $p \in \mathcal{M}$ as $x_0 \rightarrow \infty$, we get

$$d_{\infty}(w(\cdot, k\Delta t), \bar{w}) \leq C T.V.(w_0) + d_{\mathcal{M}}(p, \bar{w}).$$

By construction of the $w_{\theta, \Delta x}$, it follows that

$$p = \lim_{x \rightarrow \infty} w(x, k\Delta t) = \lim_{x \rightarrow \infty} w_0(x) \quad \forall k \in \mathbb{N},$$

so that the previous inequality yields

$$d_\infty(w(\cdot, k\Delta t), \bar{w}) \leq C \text{T.V.}(w_0) + d_\infty(w_0, \bar{w}),$$

which together with (4.32) proves (4.30).

To derive the L^1 Lipschitz estimate (4.31), we follow the proof of Corollary 19.8 in [23], only replacing the Euclidean distance function on $w(x, t_1)$ and $w(x, t_2)$ by the canonical distance function $d_{\mathcal{M}}$. In more detail, assume $t_2 > t_1$ and set $t_0 = \sup\{t \leq t_1 : t = k\Delta t, k \in \mathbb{N}\}$ and $S = \frac{(t_2 - t_0)}{\Delta t} + 1$. Now, fix some $x \in \mathbb{R}$ and define $I = [x - S\Delta x, x + S\Delta x]$, then $w(x, t_1)$ and $w(x, t_2)$ are determined by the Cauchy data $\{w(y, t_0) : y \in I\}$. Thus,

$$\begin{aligned} d_{\mathcal{M}}(w(x, t_2), w(x, t_1)) &\leq \sup\{d_{\mathcal{M}}(w(y, t_0), w(x, t_0)) : y \in I\} \\ &\leq C \text{T.V.}(\{w(y, t_0) : y \in I\}) \end{aligned}$$

and defining $I_m = [(m - S)\Delta x, (m + 1 + S)\Delta x]$, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} d_{\mathcal{M}}(w(x, t_2), w(x, t_1)) dx &= \sum_{m \in \mathbb{Z}} \int_{m\Delta x}^{(m+1)\Delta x} d_{\mathcal{M}}(w(x, t_2), w(x, t_1)) dx \\ &\leq C\Delta x \sum_{m \in \mathbb{Z}} \text{T.V.}(\{w(y, t_0) : y \in I_m\}) \\ &\leq C(1 + S)\Delta t \frac{\Delta x}{\Delta t} \text{T.V.}(w_0), \end{aligned}$$

which proves (4.31) since $S\Delta t \leq t_2 - t_1 + 2\Delta t$ and $\frac{\Delta x}{\Delta t}$ is fixed by (4.16). \square

4.4. Convergence. We first show that on each line $\{t = \text{const.}\}$ the approximates $w_{\theta, \Delta x}(\cdot, t)$ converge to a state $w(\cdot, t)$ as $\Delta x \rightarrow 0$. This follows from an extension of Helly’s Theorem [18] to our framework, recorded in the next lemma.

LEMMA 4.7. *Let $\mathcal{F} \equiv \{u : [a, b] \rightarrow \mathcal{M}\}$ be an infinite family of curves and assume there exists a constant state $\bar{w} \in \mathcal{M}$ and a constant $K > 0$ such that*

$$d_\infty(u, \bar{w}) < K \quad \text{and} \quad \text{T.V.}(u) < K, \quad \forall u \in \mathcal{F}.$$

Moreover, assume K is small enough for all $u \in \mathcal{F}$ to take values inside some coordinate patch \mathcal{U} in the sense that $\overline{\{u(x) : u \in \mathcal{F}, x \in [a, b]\}} \subset \mathcal{U}$, where the closure is taken with respect to $d_{\mathcal{M}}$.⁷ Then there exists a sequence $(w_k)_{k \in \mathbb{N}} \subset \mathcal{F}$ which converges point-wise for each $x \in [a, b]$ to a curve $w : [a, b] \rightarrow \mathcal{M}$ satisfying $\text{T.V.}(w) < K$.

Proof. Choose coordinates $y^\mu : \mathcal{U} \rightarrow \mathbb{R}^n$ such that $y^\mu(\bar{w}) = 0$ and such that the absolute value of each component of $y^\mu(p)$ is bounded by $d_{\mathcal{M}}(p, \bar{w})$ for each $p \in \mathcal{U}$. We now define for $u \in \mathcal{F}$

$$\pi_u(x) \equiv \text{T.V.} \Big|_a^x(u) \quad \text{and} \quad \Omega_u^\mu(x) \equiv \pi_u(x) - y^\mu(u(x)),$$

⁷This smallness assumption on K is not required for Helly’s Theorem in \mathbb{R}^n . Replacing this assumption by requiring $(\mathcal{M}, d_{\mathcal{M}})$ to be complete suffices to prove convergence of a sequence (w_n) on countable subsets of $[a, b]$, by using the Bolzano Weierstrass Theorem on $(\mathcal{M}, d_{\mathcal{M}})$ and a diagonal choice process, however, it is currently not clear to us how to extend this convergence to the whole interval $[a, b]$.

where the last definition is to be understood component-wise for $\mu \in \{1, \dots, n\}$ and $T.V.|_a^x$ is defined in (4.2). For each $u \in \mathcal{F}$, by the above choice of coordinates, π_u and Ω_u^μ are real-valued *increasing* functions in x . By assumption, π satisfies the bound

$$\sup_{x \in [a, b]} |\pi_u(x)| < K \quad \forall u \in \mathcal{F}$$

and, since $d_{\mathcal{M}}$ is equivalent to the Euclidean distance whenever $d_{\mathcal{M}}$ is restricted to a coordinate patch, Ω_u^μ satisfies for all $\mu = 1, \dots, n$ the bound

$$\sup_{x \in [a, b]} |\Omega_u^\mu(x)| < CK \quad \forall u \in \mathcal{F},$$

for some constant C independent of u .

According to the Lemma preceding Helly’s Theorem in [18], for an infinite family of real valued increasing functions defined on $[a, b]$ with a uniform upper bound, there exist a sequence within that family converging at each point $x \in [a, b]$. Thus, there exists a sequence $(\pi_k)_{k \in \mathbb{N}} \subset (\pi_u)_{u \in \mathcal{F}}$ and some function π such that

$$\lim_{k \rightarrow \infty} \pi_k(x) = \pi(x) \quad \forall x \in [a, b],$$

and, choosing to each π_k the corresponding $u_k \in \mathcal{F}$, there exists a subsequence of the $\Omega_k^\mu \equiv \Omega_{u_k}^\mu$ such that

$$\lim_{j \rightarrow \infty} \Omega_{k_j}^\mu(x) = \Omega^\mu(x) \quad \forall x \in [a, b], \quad \forall \mu = 1, \dots, n,$$

for some function Ω^μ . We conclude that $y_{k_j}^\mu(x) \equiv \pi_{k_j}(x) - \Omega_{k_j}^\mu(x)$ converges to the function $y_0^\mu(x) \equiv \pi(x) - \Omega^\mu(x)$. Finally, since the closure of the images of all $w \in \mathcal{F}$ is assumed to be contained in the coordinate neighborhood \mathcal{U} , y^{-1} is defined for $y_0^\mu(x)$ for all $x \in [a, b]$, and it follows that the sequence $w_j \equiv y^{-1}(y_{k_j}^\mu) \in \mathcal{M}$ converges pointwise for all $x \in [a, b]$ to $w \equiv y^{-1}(y_0^\mu)$ and that $T.V.(w) < K$. Since the convergence is point-wise, the result is independent of the choice of coordinates. \square

By (4.30), Lemma 4.7 applies to the family of $(w_{\theta, \Delta x})_{\Delta x}$ and gives us for any *bounded* interval in any of the lines $\{t = \text{const}\}$ a convergent subsequence and, by Cantor’s diagonal choice process, we can extract for each $t = \text{const}$ a subsequence which converges for all $x \in \{t = \text{const}\}$. By a further diagonal process, there exist a subsequence $(w_{\theta, \Delta x_l})_{l \in \mathbb{N}}$ of $(w_{\theta, \Delta x})$ which converges on any of the lines $\{t = \text{const} \in \mathbb{Q}\}$ to some $w(x, t) \in \mathcal{M}$ as $k \rightarrow \infty$ (with $\Delta x_l \rightarrow 0$ as $l \rightarrow \infty$). To extend this convergence to all $t \geq 0$, (as in [6]), set $w_l \equiv w_{\theta, \Delta x_l}$ and use (4.31) to compute for $q \in \mathbb{Q}, q \geq 0$,

$$\begin{aligned} & \int_a^b d_{\mathcal{M}}(w_i(x, t), w_j(x, t)) dx \\ & \leq \int_a^b d_{\mathcal{M}}(w_i(x, t), w_i(x, q)) dx + \int_a^b d_{\mathcal{M}}(w_i(x, q), w_j(x, q)) dx \\ & \quad + \int_a^b d_{\mathcal{M}}(w_j(x, q), w_j(x, t)) dx \\ & \leq C (|t - q| + \Delta t_i + \Delta t_j) T.V.(w_0) + \int_a^b d_{\mathcal{M}}(w_i(x, q), w_j(x, q)) dx. \end{aligned}$$

Now, given some $\epsilon > 0$, we choose q such that

$$C |t - q| T.V.(w_0) < \frac{\epsilon}{2}$$

and then choose i and j large enough for the remaining terms to be bounded by $\frac{\epsilon}{2}$. This shows that $d_{\mathcal{M}}(w_i(x, t), w_j(x, t))$ converges to 0 in $L^1_{\text{loc}}(\mathbb{R})$ as $i, j \rightarrow \infty$ for all $t \geq 0$ which implies convergences point-wise almost everywhere in x . Thus $(w_l(x, t))_{l \in \mathbb{N}}$ is a Cauchy sequence in $(\mathcal{U}, d_{\mathcal{M}})$ for almost every $x \in \mathbb{R}$ and, since $(\mathcal{U}, d_{\mathcal{M}})$ is a complete metric space, (assuming without loss of generality that \mathcal{U} is restricted sufficiently to be compact, completeness of $(\mathcal{U}, d_{\mathcal{M}})$ follows by the Hopf-Rinow Theorem, c.f. [5]), it follows that $(w_l(x, t))_{l \in \mathbb{N}}$ converges to some $w(x, t) \in \mathcal{M}$ for almost every $(x, t) \in \mathbb{R} \times [0, \infty)$. In summary, as a consequence of (4.30) and (4.31), we proved the following theorem:

THEOREM 4.8. *Within the family of functions $(w_{\theta, \Delta x})$ defined in (4.17), there exists a subsequence $(w_l)_{l \in \mathbb{N}}$ which converges almost everywhere in $\Omega \equiv \mathbb{R} \times \{t \geq 0\}$ to some w as $l \rightarrow \infty$. (By definition, each $w_l \equiv w_{\theta, \Delta x_l}$ corresponds to a grid length Δx_l such that $\Delta x_l \rightarrow 0$ as $l \rightarrow \infty$.) For the limit function w there exist a constant $C > 0$ such that for any $t \geq 0$*

$$T.V.(w(\cdot, t)) + d_{\infty}(w(\cdot, t), \bar{w}) \leq C (d_{\infty}(w_0, \bar{w}) + T.V.(w_0)).$$

To complete the proof of Theorem 4.1, it remains to show that the limit function w is indeed a weak solution of (1.5). To continue, we prove the following corollary.

COROLLARY 4.9. *Given a continuous function $f : \mathcal{M} \rightarrow \mathbb{R}^n$ and let $(w_l)_{l \in \mathbb{N}}$ be the convergent subsequence of $(w_{\theta, \Delta x})$ shown to exist in Theorem 4.8, then $f(w_l) \rightarrow f(w)$ in $L^1_{\text{loc}}(\Omega)$ as $l \rightarrow \infty$.*

Proof. By continuity of f , since $(w_l)_{l \in \mathbb{N}}$ converges to w point-wise almost everywhere, it follows that $(f(w_l))_{l \in \mathbb{N}}$ converges to $f(w)$ point-wise almost everywhere. Moreover, since $(w_l)_{l \in \mathbb{N}}$ is uniformly bounded with respect to l , $(f(w_l))_{l \in \mathbb{N}}$ is uniformly bounded as well. Therefore, since $f(w_l) \in L^1_{\text{loc}}(\Omega)$ for all $l \in \mathbb{N}$, the Lebesgue dominated convergence Theorem implies that $(f(w_l))_{l \in \mathbb{N}}$ converges to $f(w)$ in $L^1_{\text{loc}}(\Omega)$. \square

We continue with the following fundamental observation [6]. Given $\psi \in C^1_0(\Omega)$, define

$$J(\theta, \Delta x, \psi) \equiv \iint_{t \geq 0} (g(w_{\theta, \Delta x})\psi_t + f(w_{\theta, \Delta x})\psi_x) dx dt + \int_{-\infty}^{\infty} g(w_{0, \Delta x})\psi dx,$$

where $w_{0, \Delta x}(x)$ is some function converging to $w_0(x)$ for almost every $x \in \mathbb{R}$ such that $g(w_{0, \Delta x})$ converges to $g(w_0)$ in $L^1_{\text{loc}}(\mathbb{R})$, c.f. [6, 23]. A function $w : \Omega \rightarrow \mathcal{M}$ is a weak solution of (1.5) if and only if $J(\theta, \Delta x, \psi) = 0$ for all $\psi \in C^1_0(\Omega)$, c.f. Definition 2.6. By (4.17), $w_{\theta, \Delta x}$ is an (exact) weak solution in each time strip $k\Delta t \leq t \leq (k + 1)\Delta t$, from which we get that

$$J(\theta, \Delta x, \psi) = \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \psi(x, k\Delta t) [g(w_{\theta, \Delta x})]_k(x) dx \tag{4.34}$$

where

$$[g(w_{\theta, \Delta x})]_k(x) \equiv g(w_{\theta, \Delta x}(x, k\Delta t + 0)) - g(w_{\theta, \Delta x}(x, k\Delta t - 0)).$$

From (4.34) we obtain the non-trivial result that within the sequence $w_k = w_{\theta, \Delta x_k}$ there exists a subsequence for which $J(\theta, \Delta x, \psi) \rightarrow 0$ as $\Delta x \rightarrow 0$ for almost all θ . To formulate this final lemma, define the space of random variables

$$\Phi \equiv \{(\theta_k)_{k \in \mathbb{N}} : \theta_k \in [-1, 1] \forall k \in \mathbb{N}\}.$$

THEOREM 4.10. *There exists a set $N \subset \Phi$ of measure zero and a sequence $\Delta x_i \rightarrow 0$, as $i \rightarrow \infty$, such that for any $\theta \in \Phi \setminus N$ and any test function $\psi \in C_0^1(\Omega)$, we have $J(\theta, \Delta x_i, \psi) \rightarrow 0$ as $i \rightarrow \infty$.*

Proof. The proof follows along the line of argument given in between Lemma 19.12 and Theorem 19.14 in [23], see also [6], by replacing all functions $u_{\theta, \Delta x}$ by $g(w_{\theta, \Delta x})$ and using $J(\theta, \Delta x, \psi)$ as defined above. However, to prove that the total variation bound (4.30) implies the point-wise bound on $|J(\theta, \Delta x, \Phi)|$ in terms of Δx and $\|\Phi\|_\infty$, stated in Lemma 19.12 in [23], we also need to use the total variation bound of the next lemma, Lemma 4.11, which applies since one can always restrict \mathcal{U} enough for $\overline{\mathcal{U}}$ to be compact and to lie in some coordinate neighborhood (y, \mathcal{U}') . Finally, a L^2 estimate of $J(\cdot, \Delta x_i, \psi)$ with respect to integration over the random variable θ yields convergence almost everywhere, c.f. Theorem 19.4 in [23]. \square

LEMMA 4.11. *Let $g : \mathcal{M} \rightarrow \mathbb{R}^n$ be Lipschitz continuous and consider a curve $w : \mathbb{R} \rightarrow \mathcal{U}$, for $\mathcal{U} \subset \mathcal{M}$ being a compact set contained in a coordinate neighborhood (y, \mathcal{U}') . Then*

$$T.V.(g(w)) \leq C T.V.(w), \tag{4.35}$$

for some constant $C > 0$ which depends only on the metric tensor and the Lipschitz constant of g in coordinates y , on the domain $y(\mathcal{U})$.

Proof. Let $w_1, w_2 \in \mathcal{U}$. Choose some coordinates y on \mathcal{U} and consider $y_1 \equiv y(w_1)$ and $y_2 \equiv y(w_2) \in y(\mathcal{U})$, we then find

$$|g \circ y^{-1}(y_1) - g \circ y^{-1}(y_2)|_{\text{eucl}} \leq \text{Lip}(g \circ y^{-1}) |y_1 - y_2|_{\text{eucl}}, \tag{4.36}$$

where $\text{Lip}(g \circ y^{-1})$ denotes the Lipschitz constant of g over $y(\mathcal{U})$ and $|\cdot|_{\text{eucl}}$ denotes the Euclidean square norm. It remains to show that $|y_1 - y_2|_{\text{eucl}} \leq C d_{\mathcal{M}}(y_1, y_2)$ for some constant $C > 0$. To prove this, using that $\overline{\mathcal{U}}$ is assumed compact, we observe that there exist a constant $C > 0$ such that

$$|v|_{\mathcal{M}} \geq C |v|_{\text{eucl}}, \quad \forall v \in T_p \mathcal{M}, \quad \forall p \in \mathcal{U}. \tag{4.37}$$

(Here C can be taken to be the supremum of the largest eigenvalue of the metric tensor, which is symmetric by definition.) Now, we consider some curve $\gamma : [a, b] \rightarrow y(\mathcal{U})$ which connects y_1 with y_2 and compute, using (4.37), that

$$\int_a^b |\dot{\gamma}|_{\mathcal{M}} ds \geq C \int_a^b |\dot{\gamma}|_{\text{eucl}} ds \geq C |y_1 - y_2|_{\text{eucl}},$$

where the last inequality holds since a straight line is the shortest connection between two points in Euclidean geometry. Since the curve γ is arbitrary, we conclude that

$$d_{\mathcal{M}}(y_1, y_2) \geq C |y_1 - y_2|_{\text{eucl}}.$$

From the above inequality and the fact that $d_{\mathcal{M}}(y_1, y_2) = d_{\mathcal{M}}(w_1, w_2)$ is invariant, (4.36) implies

$$|g(w_1) - g(w_2)|_{\text{eucl}} \leq C \text{Lip}(g \circ y^{-1}) d_{\mathcal{M}}(w_1, w_2),$$

from which we conclude (4.35). \square

Lemma 4.11 completes the proof of Theorem 4.10, and the following corollary is immediate.

COROLLARY 4.12. *If $T.V.(w_0)$ and $d_{\infty}(w_0, \bar{w})$ are sufficiently small, then there exist a set of measure zero $N \subset \Phi$ and a sequence $\Delta x_i \rightarrow 0$, as $i \rightarrow \infty$, such that for any $\theta \in \Phi \setminus N$, $w_{\theta} \equiv \lim_{i \rightarrow \infty} w_{\theta, \Delta x_i}$ is a weak solution of the Cauchy problem (1.5) with initial data w_0 .*

Corollary 4.12 together with Theorem 4.8 complete the proof of Glimm’s Theorem extended to our geometric framework, that is, Theorem 4.1.

5. Riemann problems for constrained conservation laws. In this section we prove Theorem 1.3 and Proposition 1.4.

Proof of Theorem 1.3. By (1.7), that is, the assumption that

$$\det [DG(u), DC(u)] \neq 0,$$

we conclude that the linear mapping $DC(u) : \mathbb{R}^m \rightarrow \mathbb{R}^{m-n}$ is surjective at u_l . This implies that the set $\mathcal{M} \equiv C^{-1}(0)$ defines a n -dimensional submanifold in \mathbb{R}^m , c.f. [24, Theorem 5.1]. In more detail, assuming that $u_l \in \hat{\mathcal{U}}$ with $C(u_l) = 0$, then, since C is assumed to have full rank at u_l , the implicit function theorem yields that there exists an open set $\Omega \subset \mathbb{R}^n$ and a differentiable function $\varphi_{u_l} : \Omega \rightarrow \mathbb{R}^m$ invertible on its image such that φ_{u_l} parameterizes the solution space $C^{-1}(0)$ sufficiently close to u_l . Since the implicit function theorem yields for any point $u \in \hat{\mathcal{U}}$ with $C(u) = 0$ such a function φ_u parameterizing some open subset of $C^{-1}(0)$, we conclude that (1.7) implies the solution space $\mathcal{M} \equiv C^{-1}(0)$ to be a n -dimensional submanifold of \mathbb{R}^m [5, 24]. Thus, the constrained system of $n \times m$ conservation laws, (1.1) - (1.2), is a system of the form (1.5), that is, a $n \times n$ system of conservation laws with states in the manifold \mathcal{M} .

In order to apply Theorem 1.1, it remains to show that the constrained system (1.1) - (1.2) is strictly hyperbolic in the sense of Definition 2.1. For this, observe that condition (1.9), that is, $DCr_k = 0$, implies that the eigenvectors $r_k(u)$ lie in the tangent-space of \mathcal{M} whenever $u \in \mathcal{M}$, so that by our hypothesis (1.8) the resulting system is strictly hyperbolic. Now, Theorem 1.1 applies and yields the claimed existence and uniqueness of solutions, which completes the proof. \square

We now prove Proposition 1.4, which shows that the manifold \mathcal{M} is locally described by the integral curves of the eigenvectors r_k , and therefore by the rarefaction curves, c.f. (3.1). This is useful in practice, because it only involves solving linear systems but never requires solving the constraint $C(u) = 0$ explicitly which could be difficult to accomplish for many applications. One can then obtain the shock curves by substituting the local parameterization of $\mathcal{M} = C^{-1}(0)$ around u_l in the Hugoniot locus. We prove the following more precise version of Proposition 1.4.

PROPOSITION 5.1. *Assume (1.7) - (1.9) hold for all $u \in \hat{\mathcal{U}}$ for some $\hat{\mathcal{U}} \subset \mathbb{R}^m$. Then, locally, the solution space $C^{-1}(0)$ is identical to the image of the integral curves*

of the eigenvectors r_k . That is, for each $u_l \in \hat{\mathcal{U}}$ with $C(u_l) = 0$, there exist some $\mathcal{U} \subset \hat{\mathcal{U}}$ with $u_l \in \mathcal{U}$ such that

$$C^{-1}(0) \cap \mathcal{U} = \left\{ u \in \hat{\mathcal{U}} : u = S_{\epsilon_n}^n \dots S_{\epsilon_1}^1 u_l \right\}$$

where $S_{\epsilon_k}^k u' \in \mathbb{R}^m$ is the point which is reached by the integral curve of r_k emanating in u' at a parameter ϵ_k , for $k \in \{1, \dots, n\}$. Moreover, the integral curves of r_k define a local parameterization of $C^{-1}(0)$ if and only if there exists a coordinate system of Riemann invariants.

To clarify, a Riemann invariant ω_j is a scalar function which satisfies $r_i^\nu \partial_\nu \omega_j = 0$, for $i, j \in \{1, \dots, n\}$ with $i \neq j$.

Proof. Let $u_l \in C^{-1}(0)$. To begin, we solve the ODE 3.1 simultaneously with the eigenvalue problem (1.8) - (1.9), that is, we solve the system

$$\frac{du_k}{d\epsilon} = r_k(u_k(\epsilon)), \quad \text{for } u_k(0) = u_l, \tag{5.1}$$

together with

$$(\lambda_k DG - DF) r_k = 0 \quad \text{and} \quad DC r_k = 0, \tag{5.2}$$

the later two equations being evaluated at $u_k(\epsilon)$. Assuming for the moment the solution (λ_k, r_k) of (1.8) - (1.9) exist everywhere in $\hat{\mathcal{U}} \subset \mathbb{R}^m$ and not only for $u \in \hat{\mathcal{U}}$ with $C(u_l) = 0$, it follows that the above system is well-posed, which gives us the integral curves $\epsilon \mapsto u_k(\epsilon)$, with $\epsilon \in (-a, a)$ for some $a > 0$.

By the condition $DC r_k = 0$, it follows that $r_k(u) \in T_u \mathcal{M}$ whenever $u \in \mathcal{M}$. We now change coordinates in \mathbb{R}^m such that some open neighborhood of u_l in \mathcal{M} coincides with some open set in $\mathbb{R}^n \times \{0\}^{m-n}$. It follows that $T_u \mathcal{M}$, as a subspace of \mathbb{R}^m , is identical to $\mathbb{R}^n \times \{0\}^{m-n}$ for each u in that neighborhood. Thus, using $u_k(0) = u_l \in \mathcal{M}$, we conclude that the curve $\epsilon \mapsto u_k(\epsilon)$ lies in \mathcal{M} for all ϵ sufficiently small, since the $r_k(u_k(\epsilon))$ are in $\mathbb{R}^n \times \{0\}^{m-n}$ for all ϵ . (This shows that the above system is well-posed even when the solution (λ_k, r_k) of (1.8) - (1.9) exists only for $u \in C^{-1}(0)$.)

To prove that \mathcal{M} is locally “parameterized” by the integral curves u_k for $k = 1, \dots, n$, observe first that the system (5.1) - (5.2) is well-posed for any $u_l \in \mathcal{M}$. Now, for $\epsilon_k \in (-a, a)$, $k = 1, \dots, n$, we introduce the mapping $S_{\epsilon_k}^k : \mathcal{M} \rightarrow \mathcal{M}$ as $S_{\epsilon_k}^k u = u_k(\epsilon_k)$ where u_k is the integral curve of r_k (in the sense of (5.1) - (5.2)) with initial data $u_k(0) = u$. Given $u' \in \mathcal{M}$ sufficiently close to u_l , it thus remains to show that there exist $(\epsilon_1, \dots, \epsilon_n)$ such that

$$u' = S_{\epsilon_n}^n \dots S_{\epsilon_1}^1 u_l. \tag{5.3}$$

As in the proof of Theorem 1.3, this follows by the Inverse Function Theorem and the linear independence of the r_k 's.

This proves that $\mathcal{M} \equiv C^{-1}(0)$ is locally described by the integral curves of r_k . However, for this to be a proper parameterization of \mathcal{M} , that is, for $(\epsilon_1, \dots, \epsilon_n)$ to define coordinates, (5.3) must hold independently of the order of the maps $S_{\epsilon_k}^k$. By Frobenius Theorem,⁸ this independence of order holds if and only if the r_k 's commute

⁸If the integral curves of some vector fields r_k ($k = 1, \dots, n$) are coordinates, then the commutator (5.4) vanishes, since second order coordinate derivatives commute. The non-trivial part in Fobenius Theorem is the reverse implication: Given some vector fields r_k which satisfy (5.4), it follows that their integral curves define a coordinate system, c.f. [25, Chapter 1] as a reference.

in the sense that

$$[r_i, r_j](\psi) \equiv r_i(r_j(\psi)) - r_j(r_i(\psi)) = 0 \quad \forall \psi \in C^\infty(\mathcal{M}). \quad (5.4)$$

Moreover, assuming there exist a coordinate system of Riemann invariants, then their defining property $r_i^\nu \partial_\nu \omega_j = 0$ implies that $r_i^\nu \partial_\nu = \frac{\partial}{\partial w_i}$, for which the above Lie bracket vanishes. The reverse implication holds, since the parameters of integral curves of the r_k 's are Riemann invariants. \square

6. Conclusion and outlook. We introduced a geometric framework for the states of a conservation law lying in a Riemannian manifold and we proved that the methods of Lax and Glimm to construct solutions of the Riemann and Cauchy problem respectively can be extended to our geometric framework. For applications, this shows that it is not necessary to invert the accumulation function g , (which would be required to transform the conservation law to standard form), since it is possible to work in the manifolds of states. Moreover, within our framework it is possible to extend the existence result of Lax and Glimm to the case that dg is *not invertible* on a finite family of co-dimension one surfaces. In this way, it is also possible to address systems of n conservation laws of the form (1.5) which have *fewer* than n eigenvalues at co-dimension one surfaces.

Genuine non-linearity is violated by many physical systems. We expect that one can extend Liu's construction [14] to our geometric framework to obtain solutions of the Riemann problem when genuine non-linearity fails on co-dimension one surfaces. The uniqueness results of Bressan et al. [3] should apply whenever the accumulation function g is invertible. Let us remark that we chose to address the Cauchy problem with Glimm's scheme over alternative methods, since Glimm's method allows for straightforward applications and provides an algorithm adapted to numerical computations.

We believe our geometric approach to be more feasible for applying to physical conservation laws (with or without constraints), since one avoids inverting a possibly non-linear function and is only left with solving a linear generalized eigenvalue problem. Moreover, conservation laws with an accumulation function usually arise directly from physical or chemical principles while the corresponding standard form results from a desire for notational convenience in mathematics, so that the geometric approach presented here could ultimately lead to less complicated structures which are easier to work with. We thus believe that it is of advantage to use the geometric framework presented here for studying conservation laws which are not readily given in standard form by physical or chemical principles.

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