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A RELATIVISTIC VERSION OF THE EULER-KORTEWEG EQUATIONS*

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Abstract. Starting from a variational interpretation of enthalpy, this paper formulates a relativistically covariant version of the Euler-Korteweg equations of fluid dynamics. The system has a canonical Lagrangian and converges in the non-relativistic limit to Dunn and Serrin's formulation.

Key words. Relativistic Korteweg tensor, interstitial work, objectivity, covariance.

Mathematics Subject Classification. 35L65, 35Q75, 76N15.

1. Introduction. In 1901, Korteweg proposed to model capillarity of fluids by a stress tensor that depends not only on the local mass density of the fluid, but also on its first and second order spatial derivatives; he ingeniously derived this object by isotropy considerations [15]. Much later, the 'Korteweg tensor' became an object of study for Rational Mechanics and, as the crucial ingredient of the 'Euler-Korteweg system', for the theory of partial differential equations (PDE). In particular, Truesdell and Noll gave a systematic account of 'Korteweg theory' [27], Slemrod took it as a starting point for understanding the admissibility and internal structure of moving phase boundaries in van der Waals fluids [26, 25], and Dunn and Serrin solved an associated conceptual problem by introducing the so-called 'interstitial work' [7]. Benzoni-Gavage and collaborators proved wellposedness of the evolution equations, revealed an underlying Hamiltonian structure, and showed the stability of traveling-wave phase boundaries [3, 2, 1].[1] Kinetic theory has found a fundamental physical significance of the Korteweg tensor that sets it at equal footing with the Navier-Stokes viscosity tensor [14, 24].

The purpose of the present note is to formulate relativistic versions of the Korteweg tensor and of the Euler-Korteweg equations, and initiate a study of their properties. We will see that to find the tensor and the equations, one can argue quite analogously to the perfect-fluid case, reinterpreting well-known equations of motion via a variational formulation of the fluid's enthalpy. In Section 2 we do this for isentropic fluids, in Section 3 for general fluids. Section 4 investigates the non-relativistic limit of the new equations, showing that they converge to the classical Euler-Korteweg system, both in the isentropic and in the general case. In Section 5, we prove that the new equations can be derived from a Lagrangian, in exactly the way in which Ray, in [22], has shown the same for non-capillary fluids. Section 6 studies the behaviour of this Lagrangian structure in the non-relativistic limit.

The primary motivation for this paper lies in the fact that astrophysics and cosmology often use relativistic fluid dynamics in situations with considerable density gradients (cf., e.g., [13]). It seems likely that in some of these situations a dependence of the internal energy on the density gradient cannot appropriately be ignored, and it is for such possible cases that we wish to provide a covariant version of Korteweg theory. While this paper would not seem the right place to enter details of concrete

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[1]Meanwhile, a rich theory also of the 'Navier-Stokes-Korteweg' equations exists, mainly due to Kotschote [18, 17, 16]. Cf. also the recent survey [10].

such applications, we only recall the essential use of non-relativistic Korteweg theory for fluids with van-der-Waals-like pressure laws, the simplest canonical formulation of diphasicity [26, 3, 10]. For results on explicitly two-phase relativistic fluid models that are analogous to those of this paper, see [8].

A second motivation for this study is that a famous observation of Madelung implies that the classical one-particle Schrödinger equation can be written "in a hydrodynamical form" [21] which was recognized as the (non-relativistic) Euler-Korteweg equations for a particular choice of energy (cf., e.g., [5] and references therein). While our relativistic Euler-Korteweg system with a corresponding choice does not seem to directly reflect a relativistic equation for a wave function such as the Dirac or Klein-Gordon equations, it would be interesting to investigate whether it might still have something to do with a covariant version of wave mechanics in the specific case of one uncharged particle without spin.[2]

2. Relativistic Euler-Korteweg system for isentropic fluids. To begin our argumentation, we recall that the relativistic dynamics of *perfect* fluids is governed by the conservation equations for energy-momentum,

$$T^{\alpha\beta}_{,\beta} = 0, \tag{2.1}$$

and matter,

$$\left(nU^{\beta}\right)_{,\beta} = 0, \tag{2.2}$$

where

$$T^{\alpha\beta} = (\rho + p)U^{\alpha}U^{\beta} + p\, g^{\alpha\beta} \tag{2.3}$$

and nU^{β} are the energy-momentum tensor and the matter current (cf., e.g., [28], Sec. 2.10, and [4], Chap. IX). Here U^{α} denotes the 4-velocity of the fluid, and the fluid itself is specified by its specific internal energy ϵ. If, as we assume in this section, the fluid is isentropic, ϵ is a function of the particle density n alone, and the fluid's internal energy ρ and pressure p derive from this function as

$$\rho = \rho(n) = n\epsilon(n), \quad p = p(n) = n^2 \frac{d\epsilon(n)}{dn}. \tag{2.4}$$

Given (2.2), one can, using the enthalpy, or Lichnerowicz 'index', of the fluid,[3]

$$h = h(n) = \frac{d\rho(n)}{dn}, \tag{2.5}$$

re-express (2.1) by the 'equations of motion' [19, 4][4]

$$U^{\beta}\left(hU^{\alpha}\right)_{,\beta} + h^{,\alpha} = 0. \tag{2.6}$$

In this paper, we discuss modelling capillary fluids by assuming a specific energy that depends not only on n, but also on its gradient[5],

$$\epsilon = \epsilon(n, \bar{\nabla} n) = \check{\epsilon}(n, n_{,\gamma}n^{,\gamma}/2). \tag{2.7}$$

[2]In which case the Lagrangian structure might even somehow correspond to the general Lagrangian approach to quantum mechanics that was first suggested by Dirac [6].

[3]Unlike for general barotropic fluids [4], enthalpy and index coincide in the isentropic case [9].

[4]Using $h = (\rho + p)/n$ and $\mathrm{d}p = n\mathrm{d}h$, one sees that (2.1) is n times (2.6).

[5]We write $\bar{\nabla}$ for the 4-gradient and, below, ∇ for the spatial 3-gradient.

Jumping *medias in res*, we now state the key observation of this paper: Keeping relations (2.4), now as

$$\rho = \rho(n, \bar{\nabla} n) = n\epsilon(n, \bar{\nabla} n), \quad p = p(n, \bar{\nabla} n) = n^2 \frac{\partial \epsilon(n, \bar{\nabla} n)}{\partial n}, \tag{2.8}$$

and adjusting the definition of h as

$$h = \frac{\delta \rho}{\delta n}, \tag{2.9}$$

i.e.,

$$h = h(n, \bar{\nabla} n) = \frac{\partial \rho(n, \bar{\nabla} n)}{\partial n} - \left(\frac{\partial \rho(n, \bar{\nabla} n)}{\partial n_{,\gamma}} \right)_{,\gamma}, \tag{2.10}$$

makes (2.6) a counterpart of the non-relativistic Euler-Korteweg equations.

Theorem 2.1. *Asssuming* (2.2)*, equations* (2.6) *with* (2.9) *can be written as* (2.1) *with now*

$$T^{\alpha\beta} = (\rho + p) U^\alpha U^\beta + p\, g^{\alpha\beta} - K^{\alpha\beta} \tag{2.11}$$

where

$$K^{\alpha\beta} = n\big(\kappa n^{,\gamma}\big)_{,\gamma} (U^\alpha U^\beta + g^{\alpha\beta}) - \kappa n^{,\alpha} n^{,\beta} \tag{2.12}$$

with

$$\kappa \text{ the coefficient in } \frac{\partial \rho}{\partial n_{,\gamma}} = \kappa n^{,\gamma}, \text{ i.e., } \quad \kappa = n \frac{\partial \check{\epsilon}(n, \phi)}{\partial \phi}\bigg|_{\phi = n_{,\gamma} n^{,\gamma}/2}. \tag{2.13}$$

Proof. Equations (2.1) with (2.11), (2.12) read

$$0 = \big((n\epsilon + p - n\big(\kappa n^{,\gamma}\big)_{,\gamma}) U^\alpha U^\beta\big)_{,\beta} + \big((p - n\big(\kappa n^{,\gamma}\big)_{,\gamma}) g^{\alpha\beta} + \kappa n^{,\alpha} n^{,\beta}\big)_{,\beta}. \tag{2.14}$$

As now

$$h = \epsilon + n^{-1} p - \big(\kappa n^{,\gamma}\big)_{,\gamma}, \tag{2.15}$$

the right hand side of equation (2.14) is identical with

$$\big(nhU^\alpha U^\beta\big)_{,\beta} + \big(n(h - \epsilon)\big)^{,\alpha} + \big(\kappa n^{,\alpha} n^{,\beta}\big)_{,\beta}. \tag{2.16}$$

Since

$$\big(nhU^\alpha U^\beta\big)_{,\beta} = nU^\beta \big(hU^\alpha\big)_{,\beta}, \tag{2.17}$$

the equivalence of (2.1),(2.11) with (2.6) will thus be proved once we have

$$\big(n(h - \epsilon)\big)^{,\alpha} + \big(\kappa n^{,\alpha} n^{,\beta}\big)_{,\beta} = nh^{,\alpha}. \tag{2.18}$$

But

$$\big(n(h - \epsilon)\big)^{,\alpha} + \big(\kappa n^{,\alpha} n^{,\beta}\big)_{,\beta} = nh^{,\alpha} + \big(- n\epsilon^{,\alpha} + (h - \epsilon) n^{,\alpha} + n^{,\alpha} \big(\kappa n^{,\beta}\big)_{,\beta} + \kappa n^{,\beta} n^{,\alpha}_{,\beta}\big)$$

and

$$-n\epsilon^{,\alpha} + (h-\epsilon)n^{,\alpha} + n^{,\alpha}\big(\kappa n^{,\beta}\big)_{,\beta} + \kappa n^{,\beta} n^{,\alpha}_{,\beta} = -n\epsilon^{,\alpha} + \frac{p}{n} n^{,\alpha} + \kappa n^{,\beta} n^{,\alpha}_{,\beta} = 0,$$

as

$$n\epsilon^{,\alpha} = n\left(\frac{\partial\epsilon}{\partial n} n^{,\alpha} + \frac{\partial\epsilon}{\partial n_{,\beta}} n^{,\alpha}_{,\beta}\right) = \frac{p}{n} n^{,\alpha} + \kappa n^{,\beta} n^{,\alpha}_{,\beta}.$$

□

REMARK 2.1. The transition between (2.1) with (2.11), (2.12), i.e.,

$$\big((\rho+p)U^\alpha U^\beta + p\,g^{\alpha\beta} - K^{\alpha\beta}\big)_{,\beta} = 0 \tag{2.19}$$

and the non-conservative form (2.6) is quite similar to the non-relativistic case [3].

We will refer to (2.2), (2.19) as the 'relativistic Euler-Korteweg equations'.

3. Relativistic Euler-Korteweg system for general fluids. For general fluids, specific internal energy ϵ, internal energy ρ, pressure p, enthalpy h, and the capillarity coefficient κ depend also on the specific entropy s, i.e., (2.7),(2.8),(2.10), (2.13) are replaced by

$$\epsilon = \epsilon(n, \bar\nabla n, s) = \check\epsilon(n, n_{,\gamma} n^{,\gamma}/2, s), \tag{3.1}$$

$$\rho = \rho(n, \bar\nabla n, s) = n\epsilon(n, \bar\nabla n, s), \quad p = p(n, \bar\nabla n, s) = n^2 \frac{\partial \epsilon(n, \bar\nabla n, s)}{\partial n}, \tag{3.2}$$

$$h = h(n, \bar\nabla n, s) = \frac{\partial\rho(n, \bar\nabla n, s)}{\partial n} - \left(\frac{\partial\rho(n, \bar\nabla n, s)}{\partial n_{,\gamma}}\right)_{,\gamma}, \tag{3.3}$$

$$\kappa = n\frac{\partial\check\epsilon(n, \phi, s)}{\partial\phi}\bigg|_{\phi = n_{,\gamma} n^{,\gamma}/2}. \tag{3.4}$$

In the perfect fluid case (i.e., no dependence on $\bar\nabla n$), the equations of motion are (cf. [4], p. 224)

$$U^\beta (hU^\alpha)_{,\beta} + h^{,\alpha} = \theta s^{,\alpha}, \tag{3.5}$$

where θ is the temperature,

$$\theta = \frac{\partial\epsilon}{\partial s}.$$

We keep the variational definition (2.9) of enthalpy and the equations of motion (3.5) for the non-perfect case!

THEOREM 3.1. *Asssuming particle number conservation* (2.2), *equations* (3.5) *with* (3.1), (3.2), (3.3) *can, like in the isentropic case, be written as* (2.19), *with* κ *now as in* (3.4).

Proof. Equations (2.15), (2.14), (2.16), (2.17) hold as in the isentropic case. The equivalence of (2.19) with (3.5) will be proved once we show, instead of (2.18), that

$$\big(n(h-\epsilon)\big)^{,\alpha} + \big(\kappa n^{,\alpha} n^{,\beta}\big)_{,\beta} = n h^{,\alpha} - n\theta s^{,\alpha}.$$

But

$$\big(n(h-\epsilon)\big)^{,\alpha} + \big(\kappa n^{,\alpha} n^{,\beta}\big)_{,\beta} = n h^{,\alpha} + \big(- n\epsilon^{,\alpha} + (h-\epsilon) n^{,\alpha} + n^{,\alpha}\big(\kappa n^{,\beta}\big)_{,\beta} + \kappa n^{,\beta} n^{,\alpha}_{,\beta}\big)$$

and

$$-n\epsilon^{,\alpha} + (h-\epsilon)n^{,\alpha} + n^{,\alpha}\big(\kappa n^{,\beta}\big)_{,\beta} + \kappa n^{,\beta} n^{,\alpha}_{,\beta} = -n\epsilon^{,\alpha} + \frac{p}{n} n^{,\alpha} + \kappa n^{,\beta} n^{,\alpha}_{,\beta} = -n\theta s^{,\alpha},$$

as now

$$n\epsilon^{,\alpha} = n\left(\frac{\partial\epsilon}{\partial n} n^{,\alpha} + \frac{\partial \epsilon}{\partial n_{,\beta}} n^{,\alpha}_{,\beta} + \frac{\partial\epsilon}{\partial s} s^{,\alpha}\right) = \frac{p}{n} n^{,\alpha} + \kappa n^{,\beta} n^{,\alpha}_{,\beta} + n\theta s^{,\alpha}.$$

□

4. Non-relativistic limit of the equations. While all other sections use units in which the speed of light is 1, we denote it by c in this section and in Section 6, in order to see what happens when one can consider it as very big. We will study this limit first for isentropic fluids, then for general fluids. For an isentropic fluid consisting of particles of mass m, the specific energy has the form

$$\epsilon = mc^2 + \tilde{\epsilon} \quad \text{with} \quad \tilde{\epsilon} = \hat{\epsilon}(n, n^{,\gamma} n_{,\gamma}/2).$$

We introduce time t and 3-velocity $\mathbf{v} = (v^1, v^2, v^3)$ through

$$x^0 = \frac{t}{c},\ U^j = \frac{v^j}{c},\ U^0 = \sqrt{1 + \frac{v_1^2 + v_2^2 + v_3^2}{c^2}},$$

observe that

$$\tilde{\epsilon} = \hat{\epsilon}(n, |\nabla n|^2/2) + O\left(\frac{1}{c^2}\right),$$

and use the classical pressure

$$\hat{p} = n^2 (\partial\hat{\epsilon}(n,\phi)/\partial n)|_{\phi = |\nabla n|^2/2}.$$

as well as, in particular, the classical Korteweg tensor [15, 7, 26, 3]

$$\mathbf{K} = n\nabla\cdot(\hat{\kappa}\nabla n) - \hat{\kappa}\nabla n \otimes \nabla n \quad \text{with} \quad \hat{\kappa} = n(\partial\hat{\epsilon}(n,\phi)/\partial\phi)|_{\phi=|\nabla n|^2/2}.$$

THEOREM 4.1. *For $c \to \infty$, our relativistic isentropic Euler-Korteweg system converges to the classical isentropic Euler-Korteweg system*

$$\begin{aligned} (nm)_t + \nabla\cdot(nm\mathbf{v}) &= 0, \\ (nm\mathbf{v})_t + \nabla\cdot(nm\mathbf{v}\otimes\mathbf{v} + \hat{p}\mathbf{I} - \mathbf{K}) &= 0. \end{aligned} \tag{4.1}$$

Proof. [6] As $c \to \infty$, the relativistic particle number conservation law,

$$c\big(nU^\beta\big)_{,\beta} = \big(n_t + \nabla\cdot(n\mathbf{v})\big) + \frac{1}{c^2}\big(\frac{1}{2}nv^2\big)_t + O\left(\frac{1}{c^4}\right). \tag{4.2}$$

obviously recovers the classical conservation law $(4.1)_1$ for mass, and the relativistic equations

$$\begin{aligned} 0 &= \Big((\rho+p)U_iU^\beta + p\,g_i^\beta - K_i^\beta\Big)_{,\beta} \\ &= \Big\{(nmv_i)_t + \Big(nmv_iv^j + \hat p\delta_i^j - \mathbf{K}_i^j\Big)_{x^j}\Big\} + O\left(\frac{1}{c^2}\right), \quad i=1,2,3, \end{aligned} \tag{4.3}$$

yield the classical conservation law $(4.1)_2$ for momentum. □

A similar results holds for general fluids, with $\hat\epsilon, \hat p, \hat\kappa$ as before, but now additionally depending on s.

Theorem 4.2. *For $c \to \infty$, our relativistic non-isentropic Euler-Korteweg system converges to the classical non-isentropic Euler-Korteweg system*

$$\begin{aligned} (nm)_t + \nabla\cdot(nm\mathbf{v}) &= 0, \\ (nm\mathbf{v})_t + \nabla\cdot(nm\mathbf{v}\otimes\mathbf{v} + \hat p\mathbf{I} - \mathbf{K}) &= 0, \\ E_t + \nabla\cdot(((E+\hat p)\mathbf{I} - \mathbf{K})\mathbf{v} - \mathbf{w}) &= 0, \end{aligned} \tag{4.4}$$

where

$$E = n(\hat\epsilon(n, |\nabla n|^2/2, s) + \frac{1}{2}mv^2)$$

is the classical total energy and

$$\mathbf{w} = ((\partial_t + \mathbf{v}\cdot\nabla)n)\hat\kappa\nabla n.$$

Proof. Relative to the proof of Theorem 4.1, the only novelty occurs in obtaining $(4.4)_3$: In view of (4.2) and

$$cK^{0,\beta}_{,\beta} = \nabla\cdot(n\nabla(\hat\kappa\nabla n)\mathbf{v} - \hat\kappa n_t\nabla n) + O\left(\frac{1}{c^2}\right),$$

we find

$$c\left(T^{0\beta}_{,\beta} - mc^2(nU^\beta)_{,\beta}\right) = E_t + \nabla\cdot(((E+\hat p)\mathbf{I} - \mathbf{K})\mathbf{v} - \mathbf{w}) + O\left(\frac{1}{c^2}\right).$$

□

Remark 4.1. The term $\nabla\cdot\mathbf{w}$ was identified, justified, and named 'interstitial work' by Dunn and Serrin [7]. In [11], M. Kotschote and the author partly questioned the appropriateness of interstitial work and showed that in the non-relativistic non-isentropic case a formulation without it would be mathematically compatible with the second law of thermodynamics in the sense of having non-negative entropy production on all solutions. While, in contrast to that, Benzoni-Gavage et al. had already shown that the Hamiltonian structure does imply interstitial work (cf. Sec. 1 of [3]), our above findings demonstrate that also *relativistic objectivity provides evidence* for *Dunn and Serrin's interstitial work* (and thus against certain considerations in [11]).

[6] Cf. [12] for similar considerations in a different context.

5. Derivation from a Lagrangian. The derivation from a Lagrangian of the equations that govern relativistic flows of *perfect* fluids in Eulerian coordinates seems to have become known in the early 1970s, cf. [23, 22, 13]. One considers variational equations

$$0 = \frac{\delta L}{\delta Y^k} \equiv \frac{\partial L}{\partial Y^k} - \left(\frac{\partial L}{\partial Y^k_{,\beta}}\right)_{,\beta} \tag{5.1}$$

with Y^k the state variables and Lagrange multipliers. For isentropic fluids, which we will again consider first, an appropriate Lagrangian is [22]

$$L = -\rho + \varphi\big(nU^\alpha\big)_{,\alpha} + \lambda(U_\alpha U^\alpha + 1) + \mu_l U^\alpha X^l_{,\alpha}, \tag{5.2}$$

where ρ is interpreted as in (2.4), $X = (X^1, X^2, X^3)$ is a particle marker (again 'Lagrangian') variable, $\varphi, \lambda, \mu_1, \mu_2, \mu_3$ are multipliers, and the constraints

$$\begin{aligned}\big(nU^\alpha\big)_{,\alpha} &= 0,\\ U_\alpha U^\alpha + 1 &= 0,\\ U^\alpha X^l_{,\alpha} &= 0\end{aligned} \tag{5.3}$$

correspond to the conservation of particle number, the unitarity of the 4-velocity, and the preservation of particle identity[7], respectively. It turns out that this perspective readily covers the Korteweg case.

THEOREM 5.1. *For isentropic fluids, the relativistic Euler-Korteweg equations* (2.19) *derive from the Lagrangian* (5.2), *if one interprets* ρ *as in* (2.8).

Proof. We follow [22] very closely.[8] The variational equations associated with the multipliers,

$$0 = \frac{\delta L}{\delta \varphi},$$
$$0 = \frac{\delta L}{\delta \lambda},$$
$$0 = \frac{\delta L}{\delta \mu_l},$$

are equivalent to the constraints (5.3). The variational equations associated with the state variables read

$$\frac{\delta L}{\delta n} = -h - U^\alpha \varphi_{,\alpha} = 0, \tag{5.4}$$

$$\frac{\delta L}{\delta U^\alpha} = 2\lambda U_\alpha - n\varphi_{,\alpha} + \mu_l X^l_{,\alpha} = 0 \tag{5.5}$$

[7]Without this constraint, attention would be tacitly restricted to irrotational flows. The idea to use it goes back to Lin [20].

[8]For the reader's convenience, we repeat Ray's argumentation adjusted to our context, thus also suppressing those ingredients which pertain to the general relativity setting, i.e., variations of the space-time metric. Note, however, that our application of Ray's approach readily extends to the 'Euler-Korteweg-Einstein equations', i.e., the Einstein equations with the relativistic Korteweg tensor(2.12) as source.

and

$$\frac{\delta L}{\delta X^l} = \big(\mu_l U^\beta\big)_{,\beta} = 0.$$

Equations (5.4), (5.5) imply

$$2\lambda = nh,$$
$$hU_\alpha = \varphi_{,\alpha} - \frac{\mu_l}{n} X^l_{,\alpha}.$$

Now, differentiating (5.4) and using (5.5), we find

$$\begin{aligned} -h_{,\alpha} &= (U^\beta \varphi_{,\beta})_{,\alpha} \\ &= U^\beta_{,\alpha}\varphi_{,\beta} + U^\beta \varphi_{,\alpha\beta} \\ &= U^\beta_{,\alpha}\big(hU_\beta + \frac{\mu_l}{n} X^l_{,\beta}\big) + U^\beta\big(hU_\alpha + \frac{\mu_l}{n} X^l_{,\alpha}\big)_{,\beta} \\ &= U^\beta (hU_\alpha)_{,\beta} + \frac{\mu_l}{n}(U^\beta X^l_{,\beta})_{,\alpha} + U^\beta \left(\frac{\mu_l}{n}\right)_{,\beta} X^l_{,\alpha} \\ &= U^\beta (hU_\alpha)_{,\beta}, \end{aligned} \tag{5.6}$$

the latter due to (5.3), in particular as

$$nU^\beta \left(\frac{\mu_l}{n}\right)_{,\beta} = \big(\mu_l U^\beta\big)_{,\beta} = 0.$$

We have shown (2.6). □ Turning to non-isentropic fluids, one modifies the Lagrangian as

$$L = -\rho + \varphi\big(nU^\alpha\big)_{,\alpha} + \lambda(U_\alpha U^\alpha + 1) + \mu_l U^\alpha X^l_{,\alpha} + \sigma U^\alpha s_{,\alpha}, \tag{5.7}$$

where $X = (X^1, X^2, X^3), \varphi, \lambda, \mu = (\mu_1, \mu_2, \mu_3)$ denote the same Lagrange multipliers as in Section 4, and σ is an additional Lagrange multiplier corresponding to the constraint

$$U^\alpha s_{,\alpha} = 0$$

which states that entropy remains constant along particle paths.

Theorem 5.2. *For non-isentropic fluids, the relativistic Euler-Korteweg equations laws derive from the Lagrangian* (5.7), *if one interprets* ρ *as in* (3.2).

Proof. We again follow [22]. The variational equations are now

$$\begin{aligned}
0 &= \frac{\delta L}{\delta n} = -h - U^\alpha \phi_{,\alpha} \\
0 &= \frac{\delta L}{\delta U^\alpha} = 2\lambda U_\alpha - n\varphi_{,\alpha} + \mu_l X^l_{,\alpha} + \sigma s_{,\alpha} \\
0 &= \frac{\delta L}{\delta s} = -n\theta - (\sigma U^\alpha)_{,\alpha} \\
0 &= \frac{\delta L}{\delta X^l} = (\mu_l U^\alpha)_{,\alpha} \\
0 &= \frac{\delta L}{\delta \phi} = (nU^\alpha)_{,\alpha} \\
0 &= \frac{\delta L}{\delta \lambda} = U^\alpha U_\alpha + 1 \\
0 &= \frac{\delta L}{\delta \mu_l} = U^\alpha X^l_{,\alpha} \\
0 &= \frac{\delta L}{\delta \sigma} = U^\alpha s_{,\alpha}.
\end{aligned}$$

Computing as in (5.6), we now find

$$\begin{aligned}
-h_{,\alpha} &= U^\beta (hU_\alpha)_{,\beta} + U^\beta_{,\alpha}\frac{\sigma}{n}s_{,\beta} + U^\beta(\frac{\sigma}{n}s_{,\alpha})_{,\beta} \\
&= U^\beta (hU_\alpha)_{,\beta} - \theta s_{,\alpha},
\end{aligned}$$

the latter as

$$0 = (U^\beta \frac{\sigma}{n} s_{,\beta})_{,\alpha} = U^\beta_{,\alpha}\frac{\sigma}{n}s_{,\beta} + U^\beta \frac{\sigma}{n} s_{,\alpha\beta}$$

implies

$$U^\beta_{,\alpha}\frac{\sigma}{n}s_{,\beta} + U^\beta(\frac{\sigma}{n}s_{,\alpha})_{,\beta} = U^\beta(\frac{\sigma}{n})_{,\beta}s_{,\alpha} = \frac{1}{n}(\sigma U^\beta)_{,\beta}s_{,\alpha} = -\theta s_{,\alpha}.$$

We have shown (3.5). ∎

6. The Lagrangian in the classical limit. The $c \to \infty$ limit also confirms the natural Lagrangian in the classical case, which is, first considering isentropic fluids,

$$\hat{L} = -\hat{\rho} + \frac{1}{2}nmv^2 + \hat{\varphi}\big((nm)_t + \nabla\cdot(nm\mathbf{v})\big) + \hat{\mu}_l\big(\partial_t + \mathbf{v}\cdot\nabla\big)X^l. \tag{6.1}$$

where $\hat{\rho}$ is the classical piece of energy, i.e.,

$$\rho = nmc^2 + \hat{\rho} + O\left(\frac{1}{c^2}\right) \quad \text{with} \quad \hat{\rho} = n\hat{\epsilon}(n, |\nabla n|^2/2),$$

and $\hat{\phi}, \hat{\mu}_l$ are again Lagrange multipliers.

THEOREM 6.1. *Interpreting the original Lagrange multipliers as*

$$\varphi = mc(\hat{\varphi} + c^2 t), \quad \lambda = c^2\left(\frac{1}{2}nm + O\left(\frac{1}{c^2}\right)\right), \quad \mu_l = c\hat{\mu}_l, \tag{6.2}$$

the variational derivatives of the Lagrangian L converge in the limit $c \to \infty$ as

$$\frac{\delta L}{\delta n} = \frac{\delta \hat{L}}{\delta n} + O\left(\frac{1}{c^2}\right) \tag{6.3}$$

and

$$\frac{1}{c}\frac{\delta L}{\delta U^j} == \frac{\delta \hat{L}}{\delta v^j} + O\left(\frac{1}{c^2}\right), \quad j = 1, 2, 3, \tag{6.4}$$

with $\hat{L}$ as in (6.1).

Proof. Using (4.2) and (6.2), one finds

$$\begin{aligned} \frac{\delta L}{\delta n} &= -h - \left(\frac{1}{c}((1 + \frac{v^2}{2c^2})\partial_t + \mathbf{v}\cdot\nabla)\right)\varphi + O\left(\frac{1}{c^5}\right) \\ &= -h + mc^2 + \frac{1}{2}mv^2 - m\Big(\partial_t + \mathbf{v}\cdot\nabla\Big)\hat{\varphi} + O\left(\frac{1}{c^2}\right) \\ &= -\hat{h} + \frac{1}{2}mv^2 - m\Big(\partial_t + \mathbf{v}\cdot\nabla\Big)\hat{\varphi} + O\left(\frac{1}{c^2}\right) \\ &= \frac{\delta \hat{L}}{\delta n} + O\left(\frac{1}{c^2}\right) \end{aligned} \tag{6.5}$$

and

$$\frac{1}{c}\frac{\delta L}{\delta U^j} = nm\lambda v_j - n\hat{\varphi}_{,j} + \hat{\mu}_l X^l_{,j} + O\left(\frac{1}{c^2}\right) = \frac{\delta \hat{L}}{\delta v^j} + O\left(\frac{1}{c^2}\right), \tag{6.6}$$

where $\hat{h}$ denotes the classical piece of enthalpy, i.e., we interpret

$$h = mc^2 + \hat{h} + O\left(\frac{1}{c^2}\right)$$

with

$$\hat{h} = \left(\check{\epsilon} + n^2\frac{\partial\check{\epsilon}}{\partial n}\right)(n, |\nabla n|^2/2) - \nabla\cdot(\hat{\kappa}\nabla n).$$

□

For non-isentropic fluids, one uses the natural modification

$$\begin{aligned} \hat{L} = -\hat{\rho} + \frac{1}{2}nmv^2 + \hat{\varphi}\big((nm)_t + \nabla\cdot(nm\mathbf{v})\big) \\ + \hat{\mu}_l\big(\partial_t + \mathbf{v}\cdot\nabla\big)X^l + \hat{\sigma}\big(\partial_t + \mathbf{v}\cdot\nabla\big)s \end{aligned} \tag{6.7}$$

of (6.1), where

$$\hat{\rho} = n\hat{\epsilon}(n, |\nabla n|^2/2, s),$$

is again the non-relativistic piece of $\rho = nmc^2 + \hat{\rho} + O(1/c^2)$ and $\hat{\sigma}$ a Lagrange multiplier corresponding to the constraint

$$\big(\partial_t + \mathbf{v}\cdot\nabla\big)s = 0.$$

Theorem 6.2. *With* $\hat{L}$ *now as in* (6.7)*, the variational derivatives of* L *converge in the limit* $c \to \infty$ *as* (6.3),(6.4) *and*

$$\frac{\delta L}{\delta s} = \frac{\delta \hat{L}}{\delta s} + O\left(\frac{1}{c^2}\right).$$

The proof is very similar to that of Theorem 6.1.

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