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ANALYTIC INVARIANTS OF MULTIPLE POINTS*

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In memory of John Mather

Abstract. We develop an original approach in computing analytic invariants of zero-dimensional singularities, which is based essentially on the study of properties of differential forms and the cotangent complex of multiple points. Among other things, we consider a series of specific tasks and problems for zero-dimensional complete intersections, graded and gradient singularities, including the computation of cotangent homology and cohomology for certain types of such singularities. We also examine the unimodular families of gradient zero-dimensional singularities, compile an adjacency diagram and compute the primitive ideals of these families. Finally, we briefly discuss the problem of nonexistence of negative weighted derivations, some relationships between the Milnor and Tjurina numbers and estimates of these invariants in the case of zero-dimensional complete intersections.

Key words. multiple points, fat points, thick points, differential forms, derivations, deformations, cotangent complex, duality, complete intersections, gradient singularities

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Introduction. Zero-dimensional singularities, often called multiple, fat or thick points, naturally arise in almost all branches of mathematics. Quite a number of works are devoted to the study of properties of such singularities and computation of their invariants with the use of various approaches and methods. Apparently, one of the first studies in this direction goes back to the pioneering works of H. Poincaré, where he has introduced the notion of the index of a singular point of a vector field on the plane and proved the existence of versal deformations for zero-dimensional complete intersections. Then F. Macaulay has discovered a nondegenerate duality on Artinian complete intersections and described a series of remarkable applications in algebra and geometry. In the late 1960s, A. Grothendieck and his followers and successors made great progress towards understanding these ideas within the framework of the general theory of residue and duality.

Most of the current studies are based on considering combinatorial and numerical characteristics of zero-dimensional subschemes in projective spaces, i.e., *homogeneous* singularities, and the corresponding Artinian algebras, as well as on related computational algorithms adapted to work on a computer. On the other hand, deformation theory provides zero-dimensional schemes with a series of natural analytic and algebraic invariants, such as the modules of differential forms, the Poincaré-de Rham complex, the cotangent homology and cohomology, etc. Moreover, quite unexpected analogies can be used effectively in the study of deformation theory of such singularities. For instance, it is obvious that zero-dimensional schemes are, at the same time, *compact* complex spaces. Among other things, this observation leads to a description of a natural duality in the cotangent homology and cohomology of zero-dimensional singularities (see [5]). In a more general context, similar aspects of the theory are considered also in the recent paper [7].

The present article is devoted to the further development of this approach based on the systematic use of the theory of differential forms and constructions from the

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general deformation theory of varieties and complex spaces for investigating fundamental properties of zero-dimensional singularities.

In the first section we describe the starting point and discuss the basic properties of the Poincaré-de Rham complex on zero-dimensional singularities including the behavior of this complex under flat deformations, a generalization of the classical Poincaré lemma, and some related topics. In the next three sections we recall some useful properties of the cotangent homology and cohomology, the modules of relative differentials and one-dimensional singularities, which will be useful in computing the basic analytic invariants of multiple points. In the fifth section we describe some properties of differential forms and vector fields on zero-dimensional singularities, which have no analogues in the theory of singularities of positive dimension. The sections 6, 7 and 8 are devoted to solving specific tasks and problems for zero-dimensional complete intersections, graded and gradient singularities, including the computation of cotangent homology and cohomology for certain types of such singularities. Then we examine the unimodular families of gradient zero-dimensional singularities, compile an adjacency diagram and compute the primitive ideals of these families. Finally, we briefly discuss some generalizations and applications of the obtained results, including the problem of nonexistence of negative weighted derivations, some relationships between the Milnor and Tjurina numbers and estimates of these invariants in the case of zero-dimensional complete intersections, etc.

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1. Poincaré lemma and deformations. In what follows we assume that the ground field k is the field $\mathbb{C}$ of complex numbers, although many results obtained in this paper can be extended with minor changes to the case of arbitrary *perfect* fields.

Let A be a local analytic k-algebra with maximal ideal $\mathfrak{m}_A$ of Krull dimension $n \geqslant 0$. In particular, it is finitely generated, and the ground field is contained in A and isomorphic to the residue field, i.e. $\mathsf{k} \subset A$ and $\mathsf{k} \cong A/\mathfrak{m}_A$. For the sake of simplicity, we will denote a Noetherian local ring of Krull dimension *zero* by A_0. In other words, A_0 is an Artinian ring. Since $\mathfrak{m}_{A_0}^N = 0$ for sufficiently large N, the algebra A_0 is a *complete* local ring.

Let m equal the embedding dimension of the local analytic k-algebra A, i.e., $m = \dim_{\mathsf{k}} \mathfrak{m}_A/\mathfrak{m}_A^2$, and let P be the ring $\mathsf{k}\langle z_1, \ldots, z_m\rangle$ of convergent power series in m variables. Then $A \cong P/I$, where I is the ideal of the ring P generated by a suitable sequence $(f_1, \ldots, f_k)$ of elements of P. Any such k-algebra A can be regarded as the structure local algebra of a germ X of a complex space or variety. Throughout this paper we will refer to any such algebra, as well as to the corresponding germ, as a *singularity* (even if the distinguished point is nonsingular).

The module of Kähler differentials $\Omega^1_{A/\mathsf{k}}$ can be represented in the form of quotient module as follows:

$$\Omega^1_{A/\mathsf{k}} \cong \Omega^1_P/\big((f_1, \ldots, f_k)\Omega^1_P + Pdf_1 + \ldots + Pdf_k\big).$$

To simplify notation, we will often denote the module $\Omega^1_{A/\mathsf{k}}$ of Kähler differential by Ω^1_A. Similar abbreviations are also used for related objects. For all $p \geqslant 0$, we define the A-modules $\Omega^p_A \cong \wedge^p \Omega^1_A$ of regular differential p-forms; they can be represented in the form of quotient modules as follows:

$$\Omega_A^p \cong \Omega_P^p/\big((f_1,\dots,f_k)\Omega_P^p + df_1\wedge\Omega_P^{p-1}+\dots+df_k\wedge\Omega_P^{p-1}\big), \quad p\geqslant 1,$$

where $\Omega_A^0 \cong A$. Then $\Omega_A^p = 0$ for $p>m$, and there are isomorphisms $\Omega_A^p \cong \operatorname{Tors}\Omega_A^p$ for all $n<p\leqslant m$, i.e., the modules Ω_A^p are *torsion* modules. In particular, if the singularity A is *isolated*, then these modules are *finite-dimensional* vector spaces.

For any A-module M there is a canonical isomorphism $\operatorname{Hom}_A(\Omega_A^1, M) \cong \operatorname{Der}_{\mathsf{k}}(A,M)$, which is induced by the universal derivation $d_A\colon A\to\Omega_A^1$. The elements of the module $\operatorname{Der}_{\mathsf{k}}(A,A) := \operatorname{Der}_{\mathsf{k}}(A)$ of derivations are often called *vector fields* on the singularity A. The ordinary Lie bracket equips the module $\operatorname{Der}_{\mathsf{k}}(A)$ with the natural structure of a *Lie algebra*.

The usual differential $d=d_A$ endows the family $\{\Omega_A^p, p\geqslant 0\}$ with the structure of an *increasing* complex

$$0\longrightarrow A\xrightarrow{d}\Omega_A^1\xrightarrow{d^1}\dots\longrightarrow\Omega_A^{m-1}\xrightarrow{d^{m-1}}\Omega_A^m\longrightarrow 0, \tag{1}$$

where $d=d^0$ is the universal derivation of the k-algebra A; this complex is called the Poincaré, or the de Rham complex on A. We will denote its cohomology groups by $H^*_{DR}(\Omega_A^\bullet)$.

Recall that a quasihomogeneous (equivalently, graded) singularity A is determined by an ideal of P generated by *quasihomogeneous* functions $f_1,\dots,f_k$ of weighted degrees $d_1,\dots,d_k$ in weighted variables $z_1,\dots,z_m$ of weights $w_1,\dots,w_m$, $w_i\in\mathbb{Z}$, $i=1,\dots,m$. The set of weighted degrees and weights is called the type of the singularity A; it is denoted by $\pi(A)=(d_1,\dots,d_k;w_1,\dots,w_m)$. In particular, the ring A is a *graded* k-algebra which can be represented as a direct sum of homogeneous components of degree ν, i.e., $A=\oplus_{\nu\in\mathbb{Z}}A_{(\nu)}$. In this case, the modules of differential forms, the de Rham complex, its cohomology, the Lie algebra of derivations become, in the induced grading, graded A-modules, k-algebras, vector spaces and so on. For each graded A-module $M=\oplus_{\nu\in\mathbb{Z}}M_{(\nu)}$ with finite-dimensional homogeneous components, the Poincaré series is defined as follows:

$$\mathscr{P}(M;t)=\textstyle\sum_{\nu\in\mathbb{Z}}\dim_{\mathsf{k}} M_{(\nu)}.$$

One of the main results of H.-J. Reiffen [26] in the theory of contractible varieties is formulated as follows.

Proposition 1.1. *Let A be a graded singularity, i.e., a singularity whose defining ideal is generated by a sequence of quasihomogeneous functions in variables $z_1,\dots,z_m$ of weights $w_1,\dots,w_m$, respectively. Suppose that $w_i\geqslant 0$ for $i=1,\dots,m$ and $w_i>0$ for all $1\leqslant i\leqslant \ell$, where $\ell\leqslant m$. Let $\bar A = A/(z_1,\dots,z_\ell)A$. Then the natural k-linear map $H^*_{DR}(\Omega_A^\bullet)\to H^*_{DR}(\Omega_{\bar A}^\bullet)$ is bijective.*

Proof. See [27, Satz 4.6]. □

Corollary 1.2. *Under the conditions of Proposition* 1.1, *assume that $\ell=m$. Then the complex* (1) *is acyclic in positive dimensions, and $H^0_{DR}(\Omega_A^\bullet)\cong\mathsf{k}$. In other words, for such singularities, the classical Poincaré lemma holds.*

Proof. Indeed, in this case, the singularity with structure algebra $\bar A$ is an ordinary point, $\bar A\cong\mathsf{k}$, and the corresponding complex (1) is trivial: $0\to\mathsf{k}\to 0$. □

Remark 1.3. Singularities satisfying the conditions of Corollary 1.2 are often said to be *weighted homogeneous*; they are determined by quasihomogeneous polynomials in variables of positive weight, i.e., such that $w_i>0$ for all $i=1,\dots,m$.

Example 1.4. Let us consider the zero-dimensional singularity A_0 given by the triple of polynomials (x^2+y^2+xyz, y^3, z^2) in $\mathsf{k}\langle x,y,z\rangle$ with variables of weights $1, 1$ and 0, respectively. In this case, the conditions of Corollary 1.2 are not satisfied, since $\ell = 2$ and $m = 3$. Nevertheless, Proposition 1.1 implies that the complex (1) is acyclic in positive dimensions and $H^0_{DR}(\Omega^\bullet_{A_0}) \cong \mathsf{k}$, since $\bar{A} \cong \mathsf{k}\langle z\rangle/(z^2)$.

Lemma 1.5. *Let A_0 be a zero-dimensional singularity. Then, for all $p \geqslant 0$, the modules $\Omega^p_{A_0}$ are finite-dimensional vector spaces, and the Euler characteristic of the Poincaré-de Rham complex* (1) *is calculated as follows:*

$$\chi(\Omega^\bullet_{A_0}) = \sum_{i=0}^{m} (-1)^i \dim_k H^i_{DR}(\Omega^\bullet_{A_0}) = \sum_{p=0}^{m} (-1)^p \dim_k \Omega^p_{A_0}. \tag{2}$$

Proof. Finite dimensionality follows from the fact that the support of $\Omega^p_{A_0}$ is concentrated at the distinguished point, that is, $\Omega^p_{A_0} = \operatorname{Tors}(\Omega^p_{A_0})$ is a *torsion* module for all p. □

Remark 1.6. By definition, the cohomology group $H^0_{DR}(\Omega^\bullet_{A_0})$ is isomorphic to $\operatorname{Ker}(d)$, and also $\mathsf{k} \subseteq \operatorname{Ker}(d)$. If A_0 is a weighted homogeneous or quasihomogeneous singularity as in Example 1.4, then the complex (1) is acyclic in positive dimensions and $H^0_{DR}(\Omega^\bullet_{A_0}) \cong \mathsf{k}$, i.e., in these cases the Poincaré lemma holds. This also true for zero-dimensional singularities of other types, e.g., for the nonquasihomogeneous singularity given by the ideal $I = (x^2 + xy + y^3, xy^2)$.

Example 1.7. Let us consider the unimodal semiquasihomogeneous singularity given by the function $\varphi = x^4 + x^2y^3 + y^5$ of type W_{12} (see [16, §1]). Let A_0 be the zero-dimensional *gradient* singularity $\operatorname{grad}(\varphi)$ determined by the partial derivatives φ_x and φ_y of the function φ, and let $h = \operatorname{hess}(\varphi)$ be its Hessian which generates the one-dimensional *socle* of the Artinian algebra A_0, i.e., $h \in \operatorname{Soc}(A_0)$. It is not difficult to verify that $\varphi \equiv x^2y^3 \equiv h$ modulo the Jacobian ideal (φ_x, φ_y), so that $\varphi \neq 0$ in A_0 (see [33, Satz 3]). On the other hand, it is evident that $d\varphi = 0$ in $\Omega^1_{A_0}$, so that $\dim_{\mathsf{k}} \operatorname{Ker}(d) = 2$, i.e., the classical Poincaré lemma does not hold. Further calculations show that $\dim_{\mathsf{k}} A_0 = 12$, $\dim_{\mathsf{k}} \Omega^1_{A_0} = 15$, and $\dim_{\mathsf{k}} \Omega^2_{A_0} = 5$. As a result, we obtain $\chi(\Omega^\bullet_{A_0}) = 2$.

It should be noted also that if one adds the function φ to the generators of the Jacobian ideal, then the Poincaré lemma holds for the obtained *determinantal* singularity A'_0. In particular, $H^0_{DR}(\Omega^\bullet_{A'_0}) \cong \operatorname{Ker}(d) \cong \mathsf{k}$, $\dim_{\mathsf{k}} A'_0 = 11$, and the spaces of differential forms of the first and the second order on A'_0 and A_0 are the same.

Theorem 1.8. *Let A_0 be a zero-dimensional singularity, and suppose that there is a flat deformation $\widetilde{A}_0$ with reduced base space $(T, \mathfrak{o})$. Then the following assertions hold.*

(a) *The functions $T \to \mathbb{Z}$ defined by*

$$t \mapsto \dim_k \Omega^p_{\widetilde{A}_{0_t}}, \ p \geqslant 0, \quad t \mapsto \dim_k \operatorname{Der}_k(\widetilde{A}_{0_t})$$

are upper semicontinuous on T.

(b) *The function $T \to \mathbb{Z}$ given by the rule*

$$t \mapsto \chi(\Omega^\bullet_{\widetilde{A}_{0_t}}) - \dim_k \operatorname{Ker}(d_{\widetilde{A}_{0_t}}) = \sum_{i=0}^{\infty} (-1)^i \dim_k H^i_{DR}(\Omega^\bullet_{\widetilde{A}_{0_t}}) - \dim_k \operatorname{Ker}(d_{\widetilde{A}_{0_t}}),$$

is locally constant in a suitable neighborhood of the distinguished point of the base space.

Proof. Let X_t denote the multigerm associated with a suitable representative of the fiber over $t \in T$ of the flat map, and let $\widetilde{A}_{0_t}$ denote the corresponding structure semilocal ring of X_t. Since $T^0(X_t) \cong \mathrm{Der}_{\mathsf{k}}(X_t)$, then the upper semicontinuity of the second function in (a) follows from general results on the behavior of the cotangent cohomology under flat proper maps of complex spaces (see [24, Theorem 4.4.I.]). Since $T_0(X_t) \cong \Omega^1_{X_t}$, similar arguments can be applied to the first function in (a) for $p = 1$. This yields the required statement for $p > 1$, because $\Omega^p_{X_t}$ is an exterior power of the module $\Omega^1_{X_t}$, which is a finite-dimensional vector space as well.

Furthermore, it is not difficult to see that, for lower bounded complex $(L^\bullet, \partial)$ of finite-dimensional vector spaces, where $L^p = 0$ for $p < 0$, the even partial Euler characteristic

$$\sum_{i=0}^{2\ell} (-1)^i \dim_{\mathsf{k}} H^{q+i}(L^\bullet, \partial)$$

is upper semicontinuous under flat proper mappings for any $q, \ell \geqslant 0$ (cf. [24, Lemma 4.3]). Let us consider the *canonical truncation*

$$0 \longrightarrow A_0/\mathrm{Ker}(d) \xrightarrow{d} \Omega^1_{A_0} \xrightarrow{d^1} \ldots \longrightarrow \Omega^{m-1}_{A_0} \xrightarrow{d^{m-1}} \Omega^m_{A_0} \longrightarrow 0$$

of the complex (1) and denote it by $(L^\bullet, d)$. Then $H^0(L^\bullet) = 0$ and $H^i(L^\bullet) = H^i(\Omega^\bullet_{A_0})$ for all $i > 0$. In particular, $H^i(L^\bullet) = 0$ for all $i > m$, and one can prove that $\chi(L^\bullet, d)$ is locally constant by using arguments similar to those in [24, Introduction, §10 and Corollary 4.2.I.]. It remains to note that

$$\chi(L^\bullet, d) = -\dim_{\mathsf{k}} \mathrm{Ker}(d) + \sum_{p=0}^{m} (-1)^p \dim_{\mathsf{k}} \Omega^p_{A_0} = -\dim_{\mathsf{k}} \mathrm{Ker}(d) + \chi(\Omega^\bullet_{A_0})$$

in view of Lemma 1.5. □

Of course, both statements are closely related to general results about the behavior of cohomology and the Euler characteristic of coherent sheaves given on a flat family of varieties X_t, parameterized by the points of the base space T (see, e.g., [22, Ch.II, §5]).

Corollary 1.9. *The dimensions* $\dim_k \Omega^p_{A_0}$, $p \geqslant 0$, *of the modules of differential* p*-forms* $\Omega^p_{A_0}$, *as well as the dimension* $\dim_k \mathrm{Der}_k(A_0)$ *of the Lie algebra of derivations* $\mathrm{Der}_k(A_0)$ *of a zero-dimensional singularity* A_0 *are analytic invariants, and they are upper semicontinuous under deformations.*

Remark 1.10. In a similar manner, one can prove that the partial *even* Euler characteristic, i.e. the function

$$t \mapsto \sum_{i=0}^{2\ell} (-1)^i \dim_{\mathsf{k}} H^{n+i}_{DR}(\Omega^\bullet_{\widetilde{A}_{0_t}}),$$

is upper semicontinuous on T for any $n, \ell \geqslant 0$.

Corollary 1.11. *If there exists a smooth deformation of a zero-dimensional singularity (whose generic fiber consists of a set of ordinary points) over a reduced and connected base space, then*

$$\chi(\Omega^{\bullet}_{A_0}) - \dim_k \operatorname{Ker}(d) = \sum_{p=0}^{m} (-1)^p \dim_k \Omega^p_{A_0} - \dim_k \operatorname{Ker}(d) = 0 \tag{3}$$

and, consequently,

$$\dim_k \operatorname{Ker}(d) = \sum_{p=0}^{m} (-1)^p \dim_k \Omega^p_{A_0}.$$

In particular, the invariant $\dim_k \operatorname{Ker}(d)$ *is upper semicontinuous under deformations.*

Proof. The generic fiber of a smooth deformation consists of ordinary points. □

Remark 1.12. In the condition of Corollary 1.11 one can replace a set of ordinary points by a set of multiple points of the form $\mathsf{k}\langle x\rangle/(x^n)$ or of some other types for which the structure of the Poincaré-de Rham complex is known. Then, summing their contributions to the Euler characteristic of the deformed singularity, one can obtain various analogues of formula (3). Here we do not touch upon the quite nontrivial problem of the existence of such deformations for zero-dimensional singularities, because this requires other techniques (cf., e.g., [11]). However, it is easy to prove that, for *complete intersections* (for gradient singularities of functions with isolated critical points, for determinantal singularities) smooth deformations with reduced (and even with smooth base spaces) always exist. In these cases, using formula (3), one can verify the computation of the modules $\Omega^p_{A_0}$ of differential p-forms and other related objects and invariants (cf. Example 1.7).

Remark 1.13. It is not difficult to see that, if A is *reduced*, then $H^0_{DR}(\Omega^{\bullet}_A) \cong \mathsf{k}$. However, zero-dimensional singularities are nonreduced (excluding the trivial case of ordinary points), and the finite-dimensional vector space $H^0_{DR}(\Omega^{\bullet}_A)$ has a nice interpretation within the framework of the theory of J.Mather (see [21, Proposition 7.4]). More precisely, if A_0 is a zero-dimensional *complete intersection*, then the subalgebra $\operatorname{Ker}(d_{A_0}) \cap \mathfrak{m}_{A_0}$ of A_0 is a finite-dimensional vector space; it can be naturally identified with a subspace of the tangent space to the $\mathscr{K}$-orbit of a map $f\colon \mathbb{C}^m \to \mathbb{C}^m$, which determines A_0 (cf. [30, §4, 4.3]). Of course, such an orbit is always smooth; its dimension is usually called the contact *modality* of the singularity and is denoted by $m(A_0)$. It is therefore concluded that $\dim_{\mathsf{k}} \operatorname{Ker}(d_{A_0}) \leqslant m(A_0) + 1$ for the complete intersection A_0. Furthermore, taking into account Example 1.7, one can show that $\dim_{\mathsf{k}} \operatorname{Ker}(d_{\operatorname{grad}(\varphi)}) = m(\varphi) + 1$ for *gradient semiquasihomogenous* singularities; this equality can be regarded as a refined version of the above relation.

Remark 1.14. For the most part the above results have *graded* analogs in the quasihomogenous case (see Section 7 below).

2. The cotangent complex. Following [19] or [24], we denote the lower and upper cotangent functors by T_i and T^i, $i \geqslant 0$, respectively. For convenience, we recall the usual definition for $i = 0, 1$. In the notations of Section 1, for any quotient k-algebra $A = P/I$ there is an exact sequence of A-modules

$$0 \to \mathrm{T}_1(A) \to I/I^2 \xrightarrow{D} \Omega^1_P \otimes_P A \to \Omega^1_A \to 0, \tag{4}$$

where $D(\bar{f}) = d(f) \otimes_P 1$ for an element $f \in I$ and its class $\bar{f}$ in the conormal module I/I^2. Applying the functor $\mathrm{Hom}_A(-, A)$, we get the dual exact sequence

$$0 \to \mathrm{T}^0(A) \to \mathrm{Hom}_A(\Omega^1_P \otimes_P A, A) \xrightarrow{D^*} \mathrm{Hom}_A(I/I^2, A) \to T^1(A) \to 0. \tag{5}$$

In particular, there are natural isomorphisms:

$$T_0(A) \cong \Omega^1_A, \quad T^0(A) \cong \mathrm{Der}_{\mathsf{k}}(A), \quad T_1(A) \cong \textstyle\int I/I^2,$$

where $\int I$ is the *primitive* ideal of I. By definition, the primitive ideal consists of all elements $g \in I$ such that $\partial g \in I$ for all derivations $\partial \in \mathrm{Der}_{\mathsf{k}}(P)$ (see details in [5] and [7]).

Since for any ideal $I \subset A$ the I-adic completion $\widehat{A}$ is *flat* over A, the homology and cohomology functors are *compatible* with the completion (see [19, 2.3.3]). In particular, the dimensions of lower and upper cotangent modules are *analytic* invariants of the *local* algebra A. If A is an *isolated* singularity then these modules are *finite-dimensional* vector spaces over k for all $i > 0$. Moreover, if $A = A_0$ is a *multiple* point then this is true for all $i \geqslant 0$, and we will use the following notation:

$$\mathfrak{t}_i(A_0) = \dim_{\mathsf{k}} T_i(A_0), \quad \mathfrak{t}^i(A_0) = \dim_{\mathsf{k}} T^i(A_0), \quad i \geqslant 0.$$

THEOREM 2.1. *In the notation of Theorem* 1.8*, for all $i \geqslant 0$, the invariants $\mathfrak{t}^i(A_0)$ are upper semicontinuous under deformations. Moreover, if $\mathfrak{t}^j(A_0) = 0$ and $\mathfrak{t}^j((\widetilde{A}_0)_t) = 0$ for all $j > N$ in a small neighborhood of the distinguished point of a deformation with connected base space T, then the total sum $\sum(-1)^i \mathfrak{t}^i(A_0)$ is locally constant on T. Similar assertions are also true for the invariants $\mathfrak{t}_i(A_0)$.*

Proof. Indeed, for the upper cotangent functors, this is a basic statement in the theory of deformations of compact complex spaces (see [24, Theorem 4.3, Corollary 4.3]). The case of lower cotangent functors is analyzed in a similar manner. □

Next, it is well-known that, for any *reduced* Cohen-Macaulay ring A, there is an isomorphism

$$\mathrm{T}^1(A) \cong \mathrm{Ext}^1_A(\Omega^1_A, A).$$

In fact, this is also true for any singularity having *positive* depth along its singular locus. Moreover, if the local k-algebra A corresponds to an *isolated* singularity of dimension $n \geqslant 1$, then Grothendieck's local duality yields isomorphisms

$$\mathrm{Ext}^n_A(\Omega^1_A, \omega_A) \cong H^0_{\mathfrak{m}}(\Omega^1_A) \cong \mathrm{Tors}\,(\Omega^1_A),$$

where we denote the dualizing module of the singularity by ω_A.

For example, if A is a reduced Gorenstein singularity of dimension *one*, there is a natural isomorphism

$$\mathrm{T}^1(A) \cong \mathrm{Tors}\,(\Omega^1_A), \tag{6}$$

since $\omega_A \cong A$ and $n = 1$.

In particular, if A is Gorenstein and $\mathbb{Z}$-*graded*, then $\omega_A \cong A[e]$ for some $e \in \mathbb{Z}$, and the corresponding Poincaré polynomials are related as follows:

$$\mathscr{P}(\mathrm{T}^1(A); t) = t^{-e} \mathscr{P}(\mathrm{Tors}\,(\Omega^1_A); t). \tag{7}$$

In general, the above identifications with the extension modules do not exist for multiple points, because such germs are *nonreduced*, except for the trivial case of ordinary points. On the other hand, any zero-dimensional singularity over the field of complex numbers is a *compact complex space*. That is why, some useful properties of the cotangent complex and deformation theory of zero-dimensional singularities are inherited from the theory of compact complex spaces (see [5]). For instance, a general canonical duality in the tangent cohomology of compact complex spaces induces the following useful property of zero-dimensional singularities.

THEOREM 2.2. *Let A_0 be a zero-dimensional Gorenstein singularity. Then, for all $i \geqslant 0$, there are natural perfect pairings of finite-dimensional vector spaces*

$$T_i(A_0) \times T^i(A_0) \longrightarrow \mathsf{k}.$$

REMARK 2.3. The result of Theorem 2.2 is partially based on the well-known Dieudonné assertion [14, 1.43], which states that A_0 is a *self-injective* local ring. In particular, this yields a canonical isomorphism $\operatorname{Hom}_{A_0}(M, A_0) \cong \operatorname{Hom}_{\mathsf{k}}(M, \mathsf{k})$ for any A_0-module M. As a result, the exact sequence (5) takes the following form:

$$0 \to \operatorname{Hom}_{\mathsf{k}}(\Omega^1_{A_0}, \mathsf{k}) \to \operatorname{Hom}_{\mathsf{k}}(\Omega^1_P \otimes_P A_0, \mathsf{k}) \xrightarrow{D^*} \operatorname{Hom}_{\mathsf{k}}(I/I^2, \mathsf{k}) \to T^1(A_0) \to 0, \quad (8)$$

where all terms are finite-dimensional vector spaces.

3. Relative differentials. In the notation of Section 1, let $F = (f_1, \dots, f_k)$ be a sequence of elements in P. Then, for all $p \geqslant 1$, the quotient modules

$$\Omega^p_F = \Omega^p_P / \big(df_1 \wedge \Omega^{p-1}_P + \dots + df_k \wedge \Omega^{p-1}_P\big)$$

are defined. As before, the differential d is extended to the family $\{\Omega^p_F\}$, and the complex $(\Omega^\bullet_F, d)$ is also defined. If $A \cong P/(f_1, \dots, f_k)P$ and $f \in A$, then we denote the quotient module $\Omega^p_F / \big((f_1, \dots, f_k)\Omega^p_F + df \wedge \Omega^{p-1}_F\big)$ by Ω^p_f, so that

$$\Omega^p_f \cong \Omega^p_P / \big((f_1, \dots, f_k)\Omega^p_P + df_1 \wedge \Omega^{p-1}_P + \ldots + df_k \wedge \Omega^{p-1}_P + df \wedge \Omega^{p-1}_P\big);$$

this module is usually called the module of *relative* regular differential forms on A of order p, $p \geqslant 1$. It is convenient to set $\Omega^0_f = A \cong \Omega^0_A$, so that $\Omega^p_f \cong \Omega^p_A / \Omega^{p-1}_A \wedge df$. The differential d endows the family $\{\Omega^p_f\}$ with a structure of an increasing complex, which is denoted by $(\Omega^\bullet_f, d)$.

As follows from definitions, the modules Ω^p_F and Ω^p_f generally *depend* on the choice of the sequence $f_1, \dots, f_k$ and f. However, the quotient module $\Omega^p_f / f\Omega^p_f$ is an invariant of the corresponding singularity, not depending on the choice of generators of the defining ideal.

ASSERTION 3.1. *Let $\bar{A} = A/(f)A$. Then, for any $p \geqslant 0$, there is an exact sequence of A-modules*

$$0 \longrightarrow K^p_f \longrightarrow \Omega^p_f \xrightarrow{\cdot f} \Omega^p_f \longrightarrow \Omega^p_{\bar{A}} \longrightarrow 0, \quad (9)$$

where the middle map is multiplication by f and K^p_f denotes the kernel of this map. In particular, the module $\Omega^p_{\bar{A}}$ is isomorphic to the quotient $\Omega^p_f / f\Omega^p_f$.

Proof. This follows from definitions. ∎

Corollary 3.2. *If* $f \in \mathrm{Ann}_A(\Omega^p_f)$*, then* $\Omega^p_f \cong \Omega^p_{\bar{A}}$*.*

Assertion 3.3. *Let* $A = P/I$*, where* $I \subset P$ *is the ideal generated by a sequence* $(f_1, \dots, f_k)$*,* $f \in A$*, and* $\bar{A} = A/(f)A$*. Then there are natural isomorphisms of* A*-modules*

$$\Omega^1_{\bar{A}} \cong \bar{A}^m/\mathrm{Jac}^{\mathsf{T}}(F)\overline{N},$$
$$\Omega^1_f \cong \Omega^1_F \otimes_P A \cong A^m/\mathrm{Jac}^{\mathsf{T}}(F)N,$$

where $\mathrm{Jac}^{\mathsf{T}}(F)$ *denotes the transpose of the Jacobian matrix associated with the sequence* $F = (f_1, \dots, f_k, f)$*, which consists of* $k+1$ *rows and* m *columns, and* N*,* $\overline{N}$ *are the conormal modules associated with the ideal* I *and* (I, f)*, respectively.*

Proof. We use the exact sequence (4) for the singularity $\bar{A}$ determined by the ideal $(I, f) \subset P$, taking into account the fact that the corresponding conormal $\bar{A}$-module is contained in $\bar{A}^{k+1}$, while the map D is given by the matrix $\mathrm{Jac}^{\mathsf{T}}(F)$. □

Remark 3.4. With the help of the maps $\wedge^p D$ one can construct similar isomorphisms for all $p \geqslant 2$.

4. One-dimensional singularities. Let A be a local Noetherian ring of Krull dimension *one*. We recall that if A is reduced, i.e., $A = A_{\mathrm{red}}$, then the corresponding singularity is Cohen-Macaulay, or, for short, CM. A similar statement is not true in dimensions greater than one. The following properties of one-dimensional singularities are well-known:

(*i*) A is CM $\Longleftrightarrow$ $\mathrm{Hom}_A(\mathsf{k}, A) = 0$ (equivalently, A contains a non-zero divisor);

(*ii*) A is Gorenstein $\Longleftrightarrow$ A is CM and $\mathrm{Ext}^1_A(\mathsf{k}, A) \cong \mathsf{k}$;

(*iii*) A is a complete intersection $\Longleftrightarrow$ $A \cong P/I$, where $P = \mathsf{k}\langle z_1, \dots, z_m\rangle$ and the ideal I is generated by a *regular* P-sequence $(f_1, \dots, f_{m-1})$.

Remark 4.1. We recall also that a complete intersection is Gorenstein; it is an *equidimensional* germ without embedded components, or, equivalently, the corresponding local ring has no embedded primes. More generally, a complete intersection A of positive dimension is *nonreduced*, that is, $A \neq A_{red}$, if and only if any system of generators of the ideal I contains at least one function having *multiple* factors.

Lemma 4.2. *Let* A *be a one-dimensional Cohen-Macaulay ring and let* $f \in A$ *be a regular element. Then there is an exact sequence*

$$0 \longrightarrow A \xrightarrow{\wedge df} \Omega^1_A \longrightarrow \Omega^1_f \longrightarrow 0. \tag{10}$$

Proof. It suffices to show that the exterior multiplication map $\wedge df \colon A \to \Omega^1_A$ is injective. Indeed, by definition, $df \in \Omega^1_A$ and $\mathrm{Ker}(\wedge df) \cong \mathrm{Ann}(df)$. Let us first suppose that A is a *domain*. Then $\mathrm{Ann}(df) \neq 0$ if and only if $df \in \mathrm{Tors}\,(\Omega^1_A)$. However, for Cohen-Macaulay singularities the total differential of a regular element is not a torsion differential. Hence, in this case, $\mathrm{Ker}(\wedge df) = 0$.

In the general case we denote the set of minimal primes of the local ring A by $\{\wp_1, \dots, \wp_k\}$. Let Q be the ring of fractions of A. Then Q is an Artinian ring, and there exists an isomorphism $\iota \colon Q \longrightarrow A_{\wp_1} \times \cdots \times A_{\wp_k}$ such that the following triangle

$$\begin{tikzcd} & Q \arrow[d, "\iota"] \\ A \arrow[ur, "i_Q"] \arrow[r, "i"'] & \amalg A_{\wp_i} \end{tikzcd}$$

is commutative, where i_Q and i are the canonical morphisms. It is evident that the local rings $A_{\wp_i}$ are domains. Therefore, $\operatorname{Ker}(\wedge df) \subseteq \bigcap \operatorname{Ker}(\wedge df_{\wp_i}) = 0$. □

EXAMPLE 4.3. Following [7, Example 7], let X be the germ of the union of m coordinate axes in $\mathbb{C}^m$, $m \geq 2$, with the dual analytic algebra $A = P/I$. Then the ideal I is generated by the binomials $z_i z_j$ for all $i < j$. The local analytic algebra A is equipped with a $\mathbb{Z}_+$-grading in which $\deg z_i = 1$ for all i, so that X is a *weighted homogeneous* singularity. It is not difficult to compute the Poincaré series of the module Ω^p_A for all $p \geqslant 0$. Indeed, $\dim_{\mathsf{k}} A_{(0)} = 1$ and $\dim_{\mathsf{k}} A_{(\nu)} = m$ for all $\nu \geqslant 1$, so that

$$\mathscr{P}(A;t) = (1 + (m-1)t)/(1-t),$$

while $\dim_{\mathsf{k}}(\Omega^1_A)_{(\nu)} = m$ for all $\nu \geqslant 1$, $\nu \neq 2$, $\dim_{\mathsf{k}}(\Omega^1_A)_{(2)}= m^2 - \binom{m}{2} = m(m+1)/2$, and all other homogeneous components are trivial. Thus,

$$\begin{aligned} &\mathscr{P}(\Omega^1_A;t) = mt(1 + qt - qt^2)/(1-t) && \text{if } m = 2q+1, \\ &\mathscr{P}(\Omega^1_A;t) = qt(2 + (m-1)t - (m-1)t^2)/(1-t) && \text{if } m = 2q. \end{aligned}$$

Next, taking $m = 3$ and $f = z_1 + z_2 + z_3$, we obtain

$$\mathscr{P}(\Omega^1_f;t) = \mathscr{P}(\Omega^1_A;t) - t\mathscr{P}(A;t) = \{3t(1+t-t^2) - t(1+2t)\}/(1-t) = 2t + 3t^2,$$

since $\wedge df(A) \cong A$ and $\mathscr{P}(\wedge df(A);t) = \mathscr{P}(A;t)$. As a result, the space Ω^1_f is *finite-dimensional*, $\dim_{\mathsf{k}} \Omega^1_f = 5$. In fact, a direct computation shows that the vector space Ω^1_f is generated by the differential forms: $dz_1, dz_2, z_1dz_1, z_2dz_2, z_3dz_3$. Of course, a similar property holds for any regular element $f \in A$ of degree $d \geqslant 1$ in the case of m variables, $m \geqslant 1$.

REMARK 4.4. In the graded case, one can compute the Poincaré series of the module Ω^1_f of relative differentials in many other situations. For example, let A be a graded one-dimensional singularity (not necessarily reduced) and let f be an element (not necessarily regular) of weighted degree $d \geqslant 1$. Then $\mathscr{P}(\Omega^1_f;t) = \mathscr{P}(\Omega^1_A;t) - t^d\mathscr{P}(\wedge df(A);t)$. Hence, it is enough to learn to compute the Poincaré series of the module Ω^1_A and its submodule Adf (cf. examples in [7]).

EXAMPLE 4.5. In the above notation, let $I = (z_1^2, z_1z_2)$. Then $A = P/I$ is a one-dimensional plane curve singularity with an embedded component at the origin. Therefore, A is not a CM-singularity. In this case, there are no regular elements in the local ring A, which consists of zero-divisors only. Nevertheless, taking $f = z_2^2$ and $p = 1$ in the sequence (11), one can readily verify that $\operatorname{Ker}(\wedge df) \neq 0$. Indeed, in the standard homogeneous grading, we find that $\mathscr{P}(A;t) = (1+t-t^2)/(1-t)$ and $\mathscr{P}(\Omega^1_A;t) = t(2-t^2)/(1-t)$. On the other hand, since $z_1df = 0$ in Ω^1_A, we have $\operatorname{Ker}(\wedge df) = z_1 \in A$ and $\mathscr{P}(\operatorname{Ker}(\wedge df);t) = t$. As result, $\mathscr{P}(\Omega^1_f;t) = \mathscr{P}(\Omega^1_A;t) - t^2\mathscr{P}(A;t) + t^2\mathscr{P}(\operatorname{Ker}(\wedge df);t) = t(2+t)$, i.e., $\dim_{\mathsf{k}} \Omega^1_f = 3$.

Remark 4.6. More generally, let X be a singularity of *positive* dimension. Then, by definition, for any $f \in \mathcal{O}_X$, there is an exact sequence

$$\Omega_f^{p-1} \xrightarrow{\wedge df} \Omega_X^p \longrightarrow \Omega_f^p \longrightarrow 0. \tag{11}$$

If X is a complete intersection with an isolated singularity and f has an isolated critical point on X, then this sequence is left-exact for all $0 \leqslant p \leqslant c$, where c is equal to the codimension of the singular locus $\operatorname{Sing} X$ in X (cf. [23, Lemma 1.1.4]).

For completeness, we also mention another useful application related to the notion of *(homological) index* of differential forms. Let A be a one-dimensional Cohen-Macaulay ring, let $\omega \in \Omega_A^1$ be a differential form with an isolated singularity at the distinguished point and $(\widehat{\Omega}_A^\bullet, \omega)$ the *truncated* contracted de Rham complex on A (see [8]). Then, by definition,

$$\operatorname{Ind}_{\mathrm{hom},\mathfrak{o}}(\omega) = -\chi(\widehat{\Omega}_A^\bullet, \omega) = -\dim_{\mathsf{k}} H^0(\widehat{\Omega}_A^\bullet, \omega) + \dim_{\mathsf{k}} H^1(\widehat{\Omega}_A^\bullet, \omega).$$

Proposition 4.7. *Let A be a one-dimensional Cohen-Macaulay ring, and let f be a regular element of A such that $\dim_k \Omega_f^1 < \infty$. Then*

$$\operatorname{Ind}_{\mathrm{hom},\mathfrak{o}}(df) = \dim_k \Omega_f^1.$$

Proof. This assertion follows immediately from Lemma 4.2. □

Corollary 4.8. *Let A be a one-dimensional isolated complete intersection singularity, $f \in A$ a regular element and $A_0 = A/(f)A$. Let Δ be the discriminant of the minimal versal deformation of A_0. Then*

$$\operatorname{Ind}_{\mathrm{hom},\mathfrak{o}}(df) = \mu(A) + \mu(A_0) = \operatorname{mult}_{\mathfrak{o}}(\Delta),$$

where $\operatorname{mult}_{\mathfrak{o}}$ *denotes the multiplicity of a function at the distinguished point.*

Proof. Indeed, in this case $\dim_{\mathsf{k}} \Omega_f^1 < \infty$ and A_0 is a zero-dimensional complete intersection, and we can apply [8, Corollary 10.8 & Remark 10.9]). □

5. Differentials and derivations. It is well-known that the module of Kähler differentials and the Lie algebra of derivations are basic analytic invariants of singularities of positive dimension. In the zero-dimensional case these objects have quite unusual specific properties.

Assertion 5.1. *Let A_0 be a local Artinian k-algebra. Then the module $\Omega_{A_0}^1$ of Kähler differentials is a torsion module, so that $\Omega_{A_0}^1 = \operatorname{Tors}(\Omega_{A_0}^1)$. In particular, this module is a finite-dimensional vector space over the residue filed $\mathsf{k} \cong A_0/\mathfrak{m}$, and there are the following inequalities*

$$\dim_k \Omega_{A_0}^1 \leqslant (\dim_k A_0)^m, \quad \dim_k \operatorname{Der}_k(A_0) \leqslant (\dim_k A_0)^m.$$

Proof. This follows immediately from the exact sequences (4), (5) and (8). □

Theorem 5.2. *Let A_0 and A_0' be the zero-dimensional singularities determined by ideals $I \subset I'$ of P, respectively. Then, for all $p \geqslant 0$, there are natural epimorphisms $\Omega_{A_0}^p \twoheadrightarrow \Omega_{A_0'}^p$, which induce the inequalities*

$$\dim_k \Omega_{A_0'}^p \leqslant \dim_k \Omega_{A_0}^p.$$

Proof. By assumptions, there is an exact sequence $0 \to I'/I \to A_0 \to A_0' \to 0$. Next, the sequence of local ring homomorphisms $\mathsf{k} \to A_0 \to A_0'$ is defined, where the left homomorphism is an injection and the right one is an epimorphism. For any A_0'-module M, a tail part of the long exact sequence for the cotangent homology contains the short sequence [19, (0.3)]

$$\cdots \to \Omega^1_{A_0} \otimes_{A_0} M \to \Omega^1_{A_0'} \otimes_{A_0'} M \to \Omega^1_{A_0'/A_0} \otimes_{A_0'} M \to 0. \tag{12}$$

Since $A_0' \cong A_0/(I'/I)$, then the module of relative Kähler differentials vanishes, i.e., $\Omega^1_{A_0'/A_0} = 0$ (cf. [19, (0.1)-(0.2)]). Let $M = A_0'$. Then we obtain the sequence of two epimorphisms $\Omega^1_{A_0} \twoheadrightarrow \Omega^1_{A_0} \otimes_{A_0} A_0' \twoheadrightarrow \Omega^1_{A_0'}$, where the left one is induced by tensor multiplication of the canonical surjection $A_0 \twoheadrightarrow A_0'$ by the module $\Omega^1_{A_0}$ (the functor of tensor multiplication is right-exact), while the right morphism is induced by the exact sequence (12). Thus, the composition of these two surjections gives us the required epimorphism and the following relations

$$\dim_{\mathsf{k}} \Omega^1_{A_0'} \leqslant \dim_{\mathsf{k}}(\Omega^1_{A_0} \otimes_{A_0} A_0') \leqslant \dim_{\mathsf{k}}(\Omega^1_{A_0} \otimes_{A_0} A_0) = \dim_{\mathsf{k}} \Omega^1_{A_0}.$$

This completes the proof for $p = 1$. Taking an exterior product, we get the required statements in the general case as well. □

Example 5.3. Let $I = (x^2, y^2)$, and let $I' = (x^2, y^2, xy)$. It is not difficult to verify that the modules $T^1(A_0)$ and $T^1(A_0')$ are vector spaces of dimension 4. Furthermore, the cotangent homology and cohomology modules $T_i(A_0)$, $T^i(A_0)$, $T_i(A_0')$ and $T^i(A_0')$ vanish for all $i \geqslant 2$, since A_0 is a complete intersection and A_0' is a determinantal singularity of codimension 2. On the other hand,

$$\Omega^1_{A_0'} \cong \mathsf{k}\{dx, dy, ydx\}, \quad \Omega^1_{A_0} \cong \mathsf{k}\{dx, dy, ydx, xdy\}$$

that is, $\dim_{\mathsf{k}} T_0(A_0') = 3$ and $\dim_{\mathsf{k}} T_0(A_0) = 4$. Moreover, the primitive ideal $\int I$ is generated by the monomials x^3 and y^3, the ideal $\int I'$ is generated by the set (x^3, x^2y, xy^2, y^3) of four monomials, and $\dim_{\mathsf{k}} T_1(A_0) = \dim_{\mathsf{k}} T_1(A_0') = 4$.

Lemma 5.4. *Let A be a commutative ring with unity, B an arbitrary A-algebra, and $C = B/I$. Then, for all $p \geqslant 0$, there are natural surjective homomorphisms*

$$\Omega^p_{B/A} \twoheadrightarrow \Omega^p_{C/A}.$$

Proof. We use an argument similar to the one used in the proof of Theorem 5.2, which remains valid without the assumption that A_0 is zero-dimensional. □

Corollary 5.5. *Let A be an analytic k-algebra, and let $I \subset A$ be an ideal. Then the family of surjective morphisms $\Omega^p_{A/I^{n'}} \twoheadrightarrow \Omega^p_{A/I^n}$, $n' \geqslant n$, induced by the canonical epimorphisms $A/I^{n'} \twoheadrightarrow A/I^n$ form a projective system of A-modules, so that the limit*

$$\widehat{\Omega}^p_{A/k} = \varprojlim_n \Omega^p_{A/I^n}$$

exists, which determines the I-adic completion of the module of pth-order differentials of the k-algebra A.

The elements of the limit can be interpreted as *formal* differentials of the corresponding order given over the I-adic completion $\widehat{A} = \varprojlim_n A/I^n$ of the local ring A.

We note that, with the help of such constructions, abstract deformation theory is extended to the category of formal schemes, algebraic stacks, and their generalizations. However, as we will see below, in the case of zero-dimensional singularities, the ordinary operation of completion acquires unexpectedly a highly significant meaning and provides a set of new numerical invariants of a singularity.

COROLLARY 5.6. *In the notation of the above Corollary, let A be an analytic algebra, and let $I \subset A$ be an ideal such that the quotient A/I determines a zero-dimensional singularity A_0. Then, for all $p \geqslant 0$, we have the chains of inequalities*

$$\dim_k \Omega^p_{A_0} = \dim_k \Omega^p_{A/I} \leqslant \dim_k \Omega^p_{A/I^2} \leqslant \dim_k \Omega^p_{A/I^3} \leqslant \dots$$

for the dimensions of the modules of differential forms on infinitesimal neighborhoods of high orders of the singularity A_0 in the ambient variety.

REMARK 5.7. In fact, these inequalities are induced by the *natural* epimorphisms, which form a projective system for singularities of *any* dimension. In particular, both the projective limit and the corresponding nondecreasing set of positive numbers are nontrivial analytic invariants of the zero-dimensional singularity A_0 or the pair (A_0, A).

EXAMPLE 5.8. Let $A \cong P = \mathsf{k}\langle z_1, \dots, z_m\rangle$. Suppose that the ideal I is generated by the set $(z_1^{a_1}, \dots, z_m^{a_m})$ of m monomials, where $a_i \geqslant 1$, $i = 1, \dots, m$. Then $A_0 = P/I$ is a zero-dimensional *complete intersection* of embedding dimension m, and

$$\dim_{\mathsf{k}} \Omega^0_{P/I^2} = (m+1) \dim_{\mathsf{k}} \Omega^0_{A_0}.$$

PROPOSITION 5.9. *In the notation of Theorem 5.2, let $h \in \mathrm{Soc}\,(A_0)$ be an element of the socle of the local algebra A_0, and let $I' = (I, h)$. Then, for all $p \geqslant 0$, there are (noncanonical) embeddings*

$$\Omega^p_{A_0'} \subseteq \Omega^p_{A_0}$$

of finite-dimensional vector spaces.

Proof. The conditions of the proposition imply that there is an embedding $A_0' \hookrightarrow A_0$ of finite-dimensional vector spaces, which, evidently, is *proper* (i.e., strict). Next, multiplication by any element of the socle annihilates all differential forms on A_0 with irreversible coefficients (contained in the maximal ideal $\mathfrak{m}_{A_0}$), i.e., $h\Omega^1_{A_0} \cong \mathsf{k}\{hdz_1, \dots, hdz_m\}$. On the other hand, by definition, there is an isomorphism $\Omega^1_{A_0'} \cong \Omega^1_{A_0}/(h\Omega^1_{A_0} + A_0 dh)$, which induces an embedding of the corresponding vector spaces (but, certainly, not A_0-modules). The statement for $p \geqslant 2$ follows from the basic properties of the functor of exterior product. □

REMARK 5.10. It should be noted that, for the singularities considered in Example 1.7, there are isomorphisms $\Omega^p_{A_0'} \cong \Omega^p_{A_0}$ for $p = 1, 2$, so that the embeddings of Proposition (5.9) are generally *nonproper* for $p \geqslant 1$.

PROPOSITION 5.11. *In the notation of Proposition 5.9, assume that $d_{A_0}(h) = 0$ for some nonzero element $h \in \mathfrak{m}_{A_0}$, so that $\dim_k \mathrm{Ker}(d_{A_0}) \geqslant 2$. Then, for all $p \geqslant 1$, there are natural isomorphisms of A_0-modules*

$$\Omega^p_{A_0'} \cong \Omega^p_{A_0} \otimes_{A_0} A_0' \cong \Omega^p_{A_0}/h\Omega^p_{A_0}.$$

Proof. By definition, $\Omega^1_{A_0'} \cong \Omega^1_{A_0}/(h\Omega^1_{A_0} + A_0 dh)$, where $A_0' \cong A_0/(h)A_0$. In view of the vanishing condition $dh = 0$, the isomorphisms are evident for $p = 1$, etc. □

REMARK 5.12. It is appropriate to note here that, in the general situation, the dimensions $\dim_{\mathsf{k}} \operatorname{Der}_{\mathsf{k}}(A_0')$ and $\dim_{\mathsf{k}} \operatorname{Der}_{\mathsf{k}}(A_0)$ of Lie algebras of derivations cannot be compared by using the same argument as in the proofs of Theorem 5.2 or Proposition 5.9 without certain additional restrictions on the singularity. Now we will consider one condition of such a kind.

ASSERTION 5.13. *Under the conditions and in the notation of Proposition* 5.11,

$$\operatorname{Der}_k(A_0) \subseteq \operatorname{Der}_k(A_0') \quad \textit{and} \quad \dim_k \operatorname{Der}_k(A_0) \leqslant \dim_k \operatorname{Der}_k(A_0').$$

Proof. By definition, any derivation $\partial \in \operatorname{Der}_{\mathsf{k}}(A_0)$ is the composition of the universal derivation $d\colon A_0 \to \Omega^1_{A_0}$ and a suitable homomorphism $\iota_\partial\colon \Omega^1_{A_0} \to A_0$, i.e., $\partial = \iota_\partial \circ d$. Therefore, $\partial(h) = 0$, so that $\partial(I') \subseteq I'$. This means that the map ∂ defines a derivation of the k-algebra A_0'. □

PROPOSITION 5.14. *Let A_0 be a zero-dimensional Gorenstein singularity. Then there is a natural nondegenerate duality between the A_0-modules $\Omega^1_{A_0}$ and $\operatorname{Der}_k(A_0)$, as well as between the corresponding vector spaces. Consequently,*

$$\dim_k \Omega^1_{A_0} = \dim_k \operatorname{Der}_k(A_0).$$

Proof. It follows immediately from Theorem 2.2 or from Remark 2.3. More precisely, the local algebra A_0 is a *selfinjective* local ring (see [14, (1.43)]), so that the injective hulls of the residue field $\mathsf{k} = A_0/\mathfrak{m}_{A_0}$ and the ring A_0 are isomorphic to the local ring itself: $\mathscr{I}(\mathsf{k}) \cong \mathscr{I}(A_0) \cong A_0$. From this follows that, for any *Artinian* A_0-module M, there are canonical isomorphisms

$$\operatorname{Hom}_{A_0}(M, A_0) \cong \operatorname{Hom}_{A_0}(M, \mathscr{I}(A_0)) \cong \operatorname{Hom}_{A_0}(M, \mathscr{I}(\mathsf{k})) \cong \operatorname{Hom}_{\mathsf{k}}(M, \mathsf{k})$$

(see further details in [5, § 3]). In particular,

$$\operatorname{Der}_{\mathsf{k}}(A_0) \cong \operatorname{Hom}_{A_0}(\Omega^1_{A_0}, A_0) \cong \operatorname{Hom}_{\mathsf{k}}(\Omega^1_{A_0}, \mathsf{k}) \cong \Omega^1_{A_0}, \tag{13}$$

as was required. □

COROLLARY 5.15. *Suppose that zero-dimensional Gorenstein singularities A_0 and A_0' are determined by ideals $I \subset I' \subset P$, respectively. Then there is a (non-canonical) embedding $\operatorname{Der}_k(A_0') \hookrightarrow \operatorname{Der}_k(A_0)$ of finite-dimensional Lie algebras, so that*

$$\dim_k \operatorname{Der}_k(A_0') \leqslant \dim_k \operatorname{Der}_k(A_0).$$

Proof. This follows directly from Theorem 5.2 and the identifications (13). □

COROLLARY 5.16. *Under the assumptions and in the notation of the above corollary, suppose that $h \in \operatorname{Soc}(A_0)$, $h \neq 0$, $d_{A_0}(h) = 0$, and $I' = (I, h)$. Then there is an isomorphism of Lie algebras*

$$\operatorname{Der}_k(A_0') \cong \operatorname{Der}_k(A_0).$$

In particular, $\dim_k \operatorname{Der}_k(A_0') = \dim_k \operatorname{Der}_k(A_0)$.

Proof. It suffices to apply Assertion 5.13 and Corollary 5.15. □

REMARK 5.17. For completeness, it should be noted that it is not possible to construct a chain of inequalities containing the dimensions of the corresponding Lie algebras of vector fields similar to that in Corollary 5.6, because, for an arbitrary ideal $I \subset A$, an appropriate projective system of epimorphisms does not generally exist.

However, one can show (cf. [28, §1]) that, for any isolated singularity A and for all integers $n' \geqslant n \geqslant 0$, there are natural epimorphisms $\mathrm{Der}_{\mathsf{k}}(A/\mathfrak{m}_A^{n'}) \twoheadrightarrow \mathrm{Der}_{\mathsf{k}}(A/\mathfrak{m}_A^{n})$. In this case, one obtains a projective system which converges to the limit $\widehat{\mathrm{Der}}_{\mathsf{k}}(A)$, i.e., to the $\mathfrak{m}_A$-adic completion (along the *maximal* ideal of A) of the module of derivations of the singularity A; it is usually called the $\mathfrak{m}_A$-adic completion of the module of derivations or vector fields on the singularity.

6. Complete intersections. By definition, the local algebra A of a complete intersection is represented as the quotient P/I, where $P = \mathsf{k}\langle z_1, \dots, z_m\rangle$, and the ideal I is generated by a *regular* P-sequence $f_1, \dots, f_k$. It is known that $T^i(A) = 0$ and $T_i(A) = 0$ for all $i \geqslant 2$, the local ring A is Cohen-Macaulay, *equidimensional* and has no *embedded primes* (see Section 4, Remark 4.1).

ASSERTION 6.1. *Suppose that A is the local analytic k-algebra corresponding to a complete intersection germ with an isolated singularity of dimension $n \geqslant 1$ and of embedded dimension m. Then $m = n + k$, and there are natural isomorphisms*

$$T^1(A) \cong \mathrm{Ext}^1_A(\Omega^1_A, A) \cong A^k/\mathrm{Jac}(f_1, \dots, f_k)A^m.$$

ASSERTION 6.2. *In the notation of Assertion* 3.3, *let A be a local analytic k-algebra of positive dimension determined by a regular sequence $f_1, \dots, f_k$, where $f_i \in \mathsf{k}\langle z_1, \dots, z_m\rangle$ and $i = 1, \dots, k$. Let $f \in A$ be a non-zero divisor, $\bar{A} = A/(f)A$ and $F = (f_1, \dots, f_k, f)$. Then there are natural isomorphisms*

$$\Omega^1_{\bar{A}} \cong \bar{A}^m/\mathrm{Jac}^{\mathsf{T}}(F)\bar{A}^{k+1},$$
$$\Omega^1_f \cong \Omega^1_F \otimes_P A \cong A^m/\mathrm{Jac}^{\mathsf{T}}(F)A^{k+1}.$$

Proof. Following the proof of Assertion 3.3, let I denote the ideal in $\mathsf{k}\langle z_1, \dots, z_m\rangle$ generated by $f_1, \dots, f_k$ and by a suitable representative of f. Then we apply the exact sequence (4), taking into account the fact that the corresponding conormal $\bar{A}$-module I/I^2 is isomorphic to $\bar{A}^{k+1}$ and the map D is defined by the matrix $\mathrm{Jac}^{\mathsf{T}}(F)$. □

COROLLARY 6.3. *Under the assumptions and in the notation of Assertion* 6.2, *suppose that $k = m - 1$, that is, $\bar{A} = A_0$ is a zero-dimensional local k-algebra. Then there are natural isomorphisms of finite-dimensional vector spaces*

$$\Omega^1_f/f\Omega^1_f \cong \Omega^1_{A_0} \cong A_0^m/\mathrm{Jac}^{\mathsf{T}}(F)A_0^m \cong A_0^m/\mathrm{Jac}(F)A_0^m.$$

Proof. It is clear that $\Omega^1_f/f\Omega^1_f \cong \Omega^1_f \otimes_A A_0 \cong \Omega^1_{A_0}$. It remains to apply Assertion 6.2 or Assertion 3.3 for $k = m - 1$. □

LEMMA 6.4. *In the notation of Assertion* 6.2, *suppose that $k = m - 1$ and $\Delta = \det(\mathrm{Jac}(F))$ is not a zero-divisor in the one-dimensional local ring A. Then*

$$\dim_k \Omega^1_f = \dim_k A^m/\mathrm{Jac}(F)A^m = \dim_k A/\Delta A.$$

Proof. Let A be a Noetherian local ring, M a free A-module and $\varphi\colon M \to M$ an A-linear endomorphism. If $\det(\varphi)$ is not a zero-divisor in A, then φ is *injective* and the lengths $\ell_A(M/\varphi(M))$ and $\ell_A(A/\det(\varphi)A)$ are equal (see [15, Example A.2.3]). On the other hand, since A is a local k-algebra then $\ell_A(M) = \dim_{\mathsf{k}}(M)$ for any A-module M (see [15, Example A.1.1]). It remains to apply both statements for $M = A^m$ and $\varphi = \mathrm{Jac}(F)$. □

REMARK 6.5. In the notation of Lemma 6.4, suppose that $k = m - 1$ and $f_1, \dots, f_{m-1}$ is a sequence of *hypersurface sections* in the sense of [23, § 1.1, p.19]. Then the local k-algebra A is reduced and $\Delta = \det(\mathrm{Jac}(F))$ is not a zero-divisor in A. Therefore, we can apply the above lemma. However, if at least one function in a regular sequence $f_1, \dots, f_{m-1}$ has multiple factors, i.e., A is *nonreduced*, then Δ must necessarily be a zero-divisor in the one-dimensional local ring A. As a result, the conditions of Lemma 6.4 are not satisfied in this case.

PROPOSITION 6.6. *In the notation of Assertion* 6.1*, suppose that $A = A_0$ is a zero-dimensional complete intersection, that is, $k = m$. Then there are the following isomorphisms*

$$T^1(A_0) \cong A_0^m/\mathrm{Jac}(F)A_0^m, \quad T_0(A_0) \cong \Omega^1_{A_0} \cong A_0^m/\mathrm{Jac}^{\mathsf{T}}(F)A_0^m$$

of finite-dimensional vector spaces and, consequently,

$$T^1(A_0) \cong \mathrm{Ext}^0_{A_0}(\Omega^1_{A_0}, A_0) \cong \mathrm{Hom}_{A_0}(\Omega^1_{A_0}, A_0) \cong \mathrm{Der}_{\mathsf{k}}(A_0) \cong \mathrm{Hom}_{\mathsf{k}}(\Omega^1_{A_0}, \mathsf{k}) \cong T^0(A_0).$$

In particular, the vector spaces $T^1(A_0)$ and $T_0(A_0)$ are dual in the usual sense.

Proof. Let us consider the exact sequence (8). It is clear that the tensor product $\Omega^1_P \otimes_P A_0$ is isomorphic to the direct sum of m copies of A_0. Since the defining ideal of A_0 is generated by a regular sequence, then the conormal module I/I^2 is isomorphic to A_0^m too. As a result, we obtain the isomorphism $T^1(A_0) \cong A_0^m/\mathrm{Jac}(F)A_0^m$. Next, Assertion 3.3 shows that $\Omega^1_{A_0} \cong A_0^m/\mathrm{Jac}^{\mathsf{T}}(F)A_0^m$, and so on. □

COROLLARY 6.7. *Under the same assumptions, there is an isomorphism*

$$T^1(A_0) \cong T_0(A_0) \cong \mathrm{Tors}\,(\Omega^1_{A_0}),$$

which can be regarded as an analog of the isomorphism (6)*.*

Proof. In the zero-dimensional case, there is an isomorphism $\Omega^1_{A_0} \cong \mathrm{Tors}\,(\Omega^1_{A_0})$ (cf. Assertion 5.1). This yields the isomorphism $T^1(A_0) \cong T_0(A_0)$ between finite-dimensional vector spaces, which are dual in view of Proposition 6.6. □

As was noted in Remark 1.13, for a *reduced* complete intersection $T_1(A) = 0$, otherwise $T_1(A) \neq 0$. In particular, the latter condition holds for any zero-dimensional singularity containing multiple points. Moreover, one can prove the following useful property of the first *lower* cotangent module.

COROLLARY 6.8. *Under the assumptions of Corollary* 6.3*, there is a natural isomorphism*

$$T^0(A_0) \cong T_1(A_0).$$

Proof. In the notation of the proof of Proposition 6.6, the exact sequence (4) takes the form

$$0 \to T_1(A_0) \to A_0^m \xrightarrow{D} A_0^m \to \Omega^1_{A_0} \to 0. \tag{14}$$

Let us choose $\partial \in \mathrm{Der}_{\mathsf{k}}(A_0)$, so that $\partial = \partial_g = \sum g_i \partial/\partial z_i$ for some $g_i \in P$, $i = 1, \dots, m$, in the ambient ring. Since $\partial_g(F) = D(g_1, \dots, g_m)$, it follows that the image of the map $\mathrm{Der}_{\mathsf{k}}(A_0) \to A_0^m$ given by the rule $\partial_g \rightsquigarrow (g_1, \dots, g_m)$ coincides with $\mathrm{Ker}(D)$. The injectivity of this map is evident. □

REMARK 6.9. The ordinary Lie bracket equips the module of derivations $\mathrm{Der}_{\mathsf{k}}(A_0)$ with a canonical Lie algebra structure. Hence, the lower cotangent homology module $T_1(A_0)$ can be regarded as a Lie algebra as well.

PROPOSITION 6.10. *Under the assumptions and in the notation of Proposition* 6.6, *we have the chain of equalities*

$$\begin{aligned}\dim_k T^1(A_0) = \dim_k \mathrm{Der}_k(A_0) = \dim_k T_1(A_0) = \dim_k \Omega^1_{A_0} = \dim_k \mathrm{Tors}\,(\Omega^1_{A_0}) = \\ = \dim_k A_0^m/\mathrm{Jac}(F)A_0^m = \dim_k \Omega^1_f/f\Omega^1_f.\end{aligned}$$

Proof. It is easy to see that the modules in the middle of the exact sequences (4) (5) and (8) are vector spaces of the same dimension, which is equal to $m\cdot\dim_{\mathsf{k}} A_0$. Therefore, the dimensions of the last modules are equal as well. □

COROLLARY 6.11. *Under the assumptions and in the notation of Theorem* 5.2, *let A_0 and A_0' be the zero-dimensional complete intersections given by ideals $I \subset I'$ of P, respectively. Then*

$$\begin{aligned}&\mathfrak{t}_i(A_0') \leqslant \mathfrak{t}_i(A_0), \quad \mathfrak{t}^i(A_0') \leqslant \mathfrak{t}^i(A_0), \quad i = 0, 1, \quad \textit{so that}\\ &\mu(A_0') \leqslant \mu(A_0), \quad \tau(A_0') \leqslant \tau(A_0), \quad \dim_k \mathrm{Der}(A_0') \leqslant \dim_k \mathrm{Der}(A_0),\end{aligned}$$

where we denote by $\mu(A)$ and by $\tau(A)$ the Milnor and Tjurina numbers of a zero-dimensional singularity A, respectively; i.e., $\mu(A) = \dim_k A - 1$, $\tau(A) = \mathfrak{t}^1(A)$.

Proof. This is a direct consequence of Theorem 5.2 and Proposition 6.10. □.

PROPOSITION 6.12. *Let A_0 be the zero-dimensional complete intersection given by an ideal $I = (f_1, \dots, f_m)$. Assume that* $\mathrm{depth}(I;\Omega^1_F) \geqslant m-1$. *Then*

$$\dim_k \Omega^1_{A_0} = \dim_k \mathrm{Tor}_0^P(\Omega^1_F, A_0) = \dim_k \mathrm{Tor}_1^P(\Omega^1_F, A_0) = \dim_k \mathrm{Ext}^m_P(\Omega^1_F, A_0)$$

and there are natural isomorphisms

$$\Omega^1_{A_0} \cong \mathrm{Tor}_0^P(\Omega^1_F, A_0) \cong A_0^m/\mathrm{Jac}(F)A_0^m.$$

Proof. Let $K^P_\bullet(\mathbf{x}; M)$ be the ordinary Koszul complex associated with a sequence $\mathbf{x} = (x_1, \dots, x_m)$ with coefficients in the module M, and let $H_i(\mathbf{x}, M)$, $i \geqslant 0$, be its homology groups. Then it follows from properties of the Koszul homology that there is an isomorphism $H_1(F, \Omega^1_F) \cong \mathrm{Tor}_1^P(\Omega^1_F, P/I)$. Further, without loss of generality one can assume that the first $m-1$ functions form an Ω^1_F-*regular* sequence. As in Assertion 3.1, let us consider the exact sequence

$$0 \longrightarrow K^1_f \longrightarrow \Omega^1_f \xrightarrow{\cdot f} \Omega^1_f \longrightarrow \Omega^1_{A_0} \longrightarrow 0,$$

where $f = f_m$, $\Omega^1_f = \Omega^1_F/(f_1, \dots, f_{m-1})\Omega^1_F$, $K^1_f = \mathrm{Ker}(f)$ and $\Omega^1_{A_0} = \Omega^1_f/f\Omega^1_f$ by definition. Then multiplication by f is a homothety of the module Ω^1_f, and it is easy to see that $K^1_f \cong H_1(f, \Omega^1_f) \cong H_1(F, \Omega^1_F)$. As a result, one obtains a natural isomorphism $K^1_f \cong \mathrm{Tor}_1^P(\Omega^1_F, A_0)$. On the other hand, the vector spaces $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ have the same dimension, $\mathrm{Coker}(f) \cong \Omega^1_{A_0} \cong \mathrm{Tor}_0^P(\Omega^1_F, A_0)$, and the existence of isomorphisms $\mathrm{Tor}_0^P(\Omega^1_F, A_0) \cong \Omega^1_F \otimes_P A_0 \cong \dim_{\mathsf{k}} \mathrm{Ext}^m_P(\Omega^1_F, A_0)$ follows from properties of the functors Tor and Ext. □

PROPOSITION 6.13. *Let A_0 be a zero-dimensional complete intersection given by a regular sequence $f_1, \ldots, f_m \in P$. Then there are natural isomorphisms*

$$\begin{aligned} T^1(A_0) &\cong \operatorname{Tor}_0^P(\Omega^1_F, A_0) \cong \operatorname{Ext}^m_P(\Omega^1_F, A_0) \\ &\cong \operatorname{Tor}_0^P(P/(f_1, \ldots, \widehat{f}_i, \ldots, f_m),\, \Omega^1_F/f_i\Omega^1_F), \quad i = 1, \ldots, m. \end{aligned}$$

where the hat over an element of the sequence means that this element is omitted.

Proof. This follows from the above statements and properties of the functors Tor and Ext. Indeed, the last isomorphism is preserved when we delete any subsequence of length $k \leqslant m$ from the left argument of the functor Tor_0^P and transfer it to the right argument, forming the corresponding submodule of Ω^1_F. □

7. The graded case. We will discuss how to compute the Poincaré polynomials of differential forms and of cotangent homology and cohomology modules for certain types of graded complete intersections. The following *graded* variants of Theorem 1.8 and Theorem 2.2 are very useful in analysis of concrete examples.

THEOREM 7.1. *Let A_0 be a quasihomogenous zero-dimensional singularity. Then, for all $\nu \in \mathbb{Z}$ and $i \geqslant 0$, the dimensions of homogenous components*

$$\big(\Omega^p_{A_0}\big)_{(\nu)},\ H^i_{DR}(\Omega^\bullet_{A_0})_{(\nu)},\ \operatorname{Der}_{\Bbbk}(A_0)_{(\nu)},\ \operatorname{Ker}\big(d_{A_0}\big)_{(\nu)},\ T_i(A_0)_{(\nu)},\ T^i(A_0)_{(\nu)}$$

of the graded modules are numerical invariants of the singularity; they behave semi-continuously under graded deformations. Moreover, if the deformation is a smoothing of A_0, then

$$\dim_{\Bbbk} \operatorname{Ker}\big(d_{A_0}\big)_{(\nu)} = \sum_{p=0}^{m} (-1)^p \dim_{\Bbbk} \big(\Omega^p_{A_0}\big)_{(\nu)}. \tag{15}$$

Proof. It suffices to note that all arguments and constructions, used in the proofs of Theorem 1.8, Corollary 1.11, Theorem 2.1 and Corollary 6.3, are *compatible* with the grading of A_0 and all associative objects. □

THEOREM 7.2. *Let A_0 be a graded zero-dimensional Gorenstein local $\Bbbk$-algebra. Then, for all $i \geqslant 0$, the perfect pairings of Theorem 2.2 are compatible with this grading and are represented as follows:*

$$T_i(A_0) \times T^i(A_0) \longrightarrow \Bbbk[e]$$

for some $e \in \mathbb{Z}$. In particular, the Poincaré polynomials of the cotangent homology and cohomology modules are related as follows:

$$\mathscr{P}(T_i(A_0); t) = t^{-e} \mathscr{P}(T^i(A_0); t^{-1}).$$

It is well-known that, for a graded complete intersection of type $(d_1, \ldots, d_m; w_1, \ldots, w_m)$, the integer e is equal to $\sum w_i - \sum d_j$ in view of the standard representation of the dualizing module: $\omega_{A_0} \cong A_0(dz_1 \wedge \cdots \wedge dz_m / df_1 \wedge \cdots \wedge df_m)$ in the usual notation (see [5]).

PROPOSITION 7.3. *Let A_0 be a zero-dimensional complete intersection of type $(d_1, \ldots, d_m; w_1, \ldots, w_m)$. Then*

$$\mathscr{P}(\Omega^1_{A_0}; t) = \mathscr{P}(T_1(A_0); t) + \big(\sum_{i=1}^m t^{w_i} - \sum_{j=1}^m t^{d_j}\big) \prod_{j=1}^m (1 - t^{d_j}) / \prod_{i=1}^m (1 - t^{w_i}).$$

Proof. Let us consider the exact sequence (4) and note that there are natural isomorphisms of graded A_0-modules:

$$I/I^2 \cong \coprod A_0(-d_j), \quad \Omega^1_P \otimes_P A_0 \cong \coprod A_0(-w_i),$$

both modules are finite-dimensional vector spaces (of the same dimension). In the case of a zero-dimensional complete intersection, we derive the required relation for $p = 1$ with the use of the same observations as in the proof of the basic formula in [3, Lemma (3.2)], which was obtained under the condition $T_1(A_0) = 0$, taking into account the relation

$$\mathscr{P}(\Omega^1_P \otimes_P A_0; t) - \mathscr{P}(I/I^2; t) = \operatorname{res}_{\xi=0}\xi^{-2} \prod_{i=1}^{m} \frac{(1+\xi t^{w_i})}{(1-t^{w_i})} \prod_{j=1}^{m} \frac{(1-t^{d_j})}{(1+\xi t^{d_j})}$$

and the fact that all of the maps in the exact sequence (4) have weighted degree zero. □

Corollary 7.4. *Let A_0 be a zero-dimensional singularity whose defining ideal I is generated by a sequence $z_1^{d_1}, \dots, z_m^{d_m}$, $d_i \geqslant 2$, $i = 1, \dots, m$, and $\pi(A_0) = (d_1, \dots, d_m; 1, \dots, 1)$ in the standard homogeneous grading. Then*

$$\mathscr{P}(\Omega^1_{A_0}; t) = t \cdot \sum_{j=1}^{m} \big((1-t^{d_j-1}) \prod_{i=1,\, i\neq j}^{m} (1-t^{d_i})\big)/(1-t)^m,$$

so that $\tau(A_0) = \dim_k \Omega^1_{A_0} = \sum_{j=1}^m d_1 \cdots (d_j-1) \cdots d_m$.

Proof. Indeed, as follows from definitions, there is an isomorphism $\Omega^1_{A_0} \cong \oplus_{j=1}^m P dz_j/(z_1^{d_1}, \dots, z_j^{d_j-1}, \dots, z_m^{d_m})P$. We will give another proof, which can be also applied in a more general situation. First remark that, in view of Proposition 7.3, we have

$$\mathscr{P}(\Omega^1_P \otimes_P A_0; t) - \mathscr{P}(I/I^2; t) = (mt - \sum_{j=1}^{m} t^{d_j}) \prod_{j=1}^{m} (1-t^{d_j}) / \prod_{i=1}^{m} (1-t^{w_i}).$$

Next, it is not difficult to compute the *primitive* ideal $\int I$ and the quotient $\int I/I^2$, which is isomorphic to $T_1(A_0)$. Thus, in the case $m = 2$, we see that there is an isomorphism

$$T_1(A_0) \cong \int (z_1^{d_1}, z_2^{d_2})/(z_1^{d_1}, z_2^{d_2})^2,$$

and this vector space is generated over k by the set of monomials $\{z_1^i z_2^j, z_1^k z_2^r\}_{ijkr}$, where $d_1 + 1 \leqslant i \leqslant 2d_1 - 1$, $0 \leqslant j \leqslant d_2 - 1$, $0 \leqslant k \leqslant d_1 - 1$, and $d_2 + 1 \leqslant r \leqslant 2d_2 - 1$. Hence, $\dim_{\mathsf{k}} T_1(A_0) = 2d_1 d_2 - d_1 - d_2$, and, applying Proposition 7.3, we find that the Poincaré polynomial $\mathscr{P}(\Omega^1_{A_0}; t)$ has the required form. □

Remark 7.5. Let A_0 be the zero-dimensional singularity determined by the ideal $(z_1^{d_1}, \dots, z_m^{d_m})$, $d_i \in \mathbb{Z}_+$, $i = 1, \dots, m$. Then, for all $0 \leqslant p \leqslant m$, there are natural isomorphisms

$$\Omega^p_{A_0} \cong \bigoplus P dz_{i_1} \wedge \dots \wedge dz_{i_p}/(z_1^{d_1}, \dots, z_{i_1}^{d_{i_1}-1}, \dots, z_{i_p}^{d_{i_p}-1}, \dots, z_m^{d_m})P,$$

of finite-dimensional vector spaces, where the direct sum is taken over all p-tuples $1 \leqslant i_1 < \cdots < i_p \leqslant m$. It is easy to derive the Poincaré polynomials $\mathscr{P}(\Omega^p_{A_0}; t)$ as in Corollary 7.4, and to see that

$$\dim_{\mathsf{k}} \Omega^p_{A_0} = \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant m} \frac{d_1 \cdots d_m}{d_{i_1} \cdots d_{i_p}} \cdot (d_{i_1} - 1) \cdots (d_{i_p} - 1).$$

In particular, $\dim_{\mathsf{k}} A_0 = d_1 \cdots d_m$ and $\dim_{\mathsf{k}} \Omega^m_{A_0} = (d_1 - 1) \cdots (d_m - 1)$.

Example 7.6. In the above notation, let $I = (z_1^2, z_2^2, z_3^2)$. Then $\mathscr{P}(\Omega^1_{A_0}; t) = 3t(1+t)^2 = 3t + 6t^2 + 3t^3$, so that $\dim_{\mathsf{k}} \Omega^1_{A_0} = \tau(A_0) = 12$ in view of Proposition 6.10. Similarly, $\dim_{\mathsf{k}} \Omega^2_{A_0} = 6$, $\dim_{\mathsf{k}} \Omega^3_{A_0} = 1$ and $\dim_{\mathsf{k}} A_0 = 8$, i.e., $\mu(A_0) = 7$ (cf. Corollary 1.11).

Assertion 7.7. *Let A be a reduced one-dimensional complete intersection of type $(d_1, \ldots, d_{m-1}; w_1, \ldots, w_m)$ given by the ideal $I = (f_1, \ldots, f_{m-1})$, $m \geqslant 2$. Assume that a function $f \in A$ of weighted degree d_m is not a zero divisor. Then $A_0 \cong A/(f)A$ is a zero-dimensional complete intersection of type $(d_1, \ldots, d_m; w_1, \ldots, w_m)$ and*

$$\mathscr{P}(\Omega^1_f; t) = \Big(\sum_{i=1}^m t^{w_i} - \sum_{j=1}^m t^{d_j} \Big) \prod_{j=1}^{m-1} (1 - t^{d_j}) / \prod_{i=1}^m (1 - t^{w_i}).$$

Proof. We first note that if the functions $f_1, \ldots, f_{m-1}$ are *hypersurface sections* in the sense of [23, § 1.1, p.19] which determine a *one-dimensional* complete intersection, then one can compute the above Poincaré polynomial with the help of equations in [23, § 2.2] or formulas in [17, Lemma 4.4, Lemma 4.5], so that

$$\mathscr{P}(\Omega^1_f; t) = \operatorname{res}_{\xi=0} \xi^{-2} \prod_{i=1}^m \frac{(1 + \xi t^{w_i})}{(1 - t^{w_i})} \prod_{j=1}^{m-1} \frac{(1 - t^{d_j})}{(1 + \xi t^{d_j})} \frac{1}{(1 + \xi t^{d_m})}.$$

The required formula follows from this one.

Next, if we only assume that the one-dimensional singularity A is isolated (i.e., reduced), then an argument from [23, Lemma 2.3.1, Lemma 2.3.2] implies that the spaces Ω^1_A and Ω^1_f and the Milnor number $\mu(A) = \dim_{\mathsf{k}} \Omega^1_A / d(A)$ of this singularity, as well as the corresponding Poincaré series, depend *only* on the type of quasihomogeneity. Moreover, there exists a one-dimensional singularity of the same type, which is determined by an ideal generated by hypersurface sections. This completes the proof. □

Example 7.8. Let us consider the zero-dimensional complete intersection A_0 in the plane given by the monomials x^2 and y^3, for which $I = (x^2, y^3)$. It is evident that A_0 is isomorphic to the singularities determined by the ideals $(x^2 + y^3, y^3)$ or $(x^2 + y^3, x^2 - y^3)$. The generators of the second ideal are hypersurface sections, and the type of the corresponding singularity is equal to $(6, 6; 3, 2)$. Observations of this kind often allow exploiting the formula of Assertion 7.7 in concrete calculations.

Corollary 7.9. *In the notation of Assertion 7.7, assume that there exists a $\mathbb{Z}_+$-grading of A_0, in which the generators of the ideal I has the same weighted degree d (say, $d = d_1 = \cdots = d_{m-1}$). Then*

$$\mathscr{P}(\Omega^1_f; t) = \Big(\sum_{i=1}^m t^{w_i} - (m-1)t^d - t^{d_m} \Big) (1 - t^d)^{m-1} / \prod_{i=1}^m (1 - t^{w_i}).$$

Proof. Let $\alpha_{ij} \in \mathsf{k}$ be a set of constants such that $\det(|\alpha_{ij}|) \neq 0$. If this set is generic, then the functions $g_j = \sum_{i=1}^{m-1} \alpha_{ij} f_i$, $j = 1, \dots, m-1$, form a sequence of hypersurface sections; they determine a one-dimensional reduced singularity. On the other hand, the ideal $(g_1, \dots, g_{m-1}, f)$ is contact equivalent to the ideal $(f_1, \dots, f_{m-1}, f)$. Hence, one can apply the formula of Assertion 7.7. □

PROPOSITION 7.10. *In the notation of Proposition* 6.12*, let A_0 be the complete intersection given by an ideal $I = (f_1, \dots, f_m)$ of type $(d_1, \dots, d_m; w_1, \dots, w_m)$. Then*

$$\mathscr{P}(\Omega^1_{A_0}; t) = (1 - t^d)\mathscr{P}(\Omega^1_f; t) + t^d \mathscr{P}(\operatorname{Tor}_1^P(\Omega^1_F, A_0); t),$$

where $f = f_i$ and $d = d_i$ for any $i = 1, \dots, m$.

Proof. This follows from Proposition 6.12 and Assertion 3.1 for $p = 1$. □

REMARK 7.11. Thus, if $\operatorname{Tor}_1^P(\Omega^1_F, A_0) = 0$, then $\mathscr{P}(\Omega^1_{A_0}; t) = (1 - t^d)\mathscr{P}(\Omega^1_f; t)$. For example, this is so if f is an Ω^1_f-regular element, i.e., the corresponding homothety morphism of the module Ω^1_f is an injective homomorphism. Next, if $f \in \operatorname{Ann}_A(\Omega^1_f)$, then $\operatorname{Tor}_1^P(\Omega^1_F, A_0) \cong \Omega^1_f$ (cf. Corollary 3.2). In this case, $\mathscr{P}(\Omega^1_{A_0}; t) = \mathscr{P}(\Omega^1_f; t)$. In fact, this is true for the the zero-dimensional singularities considered in Corollary 7.4.

EXAMPLE 7.12. Let us consider the zero-dimensional *gradient* singularity $\operatorname{grad}(Q_{12})$ given by the partial derivatives of the function $\varphi = x^3 + yz^2 + y^5$, i.e., $f_1 = x^2$, $f_2 = y^4 + z^2$ and $f_3 = yz$. It is easy to see that the corresponding local algebra A_0 is equipped with an *infinite* number of gradings $\pi = (2w, 12, 9; w, 3, 6)$, $w \in \mathbb{Z}$. In the same notation, we set $f = f_2$. Then $\mathscr{P}(\Omega^1_f; t) = t(1 + t^2 + t^w)(1 + t + t^2)(1 + t)$, so that the dimension $\dim_{\mathsf{k}} \Omega^1_f = 18$ *does not depend* on the choice of the grading. Next, $f \in \operatorname{Ann}(\Omega^1_f)$; therefore $\mathscr{P}(\Omega^1_{A_0}; t) = \mathscr{P}(\Omega^1_f; t)$ and $\tau(\operatorname{grad}(Q_{12})) = 18$. Note also that if one set $w = 4$, then the ideal $I = (x^2, x^2 + y^4 + z^2, yz)$ determines the zero-dimensional singularity, which is isomorphic to $\operatorname{grad}(Q_{12})$. Moreover, the last two functions determine a one-dimensional singularity A, and $\tau(A) = 7$.

PROPOSITION 7.13. *In the notation of Theorem* 7.2*, let A_0 be a complete intersection of type $\pi = (d_1, \dots, d_m; w_1, \dots, w_m)$, and $c = \sum d_j - \sum w_i$. Then, for $i = 0$, 1, there are nontrivial perfect pairings*

$$T_0(A_0) \times T^0(A_0) \longrightarrow \mathsf{k}[-c],$$
$$T_1(A_0) \times T^1(A_0) \longrightarrow \mathsf{k}[-c],$$

which induce the following relations for Poincaré polynomials:

$$\mathscr{P}(\operatorname{Der}_{\mathsf{k}}(A_0); t) = t^c \mathscr{P}(\Omega^1_{A_0}; t^{-1}), \quad \mathscr{P}(T^1(A_0); t) = t^c \mathscr{P}(T_1(A_0); t^{-1}).$$

Proof. As was remarked before, in the case of complete intersections the lower and upper cotangent modules are trivial for all $i \geqslant 2$. □

8. Gradient singularities. Now we will analyze a very particular case. Let F be an analytic function, and let X be the corresponding hypersurface germ. Then the partial derivatives $f_i = \partial F / \partial z_i$, $i = 1, \dots, m$, generate the *Jacobian* ideal of F denoted by $\operatorname{Jac}(F)$. This ideal determines the singular locus $\operatorname{Sing} X \subset X$. The corresponding germ with a natural structure induced by the Jacobian ideal, as well as its dual local (analytic) k-algebra $A = P/\mathrm{Jac}(F)$ (if there is no ambiguity), is called the *gradient* singularity associated with the function F; we will denote it by $\mathrm{grad}(F)$.

If F has an *isolated* critical point at the origin, then the partial derivatives $f_1, \dots, f_m$ form a *regular* P-sequence. In this case, the hypersurface germ X has an isolated singularity and its singular locus $\mathrm{Sing}\, X$ is a *multiple* point. In our terminology, it is a zero-dimensional gradient singularity. The dual local k-algebra A of the germ $\mathrm{Sing}\, X$, often called the Milnor algebra of F, is an Artinian complete intersection of finite dimension over k, and $\dim_{\mathsf{k}} A = \mu(A) + 1$.

Recall (see [5]) that for zero-dimensional graded gradient singularities

$$\begin{aligned}
&\mathscr{P}(\mathrm{Der}_{\mathsf{k}}(A_0); t) = t^{-d}\mathscr{P}(T_1(A_0); t) = t^{c-d}\mathscr{P}(T^1(A_0); t^{-1}) = t^{c}\mathscr{P}(\Omega^1_{A_0}; t^{-1}),\\
&\mathscr{P}(T^1(A_0); t) = t^{-d}\mathscr{P}(T_0(A_0); t) = t^{-d}\ \mathscr{P}(\Omega^1_{A_0}; t),\\
&\mathscr{P}(T_1(A_0); t) = t^{d}\ \mathscr{P}(\mathrm{Der}_{\mathsf{k}}(A_0); t) = t^{c+d}\mathscr{P}(\Omega^1_{A_0}; t^{-1}),
\end{aligned} \tag{16}$$

where $d = \deg(F)$ and $c = md - 2(w_1 + \dots + w_m)$. In particular, the second relation implies an analog of the relation (7) in the one-dimensional case:

$$\mathscr{P}(\mathrm{T}^1(A_0); t) = t^{-d}\mathscr{P}(\mathrm{Tors}\,(\Omega^1_{A_0}); t), \tag{17}$$

because $\Omega^1_{A_0}$ is a *torsion* module in the zero-dimensional case (cf. Assertion 5.1).

These relations also imply that if we can compute the Poincaré polynomial for one of the cotangent modules, then all others are obtained immediately from this one.

Proposition 8.1. *Let $F = z_1^{a_1+1} + \ldots + z_m^{a_m+1}$ be a Brieskorn-Pham polynomial of type $(d; w_1, \dots, w_m)$. Assume also that $a_i \geqslant 1$, $i = 1, \dots, m$, so that $\pi(\mathrm{grad}(F)) = (d_1, \dots, d_m; w_1, \dots, w_m)$, where $d_i = d - w_i = a_i w_i$. Then*

$$\mathscr{P}(T^1(A_0); t) = t^{-d}\sum_{i=1}^{m}\Big((t^{w_i} - t^{d_i})\prod_{j\neq i}(1 - t^{d_j})\Big)\Big/\prod_{i=1}^{m}(1 - t^{w_i}).$$

In particular,

$$\tau(A_0) = \dim_k T^1(A_0) = m\Pi(a) - \textstyle\sum_{i=1}^{m}\Pi(a)/a_i,$$

where $\Pi(a) = a_1 \cdots a_m$.

Proof. It is readily seen that for $m = 2$ the module $\Omega^1_{A_0}$ is generated over k by the following monomial differential forms $\{z_1^i z_2^j dz_1, z_1^k z_2^r dz_2\}_{ijkr}$, where $0 \leqslant i \leqslant a_1 - 2$, $0 \leqslant j \leqslant a_2 - 1$, $0 \leqslant k \leqslant a_1 - 1$, $0 \leqslant r \leqslant a_2 - 2$, and so on. Another proof one can also obtain with the use of the primitive ideal of $\mathrm{Jac}(F)$ in the standard homogeneous grading, similarly to the proof of Corollary 7.4. In conclusion, we apply the identity $\mathscr{P}(T^1(A); t) = t^c\mathscr{P}(T_1(A); t^{-1})$ valid for Gorenstein germs, and taking into account that $c = md - 2\sum w_i$ for gradient complete intersections (cf. [5]). □

Example 8.2. If $F = z_1^3 + z_2^3 + z_3^3$, then $\pi(F) = (3; 1, 1, 1)$ and $\mathrm{Jac}(F) = (z_1^2, z_2^2, z_3^2)$. Hence, $\mathscr{P}(\mathrm{T}^1(A_0); t) = t^{-3}(3t + 6t^2 + 3t^3) = 3 + 6t^{-1} + 3t^{-2}$, so that $\tau(A_0) = 12$. On the other hand, $\mathscr{P}(\mathrm{T}^1(A_0); t) = t^{-3}\mathscr{P}(\Omega^1_{A_0}; t)$ in view of Example 7.6.

Assertion 8.3. *Let A_0 be the gradient singularity given by a quasihomogeneous function $F \in P$. Suppose that a subsequence of the partial derivatives of F, say $\partial F/\partial z_1, \dots, \partial F/\partial z_{m-1}$, determines a one-dimensional reduced singularity A. Let $f = \partial F/\partial z_m$. Then*

$$\delta(f|_A) = \mathscr{P}(\Omega^1_f; 1) = \dim_k \Omega^1_f, \quad \tau(A_0) = \dim_k \Omega^1_f / f\Omega^1_f,$$

where we denote by $\delta(f|_A)$ the multiplicity of the discriminant of the map $f\colon X \to \mathsf{k}$ determined by the function f.

Proof. This follows directly from Assertion 7.7 (cf. [17, Lemma 4.3]). □

Corollary 8.4. *In the notation of Assertion 8.3, let $F = z_1^d + \cdots + z_m^d$ be a homogeneous polynomial of degree $d \geqslant 3$. Then $\pi(A_0) = (d-1, \dots, d-1; 1, \dots, 1)$, and*

$$\mathscr{P}(\Omega^1_f; t) = \mathscr{P}(\Omega^1_{A_0}; t) = m{\cdot}t(1-t^{d-2})(1-t^{d-1})^{m-1}/(1-t)^m,$$

so that $\dim_{\mathsf{k}} \Omega^1_f = m(d-2)(d-1)^{m-1}$. *In this case, $f \in \operatorname{Ann}(\Omega^1_f)$ and*

$$\mathscr{P}(T^1(A_0); t) = m{\cdot}t^{1-d}(1-t^{d-2})(1-t^{d-1})^{m-1}/(1-t)^m,$$

so that $\tau(A_0) = \dim_{\mathsf{k}} \Omega^1_f$.

Proof. First remark, that the ideal I can be generated by a sequence of hypersurface sections as follows. Let ρ be a primitive root of unit of degree $d-1$, i.e., $\rho^{d-1} = 1$, $\rho \neq 1$. Then the ideal, generated by the polynomials

$$f_1 = z_1^{d-1} + \rho z_2^{d-1} + \cdots + \rho^{d-2} z_m^{d-1}, \ \dots, \ f_m = z_1^{d-1} + \rho^{d-2} z_2^{d-1} + z_3^{d-1}, \cdots + \rho^{d-3} z_m^{d-1},$$

is contact equivalent to I since the determinant of the transformation matrix does not vanish. Hence,

$$\mathscr{P}(\Omega^1_f; t) = \bigl(mt - mt^{d-1}\bigr)(1-t^{d-1})^{m-1}/(1-t)^m = mt(1-t^{d-2})(1-t^{d-1})^{m-1}/(1-t)^m.$$

The remaining part of the assertion follows from Corollary 3.2 and the relation (17). □

Example 8.5. Let $F = z_1^{d_1} + \cdots + z_m^{d_m}$. Then the ideal I, generated by a sequence of hypersurface sections $f_j = \sum \alpha_{ij} z_i^{d_j - 1}$, where $\det(|\alpha_{ij}|) \neq 0$ for a generic set of constants $\alpha_{ij} \in \mathsf{k}$, is equivalent to $\operatorname{Jac}(F)$ (cf. the proof of Corollary 7.9). In this case, $\pi(A_0) = (D, \dots, D; W_1, \dots, W_m)$, where $D = \prod_{j=1}^m (d_j - 1)$ and $W_i = D/(d_i - 1)$, $i = 1, \dots, m$. Hence,

$$\mathscr{P}(\Omega^1_f; t) = \bigl(\textstyle\sum_{i=1}^m t^{W_i} - m{\cdot}t^D\bigr)(1-t^D)^{m-1}/\prod_{i=1}^m (1-t^{W_i}),$$

where $f = f_m$. Moreover, there exists an integer $\nu \in \mathbb{Z}$ such that $\mathscr{P}(T^1(A_0); t) = t^{\nu} \mathscr{P}(\Omega^1_f; t^{-1})$. It should be underlined that the grading of A_0 does not induced by the grading of F.

Corollary 8.6. *In the notation of Assertion 8.3, let $A_0 = \operatorname{grad}(F)$ be the gradient singularity, and $f \in \operatorname{Ann}(\Omega^1_f)$. Then*

$$\mathscr{P}(T^1(A_0); t) = t^{-d}\bigl(\textstyle\sum_{i=1}^m t^{w_i} - \sum_{j=1}^m t^{d_j}\bigr) \prod_{j=1}^{m-1}(1-t^{d_j})/\prod_{i=1}^m (1-t^{w_i}),$$

where $d = \deg(F)$ and $d_j = d - w_j$, $j = 1, \dots, m$.

Proof. This follows from the relations (16). □

Example 8.7. Let $F = x^3 + y^4$ be an E_6-singularity. Then its type is defined *uniquely* up to multiples, i.e., $\pi(F) = (12; 4, 3)$. However, there is an infinite number of gradings of a $\operatorname{grad}(E_6)$-singularity because $\operatorname{Jac}(F) = (x^2, y^3)$. For example, the induced grading is $(8,9;4,3)$, the standard homogeneous grading is $(2,3;1,1)$, and so on. In the case of induced grading, we can apply Proposition 8.1

$$\mathscr{P}(\mathrm{T}^1(A_0);t) = t^{-12}(t^4(1+t^3+t^6)+t^3(1+t^3)(1+t^4)) = 2t^{-2}+2t^{-5}+t^{-6}+t^{-8}+t^{-9}.$$

On the other hand, one can consider the ideal $I=(x^2+y^3, x^2-y^3)$ of type $(6,6;3,2)$, which does not correspond to a gradient singularity. This ideal is equivalent to $\mathrm{Jac}(F)$ and satisfies the assumptions of Assertion 7.7. Therefore,

$$\mathscr{P}(\Omega^1_f;t) = (t^3+t^2-2t^6)(1-t^6)/(1-t^2)(1-t^3) = t^2+t^3+t^4+2t^5+2t^7.$$

Next, because $\int I = (x^3,y^4)+I^2 \cong \int(x^2,y^3)$, one can verify that $\mathscr{P}(\mathrm{T}_1(A_0);t) = t^8+t^9+t^{10}+2t^{11}+2t^{13}$. In this grading $c = 12-5 = 7$, hence

$$\mathscr{P}(\mathrm{T}^1(A_0);t) = t^7\mathscr{P}(\mathrm{T}_1(A_0);t^{-1}) = t^{-1}+t^{-2}+t^{-3}+2t^{-4}+2t^{-6},$$

and $\tau(A_0) = 7$ as required. Moreover, $\mathscr{P}(\mathrm{T}^1(A_0);t) = t\mathscr{P}(\Omega^1_f;t^{-1})$ similarly to Example 8.5.

Example 8.8. Let us consider a $\mathrm{grad}(S_{12})$-singularity. Then $F = x^2z+yz^2+xy^3$ and $\pi(F) = (13;4,3,5)$. In this case, $\mathrm{Jac}(F) = (y^3+2xz, 3xy^2+z^2, x^2+2yz)$ and $\pi(\mathrm{grad}(F)) = (9,10,8;4,3,5)$, i.e., the grading is defined *uniquely*. The first and the third generators determine a reduced one-dimensional germ. Let $f = 3xy^2+z^2$. In view of Assertion 7.7, we obtain

$$\mathscr{P}(\Omega^1_f;t) = t^3(1+t+t^2)(1+t^4)(1+t^3+t^6),$$

so that $\dim_{\Bbbk}\Omega^1_f = 18$ again. But, in this case, $f \notin \mathrm{Ann}(\Omega^1_f)$ and $\deg(f) = 10$, and one can verify that

$$\mathscr{P}(\Omega^1_{A_0};t) = \mathscr{P}(\Omega^1_f/f\Omega^1_f;t) = \mathscr{P}(\Omega^1_f;t) - t^{10}\mathscr{P}(\Omega^1_f;t) + t^{10}\mathscr{P}(\mathrm{Ker}(f);t),$$

so that $\mathscr{P}(\Omega^1_{A_0};t) = \mathscr{P}(\Omega^1_f;t) - t^{14}$ and $\tau(A_0) = \dim_{\Bbbk}\Omega^1_f/f\Omega^1_f = 17$. In view of the relation (17), we see that $\mathscr{P}(\mathrm{T}^1(A_0);t) = t^{-13}\mathscr{P}(\Omega^1_{A_0};t)$, since $d = 13$. In addition, $\mathscr{P}(\mathrm{T}^1(A_0);t) = t^{13}\mathscr{P}(\Omega^1_{A_0};t^{-1})$, similarly to Example 8.5.

9. Unimodular families. A list of weighted homogeneous gradient zero-dimensional singularities with modularity $\wp = 0$ and with Milnor number not exceeded 12, $\mu \leqslant 12$, is given in [9, Theorem 3]. Now we are able to make the next step forward in the classification of such singularities.

In the following table the first two columns contain the standard notation of classes (or genotypes) of singularities, their equations, types of homogeneity and restrictions on parameters. The equations and types of the corresponding *gradient* singularities are given in the third column. Then we write down *monomial vector-bases* of the first cotangent space $\mathrm{T}^1(\mathrm{grad}(F))$, the nonzero entries of the corresponding vectors, and the weights of generators. The symbol $[i]$ refers to the i-th nonzero entry of a vector-monomial generator; all other its entries are zero.

Assertion 9.1. *Any unimodular family of zero-dimensional gradient singularities is $\mathscr{K}$-equivalent to one of the following germs:*

Notation of classes grad(F)	Normal form of F π(F) Restrictions	Normal form of grad(F) π(grad(F)) Restrictions	T^1−basis weights	τ	μ
grad(P_8)	$x^3+y^3+z^3+axyz$ (3;1,1,1) $a^3+27\neq 0$	$(x^2+\alpha yz,\ y^2+\alpha xz,\ z^2+\alpha xy)$ (2,2,2;1,1,1) $\alpha(\alpha^3+1)(\alpha^3-8)\neq 0$	[1]:$1,y,z,z^2$; [2]:$1,y,z$; [3]:$1,y,z$ $-2,-1,-1,0$; $-2,-1,-1$; $-2,-1,-1$	10	7
grad(X_9)	$x^4+y^4+ax^2y^2$ (4;1,1) $a^2-4\neq 0$	$(x^3+\alpha xy^2,\ y^3+\alpha x^2y)$ (3,3;1,1) $\alpha(\alpha^2-1)(\alpha^2-9)\neq 0$	[1]:$1,x,y,x^2,xy,y^2,xy^2$; [2]:$1,x,y,y^2$ $-3,-2,-2,-1,-1,-1,0$; $-3,-2,-2,-1$	11	8
grad(J_{10})	$x^3+y^6+ax^2y^2$ (6;2,1) $4a^3+27\neq 0$	$(x^2+2\alpha xy^2,\ y^5+\alpha x^2y)$ (4,5;2,1) $\alpha(4\alpha^3+1)\neq 0$	[1]:$1,x,y,xy,y^2,y^3,y^4$; [2]:$1,x,y,y^2,y^3$ $-4,-2,-3,-1,-2,-1,0$; $-5,-3,-4,-3,-2$	12	9

Proof. In general, the modularity is no greater than the modality, i.e., $\wp \leqslant m$. In view of remarks from [13, §1], the embedding dimension of $\mathscr{K}$-unimodular families is less than 4; a complete description of such families one can find in [13] also. As a result, we obtain that the parabolic singularities of types P_8, X_9 and J_{10} form three families that depend on a complex parameter α, the modulus of the singularity. It remains to compute restrictions on the parameter. First we will examine the simplest case.

The class grad(X_9). The Jacobian ideal is generated by the two polynomials $4x^3+2axy^2$ and $4y^3+2ax^2y$; it is equivalent to the ideal (x^3+txy^2, y^3+tx^2y), where $t = a/2$. Applying the coordinate transformation $\xi(x) = ax + by$ and $\xi(y) = cx + dy$ to the generators of the defining ideal of the family and using the symmetry, it is easy to see that the monomials x^2y and xy^2 are contained in the generators with the coefficients

$$bc^2t+2acdt+3a^2b,\ 2abct+a^2dt+3c^2d,\ 2bcdt+ad^2t+3ab^2,\ b^2ct+2abdt+3cd^2. \quad (18)$$

Similarly, the monomials x^3 and y^3 have the coefficients

$$ac^2t + a^3,\ bd^2t + b^3,\ a^2ct + c^3,\ b^2dt + d^3. \quad (19)$$

If the expressions in (18) vanish then the obtained ideal is equivalent to (x^3, y^3) if and only if the determinant of the corresponding 2×2 matrix (19) does not vanish.

A routine procedure of elimination of the variables a and d (or, equivalently, b and c) from the system formed by the four equations (18) and the determinant, gives us the two (compatible) equations

$$t^5 - 10t^3 + 9t = t^2 - 1 = 0,$$

which guarantee the required vanishing conditions with the exception of at most 5 values of the parameter t. On the other hand, the first polynomial factors into 3 different factors $t(t^2 - 1)(t^2 - 9)$.

If $t^2 = 1$, then X_t is a *one-dimensional* germ. In particular, this means that the gradient family is *not flat* over these two points.

If $t = 0$ or $t^2 = 9$, then $\tau(X_t) = 12$, while $\tau = 11$ for generic values of the parameter. That is, the Tjurina number *jumps* over these three points of the base space. More precisely, the dimension of the zeroth homogeneous component of $\mathrm{T}^1(X)$ (cf. Theorem 7.2), which is isomorphic to the space $(\Omega^1_X)_{(4)}$ in virtue of the relation (16), increases by 1. By Theorem 7.2, the dimension of $(\Omega^2_X)_{(4)}$ must necessarily increase by 1.

After the coordinate change $x \longmapsto x + y$, $y \longmapsto x - y$ and $x \longmapsto x + \sqrt{-1}y$, $y \longmapsto x - \sqrt{-1}y$, we see that both singularities X_3 and X_{-3} are *contact equivalent* to X_0. For convenience of notation, we will denote the germs associated with $t=\pm 3$ and $t=0$ by $\operatorname{grad}(X_9^0)$ (cf. [9, Theorem 3]). Of course, if $t\neq 0$ the grading is defined *uniquely*.

It should be noted also that the ideal $(x^3+\alpha xy^2, y^3+\beta x^2y)$ determines the *2-parameter* family of homogeneous singularities. It is easily seen that $\tau(X_{\alpha\beta})=11$ in the generic point. This family contains the above *one-modular* family of gradient singularities (set $\alpha=\beta=t$). Moreover, if $\alpha\beta=1$, then the corresponding family of *nongradient* singularities consists of *one-dimensional* germs with an *embedded* point at the origin.

The class $\operatorname{grad}(J_{10})$. The Jacobian ideal is equal to $(x^2+\frac{2}{3}axy^2, y^5+\frac{1}{3}ax^2y)$ or, equivalently, to (x^2+2txy^2, y^5+tx^2y), where $t=a/3$. A similar procedure with the coordinate transformation $\xi(x)=ax+by$ and $\xi(y)=cx+dy+ex^2+fxy+gy^2$ gives the equation

$$t(4t^3+1)=0.$$

If $t^3=-1/4$, then X_t is a *one-dimensional* germ, i.e., the gradient family is *not flat* over these three points. If $t=0$, then $\tau(X_0)=13$, while $\tau(X_t)=12$ for generic values of the parameter. Hence, the Tjurina number jumps over the origin, in the zeroth component of $\mathrm{T}^1(X_0)_{(0)}$ again. The corresponding singularity is denoted by $\operatorname{grad}(J_{10}^0)$ in the list of [9, Theorem 3]. If $t\neq 0$, then the grading is defined *uniquely.*

There is a *2-parameter* family of quasihomogeneous singularities determined by the ideal $(x^2+\alpha xy^2, y^5+\beta x^2y)$, where $\alpha^2\beta+1\neq 0$. In the generic point $\tau(X_{\alpha\beta})=12$; this family contains the above *unimodular* family of gradient singularities (set $\frac{1}{2}\alpha=\beta=t=a/3$).

The class $\operatorname{grad}(P_8)$. A slightly different presentation $F=x^3+y^3+z^3-3\lambda xyz$ of the family P_8 is often called *Hesse normal form* of an elliptic curve; it has the property that the Hessian of any of its members is of the same form. Classically, one refers to those as the *moduli* of the cubics. We also consider a *compactification* of the Hesse family at $\lambda=\infty$, where the fiber is a 2-dimensional *free* hypersurface in the 3-dimensional complex space given by the equation $xyz=0$ (see Example 9.3 below).

It is well-known that the j-invariant of the family $x^3+y^3+z^3+axyz$ is equal to $-27a^6/4(a^3+27)$. In particular, this gives the restrictions on the parameter λ of the Hesse family as well.

In our notation, the Jacobian ideal is generated by the three polynomials $3x^2+ayz$, $3y^2+axz$ and $3z^2+axy$. First, we divide the generators by 3 and then obtain x^2+tyz, y^2+txz, z^2+txy, where $t=a/3=-\lambda$. It is useful to remark that all fibers of this family are contact 2-determined in the usual sense. Using the coordinate transformation $\xi(x)=ax+by+cz$, $\xi(y)=dx+ey+fz$ and $\xi(z)=gx+hy+iz$, we see that the monomials xy, xz and yz occur in the first generator of the family with the coefficients

$$egt+dht+2ab,\ fgt+dit+2ac,\ fht+eit+2bc. \tag{20}$$

The second and the third generators, in view of the evident symmetry, give us the six other coefficients

$$\begin{gathered} bgt+aht+2de,\ cgt+ait+2df,\ cht+bit+2ef, \\ bdt+aet+2gh,\ cdt+aft+2gi,\ cet+bft+2hi. \end{gathered}$$

As before, the equation for the parameter t can be computed explicitly by eliminating the variables aei (or bfg, or cdh) in the corresponding system of equations. As a result, we obtain

$$t(t^3+1)(t^3-8)=0.$$

If $t^3+1=0$, then X_t is a *one-dimensional* germ, i.e., the gradient family is *not flat* over the corresponding three points. If $t=0$ or $t^3=8$, then $\tau(X_t)=12$, although $\tau=10$ for generic fiber of the family. Hence, the Tjurina number *jumps* over these *four* points. Hence, the Tjurina number jumps over these four points, only in the zeroth component of $\mathrm{T}^1(X_0)_{(0)}$ as before.

Elementary calculations show that the three germs, corresponding to the roots of the equations $t^3-8=0$ are, in fact, *contact equivalent* to X_0. More precisely, let ρ denote a *primitive* cubic root of unit. The coordinate change $x\longmapsto x+y+z$, $y\longmapsto x+\rho y+\rho^2 z$ and $z\longmapsto x+\rho^2 y+\rho z$ transforms the defining ideal of $X_{t|_{t=2}}$ into $(x^2+y^2+z^2, x^2+\rho^2y^2+\rho z^2, x^2+\rho y^2+\rho^2z^2)$. This ideal is equivalent to (x^2,y^2,z^2), since the determinant of the coefficient matrix is equal to $2\rho+1\neq 0$, and so on. We will use the notation $\mathrm{grad}(P_8^0)$ for the fibers X_t, where $t(t^3-8)=0$. As before, the grading is *unique* if $t\neq 0$.

It is useful to note that the modularity of X_0 is equal to zero, $\wp(X_0)=0$. The ideal $(x^2+ayz, y^2+bxz, z^2+cxy)$ determines the *3-parameter homogeneous* family of singularities. The Tjurina number of the generic fiber equals 10. Next, $\tau=11$ if $a\neq 0$ and $b=c=0$, and $\tau=12$ if $a=b=c=0$. This family contains the above unimodular family of gradient singularities (set $a=b=c=t$). □

Remark 9.2. It should be underlined that the above three unimodular families have been investigated in [34], [35] and [29] in the context of the theory of variations of Lie algebras of derivations associated with isolated hypersurface singularities. More precisely, some exceptional values of the parameter naturally occur also as obstructions to the existence of continuous families of the Lie algebras of derivations of the *moduli* algebras (often called Milnor algebras), associated with unimodular families of hypersurface singularities. Among other things, a direct calculation shows that the dimension of the Lie algebra of derivations associated with any exceptional *zero-dimensional* fiber of such a family is *greater* than the corresponding dimension associated with the generic fiber of the family.

Example 9.3. For completeness, we will comment the last observation in more detail and compute explicitly the Lie algebras of derivations associated with fibers of the family $\mathrm{grad}(P_8)$ for *all* exceptional values of the parameter. First remark that for generic values of parameter α we have $\dim_{\mathsf{k}}\mathrm{Der}_{\mathsf{k}}(X_\alpha)=10$ (see [29, §2]). If $\alpha=0$, then the corresponding zero-dimensional moduli algebra is defined by the ideal (x^2,y^2,z^2) and $\mathrm{Der}_{\mathsf{k}}(X_0)\cong A_0\{x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}\}$ (cf. Proposition 8.1), so that the module of derivations is *finite-dimensional* and $\dim_{\mathsf{k}}\mathrm{Der}_{\mathsf{k}}(X_0)=12$ (cf. [29, Example in §2]). A similar result is true also if $\alpha^3=8$, since the corresponding fibers are contact equivalent.

Suppose now that $\alpha^3+1=0$; for these values of the parameter α, the Hesse family $F=x^3+y^3+z^3+3\alpha xyz$ of cubic hypersurfaces is degenerated, and the corresponding gradient singularity X_α is determined by the polynomials $x^2+\alpha yz$, $y^2+\alpha xz$ and $z^2+\alpha xy$, where $\alpha^3=-1$. If $\alpha=-1$, then these polynomials are the maximal minors of the Hankel matrix $\left[\begin{smallmatrix} x & y & z\\ y & z & x\end{smallmatrix}\right]$, and they define a *determinantal* one-dimensional singularity.

It is not difficult to see that the homogeneous polynomial $F|_{\alpha=-1}$ determines the divisor with *normal crossings* in the 3-dimensional space because

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(-x + \varrho y - \varrho^2 z)(-x - \varrho^2 y + \varrho z),$$

where ϱ is a cubic root of -1, $\varrho \neq -1$. Hence, one can change the coordinate functions in such a way that F transforms to the polynomial xyz and the local k-algebra A of the corresponding gradient singularity is determined by the ideal $I = (xy, xz, yz)$. That is, the germ $\mathrm{grad}(F)$ is the union of three coordinate axes, $\mu(\mathrm{grad}(F)) = 2$ and $\tau(\mathrm{grad}(F)) = 3$. In this case, the dimension of the Lie algebra of derivations $\mathrm{Der}_{\mathsf{k}}(A)$ is *infinite*; it is generated over the local k-algebra A by the three vector fields: $\mathrm{Der}_{\mathsf{k}}(A) \cong A\{x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}\}$. Hence, $\mathrm{rank}_A \mathrm{Der}_{\mathsf{k}}(A) = 3$. The same argument applies to both roots of the equation $\varrho^2 - \varrho + 1 = 0$. Finally, if $\alpha = \infty$, then $F = xyz$; this case has been just examined. For brevity, we will use the same notation $\mathrm{grad}(P_8^\infty)$ for the contact equivalent germs X_∞ and X_α, where $\alpha^3 = -1$.

Remark 9.4. In this way, Example 9.3 complements the results of [34], [35] and [29], since we describe the structure of the Lie algebra of derivations of the moduli algebras associated with *all* the exceptional values of the parameter of the family $\mathrm{grad}(P_8)$.

Of special note is a curious little-known property of the Hesse family at those exceptional points, where the corresponding gradient family is *not flat*.

Corollary 9.5. *Let $F_\alpha = x^3 + y^3 + z^3 + 3\alpha xyz$ be a family of cubic hypersurfaces in $\mathbb{C}^3$, where α runs over the projective line $\mathbb{P}^1_k$. Then the discriminant of the minimal versal deformation of any fiber X_α, where $\alpha^3 = -1$ or $\alpha = \infty$, is analytically isomorphic to the hypersurface singularity determined by the same cubic polynomial.*

Proof. Indeed, the minimal versal deformation of the *determinantal* space curve singularity, given by the ideal $I = (xy, xz, yz)$, is determined by the three equations $xy = 0$, $xz + t_1 z = 0$ and $yz + t_2 y + t_3 z = 0$, where t_1, t_2, t_3 are the coordinate functions in the 3-dimensional base space of the deformation. It is easy to verify that the discriminant of the deformation is given by the equation $t_1 t_2 t_3 = 0$.

Corollary 9.6. *There are two different (up to $\mathscr{K}$-equivalence) genotypes of the gradient zero-dimensional singularities associated with each of the hypersurface singularities P_8, X_9 and J_{10}. One corresponds to a gradient unimodular family, while another is associated with an ordinary gradient singularity with contact modality zero.*

For convenience, in the following list we collect all gradient singularities, which differ from the generic fibers of the families, by reference to the induced grading. An interesting phenomenon happens here. Indeed, the fiber at infinity of the unimodular family X_9 is *nonreduced*. In this case, the Tjurina and Milnor numbers are not defined. However, the fiber at infinity of the unimodular gradient family $\mathrm{grad}(X_9)$ is a 1-dimensional *isolated* singularity having an *embedded* component at the origin with $\tau = 3$, although in this situation the Milnor number is still undefined in the usual way. The same is also true for the family $\mathrm{grad}(J_{10})$.

Notation of classes grad(F)	Normal form of F π(F)	Normal form of grad(F) π(grad(F))	T^1−basis weights	τ	μ
grad(P_8^0)	$x^3+y^3+z^3$ (3;1,1,1)	(x^2, y^2, z^2) (2,2,2;1,1,1)	$[1]{:}1,y,z,yz$; $[2]{:}1,x,z,xz$;$[3]{:}1,x,y,xy$ $-2,-1,-1,0;-2,-1,-1,0;-2,-1,-1,0$	12	7
grad(P_8^∞)	xyz (3;1,1,1)	(yz, xz, xy) (2,2,2;1,1,1)	$(y,0,0)$; $(z,0,0)$; $(0,z,0)$ $-1;-1;-1$	3	2
grad(X_9^0)	x^4+y^4 (4;1,1)	(x^3, y^3) (3,3;1,1)	$[1]{:}1,x,y,xy,y^2,xy^2$; $[2]{:}1,x,y,xy,x^2,x^2y$ $-3,-2,-2,-1,-1,0$; $-3,-2,-2,-1,-1,0$	12	8
grad(X_9^∞)	x^2y^2 (4;1,1)	(xy^2, x^2y) (3,3;1,1)	(y,x); (xy,x^2); (y^2,xy) $-2;-1;-1$	3	–
grad(J_{10}^0)	x^3+y^6 (6;2,1)	(x^2, y^5) (4,5;2,1)	$[1]{:}1,y,y^2,y^3,y^4$; $[2]{:}1,x,xy,xy^2,xy^3,y,y^2,y^3$ $-4,-3,-2,-1,0$; $-5,-3,-2,-1,0,-4,-3,-2$	13	9

REMARK 9.7. From the preceding it is seen that, in contrast with deformation theory of *quasihomogeneous* complete intersections of positive dimension, *equisingular* families of zero-dimensional singularities with constant Milnor number may contain fibers with different Tjurina numbers. In particular, the corresponding fibers are not *analytically* isomorphic. In our setting it is possible to explain this phenomenon nonformally as follows. In the notation of Corollary 9.5, let denote by $X \to S$ the maximal *flat* subfamily (the locus of flatness) contained in a compactified family grad(P_8). Then usual arguments show (cf. [6], [20]) that the set of exceptional values of the parameters forms the locus of *nonflatness* of the relative cotangent cohomology module $\mathrm{T}^1(X/S)$, or the module of relative differentials $\Omega^1_{X/S}$, or the relative cotangent homology module $\mathrm{T}_1(X/S)$ (see also Section 11 below), etc. This is also true for the families grad(X_9) and grad(J_{10}).

10. Adjacencies. First it will be recalled some useful properties of zero-dimensional semiquasihomogeneous complete intersections.

Assume that X_0 is a complete intersection with an isolated singularity determined by a flat *quasihomogeneous* map $F_0 = (f_1, \dots, f_k)\colon \mathbb{C}^m \to \mathbb{C}^k$, $1 \leqslant k \leqslant m$, and $\pi(F_0) = (d_1, \dots, d_k; w_1, \dots, w_m)$. As usual, by the *type* of a class of semiquasihomogeneous germs of the form $F = F_0 + \sum c_i e_i$, where $c_i \in \mathbb{C}$ and $e_i \in \sum_{\nu>0} \mathrm{T}^1(X_0)_{(\nu)}$, we understand (and denote by the same symbol) the type $\pi(F_0)$ of the quasihomogeneous (principle) part F_0 of (any) representative of this class.

PROPOSITION 10.1. *In the same notation, suppose that F determines a semiquasihomogeneous zero-dimensional complete intersection. Then*

$$\mu(F_0) = \mu(F), \quad \tau(F_0) > \tau(F).$$

Proof. The proof goes similarly to the well-known line of reasoning used for complete intersections of positive dimension. Moreover, by usual arguments we may show that Theorem 7.2 can be extended to the semiquasihomogenous case as well. □

EXERCISE 10.2. The list from [9, Theorem 3] can be supplemented with the corresponding classes of gradient *semiquasihomogeneous* singularities.

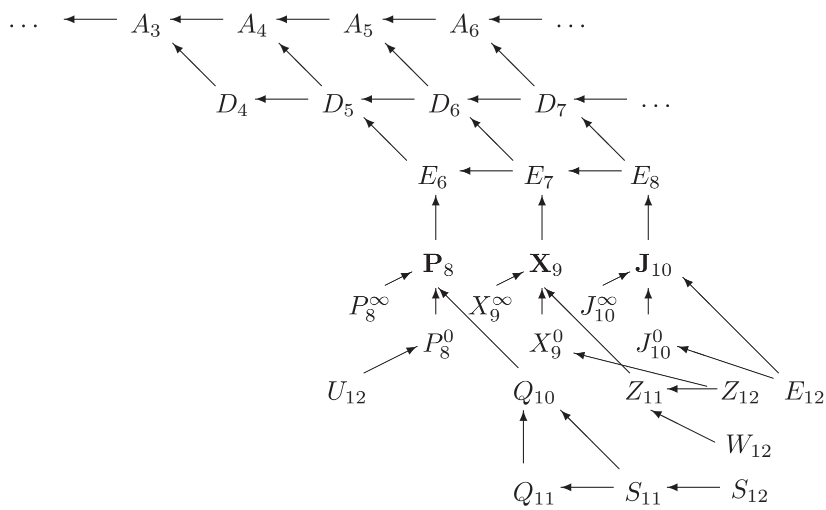
DEFINITION 10.3. *The class K of semiquasihomogeneous singularities of type π is said to be adjacent to the class K' of type π' if for any $X_0 \in K$ there exists a flat deformation $\psi\colon X \longrightarrow D$ of X_0 over a* small *disk $D \subset \mathbb{C}$ such that $\psi^{-1}(t) = X_t \in K'$ for all $t \in D \setminus \{0\}$. As usual, this property is denoted as follows: $K \longrightarrow K'$.*

In fact, there are different kinds of adjacencies between classes of singularities. In what follows, we will distinguish "generic" adjacencies denoted by the *long* arrow, and "nongeneric" ones (e.g., an adjacency of ordinary germs to the corresponding unimodular families at one point) denoted by *short* arrows (cf. [6, Proposition 5.3]).

Assertion 10.4. *An adjacency of two hypersurface singularities induces at least one adjacency of the corresponding gradient singularities.*

Example 10.5. The adjacency $X_9^0 \longrightarrow E_6$ of hypersurface germs, which is determined by the deformation $F_t = x^4 + y^4 + tx^3$, induces the "generic" adjacency $\mathrm{grad}(X_9^0) \longrightarrow \mathrm{grad}(E_6)$ of the corresponding gradient singularities. On the other hand, the "nongeneric" adjacency $\mathrm{grad}(X_9^0) \to \mathrm{grad}(X_9)$ is induced by the homogeneous deformation $F_t = x^4 + y^4 + tx^2y^2$ at the point $t = 0$. Using explicit formulas in [12], one can produce many other adjacencies for gradient semiquasihomogeneous singularities.

Herein we write down some adjacencies for *gradient* singularities and keep the notation from [9, Theorem 3]. For the sake of convenience the diagram $K \longrightarrow L$ denotes the adjacency $\mathrm{grad}(K) \longrightarrow \mathrm{grad}(L)$ of the corresponding *gradient* genotypes.



A family of isolated singularities is said to be *surrounding* (or, equivalently, *confining*) for germs with modality m if its fibers have contact modality $m+1$ and modularity $\wp \geq 1$. Such families are usually denoted by **bold** letters. In all known examples of complete intersections with isolated singularities of positive dimension, surrounding families (classes, genotypes) for singularities with modality m actually "demarcates", in the sense of the adjacency diagram, germs with modalities m and $m+1$ (see [6, Remark 5.5]).

Corollary 10.6. *The classes* $\mathrm{grad}(\mathbf{P}_8)$, $\mathrm{grad}(\mathbf{X}_9)$ *and* $\mathrm{grad}(\mathbf{J}_{10})$ *from the list of Assertion* 9.1 *are* surrounding *for gradient zero-dimensional singularities with contact modality 0.*

Proof. It is well-known that the classes P_8, X_9 and J_{10} are surrounding for the simple *hypersurface* singularities. On the other hand, it is not difficult to verify that all zero-dimensional gradient singularities, located under these families in the adjacency diagram, are *nonsimple* with contact modality at least 1. The class $\mathrm{grad}(P_8^\infty)$ is simple, but this genotype corresponds to a *one-dimensional* noncomplete intersection. □

REMARK 10.7. This raises the question of whether there is a compactification of the μ-constant family $\mathrm{grad}(\mathbf{P}_8)$ at infinity and at the three finite exceptional points, mentioned in Corollary 9.5, by *zero-dimensional* singularities which are adjacent to the family? The same question makes sense for the families $\mathrm{grad}(\mathbf{X}_9)$ and $\mathrm{grad}(\mathbf{J}_{10})$. It should be underlined that a similar question for τ-constant families of complete intersection curve singularities or isolated hypersurface singularities is a highly nontrivial problem (cf. [6, §4] or [20, §4]).

11. Cotangent homology of unimodular families. Here, we will compute the primitive ideals and the first cotangent homology modules (cf. Corollary 6.8) for the three unimodular families described above. First remark that for a *quasihomogeneous* polynomial F the primitive ideal of $\mathrm{Jac}(F)$ modulo its square contains at least one generator equal to F. We will denote other generators by φ_i, $i \geqslant 1$.

PROPOSITION 11.1. *In the notation of Assertion* 9.1 *and for all admissible values of the parameter α, the primitive ideals associated with the surrounding unimodular families of gradient singularities are represented as follows:*

$$\mathrm{grad}(\mathbf{P}_8)\colon \textstyle\int \mathrm{Jac}(F_\alpha) \approx (F_\alpha, \varphi_1, \varphi_2, \varphi_3), \text{ where } F_\alpha = x^3 + y^3 + z^3 + 3\alpha xyz,$$
$$\varphi_1 = -x^3 z + y^3 z,\ \varphi_2 = xy^2 z + (\alpha + \tfrac{2}{\alpha})x^2 z^2 + \tfrac{2}{\alpha} yz^3,$$
$$\varphi_3 = x^2 yz + (\alpha + \tfrac{2}{\alpha})y^2 z^2 + \tfrac{2}{\alpha} xz^3,$$
$$\alpha(\alpha^3 + 1)(\alpha^3 - 8) \neq 0;$$

$$\mathrm{grad}(\mathbf{X}_9)\colon \textstyle\int \mathrm{Jac}(F_\alpha) \approx (F_\alpha, \varphi_1, \varphi_2), \text{ where } F_\alpha = x^4 + y^4 + 2\alpha x^2 y^2,$$
$$\varphi_1 = x^3 y^2 + \tfrac{3}{\alpha} xy^4,\ \varphi_2 = x^2 y^3 + \tfrac{3}{5\alpha} y^5,$$
$$\alpha(\alpha^2 - 1)(\alpha^2 - 9) \neq 0;$$

$$\mathrm{grad}(\mathbf{J}_{10})\colon \textstyle\int \mathrm{Jac}(F_\alpha) \approx (F_\alpha, \varphi_1), \text{ where } F_\alpha = x^3 + y^6 + 3\alpha x^2 y^2,$$
$$\varphi_1 = x^2 y^3 - \tfrac{1}{\alpha^2} xy^5 + \tfrac{1}{7\alpha}(3 + \tfrac{5}{2\alpha^3})y^7,$$
$$\alpha(4\alpha^3 + 1) \neq 0.$$

Proof. In view of the exact sequence (4) from Section 2, it is enough to compute the corresponding primitive ideals. □

As an example, we will compute the primitive ideals of the corresponding gradient singularities for the exceptional values of the parameter. If $\alpha = 0$ then the gradient singularities, associated with the above three families, are defined by powers of variables. In this case, the primitive ideals, as well as the first cotangent homology modules, are computed in the proof of Corollary 7.4. In particular, for the Hesse family at the point $\alpha = 0$ we see that $\int \mathrm{Jac}(F_0)$ is generated modulo the square of the Jacobian ideal by the functions (x^3, y^3, z^3) or (F_0, y^3, z^3), where $F_0 = x^3 + y^3 + z^3$.

Using computations from Section 9 and Example 9.3, it is not difficult to describe the behavior of the cotangent homology of the family $\mathrm{grad}(\mathbf{P}_8)$ for the exceptional values of the parameter $\alpha \neq 0$. Thus, if $\alpha^3 = 8$, then $\alpha = 2\rho$, where ρ is a *primitive* cubic root of unity. In the notation of Proposition 11.1, we see that $\int \mathrm{Jac}(F_\rho)$ is generated modulo the square of the Jacobian ideal by the following functions:

$$F_\rho = x^3 + y^3 + z^3 + 6\rho xyz,\ \varphi_1 = xy^2 + \rho x^2 z + \rho^2 yz^2;\ \varphi_2 = x^2 y + \rho y^2 z + \rho^2 xz^2.$$

If $\alpha^3 + 1 = 0$, then $F_\alpha = x^3 + y^3 + z^3 + 3\alpha xyz$, and the germ $\mathrm{grad}(F_\alpha)$ is contact equivalent to the union of three coordinate axes; it is a *one-dimensional* noncomplete intersection. The same is also true if $\alpha = \infty$, i.e., $F_\infty = xyz$. Next, it is easy to verify that $\int \mathrm{Jac}(F_\infty) \cong (xyz)$ modulo the square of the Jacobian ideal. Hence, $\dim_{\mathsf{k}} \mathrm{T}_1(\mathrm{grad}(F_\infty)) = 1$, although $\dim_{\mathsf{k}} \mathrm{T}^1(\mathrm{grad}(F_\infty)) = 3$.

Corollary 11.2. *As in Corollary* 9.5, *let $F_\alpha = x^3 + y^3 + z^3 + 3\alpha xyz$ be a family of cubic hypersurfaces in $\mathbb{C}^3$, where α runs over the projective line $\mathbb{P}^1_k$. Then the families of the primitive ideals $\int \mathrm{Jac}(F_\alpha)$ and the first cotangent homology modules $T_1(X_\alpha)$ vary continuously over $\mathbb{P}^1_k$ with eight gaps. More precisely,* $\dim_k T_1(X_\alpha) = 10$ *for the generic fiber,* $\dim_k T_1(X_\alpha) = 12$ *if $\alpha(\alpha^3 - 8) = 0$, while* $\dim_k T_1(X_\alpha) = 1$ *if $\alpha^3 = -1$ or $\alpha = \infty$.*

The families $\mathrm{grad}(\mathbf{X}_9)$ and $\mathrm{grad}(\mathbf{J}_{10})$ are analyzed in a similar manner.

12. Gradings, derivations and further applications. It is well-known that, for isolated complete intersection singularities of *positive* dimension, the grading is determined *uniquely*, except in the case of the intersection of hypersurfaces of multiplicity two (see [3, (6.4)]). Moreover, such singularities are weighted homogenous, i.e., they are *positively* graded (in the exceptional case one should use the *normalized* grading of K.Saito). In this case, the dimensions of graded components $T^1(X)_{(\nu)}$, $\nu \in \mathbb{Z}$, as well as the Tjurina number τ, the Milnor number μ, the contact modality m and the inner modality m_0 (which is equal to the total number of diagonal and upper diagonal vector-monomials containing in the monomial basis of $T^1(X)$) are analytic *invariants* of the corresponding quasihomogeneous singularity X. For example, simple singularities have contact and inner modalities equal to zero. It is known also that the deformation theory of X possesses many remarkable properties if the space $T^1(X)$ has *negative grading*, i.e., $T^1(X)_{(\nu)} = 0$ for all $\nu > 0$.

The zero-dimensional case is quite different from that. For example, the *nongradient* singularity X, given by the ideal $(x^2 + yz, z^2 + y^3, xy)$, has type $(10, 12, 9; 5, 4, 6)$, $\mu(X) = 8$, $\tau(X) = 11$, and $\mathscr{P}(T^1(X); t) = t^{-2} + t^{-3} + 2t^{-4} + t^{-5} + t^{-6} + t^{-7} + t^{-8} + t^{-9} + t^{-10} + t^{-12}$, i.e., the space $T^1(X)$ has *negative* grading. In this case, the *contact modality* of X is equal to 1 (see [13, Table 3, Type 5]), i.e., the singularity X is a *nonsimple* germ, although $m_0(X) = 0$. Further calculations show that $\mathscr{P}(\Omega^1_X; t) = t^4(1 + t + t^2 + t^4 + t^5 + 2t^6 + 2t^7 + t^8 + t^{12})$, $\mathscr{P}(\Omega^2_X; t) = t^9(1 + t + t^2 + t^6)$, $\mathscr{P}(\Omega^3_X; t) = t^{15}$, and so on.

Now we will show that, in certain cases, one can compute the Poincaré polynomial $\mathscr{P}(T^1(X); t)$ with the use of formulas obtained above (see, e.g., Proposition 7.10 and Example 7.12).

Example 12.1. In the notation of Example 7.12, let us consider the singularity $\mathrm{grad}(S_{12})$. Then $\varphi = x^2 z + yz^2 + xy^3$, and $f_1 = y^3 + xz$, $f_2 = z^2 + 12xy^2$, $f_3 = x^2 + yz$. In this case, the grading is defined *uniquely*, $\pi(A_0) = (9, 10, 8; 4, 3, 5)$. Let $f = f_2$. Then $\mathscr{P}(\Omega^1_f; t) = t^3(1 + t + t^2)(1 + t^4)(1 + t^3 + t^6)$, so that $\dim_{\mathsf{k}} \Omega^1_f = 18$. On the other hand, $f \notin \mathrm{Ann}(\Omega^1_f)$, $\deg(f) = 10$, and one can verify that

$$\mathscr{P}(\Omega^1_{A_0}; t) = \mathscr{P}(\Omega^1_f / f\Omega^1_f; t) = \mathscr{P}(\Omega^1_f; t) - t^{10}\mathscr{P}(\Omega^1_f; t) + t^{10}\mathscr{P}(\mathrm{Ker}(f); t),$$

so that $\mathscr{P}(\Omega^1_f / f\Omega^1_f; t) = \mathscr{P}(\Omega^1_f; t) - t^{14}$ and $\tau(\mathrm{grad}(S_{12})) = \dim_{\mathsf{k}} \Omega^1_f / f\Omega^1_f = 17$.

However, all zero-dimensional singularities of the one-parameter family A_α given by the family of ideals $I_\alpha = (y^3 + xz, z^2 + \alpha xy^2, x^2 + yz)$, $\alpha \in \mathsf{k}$, have the same type of homogeneity and, consequently, the same Milnor number, $\mu(A_\alpha) = 11$. In other words, it is an *equisingular* family. Moreover, in this family $\tau(A_\alpha) = 16$, $\alpha \neq 12$, and $\tau(A_{12}) = 17$, i.e., the Tjurina number "jumps" at the point $\alpha = 12$. Hence, for $\alpha \neq 12$ all singularities are not isomorphic analytically to $\operatorname{grad}(S_{12})$. This phenomenon is typical for *modular* families of quasihomogenous zero-dimensional singularities, in particular, for $\operatorname{grad}(\mathbf{P}_8)$, $\operatorname{grad}(\mathbf{X}_9)$ and $\operatorname{grad}(\mathbf{J}_{10})$ (cf. [34]).

The next application is closely related to some problems concerning the grading structure of the module of derivations of a graded zero-dimensional complete intersection.

PROPOSITION 12.2. *Let A_0 be a zero-dimensional complete intersection given by a sequence of functions $f_1, \dots, f_m$ of type $\pi = (d_1, \dots, d_m; w_1, \dots, w_m)$. Assume that the first $m-1$ generators determine a reduced one-dimensional singularity, and denote the remaining function by f. Then the Poincaré polynomial $\mathscr{P}(\Omega^1_f; t)$ of the module Ω^1_f contains a monomial of maximal degree $t^{\upsilon_{\max}}$, where*

$$\upsilon_{\max} = \sum_{j=1}^{m-1} d_j - \sum_{i=1}^{m} w_i + \max\{d_1, \dots, d_m\}$$

Proof. We denote the Poincaré polynomial $\mathscr{P}(\operatorname{Tors}\Omega^1_A; t)$ by $\mathscr{P}_\mu(t)$, and the polynomial $\mathscr{P}(T^1(A); t)$ by $\mathscr{P}_\tau(t)$. According to formulas in [2] or [1, Lemma 4.3] for a one-dimensional singularity, we have

$$\mathscr{P}_\mu(t) = 1 + (\sum_{i=1}^{m} t^{w_i} - \sum_{j=1}^{m-1} t^{d_j} - 1) \prod_{j=1}^{m-1} (1 - t^{d_j}) / \prod_{i=1}^{m} (1 - t^{w_i}).$$

In view of the formula of Assertion 7.7, we get the identity

$$\mathscr{P}(\Omega^1_f; t) = \mathscr{P}_\mu(t) + \mathscr{P}(A_0; t) - 1, \tag{21}$$

where $\mathscr{P}(A_0; t) = \prod_{j=1}^m (1 - t^{d_j}) / \prod_{i=1}^m (1 - t^{w_i})$ and $A_0 = A/(f)A$. It is well-known that the maximal degree of a monomial in t contained in this polynomial is equal to $c_0 = \sum_{j=1}^m d_j - \sum_{i=1}^m w_i$. It is easy to see that c_0 is equal to the degree of the leading monomial in the monomial basis of the local algebra A_0. Next, local duality implies that $\mathscr{P}_\mu(t) = t^c \mathscr{P}_\tau(t^{-1})$, where $c = \sum_{j=1}^{m-1} d_j - \sum_{i=1}^m w_i$ (see [1, Theorem 3.6]), so that the minimal degree of the monomial in t occurring in $\mathscr{P}_\tau(t)$ is equal to $\upsilon_{\min} = -\max\{d_1, \dots, d_{m-1}\}$ (see [1, Corollary 4.6]). Hence the Poincaré polynomial $\mathscr{P}_\mu(t)$ contains a monomial of maximal degree

$$\upsilon_{\max} = \max\{c_0, c + \max\{d_1, \dots, d_{m-1}\}\} = \max\{c + d_m, c + \max\{d_1, \dots, d_{m-1}\}\}.$$

This gives the required statement. □

COROLLARY 12.3. *Under the conditions of Proposition* 12.2, *suppose that $d_m \geqslant \max\{d_1, \dots, d_{m-1}\}$. Then the Lie algebra of derivations* $\operatorname{Der}_{\mathsf{k}}(A_0)$ *does not contain derivations of negative weights, that is,* $\operatorname{Der}_{\mathsf{k}}(A_0)_{(\nu)} = 0$ *for all $\nu < 0$. In other words,* $\operatorname{Der}_{\mathsf{k}}(A_0)$ *has positive grading.*

Proof. Indeed, $\mathscr{P}(\operatorname{Der}_{\mathsf{k}}(A_0); t) = t^{c_0} \mathscr{P}(\Omega^1_{A_0}; t^{-1})$ in view of duality in the cotangent homology (see [5, §7]). On the other hand, $\Omega^1_{A_0} \cong \Omega^1_f / f\Omega^1_f$. Therefore, the value $\upsilon_{\max} = c + \max\{d_1, \dots, d_m\}$ for the polynomial $\mathscr{P}(\Omega^1_f / f\Omega^1_f; t)$ *does not exceed* the corresponding value for $\mathscr{P}(\Omega^1_f;t)$. As a result, we obtain the following estimate of $\upsilon_{\min}$ for the polynomial $\mathscr{P}(\mathrm{Der}_{\mathsf{k}}(A_0);t)$:

$$\upsilon_{\min} \geqslant c_0 - \upsilon_{\max} = d_m - \max\{d_1, \ldots, d_m\} \geqslant 0.$$

Thus, the weight of any derivation is not less than 0, the weight of the Euler vector field. □

Remark 12.4. For completeness, we mention that another proof of Corollary 12.3 is given in the paper [25]. In fact, this statement is a particular case of a general conjecture of the author (see [4] and [5, §7]) that there are no vector fields of negative weight on weighted homogeneous zero-dimensional complete intersections. In this way, using the dualities mentioned in Sections 7-8, one can describe the grading structure on T^1 as well. In this connection, it is useful to note also that, on quasihomogeneous isolated complete intersection singularities of positive dimension, only Hamiltonian vector fields may have negative weights (see [2, §6]). However, it is easy to see that on zero-dimensional complete intersections there are no Hamiltonian fields at all.

It should be also underlined that a series of interesting examples, concerning this conjecture and its reformulations, are examined in a series of works in a rather different context and setting (see, e.g., [36] and the references there).

Corollary 12.5. *Under the assumptions and in the notation of Proposition* 12.2*, the Tjurina number of a zero-dimensional complete intersection satisfies the following conditions:*

$$\frac{d_1 \cdots d_m}{w_1 \cdots w_m} \leqslant \tau(A_0) \leqslant \frac{d_1 \cdots d_{m-1}}{w_1 \cdots w_m}\Big(\sum_{j=1}^{m} d_j - \sum_{i=1}^{m} w_i\Big).$$

Proof. Indeed, in view of [5, Asertion 4.2] for arbitrary weighted homogeneous singularities of embedding dimensions greater than 1, we have the inequality $\dim_{\mathsf{k}} \Omega^1_{A_0} \geqslant \dim_{\mathsf{k}} A_0 = \mu(A_0) + 1$. Next, $\dim_{\mathsf{k}} A_0 = \mathscr{P}(A_0, t)|_{t=1} = \frac{d_1 \cdots d_m}{w_1 \cdots w_m}$, and $\tau(A_0) = \dim_{\mathsf{k}} \Omega^1_{A_0}$ by Proposition 6.10.

On the other hand, as before, $\dim_{\mathsf{k}} \Omega^1_{A_0} \leqslant \dim_{\mathsf{k}} \Omega^1_f$, so that it remains to calculate the value of the polynomial $\mathscr{P}(\Omega^1_f, t)$ at $t = 1$ by the formula (21), taking into account the relation

$$\mathscr{P}_\mu(t)|_{t=1} = 1 + \frac{d_1 \cdots d_{m-1}}{w_1 \cdots w_m}\Big(\sum_{j=1}^{m-1} d_j - \sum_{i=1}^{m} w_i\Big)$$

obtained in [1, Theorem 2.7]. □

We also mention that both obtained bounds are generally *unimprovable*; they turn into equalities in many cases. For example, for the singularities $\mathrm{grad}(D_k)$, $k \geq 4$, the left and right equalities hold, for the singularity $\mathrm{grad}(Q_{12})$ from Example 7.12, the right equality is attained, and so on.

It will be recalled that for complete intersections with isolated singularities of *positive* dimension, the inequality $0 \leqslant \tau \leqslant \mu$ holds. Moreover, $\tau = \mu$ is *equivalent* to quasihomogeneity in several cases: for complete intersections of dimension at least 2 (see [31]), for unobstructed Gorenstein curves (see [18]), for normal Gorenstein surfaces (see [32]) and some others. By contrast, in all known to the author examples of *smoothable* quasihomogeneous non-Gorenstein curves, as a rule, the inequality $\tau \geqslant \mu + 1$ holds (cf. [7, Example 6] or [10, Proposition 6]). In this case, the difference

$\tau - \mu$ is equal to the number of "nonvanishing cocycles" in the cohomology of the nonsingular fibre of the minimal versal deformation.

On the other hand, it is evident that any zero-dimensional singularity of embedding dimension 1 is isomorphic to a simple A_k-singularity for a suitable $k \geqslant 1$. Thus, in this case, $\tau = \mu$. Next, for a zero-dimensional singularity of embedding dimension greater than 1 with trivial group $H^0_{DR}(\Omega^\bullet_{A_0})$, the Tjurina and Milnor numbers satisfy the inequality $\tau \geqslant \mu + 1$ (see [5]). If, in addition, we also assume that the singularity A_0 is weighted homogeneous, then the equality $\tau = \mu + 1$ implies that A_0 is isomorphic to a simple $\mathrm{grad}(D_k)$-singularity, $k \geqslant 4$. Anyway, we derive the following lower and rather crude upper estimates: $\mu + 1 \leqslant \tau \leqslant (\mu + 1)^m$. Apparently, both estimates remains valid even in the *nongraded* case as well.

As follows from the results described above, the upper bound can be improved for weighted homogeneous zero-dimensional *complete intersections* with embedding dimension $m \geqslant 2$. Namely, we have $\mu + 1 \leqslant \tau \leqslant m(\mu + 1)$, or, equivalently,

$$1 \leqslant \frac{\tau}{\mu + 1} \leqslant m, \tag{22}$$

and for the class of singularities mentioned in Assertion 7.7, Corollary 12.5 gives sharp upper and lower bounds.

Remark 12.6. Let F be a weighted homogeneous polynomial with an isolated critical point of type $(1; w_1, \dots, w_m)$ in the *normalized* grading and let A_0 be the Milnor algebra of F, i.e., $A_0 = P/\mathrm{Jac}(F)P$. In the paper [37] the authors conjectured that

$$\dim_{\mathsf{k}} \mathrm{Der}_{\mathsf{k}}(A_0) \leqslant m\mu(F) - \sum_{i=1}^{m} \Big(\frac{1}{w_1} - 1\Big) \dots \widehat{\Big(\frac{1}{w_i} - 1\Big)} \dots \Big(\frac{1}{w_m} - 1\Big),$$

where the hat over an element of the product means that this element is omitted.

In support of the conjecture the case of polynomials in two variables and some other particular cases are examined in the recent papers [37], [38]. Since $\dim_{\mathsf{k}} \mathrm{Der}_{\mathsf{k}}(A_0) = \tau(A_0)$ by Proposition 6.10 and $\mu(F) = \mu(A_0) + 1$, we see that this estimate is better than the upper one in (22). In its turn, for singularities satisfying the requirements of Corollary 12.3 (including the classes $\mathrm{grad}(D_k)$, $k \geqslant 5$, $\mathrm{grad}(E_7)$, $\mathrm{grad}(Z_{12})$, $\mathrm{grad}(S_{12})$, $\mathrm{grad}(Q_{12})$, $\mathrm{grad}(E_{13})$ and $\mathrm{grad}(W_{13})$ in the notation of [9, Theorem 3]), the upper bound of Corollary 12.5 is still better than this hypothetical estimate.

Generally, in the context of our study, for any *gradient* zero-dimensional singularity A_0 associated with a weighted homogeneous polynomial F of type $(d; w_1, \dots, w_m)$ and for any $p \geqslant 0$, we can propose the following estimate

$$\dim_{\mathsf{k}} \Omega^p_{A_0} \leqslant \sum_{1 \leqslant i_1 < \dots < i_p \leqslant m} \Pi_1(\overline{i_1, \dots, i_p}) \cdot \Pi_2(i_1, \dots, i_p), \tag{23}$$

where $\overline{i_1, \dots, i_p}$ denotes the complementary subset $[1, m] \setminus \{i_1, \dots, i_p\}$, $\Pi_\ell(i_1, \dots, i_p) = (\frac{d}{w_{i_1}} - \ell) \dots (\frac{d}{w_{i_p}} - \ell)$ and $\Pi_\ell(\emptyset) = 1$.

Indeed, for $p = 0$ this relation becomes an equality, which gives the usual expression for the Milnor number of the function F, since $\mu(F) = \dim_{\mathsf{k}} A_0$. For $p = 1$ the upper estimate (23) is equivalent to the above conjecture because $\dim_{\mathsf{k}} \Omega^1_{A_0} = \dim_{\mathsf{k}} \mathrm{Der}_{\mathsf{k}}(A_0)$ by Proposition 6.10. Next, for Brieskorn-Pham polynomials the equality is attained for all $p \geqslant 1$ (cf. Remark 7.5 and Proposition 8.1), etc.

Finally, without going into detail, we note that there are several intriguing (more or less known) relationships between the Tjurina and Milnor numbers and dimensions of the modules of differential forms on zero-dimensional singularities. It makes sense to mention some of them.

Assertion 12.7. *Let A_0 be the smoothable zero-dimensional singularity determined by a sequence of functions $(f_1, \dots, f_k)$ in the ring $P = k\langle z_1, \dots, z_m\rangle$ such that $A_0 = P/(f_1, \dots, f_k)P$, where $k \geqslant m$. Then*

$$\mu(A_0) = \sum_{p=1}^{m} (-1)^{p-1} \dim_k \Omega^p_{A_0} + \dim_k \operatorname{Ker}(d_{A_0}) - 1.$$

Moreover, if $k = m$ then A_0 is a complete intersection, and we have the identity

$$\tau(A_0) - \mu(A_0) = \sum_{p=2}^{m} (-1)^{p} \dim_k \Omega^p_{A_0} - \dim_k \operatorname{Ker}(d_{A_0}) + 1.$$

Next, for a function F with an isolated critical point

$$\mu(F) = \sum_{p=1}^{m} (-1)^{p-1} \dim_k \Omega^p_{A_0} + \dim_k \operatorname{Ker}(d_{A_0}),$$

where $A_0 = P/\operatorname{Jac}(F)P$ is the Milnor algebra of F. There are similar presentations for the corresponding homogeneous components in the graded case, all these numerical invariants behave semicontinuously under deformations, and so on.

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