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MULTIPLE SOLUTIONS FOR LOGARITHMIC SCHRÖDINGER EQUATIONS WITH CRITICAL GROWTH*

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Dedicate to Professor Ling Hsiao on the occasion of her 80th birthday

Abstract. In this paper, we establish the existence of positive ground state solution and least energy sign-changing solution for the following logarithmic Schrödinger equation

$$-\Delta u + V(x)u = u\log u^2 + |u|^{2^*-2}u, \quad x \in \mathbb{R}^N.$$

It is known that the corresponding variational functional is not well defined in $H^1(\mathbb{R}^N)$. Via direction derivative and constrained minimization method, we first prove the existence of positive ground state solution and least energy sign-changing solution for the following subcritical problem

$$-\Delta u + V(x)u = u\log u^2 + |u|^{p-2}u, \quad x \in \mathbb{R}^N.$$

Then, we analyze the behavior of solutions for subcritical problem and pass the limit as the exponent p approaches to 2^*.

Key words. Logarithmic Schrödinger equations, non-smooth analysis, ground state solutions; multiple solutions, variational methods.

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1. Introduction. In this paper, we study the following logarithmic Schrödinger equation

$$-\Delta u + V(x)u = u\log u^2 + |u|^{2^*-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, the external potential $V(x) \in \mathcal{C}(\mathbb{R}^N)$. Equation (1.1) is closely related to the time-dependent logarithmic Schrödinger equation

$$i\frac{\partial \Phi}{\partial t} - \Delta\Phi + V(x)\Phi - \Phi\log|\Phi|^2 - \omega|\Phi|^{2^*-2}\Phi = 0, \text{ in } \mathbb{R}^N \times \mathbb{R}_+, \tag{1.2}$$

and the nonlinear Klein-Gordon equation

$$\frac{\partial^2 \Phi}{\partial t^2} - \Delta\Phi - \Phi\log|\Phi|^2 - \omega|\Phi|^{2^*-2}\Phi = 0, \text{ in } \mathbb{R}^N \times \mathbb{R}_+. \tag{1.3}$$

Problem (1.2) and (1.3) admit plenty of applications related to quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, open quantum systems, effective quantum gravity, theory of superfluidity and Bose-Einstein condensation, see [1, 3, 4, 7, 8, 10, 27, 29] and the references therein.

The mathematical literature concerning the logarithmic Schrödinger equation does not seem to be very extensive. In particular, if $\omega = 0$, Cazenave and Haraux [11]

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studied the existence and uniqueness of solutions for the associated Cauchy problem of (1.2) in a suitable functional framework. Guerrero et al [16] studied the global H^1 solvability of (1.2) in dimension three. The orbital stability of the ground state solution of (1.2), with respect to radial or nonradial perturbations, was investigated by Cazenave and Lions [12], Cazenave [9], Ardila [2]. Recently, Carles and Gallagher [8] studied universal dynamics for the defocusing logarithmic Schrödinger equation (1.2).

Equation (1.1) is also closely related to the classical Brezis-Nirenberg problem. Since the pioneering work of Brezis and Nirenberg [5], there have been extensive interests for semilinear problem with critical exponent. See for example, [6, 13, 15, 24] and references therein.

Equation (1.1) is formally associated with the energy functional $I : H^1(\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla u|^2 + (V(x)+1)u^2\big)dx - \frac{1}{2}\int_{\mathbb{R}^N} u^2 \log u^2 dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*} dx. \tag{1.4}$$

By using the following standard logarithmic Sobolev inequality (see Theorem 8.14 in [18])

$$\int_{\mathbb{R}^N} u^2 \log u^2 dx \le \frac{a^2}{\pi}\|\nabla u\|_2^2 + \big(\log \|u\|_2^2 - N(1+\log a)\big)\|u\|_2^2, \quad \text{for } u \in H^1(\mathbb{R}^N) \text{ and } a > 0, \tag{1.5}$$

it is easy to see that $\int_{\mathbb{R}^N} u^2 \log u^2 dx < +\infty$ for all $u \in H^1(\mathbb{R}^N)$, but there exists $u \in H^1(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} u^2 \log u^2 dx = -\infty$. Indeed, choosing smooth function

$$u(x) = \begin{cases} (|x|^{N/2}\log|x|)^{-1}, & |x| \ge 3, \\ 0, & |x| \le 2, \end{cases} \tag{1.6}$$

one can verify directly that $u \in H^1(\mathbb{R}^N)$, but $\int_{\mathbb{R}^N} u^2 \log u^2 dx = -\infty$. Thus, in general, I fails to be finite and losses $\mathcal{C}^1$ smooth on $H^1(\mathbb{R}^N)$.

Due to this loss of smoothness, the classical critical point theory cannot be applied for I. In order to study the existence of solutions, several approaches developed in the literature. In [9], Cazenave worked in an Orlicz space W endowed with Luxemburg type norm in order to make the functional $I : W \to \mathbb{R}$ well defined and $\mathcal{C}^1$ smooth. In [23], by applying non-smooth critical point theory for lower semi-continuous functionals, Squassina and Szulkin studied the following logarithmic Schrödinger equation

$$-\Delta u + V(x)u = Q(x)u \log u^2, \quad x \in \mathbb{R}^N, \tag{1.7}$$

where $V(x)$ and $Q(x)$ are spatially periodic. They showed the existence of positive ground state solution and infinitely many high energy solutions, which are geometrically distinct under $\mathbb{Z}^N$-action. See also [14, 17] for more non-smooth variational framework to logarithmic Schrödinger equation.

By using penalization technique, Tanaka and Zhang [26] obtained infinitely many multi-bump geometrically distinct solutions of equation (1.7). The authors first penalized the nonlinearity around the origin, then by taking an approach of using spatially $2L$-periodic problems ($L \gg 1$), they proved the existence of infinitely many multi-bump geometrically distinct solutions for the modified equation. Finally, via a priori estimates, they obtained the solutions for the original equation by taking the limit. We also refer the reader to the reference [16, 30] for the approach of using penalization.

Via direction derivative and constrained minimization method, Shuai [22] investigated the existence and nonexistence of positive ground state solution, least energy sign-changing solution and infinitely many nodal solutions for equation (1.7) with $Q(x) \equiv 1$ under different types of potentials.

Finally, it should be mentioned that Wang and Zhang [29] proved the positive ground state solution of the power-law equations

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}u, \quad x \in \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} u(x) = 0 \end{cases}$$

converges to the Gausson $U(x) := e^{\frac{N}{2}} e^{-\frac{\lambda}{4}|x|^2}$ as $p \to 2^+$, which, up to translations, is the unique positive solution of the logarithmic equation

$$\begin{cases} -\Delta u = \lambda u \log u^2, \quad x \in \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases}$$

The authors [29] also proved same results hold for bound state solutions.

Inspired by the above mentioned results, in this paper we study the existence of multiple solutions for equation (1.1). We assume the potential $V : \mathbb{R}^N \to \mathbb{R}$ satisfies

(V_1) $V(x) \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$, $0 < V(x) \leq \liminf\limits_{|x|\to\infty} V(x) := V_\infty < +\infty$, and $V(x) < V_\infty$ in a subset of positive Lebesgue measure.

(V_2) There are positive constants M, A and m such that

$$V(x) \leq V_\infty - \frac{A}{1+|x|^m} \quad \text{for } |x| \geq M.$$

Define

$$H := \big\{u \in H^1(\mathbb{R}^N) \;\big|\; \int_{\mathbb{R}^N} V(x)u^2 dx < +\infty\big\}$$

with the inner product and norm

$$(u,v)_H := \int_{\mathbb{R}^N} \nabla u \nabla v + V(x)uv dx, \quad \|u\|_H := (u,u)^{\frac{1}{2}},$$

and denote the $H^1(\mathbb{R}^N)$ norm by

$$\|u\| := \Big(\int_{\mathbb{R}^N} |\nabla u|^2 + V_\infty u^2 dx\Big)^{\frac{1}{2}},$$

one can verify the above two norms are equivalent. We define

$$\mathcal{D} := \big\{u \in H : \; u^2 \log u^2 \in L^1(\mathbb{R}^N)\big\} = \big\{u \in H^1(\mathbb{R}^N) : \; u^2 \log u^2 \in L^1(\mathbb{R}^N)\big\}.$$

Obviously, $\mathcal{D}$ is nonempty since $\mathcal{C}_0^\infty(\mathbb{R}^N) \subset \mathcal{D}$.

We say $u \in H^1(\mathbb{R}^N)$ is a *weak solution* of (1.1), if $u^2 \log u^2 \in L^1(\mathbb{R}^N)$ and u satisfies

$$\int_{\mathbb{R}^N} \nabla u \nabla \varphi + V(x)u\varphi dx = \int_{\mathbb{R}^N} u\varphi \log u^2 dx + \int_{\mathbb{R}^N} |u|^{2^*-2}u\varphi dx, \quad \text{for all } \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N). \tag{1.8}$$

We remark that one can not replace $\mathcal{C}_0^\infty(\mathbb{R}^N)$ by $H^1(\mathbb{R}^N)$ in the last formula. In fact, by Lemma 2.2 [26], u is a weak solution of (1.1) if and only if $u \in \mathcal{D}$ and (1.8) hold for all $\varphi \in \mathcal{D}$.

Similar as [20, 21], for $u \in \mathcal{D}$ and $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, *the derivative of I in the direction ϕ at u*, denoted by $\langle DI(u), \phi\rangle$, is defined as $\lim_{t\to 0^+}[I(u+t\phi) - I(u)]/t$. It is easy to check that

$$\langle DI(u), \phi\rangle = \int_{\mathbb{R}^N} \nabla u \nabla \phi + V(x) u\phi dx - \int_{\mathbb{R}^N} \phi u \log u^2 dx - \int_{\mathbb{R}^N} |u|^{2^*-2} u\phi dx. \quad (1.9)$$

Obviously, if $u \in \mathcal{D}$, then I has direction derivative in every direction $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ at u. Moreover, $u \in \mathcal{D}$ is a weak solution of equation (1.1) if and only if the direction derivative equals to zero in every direction $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$. We also say that $u \in H^1(\mathbb{R}^N)$ is a critical point of I if $u \in \mathcal{D}$ and $\langle DI(u), \varphi\rangle = 0$ for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$.

Our main results can be stated as follows.

THEOREM 1.1. *(i) Assume (V_1) holds and $N \geq 4$. Then equation* (1.1) *possesses a positive ground state solution $u \in \mathcal{D}$ with*

$$I(u) = c := \inf_{\mathcal{N}} I(u),$$

where

$$\mathcal{N} := \big\{u \in \mathcal{D} \setminus \{0\}, J(u) = 0\big\}$$

and

$$J(u) := \int_{\mathbb{R}^N} \big(|\nabla u|^2 + V(x)u^2\big) dx - \int_{\mathbb{R}^N} u^2 \log u^2 dx - \int_{\mathbb{R}^N} |u|^{2^*} dx.$$

(ii) Assume (V_1)–(V_2) hold and $N \geq 4$. Then equation (1.1) *possesses least energy sign-changing solution $u \in \mathcal{D}$ with*

$$I(u) = m := \inf_{\mathcal{M}} I(u),$$

where

$$\mathcal{M} := \big\{u \in \mathcal{D}, u^+ \in \mathcal{N}, u^- \in \mathcal{N}\big\},$$

and $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$.

As we mentioned above, the functional I is not well-defined in $H^1(\mathbb{R}^N)$, thus the classical variational method can not be applied for I. We prove Theorem 1.1 by three steps:

(1) Establish the existence of positive ground state solution and least energy sign-changing solution for the following subcritical problem

$$-\Delta u + V(x)u = u \log u^2 + |u|^{p-2}u, \quad x \in \mathbb{R}^N, \quad (1.10)$$

where $2 < p < 2^*$;

(2) Analyze the behavior of solutions for the subcritical problem (1.10);

(3) Pass limit as the exponent p approaches to 2^*.

Formally, the variational functional corresponding to equation (1.10) is defined by

$$I_p(u) := \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla u|^2 + (V(x)+1)u^2\big)dx - \frac{1}{2}\int_{\mathbb{R}^N} u^2 \log u^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx, \quad u \in H.$$

To study the subcritical problem (1.10), we also need to consider the following equation

$$-\Delta u + V_\infty u = u \log u^2 + |u|^{p-2}u, \quad x \in \mathbb{R}^N. \tag{1.11}$$

Similarly, we define

$$\begin{aligned} I_p^\infty(u) := \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla u|^2 + (V_\infty+1)u^2\big)dx - \frac{1}{2}\int_{\mathbb{R}^N} u^2 \log u^2 dx \\ -\frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx, \quad u \in H^1(\mathbb{R}^N). \end{aligned}$$

Our results on the subcritical problem (1.10)–(1.11) is as follows.

THEOREM 1.2. *Assume $2 < p < 2^*$ and (V_1) hold. Then*
(i) Equation (1.11) *has a positive ground state solution $u \in \mathcal{D}$ with*

$$I_p^\infty(u) = c_p^\infty := \inf_{\mathcal{N}_p^\infty} I_p^\infty(u),$$

where

$$\mathcal{N}_p^\infty := \big\{u \in \mathcal{D} \setminus \{0\}, J_p^\infty(u) = 0\big\}$$

and

$$J_p^\infty(u) := \int_{\mathbb{R}^N} \big(|\nabla u|^2 + V_\infty u^2\big)dx - \int_{\mathbb{R}^N} u^2 \log u^2 dx - \int_{\mathbb{R}^N} |u|^p dx.$$

(ii) Equation (1.10) *has a positive ground state solution $u \in \mathcal{D}$ with*

$$I_p(u) = c_p := \inf_{\mathcal{N}_p} I_p(u),$$

where

$$\mathcal{N}_p := \big\{u \in \mathcal{D} \setminus \{0\}, J_p(u) = 0\big\}$$

and

$$J_p(u) := \int_{\mathbb{R}^N} \big(|\nabla u|^2 + V(x)u^2\big)dx - \int_{\mathbb{R}^N} u^2 \log u^2 dx - \int_{\mathbb{R}^N} |u|^p dx.$$

(iii) If in addition (V_2) is satisfied, then equation (1.10) *possesses a sign-changing solution $u \in \mathcal{D}$ with*

$$I_p(u) = m_p := \inf_{\mathcal{M}_p} I_p(u),$$

where

$$\mathcal{M}_p := \big\{u \in \mathcal{D}, u^+ \in \mathcal{N}_p, u^- \in \mathcal{N}_p\big\}.$$

Moreover, we have $m_p < c_p + c_p^\infty$.

Based on Theorem 1.2, we pass limit as the exponent p approaches to the critical exponent. The main obstacle here is lack of compactness, so we have to study the following limit problem

$$-\Delta u + V_\infty u = u \log u^2 + |u|^{2^*-2} u, \quad x \in \mathbb{R}^N. \tag{1.12}$$

Formally, we define

$$I^\infty(u) := \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla u|^2 + (V_\infty+1)u^2\big)\,dx - \frac{1}{2}\int_{\mathbb{R}^N} u^2 \log u^2\,dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\,dx, \quad u \in \mathcal{D}.$$

Our results on equation (1.12) can be stated as follows.

THEOREM 1.3. *Assume $N \ge 4$, then equation* (1.12) *has a positive ground state solution $u \in \mathcal{D}$ with*

$$I^\infty(u) = c^\infty := \inf_{\mathcal{N}_\infty} I^\infty(u), \tag{1.13}$$

where

$$\mathcal{N}_\infty := \big\{u \in \mathcal{D} \setminus \{0\}, J^\infty(u) = 0\big\},$$

and

$$J^\infty(u) := \int_{\mathbb{R}^N} \big(|\nabla u|^2 + V_\infty u^2\big)\,dx - \int_{\mathbb{R}^N} u^2 \log u^2\,dx - \int_{\mathbb{R}^N} |u|^{2^*}\,dx.$$

REMARK 1.1. We remark that our results on equation (1.12) also hold for the case of periodic potentials, namely, equation

$$-\Delta u + V(x)u = u \log u^2 + |u|^{2^*-2} u, \quad x \in \mathbb{R}^N, \tag{1.14}$$

where $V(x)$ is a positive continuous function and periodic in each variables. In fact, by using translation invariance of the problem, the proof still valid and just needs slight modification.

REMARK 1.2. Since the variational functionals I, I_p, I_p^∞ and I^∞ are not smooth on H and $H^1(\mathbb{R}^N)$, thus in these cases the Nehari manifolds $\mathcal{N}$, $\mathcal{N}_p$, $\mathcal{N}_p^\infty$, $\mathcal{N}^\infty$ (and various subsets of them, say $\mathcal{M}$, $\mathcal{M}_p$) are not necessarily smooth. Nevertheless, minimizers of the action functionals on these sets are critical points of the corresponding functionals. This was proved in [22] for I in the case without the critical term, but the proof does not depend on the problem with the critical term or not. Thus we refer to [22] for this and later on whenever we find minimizers of the functionals on the Nehari manifolds (or $\mathcal{M}$, $\mathcal{M}_p$), we obtain solutions of the corresponding logarithmic equations.

Moreover, by similar arguments as Theorem 1.1 in [22], we can prove the sign-changing solutions obtained by Theorem 1.1 and Theorem 1.2 possess exactly two nodal domains.

The paper is organized as follows. In Section 2, we study the subcritical problem (1.10)–(1.11) and prove Theorem 1.2. Section 3 is devoted to the existence of positive ground state solutions for equation (1.12). In Section 4, we focus on the case of potential well where $V(x)$ satisfies (V_1)–(V_2), and prove Theorem 1.1.

2. The subcritical case. In this section, by using direction derivative and constrained minimization method, we prove the existence of positive ground state solution and least energy sign-changing solution for the subcritical problem (1.10).

The following Brézis-Lieb type lemma for $u^2 \log u^2$ is important for our proof.

LEMMA 2.1 (See Lemma 2.3, [2] or Lemma 3.1, [22]). *Let $\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^N)$ such that $u_n \to u$ a.e. in $\mathbb{R}^N$ and $\{u_n^2 \log u_n^2\}$ is a bounded sequence in $L^1(\mathbb{R}^N)$. Then, $u^2 \log u^2 \in L^1(\mathbb{R}^N)$ and*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left[u_n^2 \log u_n^2 - |u_n - u|^2 \log |u_n - u|^2\right] dx = \int_{\mathbb{R}^N} u^2 \log u^2 dx. \tag{2.1}$$

PROPOSITION 2.2. *$c_p^\infty > 0$ is achieved.*

Proof. One can easily verify that $\mathcal{N}_p^\infty$ is nonempty since $\mathcal{C}_0^\infty(\mathbb{R}^N) \subset \mathcal{D}$. Let $\{u_n\} \subset \mathcal{N}_p^\infty$ be a minimizing sequence of c_p^∞, then

$$\begin{aligned} c_p^\infty &= \lim_{n\to\infty} I_p^\infty(u_n) = \lim_{n\to\infty} [I_p^\infty(u_n) - \frac{1}{2} J_p^\infty(u_n)] \\ &= \lim_{n\to\infty} [\frac{1}{2} \int_{\mathbb{R}^N} u_n^2 dx + \big(\frac{1}{2} - \frac{1}{p}\big) \int_{\mathbb{R}^N} |u_n|^p dx]. \end{aligned}$$

Thus, $\{u_n\}$ is bounded in $L^2(\mathbb{R}^N)$ and in $L^p(\mathbb{R}^N)$. Taking $a > 0$ small enough in (1.5) yields

$$\int_{\mathbb{R}^N} u^2 \log u^2 dx \le \frac{1}{2} \|\nabla u\|_2^2 + C_1(\log \|u\|_2^2 + 1)\|u\|_2^2, \quad \text{for all } u \in H^1(\mathbb{R}^N).$$

Since $\{u_n\} \subset \mathcal{N}_p^\infty$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} \big(|\nabla u_n|^2 + V_\infty u_n^2\big) dx &\le \frac{1}{2} \|\nabla u_n\|_2^2 + C_1(\log \|u_n\|_2^2 + 1)\|u_n\|_2^2 \\ &\quad + \int_{\mathbb{R}^N} |u_n|^p dx. \end{aligned} \tag{2.2}$$

This implies that $\{u_n\}$ is bounded in H. On the other hand, from the fact that

$$\begin{aligned} \int_{\mathbb{R}^N} \big(|\nabla u_n|^2 + V_\infty u_n^2\big) dx &\le \int_{\mathbb{R}^N} \big(u_n^2 \log u_n^2\big)^+ dx + \int_{\mathbb{R}^N} |u_n|^p dx \\ &\le C_p \int_{\mathbb{R}^N} |u_n|^p dx + \int_{\mathbb{R}^N} |u_n|^p dx, \end{aligned} \tag{2.3}$$

and Sobolev embedding theorem, we deduce that

$$\int_{\mathbb{R}^N} \big(|\nabla u_n|^2 + V_\infty u_n^2\big) dx \ge C > 0. \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$\int_{\mathbb{R}^N} |u_n|^p dx \ge C_1 > 0. \tag{2.5}$$

Thus

$$c_p^\infty = \lim_{n\to\infty}[\frac{1}{2}\int_{\mathbb{R}^N} u_n^2 dx + (\frac{1}{2}-\frac{1}{p})\int_{\mathbb{R}^N} |u_n|^p dx] \geq C_2 > 0.$$

We claim that there exists $\{x_n\} \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{B_1(x_n)} |u_n|^2 dx > 0, \tag{2.6}$$

where $B_1(y) = \{z \in \mathbb{R}^N : \ |y-z| < 1\}$. Otherwise,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n|^2 dx = 0,$$

then $u_n \to 0$ strongly in $L^q(\mathbb{R}^N)$ for any $q \in (2, 2^*)$. It follows by (2.3) that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} \big(|\nabla u_n|^2 + V_\infty u_n^2\big) dx = 0,$$

which is contradict to (2.4). Therefore, (2.6) holds.

Note that $\{u_n(\cdot + x_n)\}$ is still a bounded minimizing sequence of c_p^∞. Set $v_n := u_n(\cdot + x_n)$, then, up to a subsequence, there exists $v \in H^1(\mathbb{R}^N)$ such that

$$\begin{cases} v_n \rightharpoonup v & \text{weakly in } H^1(\mathbb{R}^N), \\ v_n \to v & \text{strongly in } L^q_{loc}(\mathbb{R}^N) \text{ for } q \in (2, 2^*), \\ v_n \to v & \text{a.e. } \mathbb{R}^N. \end{cases}$$

It follows from (2.6) that $v \neq 0$. Moreover, Lemma 2.1 implies that $v \in \mathcal{D}$.

Now, we prove $J_p^\infty(v) = 0$ and $I_p^\infty(v) = c_p^\infty$. First, assume by contradiction that $J_p^\infty(v) < 0$. By elementary computations, we can deduce that there exists $0 < t < 1$ such that $J_p^\infty(tv) = 0$. Therefore

$$\begin{aligned} c_p^\infty \leq I_p^\infty(tv) &= \frac{t^2}{2}\int_{\mathbb{R}^N} v^2 dx + (\frac{t^p}{2} - \frac{t^p}{p})\int_{\mathbb{R}^N} |v|^p dx \\ &< \liminf_{n\to\infty}[\frac{1}{2}\int_{\mathbb{R}^N} v_n^2 dx + (\frac{1}{2}-\frac{1}{p})\int_{\mathbb{R}^N} |v_n|^p dx] = c_p^\infty, \end{aligned}$$

which is impossible. On the other hand, assume that $J_p^\infty(v) > 0$. It follows from Lemma 2.1 that

$$\lim_{n\to\infty}[J_p^\infty(v_n) - J_p^\infty(v_n - v) - J_p^\infty(v)] = \lim_{n\to\infty}[-J_p^\infty(v_n - v) - J_p^\infty(v)] = 0,$$

which combined with $J_p^\infty(v) > 0$ imply that $J_p^\infty(v_n - v) < 0$ for sufficiently large n. Thus, by applying the same argument as above, we have that

$$\begin{aligned} c_p^\infty &\leq \liminf_{n\to\infty}[\frac{1}{2}\int_{\mathbb{R}^N} |v_n - v|^2 dx + (\frac{1}{2}-\frac{1}{p})\int_{\mathbb{R}^N} |v_n - v|^p dx] \\ &= c_p^\infty - \frac{1}{2}\int_{\mathbb{R}^N} v^2 dx - (\frac{1}{2}-\frac{1}{p})\int_{\mathbb{R}^N} |v|^p dx, \end{aligned}$$

which is a contradiction because $\|v\|_2^2 > 0$. Thus, we conclude that $J_p^\infty(v) = 0$, and hence $v \in \mathcal{N}_p^\infty$. Furthermore, by using Fatou's Lemma we have that

$$\begin{aligned} c_p^\infty &\le \frac{1}{2}\int_{\mathbb{R}^N} v^2 dx + \big(\frac{1}{2} - \frac{1}{p}\big)\int_{\mathbb{R}^N} |v|^p dx \\ &\le \liminf_{n\to\infty}\left[\frac{1}{2}\int_{\mathbb{R}^N} v_n^2 dx + \big(\frac{1}{2} - \frac{1}{p}\big)\int_{\mathbb{R}^N} |v_n|^p dx\right] = c_p^\infty, \end{aligned}$$

which implies $I_p^\infty(v) = c_p^\infty$. □

Next, we prove c_p and m_p are achieved under condition (V_1) and (V_2). We first consider the corresponding problem in bounded domains, the solutions will be used as minimizing sequences.

Let B_R be a ball in $\mathbb{R}^N$ centered at 0 with radius R. We define

$$\begin{aligned} &\mathcal{N}_p^R = \mathcal{N}_p \cap H_0^1(B_R), &&\mathcal{M}_p^R = \mathcal{M}_p \cap H_0^1(B_R), \\ &c_p^R = \inf_{u\in\mathcal{N}_p^R} I_p(u), &&m_p^R = \inf_{u\in\mathcal{M}_p^R} I_p(u). \end{aligned} \tag{2.7}$$

Obviously, the restriction $I_p|_{H_0^1(B_R)}$ is well defined and belongs to $\mathcal{C}^1(H_0^1(B_R), \mathbb{R})$. Therefore, the following result is standard.

LEMMA 2.3. *c_p^R and m_p^R are achieved. Suppose that either $u \in \mathcal{N}_p^R$ such that $I_p(u) = c_p^R$, or $u \in \mathcal{M}_p^R$ such that $I_p(u) = m_p^R$, then u is a weak solution of the logarithmic equation* (1.1) *in the bounded domain B_R, i.e., for all $\varphi \in H_0^1(B_R)$ it holds*

$$\int_{B_R} \nabla u \nabla \varphi + V(x) u\varphi dx = \int_{B_R} u\varphi \log u^2 + |u|^{p-2} u\varphi dx.$$

LEMMA 2.4. *c_p^R, m_p^R both decrease in R and converge to c_p, m_p as $R \to +\infty$, respectively.*

Proof. The result can be proved by the same argument as that of Lemma 3.7 in [22], so we omit the details. □

LEMMA 2.5. *Let $u_R \in \mathcal{N}_p^R$ be a weak solution of equation* (1.1) *in the bounded domain B_R, i.e.*

$$\int_{B_R} \nabla u_R \nabla \varphi + V(x) u_R \varphi dx = \int_{B_R} \varphi u_R \log u_R^2 dx + \int_{B_R} |u_R|^{p-2} u_R \varphi dx, \tag{2.8}$$

for all $\varphi \in H_0^1(B_R)$. Suppose that $\int_{B_R} |\nabla u_R|^2 + V(x) u_R^2 dx$, $\int_{B_R} u_R^2 \log u_R^2 dx$ are bounded with respect to R, and for a subsequence $R_n \to +\infty$,

$$\int_{B_{R_n}} |u_n|^p dx \to \lambda \in (0, \infty),$$

where $u_n := u_{R_n}$. Then there exist $\beta \in (0, 1]$ and $\{x_n\} \subset \mathbb{R}^N$ such that for any $\varepsilon > 0$, there exists $r_\varepsilon > 0$, for any $r' \ge r \ge r_\varepsilon > 0$,

$$\liminf_{n\to\infty} \int_{B_r(x_n)} |u_n|^p dx \ge \beta\lambda - \varepsilon, \tag{2.9}$$

and

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N\setminus B_{r'}(x_n)}|u_n|^p dx \geq (1-\beta)\lambda-\varepsilon. \tag{2.10}$$

Moreover, if $\beta<1$, *then* $\liminf_{n\to\infty} I(u_n)\geq c_p+c_p^\infty$.

Proof. Since $\{u_n\}$ is bounded in H and $\int_{B_{R_n}}|u_n|^p dx$ is bounded away from zero, then the existence of such a number $\beta\in(0,1]$ follows from a result of P.-L. Lions (see Lemma I.1. [19] or Lemma 4.3 [24]).

Now suppose that $\beta<1$. Choose $\varepsilon_n\to 0$ and $r'_n\geq r_n\to+\infty$ such that, up to a subsequence, we may assume

$$\liminf_{n\to\infty}\int_{B_{r_n}(x_n)}|u_n|^p dx\geq\beta\lambda-\varepsilon_n,\quad \liminf_{n\to\infty}\int_{\mathbb{R}^N\setminus B_{r'_n}(x_n)}|u_n|^p dx\geq(1-\beta)\lambda-\varepsilon_n.$$

Let ϕ be a nonnegative cut-off function such that $\phi(s)=0$ for $s\leq 1$ and for $s\geq 4$, $\phi(s)=1$ for $2\leq s\leq 3$ and $|\phi'(s)|\leq 2$. Taking $\varphi(x)=\phi(|x-x_n|/r_n)u_n$ in (2.8), which is admissible, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}|\nabla u_n|^2\varphi+V(x)u_n^2\varphi dx-\int_{\mathbb{R}^N}\varphi(u_n^2\log u_n^2)^- dx\\
&=\int_{\mathbb{R}^N}\varphi(u_n^2\log u_n^2)^+dx+\int_{\mathbb{R}^N}u_n\nabla u_n\nabla\varphi dx+\int_{\mathbb{R}^N}|u_n|^p\varphi dx\\
&\leq C_p\int_{\mathbb{R}^N}|u_n|^p\varphi dx+\int_{\mathbb{R}^N}|u_n|^p\varphi dx+o_n(1).
\end{aligned}$$

Thus

$$\int_{B_{3r_n}(x_n)\setminus B_{2r_n}(x_n)}|\nabla u_n|^2+V(x)u_n^2+|u_n^2\log u_n^2|+|u_n|^p dx=o_n(1), \tag{2.11}$$

where we have used the estimate (2.9) and (2.10) with $r'=4r_n$, $r=r_n$. Now take another cut-off function η such that $\eta(s)=1$ for $s\leq 2$, $\eta=0$ for $s\geq 3$ and $|\eta'(s)|\leq 2$. Set

$$w_n(x):=\eta\Big(\frac{|x-x_n|}{r_n}\Big)u_n(x),\quad v_n(x):=\Big[1-\eta\Big(\frac{|x-x_n|}{r_n}\Big)\Big]u_n(x).$$

It follows from (2.9)-(2.10) that

$$\int_{\mathbb{R}^N}|w_n|^p dx\geq\beta\lambda-\varepsilon_n,\quad\int_{\mathbb{R}^N}|v_n|^p dx\geq(1-\beta)\lambda-\varepsilon_n.$$

Combining this and (2.11), we have that

$$I_p(u_n)=I_p(w_n)+I_p(v_n)+o_n(1).$$

Moreover, if we take $\phi=w_n$ in (2.8), it follows from (2.11) that

$$J_p(w_n)=\langle DI_p(u_n),w_n\rangle+o_n(1)=o_n(1).$$

Similarly, we have that $J_p(v_n)=o_n(1)$. Therefore, there are sequences $\{t_n\}$, $\{s_n\}$ satisfying $t_n\to 1$, $s_n\to 1$ as $n\to\infty$ such that

$$\bar{w}_n:=t_nw_n\in\mathcal{N}_p^{R_n}\ \text{ and }\ \bar{w}_n:=s_nv_n\in\mathcal{N}_p^{R_n}.$$

If $\{x_n\}$ is bounded, then

$$\liminf_{n\to\infty} I_p(\bar{w}_n) \geq c_p, \quad \liminf_{n\to\infty} I_p(\bar{v}_n) \geq c_p^\infty,$$

since the support of $\bar{v}_n$ is outside of the ball B_{2r_n}. If $\{x_n\}$ is unbounded, then

$$\liminf_{n\to\infty} I_p(\bar{w}_n) \geq c_p^\infty, \quad \liminf_{n\to\infty} I_p(\bar{v}_n) \geq c_p.$$

Altogether, we have

$$I_p(u_n) = I_p(w_n) + I_p(v_n) + o_n(1) = I_p(t_n w_n) + I_p(t_n v_n) + o_n(1),$$

and

$$\liminf_{n\to\infty} I_p(u_n) \geq \liminf_{n\to\infty} I_p(\bar{w}_n) + \liminf_{n\to\infty} I_p(\bar{v}_n) \geq c_p + c_p^\infty.$$

Therefore, we complete the proof. □

PROPOSITION 2.6. *If (V_1) hold, then c_p is achieved.*

Proof. We claim that $c_p < c_p^\infty$. It follows from Proposition 2.2, there exists $w \in \mathcal{N}_p^\infty$ such that $I_p^\infty(w) = c_p^\infty$. By standard arguments and elliptic regularity theory, we assume $w \geq 0$ is a classical solution. Finally, it follows from the maximum principle (see Theorem 1, [28]) that $w(x) > 0$. Thus

$$\begin{aligned} J_p(w) &= \int_{\mathbb{R}^N} \big(|\nabla w|^2 + V(x) w^2\big) dx - \int_{\mathbb{R}^N} w^2 \log w^2 dx - \int_{\mathbb{R}^N} |w|^p dx \\ &< \int_{\mathbb{R}^N} \big(|\nabla w|^2 + V_\infty w^2\big) dx - \int_{\mathbb{R}^N} w^2 \log w^2 dx - \int_{\mathbb{R}^N} |w|^p dx \\ &= J_p^\infty(w) = 0. \end{aligned} \tag{2.12}$$

Therefore, there exists $0 < t < 1$ such that $J_p(tw) = 0$. Obviously

$$c_p \leq I_p(tw) < I_p^\infty(tw) < I_p^\infty(w) = c_p^\infty. \tag{2.13}$$

Now, suppose u_n is a weak solution of the logarithmic equation (1.10) in the bounded domain B_{R_n}, which satisfying $I_p(u_n) = c_p^{R_n} \to c_p$ as $n \to \infty$. By using the facts that $I_p(u_n)$ is bounded and $J_p(u_n) = 0$, we can deduce $\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + V(x) u_n^2 dx$ and $\int_{\mathbb{R}^N} u_n^2 \log u_n^2 dx$ are all bounded in n. By Lemma 2.5, there is a sequence $\{x_n\} \subset \mathbb{R}^N$ such that for any $\varepsilon > 0$, there exists $r > 0$ satisfying

$$\liminf_{n\to\infty} \int_{B_r(x_n)} |u_n|^p dx \geq \lambda - \varepsilon,$$

where $\lambda := \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^p dx > 0$. We claim that $\{x_n\}$ must be bounded. Otherwise

$$J_p^\infty(u_n) = J_p(u_n) + o_n(1).$$

Therefore, we can find $t_n > 0$ such that $t_n \to 1$ and $J_p^\infty(t_n u_n) = 0$. Thus

$$c_p^\infty \leq \liminf_{n\to\infty} I_p^\infty(t_n u_n) = \liminf_{n\to\infty} I_p^\infty(u_n) = \liminf_{n\to\infty} I_p(u_n) = c_p, \tag{2.14}$$

which is a contradiction with (2.13). Thus, $\{x_n\}$ is bounded, and we can deduce that

$$u_n \to u \text{ strongly in } L^p(\mathbb{R}^N).$$

By a similar argument as the proof of (2.5), we have

$$\int_{\mathbb{R}^N} |u_n|^p dx \geq C_1 > 0.$$

It follows that

$$\int_{\mathbb{R}^N} |u|^p dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^p dx \geq C_1 > 0. \tag{2.15}$$

Thus, by using the weak-lower semicontinuity of norm and Fatou's Lemma, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \big(|\nabla u|^2 + V(x)u^2\big)dx - \int_{\mathbb{R}^N} \big(|u|^2 \log u^2\big)^- dx \\
&\leq \liminf_{n\to\infty} \Big[\int_{\mathbb{R}^N} \big(|\nabla u_n|^2 + V(x)u_n^2\big)dx - \int_{\mathbb{R}^N} \big(|u_n|^2 \log u_n^2\big)^- dx\Big] \\
&= \liminf_{n\to\infty} \Big[\int_{\mathbb{R}^N} \big(|u_n|^2 \log |u_n|^2\big)^+ dx + \int_{\mathbb{R}^N} |u_n|^p dx\Big] \\
&= \int_{\mathbb{R}^N} \big(|u|^2 \log u^2\big)^+ dx + \int_{\mathbb{R}^N} |u|^p dx.
\end{aligned} \tag{2.16}$$

That is

$$\int_{\mathbb{R}^N} \big(|\nabla u|^2 + V(x)u^2\big)dx \leq \int_{\mathbb{R}^N} u^2 \log u^2 dx + \int_{\mathbb{R}^N} |u|^p dx.$$

Therefore, there exists $s \in (0,1]$ such that $su \in \mathcal{N}_p$. Then

$$\begin{aligned}
c_p \leq I(su) &= \frac{1}{2}\int_{\mathbb{R}^N} |su|^2 dx + \Big(\frac{1}{2} - \frac{1}{p}\Big)\int_{\mathbb{R}^N} |su|^p dx \\
&\leq \liminf_{n\to\infty} \Big[\frac{1}{2}\int_{\mathbb{R}^N} u_n^2 dx + \Big(\frac{1}{2} - \frac{1}{p}\Big)\int_{\mathbb{R}^N} |u_n|^p dx\Big] = c_p.
\end{aligned}$$

This implies $s = 1$, i.e., $u \in \mathcal{N}_p$ satisfying $I_p(u) = c_p$. □

To complete the proof of Theorem 1.2, we need the following result.

LEMMA 2.7. *Assume* (V_1), *let* u *be a positive ground state solution of* (1.10), *then* $u \in \mathcal{C}^{2,\gamma}(\mathbb{R}^N)$ *for some* $0 < \gamma < 1$ *and satisfying the following exponential decay at infinity*

$$|u(x)| \leq Ce^{-\delta R}, \quad \int_{\mathbb{R}^N \setminus B_R} (u^2 + |\nabla u|^2)dx \leq Ce^{-\delta R}, \tag{2.17}$$

for some positive constants C, δ.

Proof. The same result has been proved by Lemma 3.10 in [22] for the following Logarithmic Schrödinger equation

$$-\Delta u + V(x)u = u \log u^2, \quad x \in \mathbb{R}^N.$$

One can imitate the proof procedures with slight modification, thus we omit the details. □

Now let $u \in \mathcal{N}_p$ with $I_p(u) = c_p$, and let $w \in \mathcal{N}_p^\infty$ with $I_p^\infty(w) = c_p^\infty$. It follows from Theorem 1.1 in [22] that u, w are positive and solve equation (1.10), (1.11), respectively. Similar as Lemma 2.7, we can show that w, ∇w are bounded, satisfying

$$\int_{\mathbb{R}^N \setminus B_R} (w^2 + |\nabla w|^2) dx \leq C e^{-\delta R}, \tag{2.18}$$

for some C, $\delta > 0$. Next, we denote $w_R(x) := w(x_1 + 2R, x')$ with $(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1}$.

Lemma 2.8. *Assume* (V_1)-(V_2) *hold, then* $m_p \leq \sup\limits_{(\alpha,\beta)\in\mathbb{R}^2} I_p(\alpha u + \beta w_R) < c_p + c_p^\infty$, *provided* R *is large enough.*

Proof. The proof is similar as Lemma 3.11 in [22], for the convenience of the reader, we give the details of proof here. Obviously,

$$\begin{aligned}
&I_p(\alpha u + \beta w_R) \\
= &\frac{\alpha^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + (V(x)+1)u^2 dx + \frac{\beta^2}{2} \int_{\mathbb{R}^N} |\nabla w_R|^2 + (V(x)+1) w_R^2 dx \\
&+ \alpha\beta \int_{\mathbb{R}^N} \nabla u \nabla w_R + (V(x)+1) u w_R dx - \frac{1}{p} \int_{\mathbb{R}^N} |\alpha u + \beta w_R|^p dx \\
&- \frac{1}{2} \int_{\mathbb{R}^N} (\alpha u + \beta w_R)^2 \log(\alpha u + \beta w_R)^2 dx.
\end{aligned}$$

We divide the proof into several steps.

(i) There exist $R_0 > 0$ and $r_0 > 0$ such that for all $R \geq R_0$, for all $\alpha^2 + \beta^2 = r^2 > r_0^2$,

$$I_p(\alpha u + \beta w_R) \leq 0.$$

Set $\bar{\alpha} = \alpha/r$, $\bar{\beta} = \beta/r$, and $\bar{\varphi} = \bar{\alpha} u + \bar{\beta} w_R$. We first find a $R' > 0$ such that for all $R > R'$, one has $\int_{\mathbb{R}^N} |\nabla \bar{\varphi}|^2 + V(x)|\bar{\varphi}|^2 dx$ and $\int_{\mathbb{R}^N} |\bar{\varphi}|^2 \log|\bar{\varphi}|^2 dx$ are both bounded from above and below by two positive constants. Our claim follows from the fact that

$$\begin{aligned}
I_p(\bar{\varphi}) = &\frac{r^2}{2} \int_{\mathbb{R}^N} |\nabla \bar{\varphi}|^2 + (V(x)+1)|\bar{\varphi}|^2 - |\bar{\varphi}|^2 \log|\bar{\varphi}|^2 dx \\
&- \frac{r^2 \log r^2}{2} \int_{\mathbb{R}^N} |\bar{\varphi}|^2 dx - \frac{r^p}{p} \int_{\mathbb{R}^N} |\bar{\varphi}|^p dx.
\end{aligned}$$

(ii) We claim there exist $R_1 > 0$ such that

$$\int_{\mathbb{R}^N} \nabla u \nabla w_R + V(x) u w_R dx = O(e^{-\delta R}) \text{ for } R > R_1,$$

where δ is similar as the one in (2.17) and (2.18).

In fact, for R large enough, by using Hölder's inequality, one has

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} \nabla u \nabla w_R + V(x) u w_R dx\right| \\
&\le C\left(\int_{B_R} |\nabla w_R|^2 + w_R^2 dx\right)^{1/2} + C\left(\int_{\mathbb{R}^N\setminus B_R} |\nabla u|^2 + u^2 dx\right)^{1/2} \\
&\le C\left(\int_{\mathbb{R}^N\setminus B_R} |\nabla w|^2 + w^2 dx\right)^{1/2} + C\left(\int_{\mathbb{R}^N\setminus B_R} |\nabla u|^2 + u^2 dx\right)^{1/2} \\
&\le Ce^{-\delta R}.
\end{aligned}$$

(iii) If α, β are bounded, there there exists $R_2 > 0$ such that, for $R > R_2$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} (\alpha u + \beta w_R)^2 \log(\alpha u + \beta w_R)^2 dx \\
&= \int_{\mathbb{R}^N} (\alpha u)^2 \log(\alpha u)^2 dx + \int_{\mathbb{R}^N} (\beta w_R)^2 \log(\beta w_R)^2 dx + O(e^{-\delta R}).
\end{aligned}$$

Since u and w_R have exponential decay, for R large enough, we have that

$$\int_{\mathbb{R}^N\setminus(B_R(0)\cup B_R(2Re_1))} (\alpha u + \beta w_R)^2 \log(\alpha u + \beta w_R)^2 dx = O(e^{-\delta R}),$$

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus B_R(0)} (\alpha u)^2 \log(\alpha u)^2 dx = O(e^{-\delta R}) \ \ \textit{and} \\
&\int_{\mathbb{R}^N\setminus B_R(2Re_1)} (\beta w_R)^2 \log(\beta w_R)^2 dx = O(e^{-\delta R}).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\Big|\int_{B_R(0)} (\alpha u + \beta w_R)^2 \log(\alpha u + \beta w_R)^2 dx - \int_{B_R(0)} (\alpha u)^2 \log(\alpha u)^2 dx\Big| \\
&\le \Big|\int_{B_R(0)} (\alpha u)^2 \big(\log(\alpha u + \beta w_R)^2 - \log(\alpha u)^2\big) dx\Big| \\
&\quad + \Big|\int_{B_R(0)} \big[2\alpha\beta u w_R + (\beta w_R)^2\big] \log(\alpha u + \beta w_R)^2 dx\Big| \\
&\le Ce^{-\delta R}.
\end{aligned} \tag{2.19}$$

Similarly, we can deduce that, for $R > 0$ large enough

$$\begin{aligned}
&\Big|\int_{B_R(2Re_1)} (\alpha u + \beta w_R)^2 \log(\alpha u + \beta w_R)^2 dx \\
&\qquad - \int_{B_R(2Re_1)} (\beta w_R)^2 \log(\beta w_R)^2 dx\Big| \le Ce^{-\delta R},
\end{aligned} \tag{2.20}$$

which gives our claim.

(iv) By using a similar arguement as step (iii), we can show that, if α, β are bounded, then there there exists $R_3 > 0$ such that, for $R > R_3$,

$$\int_{\mathbb{R}^N} |\alpha u + \beta w_R|^p dx = \int_{\mathbb{R}^N} |\alpha u|^p dx + \int_{\mathbb{R}^N} |\beta w_R|^p dx + O(e^{-\delta R}).$$

(v) We complete the estimate of $\sup_{(\alpha,\beta)\in\mathbb{R}^2} I_p(\alpha u+\beta w_R)$. By step (i), we can assume that $\alpha^2+\beta^2$ is bounded. It follows from step (ii)–(iv) that

$$I_p(\alpha u+\beta w_R)=I_p(\alpha u)+I_p(\beta w_R)+O(e^{\delta R}) \quad \text{for} \quad R>\max\{R_0,\ R_1,\ R_2,\ R_3\}.$$

We also have $J_p(w_R)\to 0$ as $R\to+\infty$. Thus, there exist $t_R>0$ such that $t_R\to 1$ as $R\to+\infty$ and $J_p(t_Rw_R)=0$. By using (V_2) we have that

$$\begin{aligned}
I_p(\alpha u+\beta w_R) &= I_p(\alpha u)+I_p(\beta w_R)+O(e^{-\delta R}) \le I_p(u)+I_p(t_Rw_R)+O(e^{-\delta R})\\
&\le I_p(u)+I_p^{\infty}(t_Rw_R)+\frac{t_R^2}{2}\int_{\mathbb{R}^N}(V(x)-V_\infty)w_R^2dx+O(e^{-\delta R})\\
&\le I_p(u)+I_p^{\infty}(w_R)-\frac{1}{3}\int_{B_1(0)}(V_\infty-V(x-Re_1))w^2(x)dx+O(e^{-\delta R})\\
&\le c_p+c_p^{\infty}-C\int_{B_1(0)}\frac{w^2(x)}{1+|x-Re_1|^m}dx+O(e^{-\delta R})\\
&\le c_p+c_p^{\infty}-\frac{C}{R^m}+O(e^{-\delta R})\\
&< c_p+c_p^{\infty},
\end{aligned}$$

provided R is large enough. Finally, one can easily check that there exist $\alpha>0$, $\beta<0$, such that $\alpha u+\beta w_R\in\mathcal{M}_p$, which implies that $m_p<c_p+c_p^{\infty}$. □

PROPOSITION 2.9. *Assume (V_1) and (V_2) hold, then m_p is achieved.*

Proof. Let $\{u_n\}$ be a sequence such that $u_n\in\mathcal{M}_p^{R_n}$ with $R_n\to+\infty$ satisfying $I_p(u_n)=m_p^{R_n}\to m_p$ as $n\to\infty$. Lemma 2.8 implies that $m_p<c_p+c_p^{\infty}$. By using the facts that $I_p(u_n)$ is bounded and $J_p(u_n)=0$, we have $\int_{\mathbb{R}^N}|\nabla u_n|^2+V(x)u_n^2dx$ and $\int_{\mathbb{R}^N}u_n^2\log u_n^2dx$ are all bounded. By Lemma 2.5, there is a sequence $\{x_n\}$ in $\mathbb{R}^N$ such that, for any $\varepsilon>0$ there exists $r>0$,

$$\liminf_{n\to\infty}\int_{B_r(x_n)}|u_n|^pdx\ge\lambda-\varepsilon,$$

where $\lambda:=\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^pdx>0$. We first claim that $\{x_n\}$ must be bounded. Otherwise, $J_p^{\infty}(u_n^{\pm})=J_p(u_n^{\pm})+o_n(1)$ and $I_p(u_n^{\pm})-I_p^{\infty}(u_n^{\pm})=o_n(1)$ as $n\to+\infty$. Therefore, we can find $s_n,\ t_n>0$ such that $s_n\to 1$, $t_n\to 1$ and $J_p^{\infty}(s_nu_n^+)=J_p^{\infty}(t_nu_n^-)=0$. Hence

$$\begin{aligned}
c_p+c_p^{\infty}\le 2c_p^{\infty} &\le \liminf_{n\to\infty}I_p^{\infty}(s_nu_n^++t_nu_n^-)\\
&=\liminf_{n\to\infty}I_p^{\infty}(u_n)=\liminf_{n\to\infty}I_p(u_n)=m_p,
\end{aligned} \tag{2.21}$$

which is a contradiction. Therefore, $\{x_n\}$ is bounded and it follows that $u_n\to u$ strongly in $L^p(\mathbb{R}^N)$. Similar as (2.15)–(2.16), we can deduce that $u\ne 0$ satisfying

$$\int_{\mathbb{R}^N}\big(|\nabla u^+|^2+V(x)|u^+|^2\big)dx\le\int_{\mathbb{R}^N}|u^+|^2\log|u^+|^2dx+\int_{\mathbb{R}^N}|u^+|^pdx$$

and

$$\int_{\mathbb{R}^N}\big(|\nabla u^-|^2+V(x)|u^-|^2\big)dx\le\int_{\mathbb{R}^N}|u^-|^2\log|u^-|^2dx+\int_{\mathbb{R}^N}|u^-|^pdx.$$

Thus, there exists $s, t \in (0,1]$ such that $\widetilde{u} := su^+ + tu^- \in \mathcal{M}_p$. It follows that

$$\begin{aligned} m_p \le I_p(\widetilde{u}) &= \frac{1}{2}\int_{\mathbb{R}^N} |su^+|^2 + |tu^-|^2 dx + \big(\frac{1}{2} - \frac{1}{p}\big)\int_{\mathbb{R}^N} |su^+|^p + |tu^-|^p dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^N} u^2 dx + \big(\frac{1}{2} - \frac{1}{p}\big)\int_{\mathbb{R}^N} |u|^p dx \\ &\le \liminf_{n\to\infty} \big[\frac{1}{2}\int_{\mathbb{R}^N} u_n^2 dx + \big(\frac{1}{2} - \frac{1}{p}\big)\int_{\mathbb{R}^N} |u_n|^p dx\big] = m_p. \end{aligned}$$

This implies $s = t = 1$, i.e., $u \in \mathcal{M}_p$ satisfying $I_p(u) = m_p$. □

3. The critical case with constant potential V_∞. In this section, by studying the behavior of the solutions for problem (1.11), we prove the existence of positive ground state solution for equation (1.12).

LEMMA 3.1. *Using the notations above, we have*

$$\limsup_{p\to 2^*} c_p \le c, \quad \limsup_{p\to 2^*} m_p \le m, \quad \limsup_{p\to 2^*} c_p^\infty \le c^\infty.$$

Proof. We prove the conclusion about m_p only, the other two inequalities can be proved similarly. Given $0 < \varepsilon < \frac{1}{2}$, we find $u \in \mathcal{M}$ such that $I(u) \le m + \varepsilon$. We first choose $T > 0$ such that $J(Tu^\pm) \le -1$. We then choose $\delta > 0$ such that

$$|I_p(tu^\pm) - I(tu^\pm)| + |J_p(tu^\pm) - J(tu^\pm)| < \varepsilon \quad \text{for } 2^* - \delta \le p \le 2^* \quad \text{and} \quad 0 \le t \le T.$$

Since $J_p(0) = 0$, $J_p(Tu^\pm) \le -\frac{1}{2}$ for all $2^* - \delta \le p \le 2^*$, there exist $t^+, t^- \in (0, T)$ such that

$$\bar{u} := t^+u^+ + t^-u^- \in \mathcal{M}_p.$$

Hence

$$\begin{aligned} m_p \le I_p(\bar{u}) &= I_p(t^+u^+) + I_p(t^-u^-) \le I(t^+u^+) + I(t^-u^-) + 2\varepsilon \\ &\le I(u) + 2\varepsilon \le m + 3\varepsilon, \end{aligned}$$

for $2^* - \delta \le p \le 2^*$. This complete the proof since ε is arbitrarily. □

PROPOSITION 3.2. *Let u_p be solutions of the subcritical problem* (1.11)*, that is*

$$\int_{\mathbb{R}^N} \nabla u_p \nabla\varphi + V_\infty u_p \varphi dx = \int_{\mathbb{R}^N} \varphi u_p \log u_p^2 dx + \int_{\mathbb{R}^N} |u_p|^{p-2} u_p \varphi dx \tag{3.1}$$

for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$. For $2 < p_n < 2^*$ with $p_n \to 2^*$, assume that $u_n := u_{p_n} \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, then $u \in \mathcal{D}$ and satisfies

$$\int_{\mathbb{R}^N} \nabla u \nabla\varphi + V_\infty u\varphi dx - \int_{\mathbb{R}^N} \varphi u \log u^2 dx - \int_{\mathbb{R}^N} |u|^{2^*-2} u\varphi dx = 0 \tag{3.2}$$

for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$. Moreover, $I^\infty(u) \le \liminf_{n\to\infty} I_{p_n}^\infty(u_n)$.

Proof. It is easy to see that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ since $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$. Hence

$$\int_{\mathbb{R}^N} \big(u_n^2 \log u_n^2\big)^+ dx \le C\int_{\mathbb{R}^N} |u_n|^{2^*} dx \le C < +\infty. \tag{3.3}$$

On the other hand, u_n satisfying

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 + V_\infty u_n^2 dx = \int_{\mathbb{R}^N} u_n^2 \log u_n^2 dx + \int_{\mathbb{R}^N} |u_n|^{p_n} dx. \tag{3.4}$$

Combining this with (3.3) implies that $\{u_n^2 \log u_n^2\}$ is bounded in $L^1(\mathbb{R}^N)$. Thus, we can deduce from Lemma 2.1 that $u^2 \log u^2 \in L^1(\mathbb{R}^N)$, which gives that $u \in \mathcal{D}$.

By using the fact $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, one can verify (3.2) easily. Since $J^\infty(u) = 0$ and $J_p^\infty(u_n) = 0$, by weakly lower semi-continuity we have

$$\begin{aligned} I^\infty(u) = I^\infty(u) - \frac{1}{2}J^\infty(u) &= \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 dx + (\frac{1}{2} - \frac{1}{2^*})\int_{\mathbb{R}^N} |u|^{2^*} dx \\ &\le \liminf_{n\to\infty} [\frac{1}{2}\int_{\mathbb{R}^N} |u_n|^2 dx + (\frac{1}{2} - \frac{1}{p_n})\int_{\mathbb{R}^N} |u_n|^{p_n} dx] \\ &\le \liminf_{n\to\infty} [I_{p_n}^\infty(u_n) - \frac{1}{2}J_{p_n}^\infty(u_n)] = \liminf_{n\to\infty} I_{p_n}^\infty(u_n). \end{aligned}$$

Thus, the proof is completed. □

LEMMA 3.3. *Let u_p with $2 < p < 2^*$ be solutions of the subcritical problem* (1.11). *Denote $u_n := u_{p_n}$ for $2 < p_n < 2^*$ with $p_n \to 2^*$. Suppose that $\lim_{n\to\infty} I_{p_n}^\infty(u_n) \in (0, \frac{1}{N}S^{\frac{N}{2}})$, where S is the best constant of Sobolev imbedding from $H^1(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$. Then, up to translations, $u_n \rightharpoonup u \neq 0$ in $H^1(\mathbb{R}^N)$.*

Proof. Since $\{I_{p_n}^\infty(u_n)\}$ is uniformly bounded and u_n are solutions for equation (1.11) with $p = p_n$, then

$$I_{p_n}^\infty(u_n) = I_{p_n}^\infty(u_n) - \frac{1}{2}J_{p_n}^\infty(u_n) = \frac{1}{2}\int_{\mathbb{R}^N} u_n^2 dx + (\frac{1}{2} - \frac{1}{p_n})\int_{\mathbb{R}^N} |u_n|^{p_n} dx.$$

Thus, the $L^2(\mathbb{R}^N)$ norm and $L^{p_n}(\mathbb{R}^N)$ norm of $\{u_n\}$ are bounded.

Taking $a > 0$ small enough in (1.5) yields

$$\int_{\mathbb{R}^N} u^2 \log u^2 dx \le \frac{1}{2}\|\nabla u\|_2^2 + C_1(\log\|u\|_2^2 + 1)\|u\|_2^2, \quad \text{for all } u \in H^1(\mathbb{R}^N).$$

Since $J_{p_n}^\infty(u_n) = 0$, we have that

$$\begin{aligned} &\int_{\mathbb{R}^N} \big(|\nabla u_n|^2 + V_\infty u_n^2\big) dx \\ &\le \frac{1}{2}\|\nabla u_n\|_2^2 + C_1(\log\|u_n\|_2^2 + 1)\|u_n\|_2^2 + \int_{\mathbb{R}^N} |u_n|^{p_n} dx. \end{aligned} \tag{3.5}$$

This implies that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. By applying the concentration-compactness principle due to P. L. Lions [19], we conclude that the following two cases may happen.

Case (i): Vanishing, i.e., for any $R > 0$, that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} u_n^2 dx = 0.$$

Case (ii): Nonvanishing, i.e., there exists $R > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n\to\infty} \int_{B_R(y_n)} u_n^2 dx = \nu > 0.$$

In case (i), we have $|u_n|_{L^r(\mathbb{R}^N)} \to 0$ for any $2 < r < 2^*$. By using the fact $J^\infty_{p_n}(u_n) = 0$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_n|^2 + V_\infty u_n^2 dx \\
&= \int_{\mathbb{R}^N} u_n^2 \log u_n^2 dx + \int_{\mathbb{R}^N} |u_n|^{p_n} dx \\
&\le C_{p_0} \int_{\mathbb{R}^N} |u_n|^{p_0} dx + \frac{p_n - p_0}{2^* - p_0} \int_{\mathbb{R}^N} |u_n|^{2^*} dx + \frac{2^* - p_n}{2^* - p_0} \int_{\mathbb{R}^N} |u_n|^{p_0} dx \\
&= \int_{\mathbb{R}^N} |u_n|^{2^*} dx + o_n(1) \\
&\le S^{\frac{N}{2-N}} \big(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \big)^{\frac{N}{N-2}} + o_n(1),
\end{aligned}$$

where $2 < p_0 < 2^* - \delta$ for some positive constant δ. From this we deduce that either $\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \to 0$ or $\liminf\limits_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \ge S^{\frac{N}{2}}$.

If $\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \to 0$, then $\int_{\mathbb{R}^N} |u_n|^{2^*} dx \to 0$, $u_n \to 0$ in $H^1(\mathbb{R}^N)$ and $I^\infty_{p_n}(u_n) \to 0$, a contradiction.

If $\liminf\limits_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \ge S^{\frac{N}{2}}$, then $\liminf\limits_{n\to+\infty} \int_{\mathbb{R}^N} |u_n|^{p_n} dx \ge S^{\frac{N}{2}}$, thus

$$\liminf_{n\to+\infty} I^\infty_{p_n}(u_n) = \liminf_{n\to+\infty} [\frac{1}{2} \int_{\mathbb{R}^N} u_n^2 dx + \big(\frac{1}{2} - \frac{1}{p_n}\big) \int_{\mathbb{R}^N} |u_n|^{p_n} dx] \ge \frac{1}{N} S^{\frac{N}{2}},$$

which is also a contradiction.

So vanishing cannot happen. Now suppose that

$$\lim_{n\to\infty} \int_{B_R(y_n)} u_n^2 dx = \nu > 0.$$

Since the functional I^∞_p and J^∞_p are translation invariant, without loss of generality, we may assume $y_n = 0$ for all n. Since $\{u_n\}$ is uniformly bounded in $H^1(\mathbb{R}^N)$, up to a subsequence, we suppose $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, and

$$\int_{B_R(0)} u^2 dx = \lim_{n\to\infty} \int_{B_R(0)} u_n^2 dx = \nu > 0.$$

This implies that $u \ne 0$. □

Lemma 3.4. *It hold* $c^\infty < \frac{1}{N} S^{\frac{N}{2}}$ *if* $N \ge 4$.

Proof. We follow the original idea of Brézis-Nirenberg [5]. We look for a positive function u such that $\sup_{t\ge0} I^\infty(tu) < \frac{1}{N} S^{\frac{N}{2}}$. Suppose $I^\infty(t_0 u) = \sup\limits_{t\ge 0} I^\infty(tu)$, then $J^\infty(t_0 u) = 0$ and $I^\infty(t_0 u) < \frac{1}{N} S^{\frac{N}{2}}$. Hence $c^\infty < I^\infty(t_0 u) < \frac{1}{N} S^{\frac{N}{2}}$.

One of the possible candidates for u is $u_\varepsilon = \phi w_\varepsilon$, where ϕ is a smooth cut-off function such that $\phi(x) = 1$ if $|x| \le 1$, $\phi(x) = 0$ if $|x| \ge 2$ and $|\nabla \phi| \le 2$; and

$$w_\varepsilon(x) = \frac{(N(N-2)\varepsilon)^{\frac{N-2}{4}}}{(\varepsilon + |x|^2)^{\frac{N-2}{2}}}, \quad \varepsilon > 0.$$

Note that the function w_ε solves the equation $\Delta w_\varepsilon + w_\varepsilon^{\frac{N+2}{N-2}} = 0$. Following [5], for sufficiently small ε, by a direct computation we estimate the terms of $I^\infty(tu_\varepsilon)$ as follows

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 dx = S^{\frac{N}{2}} + O(\varepsilon^{\frac{N-2}{2}}), \qquad \int_{\mathbb{R}^N} |u_\varepsilon|^{2^*} dx = S^{\frac{N}{2}} + O(\varepsilon^{\frac{N}{2}}),$$

$$\int_{\mathbb{R}^N} |u_\varepsilon|^2 dx = \begin{cases} d_1\varepsilon + O(\varepsilon^{\frac{N-2}{2}}) & \text{for } N \geq 5, \\ d_2\varepsilon|\ln\varepsilon| + O(\varepsilon) & \text{for } N = 4, \end{cases}$$

and

$$\int_{\mathbb{R}^N} |u_\varepsilon|^2 \log u_\varepsilon^2 dx \geq \begin{cases} d_3\varepsilon|\ln\varepsilon| & \text{for } N \geq 5, \\ d_4\varepsilon|\ln\varepsilon|^2 & \text{for } N = 4, \end{cases}$$

where ε is a positive small constant. It is easy to see that there exist $\varepsilon_0 > 0$ and $0 < T_1 < T_2$ such that for $\varepsilon \leq \varepsilon_0$, the function $t \mapsto I^\infty(tu_\varepsilon)$ achieves its maximum at some $t_0 \in [T_1, T_2]$. Hence, if $N \geq 4$ we have

$$\begin{aligned} &\sup_{t\geq 0} I^\infty(tu_\varepsilon) \\ &= \frac{t_0^2}{2}\int_{\mathbb{R}^N} \big(|\nabla u_\varepsilon|^2 + (V_\infty + 1 - \log t_0^2)u_\varepsilon^2 - u_\varepsilon^2 \log u_\varepsilon^2\big)dx - \frac{t_0^{2^*}}{2^*}\int_{\mathbb{R}^N} |u_\varepsilon|^{2^*} dx \\ &\leq \big(\frac{t_0^2}{2} - \frac{t_0^{2^*}}{2^*}\big)S^{\frac{N}{2}} - \varepsilon|\ln\varepsilon| + O(\varepsilon^{\frac{(N-2)(2-\delta)}{4}}) < \frac{1}{N}S^{\frac{N}{2}}, \end{aligned}$$

for ε small enough. Thus we complete the proof. □

LEMMA 3.5. *Assume $N \geq 4$, then equation* (1.12) *has a positive solution $u \in \mathcal{D}$ with $I^\infty(u) = c^\infty$.*

Proof. By Lemma 3.4, $c^\infty < \frac{1}{N}S^{\frac{N}{2}}$ if $N \geq 4$. It follows from Proposition 2.2 and Theorem 1.1 in [22] that, for $2 < p < 2^*$, equation (1.11) has a strictly positive solution u_p with $I_p^\infty(u_p) = c_p^\infty$ and $\limsup_{p\to 2^*} I_p^\infty(u_p) \leq c^\infty$. By Lemma 3.3 for a subsequence $p_n \to 2^*$ up to translations $u_n := u_{p_n} \rightharpoonup u \neq 0$ in $H^1(\mathbb{R}^N)$. By Lemma 3.1, Proposition 3.2, $u \in \mathcal{D}$ solves problem (1.12) and satisfies

$$I^\infty(u) \leq \limsup_{p\to 2^*} I_p^\infty(u_p) \leq c^\infty.$$

But $c^\infty \leq I^\infty(u)$ for any nontrivial critical point u of I^∞ since $J^\infty(u) = \langle DI^\infty(u), u\rangle = 0$. Finally, by standard arguments, $u \geq 0$ or $u \leq 0$. Without loss of generality, assume $u \geq 0$. By Lemma 3.10 in [22], we can deduce that u is a classical solution. It follows from the maximum principle (see Theorem 1, [28]) that $u(x) > 0$. □

4. The critical case with a potential well $V(x)$. In this section, we prove the existence of positive ground state solution and least energy sign-changing solutions for the critical problem (1.1).

The following Proposition is the counterpart of Proposition 3.2.

Proposition 4.1. *Let u_p be solutions of the subcritical problem* (1.10), *that is*

$$\int_{\mathbb{R}^N} \nabla u_p \nabla \varphi + V(x) u_p \varphi dx = \int_{\mathbb{R}^N} \varphi u_p \log u_p^2 dx + \int_{\mathbb{R}^N} |u_p|^{p-2} u_p \varphi dx \tag{4.1}$$

for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$. Let $2 < p_n < 2^*$ and $p_n \to 2^*$. Suppose that $\{y_n\} \subset \mathbb{R}^N$ and $\widetilde{u}_n := u_{p_n}(\cdot + y_n) \rightharpoonup u$ weakly in H. Then $u \in \mathcal{D}$. Moreover, if $\{y_n\}$ is bounded, then u satisfies

$$\int_{\mathbb{R}^N} \nabla u \nabla \varphi + V(x) u \varphi dx - \int_{\mathbb{R}^N} \varphi u \log u^2 dx - \int_{\mathbb{R}^N} |u|^{2^*-2} u \varphi dx = 0 \tag{4.2}$$

for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ and $I(u) \le \liminf\limits_{n\to\infty} I_{p_n}(u_n)$. If $\{y_n\}$ is unbounded, then u satisfies

$$\int_{\mathbb{R}^N} \nabla u \nabla \varphi + V_\infty u \varphi dx - \int_{\mathbb{R}^N} \varphi u \log u^2 dx - \int_{\mathbb{R}^N} |u|^{2^*-2} u \varphi dx = 0 \tag{4.3}$$

for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ and $I^\infty(u) \le \liminf\limits_{n\to\infty} I_{p_n}(u_n)$.

Proof. Similar as the proof of Proposition 3.2, we can prove that $u \in \mathcal{D}$.

If $\{y_n\}$ is bounded, we assume $y_n = 0$ for all n. By similar arguments as in the proof of Proposition 3.2, we can show that the weak limit u satisfies (4.2) and $I(u) \le \liminf\limits_{n\to\infty} I_{p_n}(u_n)$.

If $\{y_n\}$ is unbounded, we assume that $|y_n| \to +\infty$ as $n \to \infty$. Since the functions $\widetilde{u}_n = u_n(x + y_n)$ satisfies

$$\int_{\mathbb{R}^N} \nabla \widetilde{u}_n \nabla \varphi + V(x + y_n) \widetilde{u}_n \varphi dx = \int_{\mathbb{R}^N} \varphi \widetilde{u}_n \log \widetilde{u}_n^2 dx + \int_{\mathbb{R}^N} |\widetilde{u}_n|^{p_n-2} \widetilde{u}_n \varphi dx$$

for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, we can deduce that u satisfies (4.3) by taking the limit $n \to \infty$. Now up to a subsequence of $\{u_n\}$, we have

$$\begin{aligned} I^\infty(u) = I^\infty(u) - \frac{1}{2} J^\infty(u) &= \frac{1}{2} \int_{\mathbb{R}^N} u^2 dx + (\frac{1}{2} - \frac{1}{2^*}) \int_{\mathbb{R}^N} |u|^{2^*} dx \\ &\le \liminf_{n\to\infty} [\frac{1}{2} \int_{\mathbb{R}^N} u_n^2 dx + (\frac{1}{2} - \frac{1}{2^*}) \int_{\mathbb{R}^N} |u_n|^{p_n} dx] \\ &\le \liminf_{n\to\infty} [I_{p_n}(u_n) - \frac{1}{2} J_{p_n}(u_n)] = \liminf_{n\to\infty} I_{p_n}(u_n). \end{aligned}$$

□

Lemma 4.2. *Let u_p with $2 < p < 2^*$ be solutions of the subcritical problem* (1.10). *Denote $u_n := u_{p_n}$ for $2 < p_n < 2^*$ with $p_n \to 2^*$. Suppose that $\liminf\limits_{n\to\infty} I_{p_n}(u_n) \in (0, c^\infty)$, where c^∞ is defined* (1.13), *then up to translations $u_n \rightharpoonup u \neq 0$ in $H^1(\mathbb{R}^N)$.*

Proof. Since u_n are solutions for equation (1.10) with $p = p_n$, then

$$I_{p_n}(u_n) = I_{p_n}(u_n) - \frac{1}{2} J_{p_n}(u_n) = \frac{1}{2} \int_{\mathbb{R}^N} u_n^2 dx + (\frac{1}{2} - \frac{1}{p_n}) \int_{\mathbb{R}^N} |u_n|^{p_n} dx.$$

Thus, $\{u_n\}$ is bounded both in $L^2(\mathbb{R}^N)$ and $L^{p_n}(\mathbb{R}^N)$.

Taking $a > 0$ small enough in (1.5) yields

$$\int_{\mathbb{R}^N} u^2 \log u^2 dx \le \frac{1}{2}\|\nabla u\|_2^2 + C_1(\log \|u\|_2^2 + 1)\|u\|_2^2, \quad \text{for all } u \in H^1(\mathbb{R}^N).$$

It follows that

$$\begin{aligned} &\int_{\mathbb{R}^N} \big(|\nabla u_n|^2 + V(x)u_n^2\big)dx \\ &\le \frac{1}{2}\|\nabla u_n\|_2^2 + C_1(\log \|u_n\|_2^2 + 1)\|u_n\|_2^2 + \int_{\mathbb{R}^N} |u_n|^{p_n} dx, \end{aligned} \tag{4.4}$$

since $J_{p_n}(u_n) = 0$. This implies that $\{u_n\}$ is bounded in H.

We apply the concentration-compactness principle due to P. L. Lions [19], we deduce that the following two cases may happen.

Case (i): Vanishing, i.e., for any $R > 0$, that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} u_n^2 dx = 0.$$

Case (ii): Nonvanishing, i.e., there exists $R > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n\to\infty} \int_{B_R(y_n)} u_n^2 dx = \nu > 0.$$

In case (i), we have $|u_n|_{L^r(\mathbb{R}^N)} \to 0$ for any $2 < r < 2^*$. By using the fact $J_{p_n}(u_n) = 0$ we have

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla u_n|^2 + V(x)u_n^2 dx \\ &= \int_{\mathbb{R}^N} u_n^2 \log u_n^2 dx + \int_{\mathbb{R}^N} |u_n|^{p_n} dx \\ &\le C_{p_0} \int_{\mathbb{R}^N} |u_n|^{p_0} dx + \frac{p_n - p_0}{2^* - p_0} \int_{\mathbb{R}^N} |u_n|^{2^*} dx + \frac{2^* - p_n}{2^* - p_0} \int_{\mathbb{R}^N} |u_n|^{p_0} dx \\ &= \int_{\mathbb{R}^N} |u_n|^{2^*} dx + o_n(1) \\ &\le S^{\frac{N}{N-2}} \Big(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\Big)^{\frac{N}{N-2}} + o_n(1). \end{aligned}$$

From this we deduce that either $\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \to 0$ or $\liminf_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \ge S^{\frac{N}{2}}$.

If $\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \to 0$, then $\int_{\mathbb{R}^N} |u_n|^{2^*} dx \to 0$, $u_n \to 0$ in H and $I_{p_n}^{\infty}(u_n) \to 0$, a contradiction.

If $\liminf_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \ge S^{\frac{N}{2}}$, then $\liminf_{n\to+\infty} \int_{\mathbb{R}^N} |u_n|^{p_n} dx \ge S^{\frac{N}{2}}$, thus

$$\liminf_{n\to+\infty} I_{p_n}(u_n) = \liminf_{n\to+\infty} \Big[\frac{1}{2}\int_{\mathbb{R}^N} u_n^2 dx + \Big(\frac{1}{2} - \frac{1}{p_n}\Big)\int_{\mathbb{R}^N} |u_n|^{p_n} dx\Big] \ge \frac{1}{N} S^{\frac{N}{2}},$$

which is also a contradiction.

So vanishing cannot happen. Now suppose that

$$\lim_{n\to\infty} \int_{B_R(y_n)} u_n^2 dx = \nu > 0.$$

First suppose that $\{y_n\}$ is bounded and without loss of generality we may assume $y_n = 0$ for all n. Since $\|u_n\|_H$ are uniformly bounded, up to a subsequence we suppose $u_n \rightharpoonup u$ weakly in H and $\int_{B_R(0)} u^2 dx = \lim_{n\to\infty} \int_{B_R(0)} u_n^2 dx = \nu > 0$. By proposition 4.1, u is a solution of the critical problem (1.1) and $I(u) \le \liminf_{n\to\infty} I_{p_n}(u_n) < c^\infty$.

Next suppose $\{y_n\}$ is unbounded, let $\widetilde{u}_n := u_n(\cdot + y_n)$. Up to a subsequence, we suppose $\widetilde{u}_n \rightharpoonup u$ in H, and u satisfies

$$\int_{B_R(0)} u^2 dx = \lim_{n\to\infty} \int_{B_R(0)} \widetilde{u}_n^2 dx = \lim_{n\to\infty} \int_{B_R(y_n)} u_n^2 dx = \nu > 0.$$

By Proposition 4.1, u is a solution of the critical problem (1.12) and $I^\infty(u) \le \liminf_{n\to\infty} I_{p_n}(u_n) < c^\infty$. But we always have $c^\infty \le I(u)$ for a nontrivial solution of the problem (1.12), we arrive at a contradiction. Thus $\{y_n\}$ must be bounded. □

PROPOSITION 4.3. *Assume $N \ge 4$, then equation* (1.1) *has a positive solution $u \in \mathcal{D}$ with $I(u) = c$.*

Proof. By Lemma 3.5, the problem (1.12) has a strictly positive solution $u_\infty \in \mathcal{D}$ with $I^\infty(u_\infty) = c^\infty$. Then there exists $t_0 > 0$ such that $J(t_0 u_\infty) = 0$, $I(t_0 u_\infty) = \sup_{t>0} I(t u_\infty)$. By the assumption (V_1), one easily get that

$$c \le I(t_0 u_\infty) < I^\infty(t_0 u_\infty) \le I^\infty(u_\infty) = c^\infty.$$

By Lemma 3.1, we have $\limsup_{p\to 2^*} c_p \le c < c^\infty$. Proposition 2.6 implies that problem (1.10) has a positive solution u_p with $I_p(u_p) = c_p$. By Lemma 4.2, for a sequence $p_n \to 2^*$, $u_n := u_{p_n} \rightharpoonup u \ne 0$ in H. By Proposition 4.1, we deduce that $u \in \mathcal{D}$ must be a solution for equation (1.1), and $I(u) \le \liminf_{p_n\to 2^*} I_{p_n}(u_n) = \liminf_{p_n\to 2^*} c_{p_n} \le c$. On the other hand, $c \le I(u)$ for any nontrivial critical point u of I, and thus $I(u) = c$.

Finally, by standard arguments, $u \ge 0$ or $u \le 0$. Without loss of generality, assume $u \ge 0$. By Lemma 3.10 in [22], we can deduce that u is a classical solution. It follows from the maximum principle (see Theorem 1, [28]) that $u(x) > 0$. □

In the remaining part of the section we study the existence of ground state sign-changing solutions.

LEMMA 4.4. *It holds $m < c + c^\infty$.*

Proof. Let u, w be the strictly positive solutions of (1.1) and (1.12) respectively with $I(u) = c$, $I^\infty(w) = c^\infty$. We denote $w_R(x) := w(x_1 + 2R, x')$ with $(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1}$. By a similar argument as the proof of Lemma 2.8, we can show that $m \le \sup_{(\alpha,\beta)\in\mathbb{R}^2} I(\alpha u + \beta w_R) < c + c^\infty$, provided R is large enough. □

PROPOSITION 4.5. *Assume $N \ge 4$, then equation* (1.1) *has a sign-changing solution $u \in \mathcal{D}$ with $I(u) = m$, and u has exactly two nodal domains.*

Proof. By Lemma 3.1 and Lemma 4.4, we have

$$\limsup_{p\to 2^*} m_p \le m < c + c^\infty.$$

By Proposition 2.9, equation (1.10) has a sign-changing solution u_p with $I_p(u_p) = m_p$. Taking $p_n < 2^*$, $p_n \to 2^*$, $u_n = u_{p_n}$. We claim that there is a positive number ν such that

$$\int_{\mathbb{R}^N} |u_n^+|^{2^*} dx \geq \nu, \quad \int_{\mathbb{R}^N} |u_n^-|^{2^*} dx \geq \nu. \tag{4.5}$$

In fact, by $J_{p_n}(u_n^+) = 0$ and Hölder inequality we have

$$\begin{aligned}\int_{\mathbb{R}^N} \big(|\nabla u_n^+|^2 + V(x)|u_n^+|^2\big)dx &= \int_{\mathbb{R}^N} |u_n^+|^2 \log |u_n^+|^2 dx + \int_{\mathbb{R}^N} |u_n^+|^{p_n} dx \\ &\leq C_\varepsilon \int_{\mathbb{R}^N} |u_n^+|^{2^*} dx + \varepsilon \int_{\mathbb{R}^N} |u_n^+|^2 dx,\end{aligned}$$

where $0 < \varepsilon < \frac{1}{2} \inf_{\mathbb{R}^N} V(x)$. It follows from the above inequality and Sobolev embedding theorem that

$$\int_{\mathbb{R}^N} |u_n^+|^{2^*} dx \geq \nu.$$

Similarly,

$$\int_{\mathbb{R}^N} |u_n^-|^{2^*} dx \geq \nu.$$

We distinguish two cases here.
Case 1. $\{u_n\}$ is vanishing: for any $R > 0$,

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^2 dx = 0.$$

Case 2. $\{u_n\}$ is non-vanishing: there exist $R > 0$, $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n\to\infty} \int_{B_R(y_n)} |u_n|^2 dx = \delta > 0.$$

For Case 1, as we did in the proof of Lemma 4.2, by $J_{p_n}(u_n^+) = 0$, one gets

$$\int_{\mathbb{R}^N} \big(|\nabla u_n^+|^2 + V(x)|u_n^+|^2\big)dx \leq \int_{\mathbb{R}^N} |u_n^+|^{2^*} dx + o_n(1).$$

From this we have $\liminf_{n\to\infty} \int_{\mathbb{R}^N} |u_n^+|^{2^*} dx \geq S^{\frac{N}{2}}$ and $\liminf_{n\to\infty} I_{p_n}(u_n^+) \geq \frac{1}{N} S^{\frac{N}{2}}$. Similarly, $\liminf_{n\to\infty} I_{p_n}(u_n^-) \geq \frac{1}{N} S^{\frac{N}{2}}$. This leads to

$$m \geq \lim_{n\to\infty} m_{p_n} = \lim_{n\to\infty} I_{p_n}(u_n) \geq \frac{2}{N} S^{\frac{N}{2}} > c + c^\infty,$$

which is a contradiction.

For Case 2, suppose $\widetilde{u}_n = u_n(\cdot + y_n) \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, which implies $u \neq 0$. Let $v_n = u_n - u(\cdot - y_n)$. We further divide Case 2 into two subcases.

Subcase 2.a. $\{v_n\}$ is non-vanishing: there exist $R' > 0$, $\{z_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n\to\infty} \int_{B_{R'}(z_n)} |v_n|^2 dx = \delta' > 0.$$

Subcase 2.b. $\{v_n\}$ is vanishing: for any $R>0$,

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}^N}\int_{B_R(y)}|v_n|^2dx=0.$$

If Subcase 2.a occurs, we claim $|z_n-y_n|$ is unbounded. In fact, for all $R>0$,

$$\lim_{n\to\infty}\int_{B_R(0)}|u_n(x+y_n)-u(x)|^2dx\to 0.$$

But,

$$\begin{aligned}\int_{B_{R'}(z_n-y_n)}|u_n(x+y_n)-u(x)|^2dx&=\int_{B_{R'}(z_n)}|u_n(x)-u(x-y_n)|^2dx\\&=\int_{B_{R'}(z_n)}v_n^2dx=\delta'>0.\end{aligned}\tag{4.6}$$

Hence $|z_n-y_n|$ must be unbounded. Suppose $\widehat{u}_n=u_n(\cdot+z_n)\rightharpoonup w$ in $H^1(\mathbb{R}^N)$. Since $u(x-y_n+z_n)\to 0$ in $L^2(B_{R'}(0))$, we have

$$\begin{aligned}\int_{B_{R'}(0)}w^2dx&=\lim_{n\to\infty}\int_{B_{R'}(0)}|u_n(x+z_n)|^2dx\\&=\lim_{n\to\infty}\int_{B_{R'}(0)}|v_n(x+z_n)+u(x+z_n-y_n)|^2dx\\&=\lim_{n\to\infty}\int_{B_{R'}(z_n)}v_n^2dx=\delta'>0.\end{aligned}\tag{4.7}$$

Therefore $w\neq 0$. Since at least one of the sequences $\{y_n\}$ and $\{z_n\}$ is unbounded, there are three cases according to that either one of the two sequences is bounded and the other is unbounded or both sequences are unbounded. For definiteness, we assume that $\{y_n\}$ is bounded and $\{z_n\}$ is unbounded, the other two cases can be dealt similarly. By Proposition 4.1, u and w are solutions of the problems (1.1) and (1.12) respectively. We now estimate $I_{p_n}(u_n)$. Given $R>0$, for n large enough $B_R(y_n)\cap B_R(z_n)=\emptyset$, we get

$$\begin{aligned}I_{p_n}(u_n)&=I_{p_n}(u_n)-\frac{1}{2}\langle DI_{p_n}(u_n),u_n\rangle\\&=\frac{1}{2}\int_{\mathbb{R}^N}u_n^2dx+(\frac{1}{2}-\frac{1}{p_n})\int_{\mathbb{R}^N}|u_n|^{p_n}dx\\&\geq\frac{1}{2}\int_{B_R(y_n)}u_n^2dx+(\frac{1}{2}-\frac{1}{p_n})\int_{B_R(y_n)}|u_n|^{p_n}dx\\&\quad+\frac{1}{2}\int_{B_R(z_n)}u_n^2dx+(\frac{1}{2}-\frac{1}{p_n})\int_{B_R(z_n)}|u_n|^{p_n}dx.\end{aligned}$$

Letting $n\to\infty$, by Fatou's Lemma, we have

$$\begin{aligned}\lim_{n\to\infty}I_{p_n}(u_n)\geq&\frac{1}{2}\int_{B_R(0)}u^2dx+(\frac{1}{2}-\frac{1}{2^*})\int_{B_R(0)}|u|^{2^*}dx\\&+\frac{1}{2}\int_{B_R(0)}w^2dx+(\frac{1}{2}-\frac{1}{2^*})\int_{B_R(0)}|w|^{2^*}dx.\end{aligned}$$

Letting $R \to +\infty$, we obtain

$$\lim_{n\to\infty} I_{p_n}(u_n) \geq I(u) - \frac{1}{2}\langle DI(u), u\rangle + I^{\infty}(w) - \frac{1}{2}\langle DI^{\infty}(w), w\rangle \geq I(u) + I^{\infty}(w)$$

and

$$m = \lim_{p\to 2^*} m_p = \lim_{n\to\infty} I_{p_n}(u_n) \geq I(u) + I^{\infty}(u) \geq c + c^{\infty},$$

which is a contradiction. Thus, Subcase 2.a can not occur.

For Subcase 2.b, we have three subcases.

Subcase 2.b_1: v_n converges to zero in $L^{2^*}(\mathbb{R}^N)$, and $\{y_n\}$ is bounded;
Subcase 2.b_2: v_n converges to zero in $L^{2^*}(\mathbb{R}^N)$, and $\{y_n\}$ is unbounded;
Subcase 2.b_3: v_n does not converge to zero in $L^{2^*}(\mathbb{R}^N)$.

We next examine case by case to show all cases are not possible except Subcase 2.b_1, which leads to the desired existence result.

For Subcase 2.b_1, by Proposition 4.1, $u \in \mathcal{D}$ and solves equation (1.1), and

$$I(u) \leq \liminf_{n\to\infty} I_{p_n}(u_n) = \lim_{n\to\infty} m_{p_n} \leq m.$$

Since $v_n \to 0$ in $L^{2^*}(\mathbb{R}^N)$, $u(\cdot + y_n) \to u$ in $L^{2^*}(\mathbb{R}^N)$, by (4.5), $u^+ \neq 0$, $u^- \neq 0$, hence u is a sign-changing solution of problem (1.1). As a sign-changing solution of problem (1.1), $J(u^+) = J(u^-) = 0$, $I(u) \geq m$. Thus, $I(u) = m$.

For Subcase 2.b_2, by Proposition 4.1, $u \in \mathcal{D}$ and solves equation (1.12), and

$$I^{\infty}(u) \leq \liminf_{n\to\infty} I_{p_n}(u_n) = \lim_{n\to\infty} m_{p_n} \leq m. \tag{4.8}$$

As in Subcase 2.b_1, $u^+ \neq 0$, $u^- \neq 0$, hence u is a sign-changing solution of problem (1.12). As a sign-changing solution of problem (1.12), $J(u^+) = J(u^-) = 0$, hence

$$I^{\infty}(u) = I^{\infty}(u^+) + I^{\infty}(u^-) \geq c^{\infty} + c^{\infty} > c + c^{\infty}.$$

Combining this and (4.8), we get a contradiction since $m < c + c^{\infty}$.

For Subcase 2.b_3, since $\{v_n\}$ is vanishing: for any $R > 0$,

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^N} \int_{B_R(y)} |v_n|^2 dx = 0.$$

Thus, $v_n \to 0$ in $L^r(\mathbb{R}^N)$ for all $2 < r < 2^*$. Then, by using the fact $u_n(\cdot + y_n) \to u$ in $L^r_{loc}(\mathbb{R}^N)$ for $2 < r < 2^*$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |u_n|^r dx &= \int_{\mathbb{R}^N} |u_n(x+y_n)|^r dx \\ &= \int_{\mathbb{R}^N} |u_n(x+y_n) - u(x)|^r dx + \int_{\mathbb{R}^N} |u(x)|^r dx + o_n(1) \\ &= \int_{\mathbb{R}^N} |u(x)|^r dx + o_n(1).\end{aligned}$$

Similarly,

$$\int_{\mathbb{R}^N} |u_n|^{2^*} dx = \int_{\mathbb{R}^N} |v_n|^{2^*} dx + \int_{\mathbb{R}^N} |u(x)|^{2^*} dx + o_n(1).$$

By using Lemma 2.1, we also have

$$\int_{\mathbb{R}^N} u_n^2 \log u_n^2 dx = \int_{\mathbb{R}^N} v_n^2 \log v_n^2 dx + \int_{\mathbb{R}^N} u^2 \log u^2 dx + o_n(1).$$

Now suppose $\{y_n\}$ is bounded. It follows from $J_{p_n}(u_n) = 0$ that

$$\begin{aligned}
0 &= \int_{\mathbb{R}^N} \big(|\nabla u_n|^2 + V(x)u_n^2\big)dx - \int_{\mathbb{R}^N} u_n^2 \log u_n^2 dx - \int_{\mathbb{R}^N} |u_n|^{p_n} dx \\
&\geq \int_{\mathbb{R}^N} \big(|\nabla u_n|^2 + V(x)u_n^2\big)dx - \int_{\mathbb{R}^N} u_n^2 \log u_n^2 dx \\
&\quad - \int_{\mathbb{R}^N} \frac{p_n - 2}{2^* - 2}|u_n|^{2^*} + \frac{2^* - p_n}{2^* - 2}|u_n|^2 dx \\
&\geq J(u) + \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - \int_{\mathbb{R}^N} v_n^2 \log v_n^2 dx - \int_{\mathbb{R}^N} |v_n|^{2^*} dx + o_n(1) \\
&\geq \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - \int_{\mathbb{R}^N} |v_n|^{2^*} dx + o_n(1). \qquad (4.9)
\end{aligned}$$

From this and Sobolev embedding theorem, we have $\lim\limits_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \geq S^{\frac{N}{2}}$. Hence

$$\begin{aligned}
I_{p_n}(u_n) &= I_{p_n}(u_n) - \frac{1}{p_n}\langle DI_{p_n}(u_n), u_n\rangle \\
&= \big(\frac{1}{2} - \frac{1}{p_n}\big) \int_{\mathbb{R}^N} |\nabla u_n|^2 + V(x)u_n^2 dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N} u_n^2 dx - \big(\frac{1}{2} - \frac{1}{p_n}\big) \int_{\mathbb{R}^N} u_n^2 \log u_n^2 dx \\
&\geq I(u) + \big(\frac{1}{2} - \frac{1}{p_n}\big) \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - \big(\frac{1}{2} - \frac{1}{p_n}\big) \int_{\mathbb{R}^N} (v_n^2 \log v_n^2)^+ dx + o_n(1) \\
&\geq I(u) + \big(\frac{1}{2} - \frac{1}{p_n}\big) \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - C_{p_0} \int_{\mathbb{R}^N} |v_n|^{p_0} dx + o_n(1) \\
&\geq I(u) + \big(\frac{1}{2} - \frac{1}{p_n}\big) S^{\frac{N}{2}} + o_n(1) = I(u) + \frac{1}{N} S^{\frac{N}{2}} + o_n(1), \qquad (4.10)
\end{aligned}$$

where $2 < p_0 < 2^*$. Thus

$$m \geq \lim_{n\to\infty} m_{p_n} = \lim_{n\to\infty} I_{p_n}(u_n) \geq I(u) + \frac{1}{N}S^{\frac{N}{2}} \geq c + \frac{1}{N}S^{\frac{N}{2}} \geq c + c^{\infty},$$

a contradiction.

If $\{y_n\}$ is unbounded, then u solves problem (1.12) and $\lim\limits_{n\to\infty} \int_{\mathbb{R}^N} V(x)u_n^2 dx \geq \int_{\mathbb{R}^N} V_\infty u^2 dx$. Instead of (4.9) and (4.10), we have

$$\begin{aligned}
0 &= \int_{\mathbb{R}^N} \big(|\nabla u_n|^2 + V(x)u_n^2\big)dx - \int_{\mathbb{R}^N} u_n^2 \log u_n^2 dx - \int_{\mathbb{R}^N} |u_n|^{p_n} dx \\
&\geq J^\infty(u) + \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - \int_{\mathbb{R}^N} v_n^2 \log v_n^2 dx - \int_{\mathbb{R}^N} |v_n|^{2^*} dx + o_n(1) \\
&\geq \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - \int_{\mathbb{R}^N} |v_n|^{2^*} dx + o_n(1).
\end{aligned}$$

Again we have $\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla v_n|^2dx \geq S^{\frac{N}{2}}$ and

$$\begin{aligned}
I_{p_n}(u_n) &= I_{p_n}(u_n) - \frac{1}{p_n}\langle DI_{p_n}(u_n), u_n\rangle \\
&= \big(\frac{1}{2}-\frac{1}{p_n}\big)\int_{\mathbb{R}^N}|\nabla u_n|^2 + V(x)u_n^2dx \\
&\quad +\frac{1}{2}\int_{\mathbb{R}^N}u_n^2dx - \big(\frac{1}{2}-\frac{1}{p_n}\big)\int_{\mathbb{R}^N}u_n^2\log u_n^2dx \\
&\geq I^\infty(u) + \big(\frac{1}{2}-\frac{1}{p_n}\big)\int_{\mathbb{R}^N}|\nabla v_n|^2dx - \big(\frac{1}{2}-\frac{1}{p_n}\big)\int_{\mathbb{R}^N}(v_n^2\log v_n^2)^+dx + o_n(1) \\
&\geq I^\infty(u) + \big(\frac{1}{2}-\frac{1}{p_n}\big)\int_{\mathbb{R}^N}|\nabla v_n|^2dx - C_{p_0}\int_{\mathbb{R}^N}|v_n|^{p_0}dx + o_n(1) \\
&\geq I^\infty(u) + \big(\frac{1}{2}-\frac{1}{p_n}\big)S^{\frac{N}{2}} + o_n(1) = I^\infty(u) + \frac{1}{N}S^{\frac{N}{2}} + o_n(1).
\end{aligned}$$

Thus,

$$m \geq \lim_{n\to\infty} m_{p_n} = \lim_{n\to\infty} I_{p_n}(u_n) \geq I^\infty(u) + \frac{1}{N}S^{\frac{N}{2}} \geq c^\infty + \frac{1}{N}S^{\frac{N}{2}} \geq c + c^\infty,$$

which is also a contradiction.

Thus, we have treated all the cases and the proof of the Proposition is finished. □

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