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HSIAO-LIU CORRECTION FUNCTIONS AND THEIR APPLICATIONS*

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Dedicated to Professor Ling Hsiao on the occasion of her 80th birthday

Abstract. In [19], Hsiao-Liu correction functions were first introduced to show that the solutions of a system of hyperbolic conservation laws with damping time-asymptotically tend to the nonlinear diffusion waves governed by the classical Darcy's law. This is a survey paper on Hsiao-Liu correction functions and their applications.

Key words. Correction functions, nonlinear diffusion waves, Darcy's law, asymptotic stability.

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1. Introduction. We start with the discussion of the following hyperbolic conservation laws with damping

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u, \end{cases} \tag{1.1}$$

which models the compressible flow through porous media. This system can be viewed as isentropic Euler equations in the Lagrangian coordinates with friction term $-\alpha u$ for the momentum equation. Here, $v(x,t) > 0$ is the specific volume, $u(x,t)$ is the velocity, the pressure p is a smooth function of v with $p > 0,\ p' < 0$, and α is a positive constant.

In [19], Hsiao and Liu guessed, according Darcy's law, that the solution (v,u) to system (1.1) are expected to behave time asymptotically as the solutions $(\bar{v},\bar{u})$ to the following parabolic (porous media) equations

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ p(\bar{v})_x = -\alpha \bar{u}, \end{cases} \tag{1.2}$$

or

$$\begin{cases} \bar{v}_t = -\dfrac{1}{\alpha} p(\bar{v})_{xx}, \\ p(\bar{v})_x = -\alpha \bar{u}. \end{cases} \tag{1.3}$$

We consider the solutions $(v(x,t), u(x,t))$ to the Cauchy problem for (1.1) with the initial data

$$(v,u)(x,0) = (v_0(x), u_0(x)), \qquad x \in \mathbb{R}, \tag{1.4}$$

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and the far-field behavior

$$(v_0(x), u_0(x)) \to (v_\pm, u_\pm), \qquad \text{as} \quad x \to \pm\infty,$$

where $v_\pm$ and $u_\pm$ are constant states. Denote $\bar{v}$ by any solutions of (1.3) with the same end states as $v(x,0)$:

$$\bar{v}(\pm\infty, t) = v_\pm$$

and set

$$\bar{u} = -\frac{1}{\alpha} p(\bar{v})_x.$$

Then, it holds that $\bar{u}(\pm\infty, t) = 0$. Next, we investigate the asymptotic behavior of $(v,u)(x,t)$ at $x = \pm\infty$. Taking the limits as $x = \pm\infty$ in (1.1) and noting that u_x and $p(v)_x$ will vanish at $x = \pm\infty$, one has

$$\begin{cases} \dfrac{d}{dt} v(\pm\infty, t) = 0, \\ \dfrac{d}{dt} u(\pm\infty, t) = -\alpha u(\pm\infty, t), \end{cases}$$

which leads to

$$\begin{cases} v(\pm\infty, t) = v_\pm, \\ u(\pm\infty, t) = u_\pm e^{-\alpha t}. \end{cases}$$

It is easy to see that

$$(v - \bar{v},\ u - \bar{u}) \to (0,\ u_\pm e^{-\alpha t}), \qquad \text{as} \quad x \to \pm\infty.$$

Hence, when $u_\pm \neq 0$, the energy method used to get L^2-estimates of the solutions can not be applied directly to the disturbing functions $(v(x,t) - \bar{v}(x,t), u(x,t) - \bar{u}(x,t))$. To overcome this difficulty, some auxiliary functions are needed to eliminate the gap yield by u and $\bar{u}$ at $x = \pm\infty$. That is, we need to introduce a pair of correction functions $(\hat{v}(x,t), \hat{u}(x,t))$ such that

$$(v - \bar{v} - \hat{v},\ u - \bar{u} - \hat{u}) \to (0,0), \qquad \text{as} \quad x \to \pm\infty. \tag{1.5}$$

The correction functions were first ingeniously constructed by Hsiao and Liu in [19, 20]. They have an important observation that, in addition to (1.5), the appropriate correction functions need to satisfy the following two properties:

- the time-decay rates of the correction functions $(\hat{v}, \hat{u})$ must be faster than those of $(v - \bar{v}, u - \bar{u})$;
- the equations of the correction functions $(\hat{v}, \hat{u})$ should have a coherent structure with (1.1) and (1.2).

Based on this observation, we turn to the concrete structure of $(\hat{v}, \hat{u})$. Let $\hat{u}(x,t)$ be the solution to the following problem:

$$\begin{cases} \dfrac{d}{dt} \hat{u}(x,t) = -\alpha \hat{u}(x,t), \\ \hat{u}(\pm\infty, t) = u_\pm e^{-\alpha t}. \end{cases}$$

Then it can be solved as

$$\hat{u}(x,t) = m(x)e^{-\alpha t},$$

where $m(\pm\infty) = u_\pm$. To do this, we construct it as

$$m(x) = u_- + (u_+ - u_-)\int_{-\infty}^{x} m_0(y)dy.$$

Here

$$m_0(x) \in C_0^\infty(\mathbb{R}) \quad \text{and} \quad \int_{-\infty}^{\infty} m_0(x)dx = 1.$$

Now, let $\hat{v}(x,t)$ satisfy

$$\hat{v}_t = \hat{u}_x,$$

which immediately leads to

$$\hat{v}(x,t) = -\frac{u_+ - u_-}{\alpha} m_0(x)e^{-\alpha t}.$$

Thus, we obtain the following Hsiao-Liu correction functions (see in [19])

$$\begin{cases} \hat{v}(x,t) = -\dfrac{u_+ - u_-}{\alpha} m_0(x)e^{-\alpha t}, \\ \hat{u}(x,t) = e^{-\alpha t}\left(u_- + (u_+ - u_-)\displaystyle\int_{-\infty}^{x} m_0(y)dy\right), \end{cases}$$

which satisfies

$$\begin{cases} \hat{v}_t = \hat{u}_x, \\ \hat{u}_t = -\alpha\hat{u}, \\ (\hat{v},\hat{u})(\pm\infty,t) = (0, u_\pm e^{-\alpha t}). \end{cases} \tag{1.6}$$

We have from (1.1), (1.2) and (1.6) that

$$\begin{cases} (v - \bar{v} - \hat{v})_t - (u - \bar{u} - \hat{u})_x = 0, \\ (u - \bar{u} - \hat{u})_t + (p(v) - p(\bar{v}))_x + \bar{u}_t + \alpha(u - \bar{u} - \hat{u}) = 0, \\ (v - \bar{v} - \hat{v},\ u - \bar{u} - \hat{u}) \to (0,0), \quad \text{as} \quad x \to \pm\infty. \end{cases}$$

Defining

$$\begin{cases} V(x,t) = \displaystyle\int_{-\infty}^{x} (v(y,t) - \bar{v}(y + x_0, t) - \hat{v}(y,t))dy, \\ U(x,t) = u(x,t) - \bar{u}(x + x_0, t) - \hat{u}(x,t), \end{cases}$$

where x_0 is a constant uniquely determined by

$$\int_{-\infty}^{\infty} (v(x,0) - \bar{v}(x + x_0, 0))dx = -\frac{u_+ - u_-}{\alpha}.$$

Thus, $(V(x,t),U(x,t))$ solves the following Cauchy problem:

$$\begin{cases} V_t - U = 0, \\ U_t + (p(V_x + \bar{v} + \hat{v}) - p(\bar{v}))_x + \alpha U = \dfrac{1}{\alpha} p(\bar{v})_{xt}, \\ (V(x,0),U(x,0)) = (V_0(x),U_0(x)) \to (0,0), \quad \text{as} \quad x \to \pm\infty, \end{cases} \tag{1.7}$$

i.e.,

$$\begin{cases} V_t - U = 0, \\ U_t + (p'(\bar{v})V_x)_x + \alpha U = f_1 + f_2, \\ (V(x,0),U(x,0)) = (V_0(x),U_0(x)) \to (0,0), \quad \text{as} \quad x \to \pm\infty, \end{cases}$$

where

$$\begin{cases} f_1 = \dfrac{1}{\alpha} p(\bar{v})_{xt}, \\ f_2 = -[p(V_x + \bar{v} + \hat{v}) - p(\bar{v}) - p'(\bar{v})V_x]_x. \end{cases}$$

The main results obtained by Hsiao and Liu in [19] can be stated as follows:

Theorem 1.1 (Hsiao and Liu, [19]). *If $V_0(x) = V(x,0) \in H^3(\mathbb{R})$, $U_0(x) = V_t(x,0) \in H^2(\mathbb{R})$ and*

$$|u_+ - u_-| + |v_+ - v_-| + \|V_0\|_{H^3} + \|U_0\|_{H^2} \leqslant \varepsilon_0,$$

for some sufficiently small ε_0, then there exists a global in time solution

$$V(x,t) \in L^\infty\left([0,\infty), H^3(\mathbb{R})\right), \qquad U(x,t) \in L^\infty\left([0,\infty), H^2(\mathbb{R})\right),$$

of (1.7)*, which satisfies*

$$\|(V_x,U)(t)\|_{L^2} + \|(V_x,U)(t)\|_{L^\infty} = O(1)\varepsilon_0(1+t)^{-\frac{1}{2}}.$$

The proposition of Hsiao-Liu correction functions has attracted considerable attentions of mathematicians. This function becomes an important theoretical tool for studying the stability of nonlinear diffusion waves. There are several improvements of [19]. However, the more important issue is to generalize [19] to more general cases. Apparently, there are two lines for such kind of generalizations. The first one is to focus on p-system itself and get more and more profound research results. The second one is to extend the results of [19] to various other models to solve more interesting problems.

The plan of this paper is organized as follows. In Section 2, we will focus on some research progress of the p-system with damping itself. The successful application of Hsiao-Liu correction functions to various other related models will be introduced in Section 3.

2. p-system with damping. In this section, we will introduce the related research of p-system with damping by classification.

2.1. p-system with linear damping. Then, by taking more detailed but elegant energy estimates, Nishihara [60] succeeded in improving the convergence rates as $\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty}=O(1)((1+t)^{-3/4},(1+t)^{-5/4})$.

Theorem 2.1 (Nishihara, [60]). *Under the conditions of Theorem 1.1, there exists a global in time solution of* (1.7) *which satisfies*

$$V(x,t)\in W^{\bar{k},\infty}\left([0,\infty);H^{3-\bar{k}}\right),\qquad U(x,t)\in W^{k,\infty}\left([0,\infty);H^{2-k}\right)$$

for $\bar{k}=0,1,2,3;\ k=0,1,2,$ *and*

$$\begin{aligned}&\left\|\partial_x^k V_x(t)\right\|_{L^2}=O(1)\varepsilon_0(1+t)^{-\frac{k+1}{2}},\\&\left\|\partial_x^k U(t)\right\|_{L^2}=O(1)\varepsilon_0(1+t)^{-\frac{k+2}{2}},\\&\|V_x(t)\|_{L^\infty}=O(1)\varepsilon_0(1+t)^{-\frac{3}{4}},\qquad \|U(t)\|_{L^\infty}=O(1)\varepsilon_0(1+t)^{-\frac{5}{4}}.\end{aligned}$$

Moreover, if $v_+=v_-,\ u_+=u_-=0$ *and* $V_0,\ U_0\in L^1(\mathbb{R})$ *with*

$$\int_{-\infty}^{\infty}(v_0(x)-v_-)\,dx=0,$$

then

$$\|V_x(t)\|_{L^\infty}=O(1)\varepsilon_0(1+t)^{-1},\qquad \|U(t)\|_{L^\infty}=O(1)\varepsilon_0(1+t)^{-\frac{3}{2}}.$$

Furthermore, when the initial perturbation is in $H^3\cap L^1$, by constructing an approximate Green function with the energy method together, Nishihara, Wang and Yang [63] improved the convergence rates as $\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty}=O(1)((1+t)^{-1},(1+t)^{-3/2})$, which are optimal in the sense comparing with the decay of the solution to the heat equation.

Theorem 2.2 (Nishihara, Wang and Yang, [63]). *If* $V_0(x)\in H^3(\mathbb{R})\cap L^1(\mathbb{R})$, $U_0(x)\in H^2(\mathbb{R})\cap L^1(\mathbb{R})$, *and*

$$|u_+-u_-|+|v_+-v_-|+\|V_0\|_{H^3}+\|U_0\|_{H^2}+\|V_0\|_{L^1}+\|U_0\|_{L^1}\leqslant\varepsilon_0$$

for some sufficiently ε_0, *then there exists a global in time solution* $(V(x,t),U(x,t))$ *of* (1.7), *which satisfies*

$$\begin{aligned}&\left\|\partial_x^k V_x(t)\right\|_{L^p}=O(1)\varepsilon_0(1+t)^{-\frac{p-1}{2p}-\frac{k+1}{2}},\\&\left\|\partial_x^k U(t)\right\|_{L^p}=O(1)\varepsilon_0(1+t)^{-\frac{p-1}{2p}-\frac{k+2}{2}},\end{aligned}$$

for any $k\leqslant 2$ *if* $p=2$ *and* $k\leqslant 1$ *if* $p\in(2,+\infty]$.

These results in above theorems need the initial perturbation around the specified diffusion wave and the wave strength both to be sufficiently small. Such restrictions were then partially released by Zhao [78], where the initial perturbation in the L^∞ sense can be arbitrarily large. Before stating the main results in [78], we first assume that $p(v)$ satisfies one of the following two requirements:

(H_1) $p(v)\in C^4(0,\infty),\ p'(v)<0,\ p''(v)>0,\ 4p'(v)p'''(v)\geqslant 5(p''(v))^2,$ for all $v>0$.

(H$_2$) $\lim\limits_{v\to 0}\int_v^1\sqrt{-p'(\tau)}d\tau=+\infty$, $\lim\limits_{v\to 0}p'(v)=0$, $p(v)\in C^4(\mathbb{R})$, $p'(v)<0$, for all $v\in\mathbb{R}$.

THEOREM 2.3 (Zhao, [78]). *Let the assumption (H$_1$) (respectively (H$_2$)) holds and for arbitrarily given positive constants v_1, v_2 (respectively M_1) and M_2, there exists a sufficiently small positive constant M_3 such that $r_0(x)$, $s_0(x)\in C_b^1(\mathbb{R})$ with*

$$\begin{cases} v_1\leqslant v_0(x)\leqslant v_2 \text{ (respectively } |v_0(x)|\leqslant M_1), \qquad |u_0(x)|\leqslant M_2,\\ |r_0'(x)|\leqslant \alpha M_3, \qquad |s_0'(x)|\leqslant \alpha M_3,\end{cases}$$

where

$$r_0(x)=u_0(x)+\Phi(v_0(x)), \qquad s_0(x)=u_0(x)-\Phi(v_0(x)).$$

Then the Cauchy problem (1.1)-(3.2) *admits a unique global smooth solution $(v(t,x),u(t,x))$.*

Moreover, if $V_0(x)\in H^3(\mathbb{R})$, $U_0(x)\in H^2(\mathbb{R})$, we can get that, as the time t goes to infinity, such a solution $(v(t,x),u(t,x))$ tends to the similarity solution $(\bar v(t,x+x_0),\bar u(t,x+x_0))$ of (1.3) *and the following decay estimates hold*

$$\begin{aligned}&\sum_{k=0}^{3}(1+t)^k\left\|\partial_x^k V(t)\right\|^2+\sum_{k=0}^{2}(1+t)^{k+2}\left\|\partial_x^k U(t)\right\|^2+(1+t)^4\left\|\partial_t U(t)\right\|^2\\ &+\int_0^t\left[\sum_{j=1}^{3}(1+\tau)^{j-1}\left\|\partial_x^j V(\tau)\right\|^2+\sum_{j=0}^{2}(1+\tau)^{j+1}\left\|\partial_x^j U(\tau)\right\|^2\right]d\tau\\ \leqslant\ & O(1)\left(\|V_0\|_3^2+\|U_0\|_2^2+1\right),\end{aligned}$$

and

$$\begin{aligned}&(1+t)^5\left(\|U_{tt}(t)\|^2+\|U_{xt}(t)\|^2\right)+\int_0^t(1+\tau)^5\left\|U_{tt}(\tau)\right\|^2d\tau\\ \leqslant\ & O(1)\left(\|V_0\|_3^2+\|U_0\|_2^2+1\right).\end{aligned}$$

Furthermore, under the additional assumption that $V_0(x)\in L^1(\mathbb{R})$, $U_0(x)\in L^1(\mathbb{R})$, the following optimal L^p $(2\le p\le\infty)$ decay estimates are true

$$\begin{aligned}\left\|\partial_x^k V_x(t)\right\|_{L^p}&=O(1)\varepsilon_0(1+t)^{-\frac{p-1}{2p}-\frac{k+1}{2}},\\ \left\|\partial_x^k U(t)\right\|_{L^p}&=O(1)\varepsilon_0(1+t)^{-\frac{p-1}{2p}-\frac{k+2}{2}},\end{aligned}$$

for any $k\leqslant 2$ if $p=2$ and $k\leqslant 1$ if $p\in(2,\infty]$.

Notice that, for the Darcy's law, with different initial data, the solution $(\bar v,\bar u)(x,t)$ is different. That is, the asymptotic profiles for the p-system with linear damping are not unique. Based on this observation, Nishihara [24] and Mei [21] found the best asymptotic profiles for the cases $v_-=v_+$ and $v_-\neq v_+$ respectively, and both of them obtained the convergence rates as $\|(v-\bar v,u-\bar u)(t)\|_{L^\infty}=O(1)((1+t)^{-3/2}\log t,(1+t)^{-2}\log t)$. And finally, the convergence rates were improved to $O((1+t)^{-3/2},(1+t)^{-2})$ by Geng and Wang [13].

2.2. ***p*-system with boundary effect.** On the other hand, Nishihara and Yang [64] considered the initial-boundary value problem on $\mathbb{R}^+$ to p-system with linear damping

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u, \quad x \in \mathbb{R}^+, \quad t > 0, \end{cases} \tag{2.1}$$

and with the initial data

$$(v,u)(x,0) = (v_0(x), u_0(x)) \to (v_+, u_+), \quad v_+ > 0, \quad \text{as} \ x \to \infty, \tag{2.2}$$

and the Dirichlet boundary condition

$$u|_{x=0} = 0, \tag{2.3}$$

or the Neumann boundary condition

$$u_x|_{x=0} = 0. \tag{2.4}$$

They obtained the asymptotic behavior and the convergence rates by perturbing the initial date around the linear diffusion wave $(\tilde{v}, \tilde{u})(x,t)$ which satisfies

$$\begin{cases} \tilde{v}_t - \tilde{u}_x = 0, \\ p'(v_+)\tilde{v}_x = -\alpha \tilde{u}, \quad x \in \mathbb{R}^+, \quad t > 0, \\ \tilde{u}|_{x=0} = 0, \quad (\tilde{v}, \tilde{u})(x,t)|_{x=\infty} = (v_+, 0). \end{cases}$$

And the correction functions are defined in a similar fashion to those in Hsiao and Liu [19]

$$\begin{cases} \hat{v}_1(x,t) = -\dfrac{u_+ m_0(x)}{\alpha} e^{-\alpha t}, \\ \hat{u}_1(x,t) = u_+ \displaystyle\int_0^x m_0(y)dy e^{-\alpha t}, \end{cases}$$

which satisfies

$$\begin{cases} \hat{v}_{1t} = \hat{u}_{1x}, \\ \hat{u}_{1t} = -\alpha \hat{u}_1, \\ \hat{u}_1|_{x=0} = 0, \quad (\hat{v}_1, \hat{u}_1)|_{x=\infty} = (0, u_+ e^{-\alpha t}). \end{cases}$$

Here

$$m_0(x) \in C_0^\infty(\mathbb{R}^+) \quad \text{and} \quad \int_0^\infty m_0(x)dx = 1.$$

Set

$$\begin{cases} V_1(x,t) = \displaystyle\int_{-\infty}^x (v(y,t) - \tilde{v}(y+x_0,t) - \hat{v}_1(y,t))dy, \\ U_1(x,t) = u(x,t) - \tilde{u}(x+x_0,t) - \hat{u}_1(x,t). \end{cases}$$

The main results obtained by Nishihara and Yang in [64] can be stated as follows:

THEOREM 2.4 (Nishihara and Yang, [64]). *Suppose that* $v_0 - v_+ \in L^1(\mathbb{R}^+)$, $(V_{10}, U_{10}) \in H^3(\mathbb{R}^+) \times H^2(\mathbb{R}^+)$, *and that both* $\|v_0 - v_+\|_{L^1(\mathbb{R}^+)} + \|V_{10}\|_{H^3(\mathbb{R}^+)} + \|U_{10}\|_{H^2(\mathbb{R}^+)}$ *and* u_+ *are sufficiently small. Then there exists a unique time-global solution* (V_1, U_1), *which satisfies*

$$V_1 \in C^i\left([0,\infty); H^{3-i}(\mathbb{R}^+)\right), \qquad U_1(x,t) \in C^j\left([0,\infty); H^{2-j}(\mathbb{R}^+)\right)$$

for $i = 0,1,2,3; j = 0,1,2,$ *and moreover*

$$\begin{aligned}
&\sum_{k=0}^{3}(1+t)^k \left\|\partial_x^k V_1(t)\right\|^2_{L^2(\mathbb{R}^+)} + \sum_{k=0}^{2}(1+t)^{k+2}\left\|\partial_x^k U_1(t)\right\|^2_{L^2(\mathbb{R}^+)} \\
&+ \int_0^t \left[\sum_{j=1}^{3}(1+\tau)^{j-1}\left\|\partial_x^j V_1(\tau)\right\|^2_{L^2(\mathbb{R}^+)} + \sum_{j=0}^{2}(1+\tau)^{j+1}\left\|\partial_x^j U_1(\tau)\right\|^2_{L^2(\mathbb{R}^+)}\right] d\tau \\
\leqslant\ & O(1)\left(\|V_0\|^2_{H^3(\mathbb{R}^+)} + \|U_0\|^2_{H^2(\mathbb{R}^+)} + |\delta_0|\right),
\end{aligned}$$

and

$$\begin{aligned}
&(1+t)^4\|U_{1t}(t)\|^2_{L^2(\mathbb{R}^+)} + (1+t)^5\left(\|U_{1xt}(t)\|^2_{L^2(\mathbb{R}^+)} + \|U_{1tt}(t)\|^2_{L^2(\mathbb{R}^+)}\right) \\
&+ \int_0^t \left((1+\tau)^4\|U_{1xt}(\tau)\|^2_{L^2(\mathbb{R}^+)} + (1+\tau)^5\|U_{1tt}(\tau)\|_{L^2(\mathbb{R}^+)}\right) d\tau \\
\leqslant\ & O(1)\left(\|V_{10}\|^2_{H^3(\mathbb{R}^+)} + \|U_{10}\|^2_{H^2(\mathbb{R}^+)} + |\delta_0|\right),
\end{aligned}$$

where

$$\delta_0 = 2\left(\int_0^\infty (v_0(x) - v_+)dx - \frac{u_+}{\alpha}\right).$$

Furthermore, if $\psi(x,t)$ is the solution of

$$\begin{cases}
\psi_t - \kappa\psi_{xx} = \frac{1}{\alpha}\left(p'(v_+) - p'(\tilde{v})\right)\tilde{v}_x, & (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\
\psi(x,0) = V_{10}(x) + \frac{1}{\alpha}U_{10}(x), & \psi(0,t) = 0,
\end{cases}$$

Nishihara and Yang in [64] also obtain the following asymptotic behavior and convergence rates of V as $t \to \infty$.

THEOREM 2.5 (Nishihara and Yang, [64]). *Suppose that* $(V_{10}, U_{10}) \in L^1(\mathbb{R}^+)$, *then the solution obtained in Theorem 2.4 satisfies*

$$\left\|\left(V_1 - \psi, (V_1-\psi)_x, (V_1-\psi)_t\right)(\cdot,t)\right\|_{L^\infty(\mathbb{R}^+)} = O\left(t^{-1}\,\mathrm{lnt}, t^{-3/2}\,\mathrm{lnt}, t^{-2}\ln t\right), \quad t \to \infty.$$

Moreover, Nishihara and Yang [64] also obtained the asymptotic behavior and convergence rates of the solution of (2.1), (2.2) and (2.4). Marcati, Mei and Rubino [56] also considered (2.1)-(2.3), and they got the asymptotic behavior and improved the convergence rates in [64] by perturbing the initial value around the nonlinear diffusion waves which satisfies the following equation

$$\begin{cases}
\bar{v}_t - \bar{u}_x = 0, \\
p(\bar{v})_x = -\alpha\bar{u}, \quad x \in \mathbb{R}^+, \quad t > 0, \\
\bar{u}|_{x=0} = 0, \quad (\bar{v},\bar{u})(x,t)|_{x=\infty} = (v_+, 0).
\end{cases}$$

In [45], Jiang and Zhu considered (2.1)-(2.3) and obtained the same asymptotic behavior and the convergence rates as in [56] but only under a rather weaker smallness assumption on the initial oscillation. Later, Ma and Mei in [53] found that the best asymptotic profile for the original solution is the parabolic solution of the initial-boundary value problem for the corresponding porous media equation with a specified initial data. Furthermore, they showed the convergence rates of the original solution to its best asymptotic profile, which are much better than those in [56]. For other studies related to this topic, we refer to [14, 55, 58] and the references therein.

2.3. p-system with nonlinear damping. Jiang and Zhu in [81] first consider the Cauchy problem of the p-system with nonlinear damping

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -f(u), \qquad x \in \mathbb{R}, \quad t > 0, \end{cases} \tag{2.5}$$

with the initial data

$$(v,u)(x,0) = (v_0(x), u_0(x)), \qquad x \in \mathbb{R}, \tag{2.6}$$

and the far-field behavior

$$((v_0(x), u_0(x)) \to (v_\pm, u_\pm), \qquad v_- \neq v_+ \quad \text{as} \quad x \to \pm\infty,$$

where $f(u)$ is a superposition of the linear and nonlinear damping parts, i.e. $f(u) = \alpha u + g(u)$. Precisely, we obtained that the Cauchy problem (2.5)-(2.6) admits a unique global solution and such a solution tends time-asymptotically to the corresponding nonlinear diffusion wave governed by the classical Darcy's law under the condition $u_+ = u_- = 0$.

Later, for the case of $u_+ \neq u_-$, Mei in [59] considered the following p-system with nonlinear damping

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u - \beta |u|^{q-1} u, \qquad x \in \mathbb{R}, \quad t > 0. \end{cases}$$

Inspired by Hsiao-Liu correction functions, they succeed in constructing such a pair of correction functions

$$\begin{cases} \hat{u}(x,t) = \dfrac{m(x) e^{-\alpha t}}{\left(1 - \frac{\beta}{\alpha} \left[|m(x)| e^{-\alpha t}\right]^{q-1}\right)^{\frac{1}{q-1}}}, \\[2ex] \hat{v}(x,t) = - \dfrac{m'(x) e^{-\alpha t}}{\alpha \left(1 - \frac{\beta}{\alpha} \left[|m(x)| e^{-\alpha t}\right]^{q-1}\right)^{\frac{1}{q-1}}}, \end{cases}$$

which satisfies

$$\begin{cases} \hat{v}_t - \hat{u}_x = 0, \\ \hat{u}_t = -\alpha \hat{u} - \beta |\hat{u}|^{q-1} \hat{u}. \end{cases}$$

With the above correction functions in hand, they can obtained the convergence to the nonlinear diffusion waves without the restriction $u_+ = u_- = 0$. For other studies related to this topic such as the p-system with nonlinear damping and boundary effect, etc., we refer to [5, 15, 43, 44, 76] and the references therein.

2.4. p-system with time-depending damping. Recently, Cui-Yin-Zhang-Zhu [3] studied the system of Euler equations with time-depending damping

$$\begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u + \partial_x p(v) = -\dfrac{\alpha}{(1+t)^\lambda} u, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \end{cases}$$

with initial data

$$(v,u)\,|_{t=0} = (v_0,u_0)(x) \to (v_\pm, u_\pm), \quad \text{as} \quad x \to \pm\infty \quad \text{and} \quad v_+ \neq v_-,$$

where the external term $-\frac{\alpha}{(1+t)^\lambda} u$ with physical coefficients $\alpha > 0$ and $\lambda \geq 0$, is called a time-depending damping. The asymptotic profiles are expected to be diffusion waves that satisfy the time-dependent porous media equation:

$$\begin{cases} \partial_t \bar{v} - \partial_x \bar{u} = 0, \\ \partial_x p(\bar{v}) = -\dfrac{\alpha}{(1+t)^\lambda} \bar{u}, \end{cases}$$

or

$$\begin{cases} \partial_t \bar{v} = -\dfrac{(1+t)^\lambda}{\alpha} \partial_{xx} p(\bar{v}), \\ \partial_x p(\bar{v}) = -\dfrac{\alpha}{(1+t)^\lambda} \bar{u}. \end{cases}$$

Based on Hsiao-Liu's idea in [19], we set

$$\hat{v}(x,t) = B(t) m_0(x)(u_+ - u_-),$$

and

$$\hat{u}(x,t) = \beta(t) \Big\{ u_- + (u_+ - u_-) \int_{-\infty}^x m_0(y)dy \Big\}.$$

Here $B(t) = -\displaystyle\int_t^\infty \beta(\tau)d\tau$ and

$$\beta(t) = \begin{cases} e^{-\frac{\alpha}{1-\lambda}[(1+t)^{1-\lambda}-1]}, & \text{if} \quad \lambda \in [0,1), \\ (1+t)^{-\alpha}, & \text{if} \quad \lambda = 1. \end{cases}$$

Then

$$\begin{cases} \partial_t \hat{v} - \partial_x \hat{u} = 0, \\ \partial_t \hat{u} = -\dfrac{\alpha}{(1+t)^\lambda} \hat{u}. \end{cases}$$

Let

$$\begin{cases} V(x,t) = \displaystyle\int_{-\infty}^x (v(y,t) - \bar{v}(y+x_0,t) - \hat{v}(y,t))dy, \\ z(x,t) = u(x,t) - \bar{u}(x+x_0,t) - \hat{u}(x,t), \end{cases}$$

where

$$x_0 = \frac{1}{v_+ - v_-} \int_{-\infty}^{\infty} \big(v_0(x) - \bar{v}(x,0) - B(0)m_0(x)(u_+ - u_-)\big)dx.$$

The followings are the main results in [3].

THEOREM 2.6 (The case of $0 \leqslant \lambda < \frac{1}{7}$). *Assume that both the wave strength $\delta = |v_+ - v_-| + |u_+ - u_-|$ and $\|V_0\|_{H^3} + \|z_0\|_{H^2}$ are sufficiently small. Furthermore, we assume the pressure satisfies $p(v) \in C^4(\mathbb{R}^+)$, $p'(v) < 0$ for any $v \in \mathbb{R}^+$. Then, there exists a unique time-global solution satisfying*

$$V \in C^k((0,\infty), H^{3-k}(\mathbb{R})), \qquad k = 0,1,2,3,$$

$$V_t \in C^k((0,\infty), H^{2-k}(\mathbb{R})), \qquad k = 0,1,2,$$

furthermore, we have

$$\begin{aligned}
&\sum_{k=0}^{3}(1+t)^{(\lambda+1)k}\|\partial_x^k V(\cdot,t)\|_{L^2}^2 + \sum_{k=0}^{2}(1+t)^{(\lambda+1)k+2}\|\partial_x^k V_t(\cdot,t)\|_{L^2}^2 \\
&+ \int_0^t \left[\sum_{j=1}^{3}(1+s)^{(\lambda+1)j-1}\|\partial_x^j V(\cdot,s)\|_{L^2}^2 + \sum_{j=0}^{2}(1+s)^{(\lambda+1)j+1}\|\partial_x^j V_t(\cdot,s)\|_{L^2}^2\right] ds \\
&\leqslant C(\|V_0\|_{H^3}^2 + \|z_0\|_{H^2}^2 + \delta).
\end{aligned}$$

THEOREM 2.7 (The case of $\frac{1}{7} < \lambda < 1$). *Assume that both the wave strength $\delta = |v_+ - v_-| + |u_+ - u_-|$ and $\|V_0\|_{H^3} + \|z_0\|_{H^2}$ are sufficiently small. Furthermore, we assume the pressure satisfies $p(v) \in C^4(\mathbb{R}^+)$, $p'(v) < 0$ for any $v \in \mathbb{R}^+$. Then, there exists a unique time-global solution satisfying*

$$V \in C^k((0,\infty), H^{3-k}(\mathbb{R})), \qquad k = 0,1,2,3,$$

$$V_t \in C^k((0,\infty), H^{2-k}(\mathbb{R})), \qquad k = 0,1,2,$$

furthermore, we have

$$\begin{aligned}
&\sum_{k=0}^{3}(1+t)^{(\lambda+1)k+\frac{1}{2}-\frac{7\lambda}{2}}\|\partial_x^k V(\cdot,t)\|_{L^2}^2 + \sum_{k=0}^{2}(1+t)^{(\lambda+1)k+\frac{5}{2}-\frac{7\lambda}{2}}\|\partial_x^k V_t(\cdot,t)\|_{L^2}^2 \\
&\leqslant C(\|V_0\|_{H^3}^2 + \|z_0\|_{H^2}^2 + \delta),
\end{aligned}$$

and for any $\beta \in (\frac{1}{2} - \frac{5\lambda}{2}, \lambda)$, we have

$$\begin{aligned}
&\int_0^t \left[\sum_{j=0}^{3}(1+s)^{(\lambda+1)(j-1)+\beta}\|\partial_x^j V(\cdot,s)\|_{L^2}^2 + \sum_{j=0}^{2}(1+s)^{(\lambda+1)j+\beta-\lambda+1}\|\partial_x^j V_t(\cdot,s)\|_{L^2}^2\right] ds \\
&\leqslant C(1+t)^{\beta+\frac{5\lambda}{2}-\frac{1}{2}}(\|V_0\|_{H^3}^2 + \|z_0\|_{H^2}^2 + \delta).
\end{aligned}$$

THEOREM 2.8 (The case of $\lambda = \frac{1}{7}$). *Assume that both the wave strength $\delta = |v_+ - v_-| + |u_+ - u_-|$ and $\|V_0\|_{H^3} + \|z_0\|_{H^2}$ are sufficiently small. Furthermore, we* *assume the pressure satisfies $p(v) \in C^4(\mathbb{R}^+)$, $p'(v) < 0$ for any $v \in \mathbb{R}^+$. Then, there exists a unique time-global solution satisfying*

$$V \in C^k((0,\infty), H^{3-k}(\mathbb{R})), \qquad k = 0,1,2,3,$$

$$V_t \in C^k((0,\infty), H^{2-k}(\mathbb{R})), \qquad k = 0,1,2,$$

furthermore, we have for any sufficiently small $\varepsilon > 0$

$$\begin{aligned} &\sum_{k=0}^{3}(1+t)^{\frac{8k}{7}}\|\partial_x^k V(\cdot,t)\|_{L^2}^2 + \sum_{k=0}^{2}(1+t)^{\frac{8k}{7}+2}\|\partial_x^k V_t(\cdot,t)\|_{L^2}^2 \\ &+ \int_0^t \left[\sum_{j=1}^{3}(1+s)^{\frac{8j}{7}-1}\|\partial_x^j V(\cdot,s)\|_{L^2}^2 + \sum_{j=0}^{2}(1+s)^{\frac{8j}{7}+1}\|\partial_x^j V_t(\cdot,s)\|_{L^2}^2\right] ds \\ \leqslant\ & C(1+t)^{\varepsilon}(\|V_0\|_{H^3}^2 + \|z_0\|_{H^2}^2 + \delta). \end{aligned}$$

For other related research in this area, we refer to [1, 4, 16, 17, 46, 47] and the references therein.

3. The other related models. In this section, we will introduce the application of Hsiao-Liu correction functions in various other models.

3.1. Nonisentropic p-system with damping. The nonisentropic p-system with damping can be written as

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v,s)_x = -\alpha u, \qquad \alpha > 0, \\ s_t = 0, \end{cases} \tag{3.1}$$

which can be used to model the adiabatic gas flow through porous media. Here v is the specific volume, u denotes the velocity, s stands for the entropy, p denotes the pressure.

We consider the solution $(v(x,t), u(x,t), s(x,t))$ to the Cauchy problem for (3.1) with the initial data

$$(v,u,s)(x,0) = (v_0(x), u_0(x), s_0(x)), \qquad x \in \mathbb{R}, \tag{3.2}$$

and the far-field behavior

$$(v_0(x), u_0(x), s_0(x)) \to (v_\pm, u_\pm, s_\pm), \qquad \text{as} \qquad x \to \pm\infty.$$

According Darcy's law, the solutions of (3.1) are expected to converge to the corresponding solutions of the following equations

$$\begin{cases} \tilde{v}_t = -p(\tilde{v},s)_{xx}, \\ \tilde{u} = -\dfrac{1}{\alpha}p(\tilde{v},s)_x, \\ s_t = 0, \end{cases}$$

or

$$\begin{cases} \tilde{v}_t - \tilde{u}_x = 0, \\ \tilde{u} = -\dfrac{1}{\alpha} p(\tilde{v}, s)_x, \\ s_t = 0, \end{cases}$$

with initial data

$$(\tilde{v}, s)(x, 0) = (\tilde{v}_0(x), s_0(x)) \to (v_\pm, s_\pm), \qquad \text{as} \quad x \to \pm\infty.$$

In order to eliminate the gap at infinity due to $u_- \neq u_+$, a pair of correction functions need to be introduced as in [19]. With the correction functions, the asymptotic stability has been obtained in [29, 28] in the case of $v_- = v_+,\ s_- = s_+$. Later, Nishihara and Nishikawa [62] get a better decay rate in this case. In the case of $v_- \neq v_+,\ s_- = s_+$, Hsiao and Luo [24] set the adiabatic flow as the perturbation near the isentropic flows and obtain the stability theorem. In Marcati and Pan [57], the stability results with convergence rates have been obtained in the following two cases: (1) $s_+ = s_-$, (2) $p(v_-, s_-) = p(v_+, s_+)$. The more general case of $v_- \neq v_+,\ s_- \neq s_+$ has also been solved by Pan in [66] and the convergence rates have also been obtained. By suitably choosing initial data, Geng and Wang [13] find the best asymptotic profile of the system (3.1) and the authors further obtain the convergence rates as $\|(v - \tilde{v}, u - \tilde{u})(t)\|_{L^\infty} = O(1)((1+t)^{-3/2}, (1+t)^{-2})$ which are much better than the rates obtained in [62]. For the other interesting results in this topic, we refer to [23, 26, 71] and the references therein.

3.2. Bipolar Euler-Poisson equations with damping. The bipolar hydrodynamic models, generally used in description of the charged fluid particles such as electrons and holes in bipolar semiconductor devices or positively and negatively charged ions in a plasma, are the Euler-Poisson equations with damping as follows:

$$\begin{cases} n_{1t} + J_{1x} = 0, \\ J_{1t} + \left(\dfrac{J_1^2}{n_1} + p(n_1) \right)_x = n_1 E - \dfrac{J_1}{\tau}, \\ n_{2t} + J_{2x} = 0, \\ J_{2t} + \left(\dfrac{J_2^2}{n_2} + p(n_2) \right)_x = -n_2 E - \dfrac{J_2}{\tau}, \\ E_x = n_1 - n_2 - D(x). \end{cases} \tag{3.3}$$

Here $n_1(x,t)$ and $n_2(x,t)$ represent the densities of electrons and holes for bipolar semiconductor devices, $J_1(x,t)$ and $J_2(x,t)$ denote the current densities for electrons and holes, respectively. $E(x,t)$ is the electric field. $D(x) > 0$ is the doping profile standing for the density of impurities in the semiconductor device. $p(s)$ is the pressure function for both of electrons and holes. $\tau > 0$ is the relaxation-time.

We consider the Cauchy problem of (3.3) with initial data given by

$$(n_1, n_2, J_1, J_2)(x, 0) = (n_{10}, n_{20}, J_{10}, J_{20})(x), \qquad x \in \mathbb{R},$$

where

$$(n_{10}, n_{20}, J_{10}, J_{20})(x) \to (n_\pm, n_\pm, J_{1\pm}, J_{2\pm}), \qquad \text{as} \quad x \to \pm\infty. \tag{3.4}$$

Using the idea of Hsiao-Liu correction function [19], Gasser, Hsiao and Li [11] introduced a correction function similar to that in [19] to remove the gap between the original solution and the far-field diffusion wave, and showed that, the smooth solutions of (3.3)-(3.4) tend to the diffusion waves with algebraic convergence rate. Furthermore, Huang, Mei and Wang [38] investigated the existence and the stability of the diffusion wave when the states at the far field are more general. Donatelli, et al. [9] considered the global existence and asymptotic behavior of solutions with the different pressure functions. Huang, Mei, Wang and Yang [40] investigated the existence of diffusion wave and its the large-time behavior of solution to the bipolar hydrodynamic model of semiconductors with boundary. For the other interesting results in hydrodynamic model of semiconductors, we refer to [21, 39, 18, 27, 31, 32, 33, 34, 22, 54] and the references therein.

3.3. Dissipative nonlinear evolution equations with ellipticity. In this subsection, we are concerned with the Cauchy problem for a set of nonlinear equations with ellipticity and dissipative effects, which reads, cf. [35, 36]

$$\begin{cases} \psi_t = -(\sigma - \alpha)\psi - \sigma\theta_x + \alpha\psi_{xx}, \\ \theta_t = -(1-\beta)\theta + \nu\psi_x + 2\psi\theta_x + \beta\theta_{xx}, \end{cases} \tag{3.5}$$

with initial data

$$(\psi(x,0), \theta(x,0)) = (\psi_0(x), \theta_0(x)), \tag{3.6}$$

where α, β, σ and ν are positive constants such that $\alpha < \sigma$ and $\beta < 1$.

As in [19], it is natural to expect the solution of (3.5) time-asymptotically behave as those of the following system

$$\begin{cases} \bar\psi_t = -(1-\alpha)\bar\psi - \bar\theta_x + \alpha\bar\psi_{xx}, \\ -(1-\alpha)\bar\theta + \nu\bar\psi_x = 0, \end{cases}$$

or

$$\begin{cases} \bar\psi_t = -(1-\alpha)\bar\psi + \left(\alpha - \dfrac{\nu}{1-\alpha}\right)\bar\psi_{xx}, \\ \bar\theta = \dfrac{\nu}{1-\alpha}\bar\psi_x. \end{cases}$$

Motivated by the method of obtaining Hsiao-Liu correction functions for p-system with damping in [19], we [82] introduce the following correction functions

$$\hat\theta(x,t) = e^{-(1-\alpha)t}\left(\theta_- + (\theta_+ - \theta_-)\int_{-\infty}^{x} m_0(y)dy\right).$$

With this correction function in hand, we can use the L^2 energy estimates to show that

$$\sup_{x\in\mathbb{R}}\left(|(\psi,\theta)(x,t)| + |(\psi_x,\theta_x)(x,t)|\right) \to 0, \qquad \text{as} \qquad t\to\infty,$$

and the solutions decay with exponential rates. Later, Wang [70] derived the optimal decay rates of solution to the Cauchy problem of (3.5)-(3.6). For various asymptotic profiles, the stability and decay rates have been studied in [7, 8, 61, 67, 69] and the references therein.

3.4. p-system with relaxation. In this part, we consider the asymptotic behavior of solutions for the Cauchy problem of p-system with relaxation

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \dfrac{1}{\varepsilon}(f(v) - u), \end{cases} \tag{3.7}$$

with initial data

$$(v,u)(x,0) = (v_0(x), u_0(x)) \to (v_\pm, u_\pm), \quad v_\pm > 0, \quad \text{as} \quad x \to \pm\infty. \tag{3.8}$$

For the case of $u_\pm = f(v_\pm)$, Liu [52] studied the stability of rarefaction waves and travelling waves for the general 2×2 hyperbolic conservation laws with relaxation. Zhu [79] obtained the stability results for the case of $u_\pm \neq f(v_\pm)$. More precisely, when $u_\pm \neq f(v_\pm)$, the energy method used to get L^2-estimates of the solutions in [52, 48] can not be applied directly. To overcome this difficulty, as in [19], we introduce a pair of correction functions $(\hat{v}(x,t), \hat{u}(x,t))$ as follows:

$$\begin{cases} \hat{v}(x,t) = -(a_+ - a_-)e^{-t} m_0(x), \\ \hat{u}(x,t) = e^{-t}\left(a_- + (a_+ - a_-)\displaystyle\int_{-\infty}^{x} m_0(y)dy\right), \end{cases} \tag{3.10}$$

where $a_\pm = u_\pm - f(v_\pm)$ and $m_0(x)$ is a smooth function with compact support satisfying

$$\int_{-\infty}^{\infty} m_0(x)dx = 1.$$

With this, we can apply the energy method to get L^2-estimates of the solutions and prove the solution of (3.7), (3.8) exists and tends to the equilibrium rarefaction wave and the travelling wave. For other studies related to this topic such as the stability of the strong nonlinear diffusion waves, as well as the convergence with boundary effect, etc., we refer to [10, 51, 65, 72, 73, 77, 83] and the references therein.

Due to space limitations, we can only list the above applications of Hsiao-Liu correction functions. For other studies related to this topic such as the convergence in weak sense, as well as the system with vacuum, etc., we refer to [12, 25, 37, 41, 42, 30, 68, 74, 80] and the references therein. Some recent studies on Hsiao-Liu correction functions, we refer to [1, 2, 6, 47, 49, 50, 75] and the references therein.

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