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$\mathbf{CP^2}$-STABLE THEORY

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Abstract. In the topological category, it is shown that the dimension 4 disk theorem holds without fundamental group restriction after stabilizing with many copies of complex projective space. As corollaries, a stable 4-dimensional surgery theorem and a stable 5-dimensional s-cobordism are obtained. These results contrast with the smooth category where the usefulness of adding CP^2's depends on chirality.

Surgery is the fundamental tool for constructing manifolds of dimension $n \geq 5$ and the s-cobordism theorem (in dimension $n+1$) is the fundamental tool for constructing isomorphism between n-manifolds. When the category is TOP both techniques extend to the case $n = 4$ provided the fundamental group is "good", [F2], [FQ2] ("good" is the closure of finite groups and Abelian groups under the operations: (1) subgroup, (2) quotient, (3) extension, and (4) direct limit. It was later noticed that for finitely generated groups the condition "good" is identical with the notion elementary amenable. See [St], [C].) In categories Diff or PL both theorems fail when $n = 4$ ([D1], [D2]), even in the simply connected case. The necessity of the "good" hypothesis on π_1 in the topological category has been the central unsolved problem in the subject for the last decade.

Because there has been progress only on special cases of this question ([F3], [F4]) it is natural to consider "easier" stable questions. It was known very early ([CS2], [FQ1], [L], [Q]) that standard versions of the 4-dimensional surgery theorem and the 5-dimensional s-cobordism are true (in all categories) after appropriate stabilization by connected sum with copies of $S^2 \times S^2$. (In the case of surgery one stabilizes the normal map f to $f \sharp id \sharp_k S^2 \times S^2$. In the case of the s-cobordism theorem one must form a

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"connected sum $\times[0,1]$" with $[0,1]\times\sharp_k S^2\times S^2$.) In this paper we consider stabilization by copies of CP^2 all with a fixed orientation. The identity:

$$CP^2\sharp CP^2\sharp - CP^2 = S^2\times S^2\sharp CP^2$$

shows that for a stabilization which is careless of orientations the early stable results are adequate.

In the smooth category chirality plays an important role. For example, the negative definite form $E_8\oplus E_8$ cannot be realized even after stabilization by copies of $< -1 >$ (corresponding to $-\sharp_k CP^2$) [D1] and if two manifolds are distinguished by a Donaldson invariant derived from anti-self dual connections then after blowing up points, i.e., sum with $-CP^2$ they will continue to be distinguished by a related Donaldson invariant [DK]. On the other hand, stabilization of $E_8\oplus E_8$ by $< +1 >$ is realizable (as $CP^2\sharp_{16} - CP^2$). Also, any algebraic surface becomes a standard smooth manifold after connected sum with CP^2 [Ma].

No similar chirality is found in the topological category. We prove that any unobstructed 4-dimensional surgery problem can be solved after connected sum with copies of CP^2 (or with copies of $-CP^2$), and a stable 5-dimensional s-cobordism.

The basic result is the following disk theorem:

Theorem 1. *Suppose $A\longrightarrow M^4$ is an immersion of a union of disks, with algebraically transverse spheres whose algebraic intersections and self-intersections numbers are 0 in $Z[\pi_1 M]$, then there is a topologically embedded union of disks with the same framed boundary as A after stabilizing M with many copies of CP^2.*

For specificity we may take the complex orientation for CP^2, but the other would work equally well.

The disk theorem is known ([F2], [FQ2]) to imply both the surgery and s-cobordism conjectures, since it allows for the construction of flat Whitney disks wherever these are needed in the proofs. So as corollaries:

Theorem 2 ($\mathbf{CP^2}$-stable surgery). *Let $f:(M,\partial M)\longrightarrow(X,\partial X)$ be a degree one normal map from a topological 4-manifold to a Poincare pair which induces a $Z[\pi_1 X]$-homology isomorphism over ∂X. Suppose that the surgery obstruction vanishes $\mathcal{O}(f)=0\in L_4^{(s)}(\pi_1 X)$ in the (simple) L-group. Let X be represented as in [W] by a space whose fundamental class is carried by a top cell D^4. Define $f_k:(M\sharp_k CP^2,\partial M)\longrightarrow$*

$(X\sharp_{k,alongD^4}CP^2, \partial X)$ as $f\sharp id(\sharp_k CP^2)$ and extend the bundle map $b : \nu_M \to \xi$ by connected sum with $id(\nu\sharp_k CP^2)$. For k sufficiently large, f_k is normally bordant (rel ∂) to a (simple) homotopy equivalence.

Theorem 3 ($\mathbf{CP^2}$-stable s-cobordism theorem). *Suppose $(W^5; M_0, M_1)$ is a compact s-cobordism which is a product over ∂M_0. Define $W\sharp_I(CP^2\times I)$ to be the s-cobordism obtained by deleting a regular neighbourhood of an arc from M_0 to M_1, and substituting $(CP^2 - ball)\times I$. Note that it changes M_0 and M_1 by connected sum with CP^2. Then there is a topological product structure on $W\sharp_I k(CP^2 \times I)$, for some k extending the product structure over ∂M_0. In particular, it implies that $M_0\sharp kCP^2$ is homeomorphic to $M_1\sharp kCP^2$.*

The key to the proof of Theorem 1 is the following observation: inside CP^2, there is a pair of 2-spheres which intersect transversely at exactly one point. Let $A = \{A_i\} \longrightarrow M$ be a framed immersion of a union of disks, and p_{ij} be an intersection point of A_i and A_j. After stabilizing M with CP^2, we sum one embedded 2-sphere to A_i in $M\sharp CP^2$, then the other embedded 2-sphere is a dual (not framed) 2-sphere to $A_i\sharp S^2$. By summing the dual to A_j, p_{ij} is removed. We can remove all intersection points among A in this way and have an embedded union of disks with the same boundary as A. But these embedded disks are not directly useful for Whitney moves because the relative framings have not been controlled. Actually, the framing changes as follows: if $i \neq j$, then the framing of both A_i and A_j change by ± 1, if $i = j$, then the disk changes framing by 0 or 4 depending on the sign of p_{ii}.

Proof of Theorem 1. By Lemma 3.3 of [FQ2], it is sufficient to get a π_1-null 3-stage capped grope. Raise the grope to height 4 as in 2.7 [FQ2] and use the preceding observation to remove all double points of the caps. Twist if necessary to correct the framing. Then contract the top stage to obtain a 3-staged capped grope which satisfies:

(1) all new double points are π_1-trivial and
(2) framing is 0.

This completes the proof of Theorem 1. □

Addendum. In theorems 1, 2, and 3, CP^2 may be replaced by any oriented closed 1-connected topological four manifold $M \ncong S^4$.

Proof. Let $\alpha : S^2 \looparrowright M$ be an immersion representing an indivisible homology class. After complicating the immersion by "finger moves" α will have an immersed geometric dual $\beta : S^2 \looparrowright M$. Now follow the preceding argument with α, β in M replacing the two copies of CP^1 in CP^2.

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