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GENERAL HYPERPLANE SECTIONS OF NONSINGULAR FLOPS IN DIMENSION 3

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Let X be a 3-dimensional complex manifold, and $f : X \to Y$ a proper bimeromorphic morphism to a normal complex space which contracts an irreducible curve $C \subset X$ to a singular point $Q \in Y$ while inducing an isomorphism $X \setminus C \simeq Y \setminus \{Q\}$. We assume that the intersection number with the canonical divisor $(K_X \cdot C)$ is zero. In this case, it is known that the singularity of Y is Gorenstein terminal, and there exists a *flop* $f^\# : X^\# \to Y$ ([R]), which we call a *nonsingular flop* because X is nonsingular.

In order to investigate f analytically, we replace Y by its germ at Q and consider a general hyperplane section H of Y through Q. Then H has only a rational double point, its pull-back $L \subset X$ by f is normal, and the induced morphism $f_H : L \to H$ factors the minimal resolution $g : M \to H$ ([R]). The dual graph Γ of the exceptional curves of g is a Dynkin diagram of type A_n, D_n or E_n. Let $F = \sum_{k=1}^n m_k C_k$ be the *fundamental cycle* for g on M. The natural morphism $h : M \to L$ is obtained by contracting all the exceptional curves of g except the strict transform C_{k_0} of C.

Kollár defined an invariant of f called the *length* as the length of the scheme theoretic fiber $f^{-1}(Q)$ at the generic point of C. It coincides with the multiplicity m_{k_0} of the fundamental cycle at C_{k_0}.

Katz and Morrison proved the following theorem ([KM, Main Theorem]). The purpose of this paper is to give its simple geometric proof.

Theorem. *Let $f : X \to Y$ be as above. Then the singularity of the general hyperplane section H and the partial resolution $f_H : L \to H$ are determined by the length ℓ of f. More precisely, H has a rational double point of type A_1, D_4, E_6, E_7, E_8, or E_8, if $\ell = 1$, 2, 3, 4, 5 or 6, respectively.*

We note that there is only one irreducible component of $g^{-1}(Q)$ whose multiplicity in F coincides with ℓ in the above cases.

Proof. Let H' be another general hyperplane section of Y through Q, and $f_{H'} : L' \to H'$ the induced morphism. H and H' have the same type of

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singularities and so do L and L'. Let P_i and P_i' be the singular points of L of L', respectively.

Let D be the effective Cartier divisor on L given by $L' \cap L$. Then D is a general member of the linear system of effective Cartier divisors on L which contain C and such that $(D \cdot C) = 0$. In fact, if s_0 is the global section of $\mathcal{O}_L(-C) \subset \mathcal{O}_L$ corresponding to D, then from an exact sequence

$$0 \to \mathcal{O}_X(-L) \to \mathcal{O}_X \to \mathcal{O}_L \to 0$$

there exists a lifting $s \in H^0(X, \mathcal{O}_X)$ of s_0 which defines L', because $H^1(X, \mathcal{O}_X(-L)) = 0$.

Let $\tilde{D}$ be the total transform of D on M. Then we can write $\tilde{D} = F + D'$ for some D' which is reduced, nonsingular and has no common irreducible components with F. If Γ is of type A_n, then D' has 2 irreducible components each of which intersects transversally one of the end components of F. Otherwise, D' is irreducible, and intersects transversally a uniquely determined component C_{k_1} of F, for which $m_{k_1} = 2$ holds (k_1 may be equal to k_0).

Let t be the global section of $\mathcal{O}_X$ corresponding to L. Then $s + ct$ is also a lifting of s_0 for any $c \in \mathbb{C}$. Let $L'(c)$ be the corresponding divisor on X.

Let P be a point on C which is different from the P_i. For local analytic coordinates $\{x, y, t\}$, we can write $s + ct = F(x, y) + t(G(x, y, t) + c)$. For a general choice of c, $(G(x, y, t) + c)|_C$ does not vanish at the singular points P_i' of $L' = L'(0)$ other than the P_i, and has only simple zeroes at some points P_j''. Then $L'' = L'(c)$ has singularities only at the P_j'', besides possibly at the P_i, with equations of the type $x^\ell + ty = 0$.

If we replace L' by L'', we conclude that L' has only singularities of type $A_{\ell-1}$ outside the P_i. We shall investigate the singularities of L' at the P_i case by case.

Let Γ_i be the dual graph of the exceptional curves of h over P_i, and F_i the corresponding fundamental cycle. From the description of $\tilde{D}$ above, we can calculate the multiplicity d_i of D at the point P_i by $d_i = ((m_{k_0} C_{k_0} + D') \cdot F_i)$.

If $\ell = 1$ or 2, then we can check that $d_i \le 3$. If 2 singular surfaces meet at a common singular point, then the intersection curve has multiplicity at least 4 there. Hence L' is nonsingular at the P_i. Then it follows that $\Gamma = A_1$ or D_4, respectively.

But if $\ell \ge 3$, then d_i can be bigger, and we should look at the singularity of L' more closely.

We assume first that $\ell = 3$. If $\Gamma = E_6$, then there is nothing to prove. We have to prove that $\Gamma \ne E_7, E_8$. If $\Gamma = E_7$, then L has two singular points P_1 and P_2, where F_1 meets D'. We have 2 cases; $\Gamma_1 = A_1$ and

$\Gamma_2 = A_5$, or $\Gamma_1 = A_4$ and $\Gamma_2 = A_2$. In the former case, L' has at most A_1 singularity at P_1 because of the symmetry of L and L', while being nonsingular at P_2, since $d_2 = 3$. Therefore, L' has simpler singularities than L, a contradiction. In the latter case, it has at most A_2 at P_2. We shall prove that L' has A_1 at P_1.

Let $\mu : X^{(1)} \to X$ be the blowing-up at P_1, $E \simeq \mathbb{P}^2$ the exceptional divisor, and $L^{(1)}$ (resp. $L^{(1)\prime}$) the strict transform of L (resp. L'). $B = L^{(1)} \cap E$ consists of 2 lines B_1 and B_2 which correspond to the 2 end components of F_1. Their multiplicities in $\mu^* L' \cap L^{(1)}$ are equal to those in F, which are 2. If L' has multiplicity n at P_1, then we have $\mu^* L' = L^{(1)\prime} + nE$ and $L^{(1)\prime} \cap E$ is a plane curve of degree n. Since $L^{(1)} \cdot E = B$, we deduce that $\mu^* L' = L^{(1)\prime} + 2E$, and neither of the B_i are contained in $B' = L^{(1)\prime} \cap E$. Thus the intersection of 2 conics B and B' is equal to $(L^{(1)} \cap L^{(1)\prime}) \cap E$. We see from the description of $\tilde{D}$ that it consists of 2 points, one at $B_1 \cap B_2$ and the other on one component B_1. Then B' must be a nonsingular conic, and L' has A_1 singularity at P_1.

If $\Gamma = E_8$, then we have again 2 cases; $\Gamma_1 = A_1$ and $\Gamma_2 = E_6$, or $\Gamma_1 = A_7$. In the former case, L' has at most A_1 singularity at P_1, while being nonsingular at P_2, since $d_2 = 3$. In the latter case, it has A_1 at P_1 as in the case of E_7.

Next we assume that $\ell = 4$. If $\Gamma = E_7$, then there is nothing to prove. If $\Gamma = E_8$, then we have 2 cases: $\Gamma_1 = D_5$ and $\Gamma_2 = A_2$, or $\Gamma_1 = A_6$ and $\Gamma_2 = A_1$.

In the former case, L' has at most A_2 singularity at P_2. By the symmetry of L and L', L' has D_5 at P_1. Let $\mu : X^{(1)} \to X$, E, $L^{(1)}$ and $L^{(1)\prime}$ as before. $B = L^{(1)} \cap E$ is a line, and $L^{(1)} \cdot E = 2B$. Since the corresponding curve has multiplicity 4 in F, we have $\mu^* L' = L^{(1)\prime} + 2E$, and B is not contained in $B' = L^{(1)\prime} \cap E$. $L^{(1)}$ has 2 singular points $P_1^{(1)}$ and $P_2^{(1)}$ on B which are of types A_3 and A_1, respectively. We have $B \cap B' = P_1^{(1)}$ by the description of $\tilde{D}$.

Let $\nu : X^{(2)} \to X^{(1)}$ be the blowing-up at $P_1^{(1)}$, $E^{(1)} \simeq \mathbb{P}^2$ the exceptional divisor, and $L^{(2)}$ (resp. $L^{(2)\prime}$) the strict transform of $L^{(1)}$ (resp. $L^{(1)\prime}$). $B^{(1)} = L^{(2)} \cap E^{(1)}$ consists of 2 lines, and one of the corresponding curves on M has multiplicity 3 in F, hence $\nu^* L^{(1)\prime} = L^{(2)\prime} + E^{(1)}$, and $L^{(1)\prime}$ is nonsingular at $P_1^{(1)}$. But this contradicts the symmetry of L and L'.

In the latter case, L' has at most A_1 singularity at P_2. Let $\mu : X^{(1)} \to X$, etc., as before. $B = L^{(1)} \cap E$ consists of 2 lines B_1 and B_2, which correspond to the 2 end components of F_1. Since their multiplicities in F are 3 and 2, B_1 is contained in $B' = L^{(1)\prime} \cap E$, while B_2 is not. Thus we have $B' = B_1 + B_2'$ with $B_2 \neq B_2'$. Since C is nonsingular, its strict transform on $X^{(1)}$ intersects E at only one point. It is also the image of C_{k_0} on $X^{(1)}$, so it passes through the point $B_1 \cap B_2$, hence not $B_1 \cap B_2'$, a contradiction to the symmetry.

Finally, if $\ell \geq 5$, the assertion of the theorem is clear. Q.E.D.

References

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