Mathematical Research Letters **1**, 65–73 (1994)

L^2-ESTIMATES FOR A CLASS OF SINGULAR OSCILLATORY INTEGRALS

Andreas Seeger

Abstract. We prove L^2 estimates for singular integral operators with oscillatory terms and consequently obtain new results for Hilbert transforms along variable curves in the plane. In particular we extend a result of Nagel, Vance, Wainger and Weinberg for translation invariant even curves.

1. Introduction

We study the L^2 boundedness of singular integral operators acting on functions on the real line which are of the form

$$T^\lambda f(x) = \lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon} e^{i\lambda S(x,y)}K(x,y)f(y)dy \tag{1.1}$$

where $S \in C^2(X \times X)$, $X \subset \mathbb{R}$ open and where the singular kernel K is compactly supported in $X \times X$. We assume that K is the kernel of a classical pseudodifferential operator $\mathcal{K}$ of order 0; that is, $\mathcal{K} \in OPS^0_{1,0}(\mathbb{R})$ has a principal symbol which is homogeneous of degree 0. In other words, in local coordinates

$$K(x,y) = \frac{\beta_0(x,y)}{x-y} + \beta_1(x,y)\delta_\Delta + K_{-1}(x,y)$$

where δ_Δ is a measure supported on the diagonal (the pullback $\rho^*\delta$ of the Dirac measure δ in $\mathbb{R}$ under $\rho(x,y) = x-y$) and $K_{-1}(x,y) = O(\log|x-y|)$ for x near y. For obtaining L^p estimates the only nontrivial term to be considered is $(x-y)^{-1}$.

The operators T^λ and their generalizations in higher dimensions have been studied by Phong and Stein [8], Ricci and Stein [10], and Pan [7]. As shown in [8] uniform $L^2(\mathbb{R})$ estimates for T^λ can be applied to show $L^2(\mathbb{R}^2)$

Received September 23, 1993.
Research supported in part by a grant from the National Science Foundation

boundedness of Hilbert transforms $\mathfrak{H}$ along variable curves, in certain model cases. Here $\mathfrak{H}$ is defined a priori on functions in $C_0^\infty(\mathbb{R}^2)$ by

$$\mathfrak{H}f(x) \;=\; \eta(x)\,\text{p.v.}\int_{-\delta}^{\delta} f(x_1, x_2 - S(x_1, x_1 - t))\frac{dt}{t}$$

where η is a cutoff-function and $\delta > 0$ is suitably small.

The behavior of T^λ is well understood if one imposes a weak finite type condition (namely that the mixed derivative S''_{xy} does not vanish of infinite order at any point (x_0, y_0), see [7]). In this paper we consider some cases where this finite type condition is not satisfied. In a translation invariant model case, which corresponds to $S(x,y) = \gamma(x-y)$, Nagel, Vance, Wainger and Weinberg [6] proved a completely satisfactory result in the case that γ is *even* and *convex*. In this case the associated Hilbert transform is bounded if and only if γ' satisfies a doubling condition; *i.e.* $\gamma'(Bt) \geq 2\gamma'(t)$ for $t \geq 0$ and a suitable constant B. In the odd case they have an even weaker necessary and sufficient condition.

In order to formulate a diffeomorphism invariant result for T_λ one would like to introduce a concept of generalized convexity which is invariant under a C^2 change of variable.

Definition. A C^1-function $F : I \to \mathbb{R}$ is *κ-quasimonotone* if $F'(t) = a(t) + b(t)F(t)$ where a does not change sign in I and $|b(t)| \leq \kappa$ for all $t \in I$.

Now for $\gamma \in C^2(I)$ a weak convexity condition would be that the derivative γ' is a κ-quasimonotone function in I, for some $\kappa > 0$. This condition is invariant under a C^2 change of variable. To see this suppose that $\chi : I' \to I$ is a C^2 diffeomorphism and that we can split $\gamma'' = a + b\gamma'$ as in the definition. Then

$$(\gamma\circ\chi)''(t) = (\chi'(t))^2 a(\chi(t)) + \left[\chi'(t)b(t) + \frac{\chi''(t)}{\chi'(t)}\right](\gamma\circ\chi)'(t)$$

which shows the κ'-quasimonotonicity of $(\gamma\circ\chi)'$ on I' for a suitable $\kappa' \geq 0$.

Theorem 1.1. *Suppose that $\kappa \geq 0$, and suppose that for any y the functions*

$$x \mapsto S'_x(x,y) - S'_x(x,x)$$

are κ-quasimonotone in $\{x : x > y\}$ and in $\{x : x < y\}$; moreover suppose that for any u the functions

$$y \mapsto S'_y(u,y) - S'_y(y,y)$$

are κ-quasimonotone in $\{y : y > u\}$ and in $\{y : y < u\}$.

Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function satisfying

$$g(Bt) \geq 2g(t) \tag{1.2}$$

and suppose that

$$A^{-1}g(|x-y|) \leq |S'_x(x,y) - S'_x(x,x)| \leq Ag(|x-y|) \tag{1.3}$$

$$A^{-1}g(|x-y|) \leq |S'_y(x,y) - S'_y(y,y)| \leq Ag(|x-y|) \tag{1.4}$$

Then T^λ is bounded on $L^2(\mathbb{R})$ with bound independent of λ.

Using the Fourier transform in the x_2 variable and Plancherel's theorem as in [8, p.117] one obtains the following Corollary for Hilbert transforms on variable curves (as defined above).

Corollary 1.2. *Suppose that S satisfies the assumptions of Theorem 1.1. Then $\mathfrak{H}$ is bounded on $L^2(\mathbb{R}^2)$.*

Remarks.

(1) The hypotheses are satisfied for certain flat cases; for example one may take $S(x,y) = e^{-(x-y)^{-2}}g_1(x,y)$ or $S(x,y) = e^{-g_2(x,y)(x-y)^{-2}}$ where $g_1, g_2 \in C^2$ and $g_1(x,x) \neq 0$, $g_2(x,x) > 0$ (of course it suffices to check the hypotheses near the diagonal).

(2) In the translation-invariant case we have $S(x,y) = \gamma(x-y)$. Assume that $\gamma(t)$ is even or odd, convex for $t > 0$, and $|\gamma'(t) - \gamma'(0)|$ is doubling in $\mathbb{R}_+$. Then the hypotheses of Theorem 1.1 are satisfied. In particular we recover the result of [6] for even curves.

(3) The nondegenerate case $S''_{xy} \neq 0$ is covered by Theorem 1.1. The derivative of $x \mapsto S'_x(x,y) - S'_x(x,x)$ is $-S''_{xy}(x,x) + O(|x-y|)$ near the diagonal. This shows the quasimonotonicity and (1.3) with $g(t) = t$. The same argument applies to $y \mapsto S'_y(x,y) - S'_y(y,y)$.

(4) Our hypotheses do not depend on the size of the second derivatives of S. Therefore the assumption $S \in C^2$ may be relaxed; for example one can cover certain piecewise linear examples.

(5) Clearly the hypotheses of Theorem 1.1 are invariant under changes of variables; in the sense that if the hypotheses apply to T^λ and χ is a change of variable then they also apply to $f \mapsto [T^\lambda(f \circ \chi)] \circ \chi^{-1}$. This can be considered a first step towards a diffeomorphism invariant version of Corollary 1.2. We note that under finite type conditions a diffeomorphism invariant theory for Hilbert transforms on curves has been worked out by Christ, Nagel, Stein and Wainger ([2], [3]).

(6) Using a different method, Carbery, Wainger and Wright earlier obtained L^p estimates for Hilbert transforms on a class of flat curves on the Heisenberg group, which are dilation invariant under certain group automorphisms (see [1]). The orthogonality argument of the present paper however has its root in [11, §4].

(7) Greenleaf and Uhlmann [4] showed that singular Radon transforms can be considered as members of more general classes $I^{p,l}$ of Fourier integral operators associated with two cleanly intersecting Lagrangians. For general finite type or even flat cases estimates for operators in $I^{p,l}$ classes are presently not well understood.

In what follows c and C will always be positive constants independent of λ which may assume different values in different lines.

2. Almost orthogonality arguments

As pointed out above we may assume that $K(x,y) = \beta(x,y)(x-y)^{-1}$ where β vanishes if $|x-y| \geq 1/2$. Let $\chi \in C_0^\infty(\mathbb{R})$ be an *even* function supported in $(1/2,2)\cup(-2,-1/2)$ with the property that $\sum_{l=-\infty}^{\infty} \chi(2^l s) = 1$ for all $s \neq 0$. Let

$$K_j(x,y) \,=\, \chi(2^j|x-y|)K(x,y)$$

and denote by T_j the integral operator with kernel K_j. Observe that the operators T_j are uniformly bounded on L^p, $1 \leq p \leq \infty$; moreover $T^\lambda = \sum T_j$ as operators acting on C_0^∞ functions. In order to show that T^λ extends to a bounded operator on $L^2(\mathbb{R})$ we consider the operator $\mathcal{T} = \sum T_j$, where the sum involves only finitely many terms, and show that it is bounded with operator norm independent of λ and the number of terms in the sum. The uniform boundedness of T_λ then follows by a limiting argument.

We now describe the first basic decomposition of the operator. Let N be such that

$$2^{N-2} \leq B < 2^{N-1}$$

and let

$$t_0 \,=\, \min\bigl\{\max\{t : tg(t) \leq \lambda^{-1}\}, 2^{-2N}\bigr\}$$

Let $M \in \mathbb{N}$ be defined by

$$2^{-M-1} < t_0 \leq 2^{-M}.$$

We decompose

$$\begin{aligned}\mathcal{T} \,&=\, \sum_{j>M+2N} T_j + \sum_{M-2N\leq j\leq M+2N} T_j + \sum_{j<M-2N} T_j \\ &=\, \mathcal{T}^1 + \mathcal{T}^2 + \mathcal{T}^3\end{aligned}$$

Since the sum defining $\mathcal{T}^2$ involves only $4N+1$ terms it is a bounded operator on L^2 with operator norm $O(\log B)$. We use the orthogonality lemma of Cotlar and Stein (see [5]) and the L^2-boundedness of $\mathcal{T}^1$ and $\mathcal{T}^3$ will follow from

$$\|T_j^* T_l\| + \|T_j T_l^*\| \le C 2^{-(l-j)\gamma}$$

where $l > j+2N$ and either $j, l \ge M+2N$ or $j, l \le M-2N$ and $\gamma > 0$. Note that the case $l < j-2N$ follows by duality and that the case $|l-j| \le 2N$ follows from the uniform boundedness of the T_j.

The following estimate is crucial in our proof.

Lemma 2.1. *Suppose (1.2-4) hold and suppose that* $|x-z| \approx 2^{-l}$, $|x-y| \approx 2^{-j}$, $j < l-N-2$. *Then there is* $\gamma > 0$ *such that the inequality*

$$|S'_x(x,z) - S'_x(x,x)| \le C 2^{-\gamma(l-j)} |S'_x(x,y) - S'_x(x,x)|$$

holds..

Proof. The assertion with $\gamma \ge c[\log(1+B)]^{-1}$ is a straightforward consequence of the doubling condition (1.2) for g. $\square$

Estimation of $\mathcal{T}^1$. kernel H_{jl} of $T_j^* T_l$ which is given by

$$H_{jl}(y,z) = \int e^{i\lambda(S(x,z)-S(x,y))} \overline{K_j(x,y)} K_l(x,z)\, dx. \tag{2.1}$$

We shall first assume that $j, l \le M-2N$ (and $l > j+2N$). Note that if this case comes up at all then $t_0 g(t_0)\lambda \ge 1$. An integration by parts yields

$$\begin{aligned} H_{jl}(y,z) &= \frac{i}{\lambda} \int e^{-i\lambda(S(x,y)-S(x,z))} \frac{S''_{xx}(x,y) - S''_{xx}(x,z)}{(S'_x(x,y) - S'_x(x,z))^2} \overline{K_j(x,y)} K_l(x,z)\, dx \\ &\quad - \frac{i}{\lambda} \int e^{-i\lambda(S(x,y)-S(x,z))} \frac{\frac{d}{dx}\left[\overline{K_j(x,y)} K_l(x,z)\right]}{S'_x(x,y) - S'_x(x,z)}\, dx \\ &= I(y,z) + II(y,z). \end{aligned}$$

Let $J_l(z) = \{x : 2^{-l-1} \le |x-z| \le 2^{-l+1}\}$ and split $J_l(z) = J_l^+(z) \cup J_l^-(z)$ where $x > z$ if $x \in J_l^+(z)$ and $x < z$ if $x \in J_l^-(z)$. We split $I = I^+ \cup I^-$ where I^+, I^- are defined by restricting the x-integration in (2.1) to $J_l^+(z)$ and $J_l^-(z)$, resp. In the same way split II and for the present case $j, l > M+2N$ we only consider I^+ and II^+ (the analysis for the three remaining terms is the same).

Define

$$\Psi(x,y) = S'_x(x,y) - S'_x(x,x).$$

Then

$$|I^+(y,z)| \le \frac{2^{j+l}}{\lambda}\int_{J_l^+(z)} \frac{|\Psi_x'(x,y)|+|\Psi_x'(x,z)|}{(\Psi(x,y)-\Psi(x,z))^2}\,dx$$

In view of the doubling property of Ψ (Lemma 2.1) we have $x \in J_l^+(z)$

$$|\Psi(x,y)-\Psi(x,z)| \ge \frac{1}{2}|\Psi(x,y)| \ge c2^{(l-j)\gamma}|\Psi(x,z)| \tag{2.2}$$

and by the monotonicity of $tg(t)$ (which follows from the monotonicity of $g(t)$)

$$|\Psi(x,z)| \ge cA^{-1}g(|x-z|) \ge c'A^{-1}\frac{t_0 g(t_0)}{|x-z|}. \tag{2.3}$$

By assumption the functions $x \to \Psi(x,y)$ are κ-quasimonotone in $J_l^+(z)$. Therefore we may decompose

$$\Psi_x'(x,y) = a_y(x) + b_y(x)\Psi(x,y) \tag{2.4}$$

where a_y does not change sign and $|b_y(x)| \le \kappa$. Using the first inequality in (2.2) and using (2.4) twice we estimate

$$\int_{J_l^+(z)} \frac{|\Psi_x'(x,y)|}{(\Psi(x,y)-\Psi(x,z))^2}\,dx \le C\Big[\int_{J_l^+(z)} \frac{|a_y(x)|}{(\Psi(x,y))^2}dx + \int_{J_l^+(z)} \frac{|b_y(x)|}{|\Psi(x,y)|}dx\Big]$$

$$\begin{aligned} &\le C\Big[\Big|\int_{J_l^+(z)} \frac{\Psi_x'(x,y)-b_y(x)\Psi(x,y)}{(\Psi(x,y))^2}dx\Big| + \kappa\int_{J_l^+(z)} \frac{1}{|\Psi(x,y)|}dx\Big] \\ &\le C\Big[\Big|\int_{J_l^+(z)} \frac{\partial}{\partial x}\frac{1}{\Psi(x,y)}dx\Big| + 2\kappa\int_{J_l^+(z)} \frac{1}{|\Psi(x,y)|}dx\Big] \end{aligned} \tag{2.5}$$

By (2.5), the second inequality in (2.2) and (2.3) we obtain

$$\int_{J_l^+(z)} \frac{|\Psi_x'(x,y)|}{(\Psi(x,y)-\Psi(x,z))^2}\,dx \le C2^{-l}2^{-(l-j)\gamma}[t_0 g(t_0)]^{-1}. \tag{2.6}$$

Similarly

$$\begin{aligned}
&\int_{J_l^+(z)} \frac{|\Psi'_x(x,z)|}{(\Psi(x,y)-\Psi(x,z))^2}\,dx \;\le\; C\int_{J_l^+(z)} \frac{|\Psi'_x(x,z)|}{(\Psi(x,y))^2}\,dx \\
&\le\; C2^{-2(l-j)\gamma}\int_{J_l^+(z)} \frac{|\Psi'_x(x,z)|}{(\Psi(x,z))^2}\,dx \\
&\le\; C2^{-2(l-j)\gamma}\Big[\int_{J_l^+(z)} \frac{|a_z(x)|}{(\Psi(x,z))^2}dx + \kappa\int_{J_l^+(z)} \frac{|b_z(x)|}{|\Psi(x,z)|}dx\Big] \\
&\le\; C2^{-2(l-j)\gamma}\Big[\Big|\int_{J_l^+(z)} \frac{\partial}{\partial x}\frac{1}{\Psi(x,y)}dx\Big| + 2\kappa\int_{J_l^+(z)} \frac{1}{|\Psi(x,y)|}dx\Big] \\
&\le\; C2^{-2(l-j)\gamma}2^{-l}[t_0 g(t_0)]^{-1}.
\end{aligned} \tag{2.7}$$

From (2.6) and (2.7) and the definition of t_0 it follows that

$$I^+(y,z) \;\le\; C2^j\frac{2^{-(l-j)\gamma}}{\lambda t_0 g(t_0)} \;\le\; C2^j 2^{-(l-j)\gamma}$$

Finally, using (2.2) and (2.3) again we see that

$$II^+(y,z) \;\le\; C2^{2j+l}\lambda^{-1}\int_{J_l^+(z)} |\Psi(x,y)-\Psi(x,z)|^{-1}\,dx \le\; C2^j 2^{-(l-j)\gamma}.$$

Analogous estimates hold for I^- and II^-. Now $I^\pm$ and $II^\pm$ vanish whenever $|y-z|\ge 2^{-j+2}$. Therefore we obtain

$$\int |H_{jl}(y,z)|\,dz \;+\; \int |H_{jl}(y,z)|\,dy \;\le\; C'2^{-(l-j)\gamma}$$

which implies $\|T_j^*T_l\| = O(2^{-(l-j)\gamma})$ if $l > j+2N$ and $j,l \le M-2N$. The same conclusion for $\|T_jT_l^*\|$ follows in exactly the same way if we use the hypothesis (1.3) instead of (1.2). This yields the desired estimate for $\|\mathcal{T}^1\|$. □

Estimation of $\mathcal{T}^3$. In order to bound $\|\mathcal{T}^3\|$ we have to bound $\|T_jT_l^*\|$ for $j,l \ge M+2N$ and $l > j+2N$. Since χ is an even function we have

$$\int \frac{\chi(2^l(x-z))}{x-z}dx \;=\; 0$$

and therefore

$$H_{jl}(x,z) \;=\; \int_{J_l(z)} [R_{j\lambda}(x,y,z) - R_{j\lambda}(z,y,z)]\frac{\chi(2^l(x-z))}{x-z}dx$$

where

$$R_{j\lambda}(x,y,z) \;=\; e^{-i\lambda(S(x,y)-S(x,z))}\beta(x,z)\overline{K_j(x,y)}.$$

Now whenever $|x'-z|\le 2^{-l+1}$ we have

$$\begin{aligned}
\lambda|S'_x(x',y)-S'_x(x',z)| \;&\le\; \lambda|S'_x(x',y)-S'_x(x',x')| \\
&\qquad + \lambda|S'_x(x',z)-S'_x(x',x')| \\
&\le\; 2\lambda|S'_x(x',y)-S'_x(x',x')| \\
&\le\; 2A\lambda g(|x'-y|) \\
&\le\; A2^{j+2}\lambda|x-y|g(|x-y|) \\
&\le A2^{j+2}\lambda\frac{t_0}{2}g(\frac{t_0}{2}) \\
&\le\; C2^j.
\end{aligned}$$

From this one easily sees that $\partial_x R_{j\lambda}(x',y,z)=O(2^{2j})$ and therefore

$$|R_{j\lambda}(x,y,z)-R_{j\lambda}(z,y,z)| \;\le\; C2^{2j-l}$$

if $|x-z|\le 2^{-l+1}$. This implies the bound

$$\int |H_{jl}(y,z)|\,dz \;+\; \int |H_{jl}(y,z)|\,dy \;\le\; C2^{-(l-j)}$$

and shows that $\|T_j^*T_l\| \;=\; O(2^{-(l-j)})$ if $l>j+2N$ and $j,l\ge M+2N$. Since the analogous conclusion for $\|T_jT_l^*\|$ follows similarly we can deduce the boundedness of $\mathcal{T}^3$. $\square$

This completes the proof of Theorem 1.1.

Acknowledgement

I would like to thank Stephen Wainger whose comments on the translation invariant case led to a simplification of the original hypotheses.

References

1. A. Carbery, S. Wainger and J. Wright, *Hilbert transforms and maximal functions associated to flat curves on the Heisenberg group*, preprint 1993.
2. M. Christ, *Hilbert transforms along curves III: Rotational curvature*, Unpublished manuscript.
3. M. Christ, A. Nagel, E. M. Stein and S. Wainger, *Singular and maximal Radon transforms*, Princeton Univ. Press (to appear).

4. A. Greenleaf and G. Uhlmann, *Estimates for singular Radon transforms and pseudodifferential operators with singular symbols*, J. Funct. Anal. **89** (1990), 202–232.
5. A. Knapp and E.M. Stein, *Intertwining operators for semisimple Lie groups*, Annals of Math. **93** (1971), 489–578.
6. A. Nagel, J. Vance, S. Wainger and D. Weinberg, *Hilbert transforms for convex curves*, Duke Math. J. **50** (1983), 735–744.
7. Y. Pan, *Uniform estimates for oscillatory integral operators*, J. Funct. Anal. **100** (1991), 207-220.
8. D. H. Phong and E.M. Stein, *Hilbert integrals, singular integrals and Radon transforms I*, Acta Math. **157** (1986), 99–157.
9. ———, *Singular Radon transforms and oscillatory integrals*, Duke Math. J. **58** (1989), 347–369.
10. F. Ricci and E. M. Stein, *Harmonic analysis on nilpotent groups and singular integrals I. Oscillatory integrals*, J. Funct. Anal. **73** (1987), 179–194.
11. A. Seeger, *Degenerate Fourier integral operators in the plane*, Duke Math. J. **71** (1993), no. 3 (to appear).

University of Wisconsin, Madison, WI 53706
E-mail address: seeger@math.wisc.edu