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# AN ALGEBRA OF PSEUDODIFFERENTIAL OPERATORS

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## 0. Introduction

In this note we introduce, without explicit reference to any derivatives, a normed space of symbols on $\mathbb{R}^{2n}$ which is contained in the classical symbol space $S^0_{0,0}$ of smooth functions which are bounded with all their derivatives. We show that the corresponding space of pseudors (using the abbreviation pseudor = pseudodifferential operator)

a) is independent of the choice of the parameter $t \in [0,1]$ in the quantization:

$$\mathrm{Op}_t(a)u(x) = \frac{1}{(2\pi)^n} \iint e^{i(x-y)\cdot\xi} a(tx + (1-t)y, \xi) u(y)\, dy\, d\xi,$$

b) is stable under composition,

c) is contained in the space of $L^2$- bounded operators.

We thank J. M. Bony and N. Lerner for a stimulating discussion about the possible prospects of all this.

## 1. The symbol class and invariance under change of quantization

If $e_1, \dots, e_m$ is a basis in $\mathbb{R}^m$ we say that $\Gamma = \bigoplus_1^m \mathbb{Z}e_j$ is a lattice. Let $\Gamma$ be such a lattice and let $\chi_0 \in C_0^\infty(\mathbb{R}^m)$ have the property that $1 = \sum_{j\in\Gamma} \chi_j$, where $\chi_j(x) = (\tau_j \chi_0)(x) = \chi_0(x - j)$. Then we let $S(1)$ be the space of $u \in \mathcal{S}'(\mathbb{R}^m)$ with the property that

$$\sup_{j\in\Gamma} |\mathcal{F}(\chi_j u)(\xi)| \in L^1(\mathbb{R}^m).$$

Here $\mathcal{F}$ denotes the standard Fourier transformation: $\mathcal{F}u(\xi) = \widehat{u}(\xi) = \int e^{-ix\cdot\xi} u(x)\, dx$. We equip $u \in S(1)$ with the norm

$$\|u\|_{\Gamma,\chi_0} = \int \sup_j |\mathcal{F}(\chi_j u)(\xi)|\, d\xi. \tag{1.1}$$

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It is easy to prove directly, and this will also follow from Theorem 1.1 below, that $S(1)$ does not depend on the choice of $\Gamma$, $\chi_0$, and that two different choices give rise to equivalent norms. We also notice that $S(1)$ is contained in the space of bounded continuous functions.

The convergence in the norm of $S(1)$ is too strong for our purposes, since $\mathcal{S}$ is not dense in $S(1)$ for the norm. We say that a sequence $u_\nu \in S(1)$ converges narrowly to $u$ if $u_\nu \to u$ in $\mathcal{S}'$ (weakly) and if there exists $0 \leq U \in L^1(\mathbb{R}^m)$ such that $\sup_j |\widehat{\chi_j u_\nu}(\xi)| \leq U(\xi)$. Since $\widehat{\chi_j u}(\xi) = \lim_{\nu\to\infty} \widehat{\chi_j u_\nu}(\xi)$, it follows that $u$ belongs to $S(1)$ and that $u_\nu \to u$ point wise. We also notice that $\mathcal{S}$ is dense in $S(1)$ for narrow convergence. In fact, let $u \in S(1)$ and put $u_\nu = \psi(\frac{x}{\nu})(\phi_{\frac{1}{\nu}} * u)$ (with $*$ denoting convolution) where $\psi \in \mathcal{S}$, $\psi(0) = 1$, $\phi \in C_0^\infty$, $\int \phi = 1$, $\phi_{\frac{1}{\nu}} = \nu^m \phi(\nu x)$. We have

$$\chi_j(\phi_{\frac{1}{\nu}} * u) = \sum_{|k-j|\leq C} \chi_j(\phi_{\frac{1}{\nu}} * (\chi_k u)),$$

for some constant $C$, so

$$\mathcal{F}(\chi_j(\phi_{\frac{1}{\nu}} * u)) = \frac{1}{(2\pi)^m} \sum_{|k-j|\leq C} \widehat{\chi}_j * (\widehat{\phi}_{\frac{1}{\nu}} \widehat{\chi_k u}),$$

and since $|\widehat{\phi}_{\frac{1}{\nu}}| \leq \text{Const.}$, $|\widehat{\chi}_j| = |\widehat{\chi}_0|$, we see that

$$|\mathcal{F}(\chi_j(\phi_{\frac{1}{\nu}} * u)(\xi))| \leq C|\widehat{\chi}_0| * \sup_k |\widehat{\chi_k u}|.$$

Here and in the following $C$ will denote a new constant in every new formula. Consequently,

$$|\widehat{\chi_j u_\nu}(\xi)| \leq C|\nu^m \widehat{\psi}(\nu\cdot)| * |\widehat{\chi}_0| * \sup_k |\widehat{\chi_k u}(\xi)|,$$

which is bounded by some fixed $L^1$ function, since this is the case for $|\nu^m \widehat{\psi}(\nu\cdot)| * |\widehat{\chi}_0|$. Since $u_\nu \to u$ in $\mathcal{S}'$, we conclude that $u_\nu \to u$ narrowly. Notice also that we have found a sequence $u_\nu$ with the additional property $\|u_\nu\|_{\Gamma,\chi_0} \leq C\|u\|_{\Gamma,\chi_0}$, where $C$ does not depend on $u$.

**Theorem 1.1.** *Let $\Phi(x)$ be a non-degenerate quadratic form on $\mathbb{R}^m$. Then the convolution operator $u \mapsto e^{i\Phi} * u$ is bounded from $S(1)$ to $S(1)$, and is continuous in the sense of narrow convergence.*

*Proof.* By our density remarks, it is enough to consider $e^{i\Phi} * u$ in the case when $u \in \mathcal{S}$. Let $\Gamma'$ be a second lattice and let $\chi_0' \in C_0^\infty$ have the property: $1 = \sum_{j\in\Gamma'} \chi'_j$, with $\chi'_j = \tau_j \chi'_0$. Let $\widetilde{\chi}_0$ satisfy: $\widetilde{\chi}_0\chi_0 = \chi_0$ and put $\widetilde{\chi}_k = \tau_k\widetilde{\chi}_0$. Then for $j \in \Gamma'$, $k \in \Gamma$, we have

$$\begin{aligned}\mathcal{F}(\chi'_j(e^{i\Phi} * \chi_k))(\xi) &\\ &= \iiint e^{i(-x\cdot\xi+\Phi(x-y)+y\cdot\eta)}\chi'_j(x)\widetilde{\chi}_k(y)\widehat{\chi_k u}(\eta)\,d\eta\,dy\,dx/(2\pi)^m \\ &= \frac{e^{iF(j,k)}}{(2\pi)^m}\iiint e^{i(-x\cdot(\xi-\partial_x\Phi(j-k))+y\cdot(\eta-\partial_x\Phi(j-k)))}\chi_{j,k}(x,y) \\ &\qquad\qquad \widehat{\chi_k u}(\eta)\,dy\,dx\,d\eta,\end{aligned}$$

where $F$ is real-valued and where

$$\chi_{j,k}(x,y) = \chi_j(x)\widetilde{\chi}_k(y)e^{i\Phi((x-j)-(y-k))}.$$

Here we have also used the Taylor sum formula:

$$\begin{aligned}\Phi(x-y) = \Phi(j-k) + \partial_x\Phi(j-k)\cdot(x-j) - \partial_y\Phi(j-k)\cdot(y-k) &\\ + \Phi((x-y)-(j-k)).&\end{aligned}$$

Notice that the modulus of any derivative of $\chi_{j,k}(x,y)$ can be bounded by a constant which is independent of $x$, $y$, $j$, $k$.

We make $2N$ integrations by parts, using the operators

$$\frac{1-(\xi-\partial_x\Phi(j-k))\cdot D_x}{\langle\xi-\partial_x\Phi(j-k)\rangle^2}, \qquad \frac{1+(\eta-\partial_x\Phi(j-k))\cdot D_y}{\langle\eta-\partial_x\Phi(j-k)\rangle^2},$$

with the notation $D_x = \frac{1}{i}\partial_x$, $\langle x\rangle = \sqrt{1+x^2}$. After estimating the resulting $x$, $y$ integrals in a straight forward way, we get:

$$\begin{aligned}(1.2)\quad \mathcal{F}(\chi'_j(e^{i\Phi} * \chi_k))(\xi) &\\ = \mathcal{O}_N(1)\int \langle\xi-\partial_x\Phi(j-k)\rangle^{-N}\langle\eta-\partial_x\Phi(j-k)\rangle^{-N}|\widehat{\chi_k u}(\eta)|\,d\eta.&\end{aligned}$$

Since $\Phi$ is non-degenerate, $\langle\xi-\partial_x\Phi(j-k)\rangle$ is of the same order of magnitude as $\langle\Phi''^{-1}\xi - j + k\rangle$ and similarly for $\langle\eta-\partial_x\Phi(j-k)\rangle$. With $N > m$, we then get:

$$(1.3)\qquad \sum_{k\in\Gamma}\langle\xi-\partial_x\Phi(j-k)\rangle^{-N}\langle\eta-\partial_x\Phi(j-k)\rangle^{-N} \le C_N\langle\xi-\eta\rangle^{-N}.$$

Summing over $k$ in (1.2) we get

$$|\mathcal{F}(\chi_j'(e^{i\Phi} * u))(\xi)| \leq C_N \int \langle \xi - \eta \rangle^{-N} \sup_k |\widehat{\chi_k u}(\eta)|\, d\eta \tag{1.4}$$

and consequently $\sup_j |\mathcal{F}(\chi_j'(e^{i\Phi} * u))(\xi)|$ is also bounded by the left hand hand side of (1.4) (which is an $L^1$ function).

It remains to establish the narrow continuity. Let $u_\nu \to 0$ narrowly in $S(1)$. Then it follows from the preceding estimates that

$$|\mathcal{F}(\chi_j'(e^{i\Phi} * u_\nu))| \leq V,$$

for some $L^1$ function $V$ independent of $\nu$. We consider the convergence of $\mathcal{F}(\chi_j'(e^{i\Phi} * u_\nu))$ for some fixed $j$, say $j = 0$. Since $\mathcal{F}(\chi_k u_\nu) \to 0$ in $L^1$ for every fixed $k$ (by dominated convergence), we see from (1.2) that for $R > 0$:

$$\begin{aligned} &\overline{\lim} \left|\mathcal{F}(\chi_0'(e^{i\Phi} * u_\nu))(\xi)\right| \leq \overline{\lim} \sum_{|k|\geq R} \left|\mathcal{F}(\chi_0'(e^{i\Phi} * \chi_k u_\nu))(\xi)\right| \\ &\leq C_N\, \overline{\lim} \int K_{N,R}(\xi,\eta) \sup_k |\mathcal{F}(\chi_k u_\nu)(\eta)|\, d\eta \leq C_N \int K_{N,R}(\xi,\eta) U(\eta)\, d\eta, \end{aligned}$$

where $|\widehat{\chi_k u}| \leq U \in L^1$ and

$$K_{N,R}(\xi,\eta) = \sum_{|k|\geq R} \langle \xi - \partial_x \Phi(j-k)\rangle^{-N} \langle \eta - \partial_x \Phi(j-k)\rangle^{-N}$$

can be estimated by the right hand side of (1.3). Since $K_{N,R}(\xi,\eta) \to 0$ pointwise when $R \to \infty$, it follows that $\overline{\lim}\left|\mathcal{F}(\chi_0'(e^{i\Phi} * u_\nu))(\xi)\right| = 0$, so $\mathcal{F}(\chi_0'(e^{i\Phi} * u_\nu)) \to 0$ in $L^1$ by dominated convergence. Consequently $e^{i\Phi} * u_\nu \to 0$ in $\mathcal{S}'$, so $e^{i\Phi} * u_\nu \to 0$ narrowly. $\square$

**Corollary 1.2.** *Let $t$, $s \in [0,1]$, $a_t$, $a_s \in \mathcal{S}'(\mathbb{R}^{2n})$, and assume that $\mathrm{Op}_t(a_t) = \mathrm{Op}_s(a_s)$. Then $a_t \in S(1)$ iff $a_s \in S(1)$; moreover, the correspondence $S(1) \ni a_s \mapsto a_t \in S(1)$ is bounded and narrowly continuous.*

*Proof.* The case $t = s$ is trivial and if $t \neq s$, we have

$$a_t = e^{i(t-s)D_x\cdot D_\xi} a_s = C_n e^{i\Phi} * a_s,$$

with $\Phi = \Phi_{t-s}$ as in the preceding theorem. $\square$

## 2. Composition of symbols

We work on $\mathbb{R}^{2n}$ and denote the variables there by $x$, $y$, $z$, rather than by $(x,\xi)$, $(y,\eta)$, $(z,\zeta)$ etc. For simplicity, we concentrate on the composition in the Weyl quantization ($t = \frac{1}{2}$). Corollary 1.2 implies that our results below extend to the other quantizations. If $u, v \in \mathcal{S}'(\mathbb{R}^{2n})$, we recall that the Weyl composition $w = u\sharp v$, defined by $\mathrm{Op}_{\frac{1}{2}}(w) = \mathrm{Op}_{\frac{1}{2}}(u) \circ \mathrm{Op}_{\frac{1}{2}}(v)$ is given by

(2.1)

$$u\sharp v(x) = (e^{\frac{i}{2}\omega(D_x,D_y)}u(x)v(y))_{y=x} = C_n \iint e^{-i\sigma(x-y,x-z)}u(y)v(z)\,dy\,dz,$$

where $\omega$ denotes the standard symplectic two-form and we put $\sigma = \frac{1}{2}\omega$.

**Theorem 2.1.** *The Weyl composition extends (uniquely) to a bilinear map $S(1) \times S(1) \to S(1)$ which is norm continuous and preserves narrow convergence of sequences.*

*Proof.* Let $\chi_0$, $\Gamma$, $\widetilde{\chi}_0$ be as in section 1. For $u, v \in \mathcal{S}(\mathbb{R}^{2n})$, we get, writing $\sigma(x,y) = x \cdot Jy$:

$$\begin{aligned}
&\mathcal{F}(\chi_j(\chi_k u\sharp\chi_\ell u))(\xi) \\
&\quad = \int\!\!\!\int\!\!\!\int\!\!\!\int\!\!\!\int e^{i(-x\cdot\xi-\sigma(x-y,x-z)+y\cdot\eta+z\cdot\zeta)}\chi_j(x)\widetilde{\chi}_k(y)\widetilde{\chi}_\ell(z) \\
&\qquad\qquad \widehat{\chi_k u}(\eta)\widehat{\chi_\ell v}(\zeta)\,dx\,dy\,dz\frac{d\eta\,d\zeta}{(2\pi)^{4n}} \\
&\quad = e^{i\sigma(j-k,j-\ell)}\int^5 e^{i(x\cdot(-\xi-J(k-\ell))+y\cdot(\eta-J(\ell-j))+z\cdot(\zeta-J(j-k))} \\
&\qquad\qquad \chi^0_{j,k,\ell}(x,y,z)\widehat{\chi_k u}(\eta)\widehat{\chi_\ell v}(\zeta)\,dx\,dy\,dz\frac{d\eta\,d\zeta}{(2\pi)^{4n}}.
\end{aligned}$$

Here

$$\chi^0_{j,k,\ell}(x,y,z) \overset{\mathrm{def}}{=} \chi_j(x)\widetilde{\chi}_k(y)\widetilde{\chi}_\ell(z)e^{-i\sigma((x-j)-(y-k),(x-j)-(z-\ell))}$$

and all its derivatives can be bounded by constants that are independent of $j,k,\ell,x,y,z$. Repeated integrations by parts in each of the variables $x,y,z$ give

$$\begin{aligned}
(2.2)\quad \mathcal{F}(\chi_j(\chi_k u\sharp\chi_\ell v))(\xi) = \int^5 &\langle\xi + J(k-\ell)\rangle^{-N}\langle\eta + J(j-\ell)\rangle^{-N} \\
&\langle\zeta + J(k-j)\rangle^{-N}\chi^N_{j,k,\ell}(x,y,z)\widehat{\chi_k u}(\eta)\widehat{\chi_\ell v}(\zeta)\,dx\,dy\,dz\,d\eta\,d\zeta
\end{aligned}$$

where $\operatorname{supp}\chi^N_{j,k,\ell} \subset \operatorname{supp}\chi^0_{j,k,\ell}$ and $|\chi^N_{j,k,\ell}|$ is bounded by some constant which is independent of $x, y, z, j, k, l$.

Choose $N \geq 2n+1$. Using that $J$ is bijective, we get as in the proof of Theorem 1.1, first that

$$\sum_k \langle \xi + J(k-\ell)\rangle^{-N} \langle \zeta + J(k-j)\rangle^{-N} = \mathcal{O}_N(1)\langle \xi - \zeta + J(j-\ell)\rangle^{-N},$$

and then after summing also in $\ell$ that

$$\sum_{k,\ell} \langle \xi + J(k-\ell)\rangle^{-N} \langle \eta + J(j-\ell)\rangle^{-N} \langle \zeta + J(k-j)\rangle^{-N} = \mathcal{O}_N(1)\langle \xi - \zeta - \eta\rangle^{-N}.$$

If we sum over $k, \ell$ and estimate as in section 1, we get

$$|\mathcal{F}(\chi_j(u\sharp v))(\xi)| \leq \mathcal{O}_N(1) \iint \langle \xi - (\zeta+\eta)\rangle^{-N} (\sup_k |\widehat{\chi_k u}(\eta)|)(\sup_\ell |\widehat{\chi_\ell v}(\zeta)|)\, d\eta\, d\zeta, \tag{2.3}$$

and taking the supremum in $j$ and integrating in $\xi$, we get

$$\|u\sharp v\|_{\Gamma,\chi_0} \leq C\|u\|_{\Gamma,\chi_0}\|v\|_{\Gamma,\chi_0}. \tag{2.4}$$

It is now clear that for $u, v \in S(1)$, we can can define define $u\sharp v$ by the above procedure, so that $u\sharp v$ is bilinear and satisfies (2.4). In other words, we have a bilinear continuous extension of $\sharp$ to: $S(1) \times S(1) \ni (u, v) \mapsto u\sharp v \in S(1)$.

It remains to prove the narrow continuity (which by density will imply that our extension is unique). Let $u_\nu$, $v_\nu$ be sequences in $S(1)$ which tend narrowly to $u$, $v$. Then

$$|\mathcal{F}(\chi_j u_\nu)(\xi)| \leq U, \qquad |\mathcal{F}(\chi_j v_\nu)(\xi)| \leq V,$$

where $U, V$ are $L^1$ functions independent of $\nu, j$. The proof above shows that

$$|\mathcal{F}(\chi_j(u_\nu\sharp v_\nu))(\xi)| \leq W$$

for some $W \in L^1$, independent of $j, \nu$. To show that $u_\nu\sharp v_\nu \to u\sharp v$ narrowly, it then suffices to show that $\mathcal{F}(\chi_j(u_\nu\sharp v_\nu))(\xi) \to \mathcal{F}(\chi_j(u\sharp v))(\xi)$ for all fixed $j, \xi$, and we may assume for simplicity that $j = 0$. Since $\mathcal{F}(\chi_k u_\nu) \to \mathcal{F}(\chi_k u)$ and $\mathcal{F}(\chi_\ell v_\nu) \to \mathcal{F}(\chi_\ell v)$ in $L^1$ for all fixed $k, \ell$, we see that for every $R > 0$:

$$\begin{aligned}
&\overline{\lim}_{\nu\to\infty} |\mathcal{F}(\chi_0(u_\nu\sharp v_\nu))(\xi) - \mathcal{F}(\chi_0(u\sharp v))(\xi)| \\
&\quad \leq C_N \iint \Big( \sum_{|(k,\ell)|\geq R} \langle \xi + J(k-\ell)\rangle^{-N} \langle \eta + J(j-\ell)\rangle^{-N} \\
&\qquad\qquad \langle \zeta + J(k-j)\rangle^{-N} \Big) U(\xi) V(\eta)\, d\xi\, d\eta.
\end{aligned}$$

Here the double sum is bounded by $C_N\langle\xi-(\zeta+\eta)\rangle^{-N}$ (as we have already seen) and it is easy to see that it tends to 0 pointwise when $R\to\infty$. It is then clear that $\overline{\lim}_{\nu\to\infty}|\mathcal{F}(\chi_0(u_\nu\sharp v_\nu))(\xi)-\mathcal{F}(\chi_0(u\sharp v))(\xi)|=0$. $\Box$

## 3. $L^2$ boundedness

We follow the idea of the classical proof of Kohn-Nirenberg [KN]. After multiplying our norm on $S(1)$ by some sufficiently large constant, we obtain a norm $|||\cdot|||$ on $S(1)$ with the property: $|||a\sharp b|||\leq|||a|||\,|||b|||$. Notice also that $|||\overline{a}|||=|||a|||$ if we take $\chi_j$ real.

Let first $a\in\mathcal{S}$ with $|||a|||<1$. Then $|||\overline{a}\sharp a|||<1$. Now recall that for $|t|<1$ we have the convergent power series representation:

$$(1-t)^{\frac{1}{2}}=1-\frac{\frac{1}{2}}{1}t+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1\cdot 2}t^2-\cdots\ =f(t).$$

Since $S(1)$ is complete, we can define $(1-\overline{a}\sharp a)^{\frac{1}{2}}=b$ as $f(\overline{a}\sharp a)$ in the functional sense. Since $\overline{a}\sharp a$ is real, $b$ is real. We have $1-\overline{a}\sharp a=b\sharp b$. Let $b_j\in\mathcal{S}$ be a sequence of real functions converging to $b$ narrowly. Then $b_j\sharp b_j\to b\sharp b$ narrowly, so if $u\in\mathcal{S}(\mathbb{R}^n)$, we have

$$(\mathrm{Op}(b\sharp b)u|u)=\lim\,(\mathrm{Op}(b_j\sharp b_j)u|u)=\lim\,\|\mathrm{Op}(b_j)u\|^2\geq 0.$$

Here we write $\mathrm{Op}=\mathrm{Op}_{\frac{1}{2}}$ and $\|\cdot\|$ and $(\cdot|\cdot\cdot)$ are the standard norms and scalar products in $L^2(\mathbb{R}^n)$. It follows that $((1-\mathrm{Op}(a)^*\,\mathrm{Op}(a))u|u)\geq 0$, so $\|\mathrm{Op}(a)u\|\leq\|u\|$. Hence for general $a\in\mathcal{S}$: $\|\mathrm{Op}(a)\|_{\mathcal{L}(L^2,L^2)}\leq|||a|||$. *For a general* $a\in S(1)$, *we then get that* $\mathrm{Op}(a)\in\mathcal{L}(L^2,L^2)$, *and*

$$\|\mathrm{Op}(a)\|_{\mathcal{L}(L^2,L^2)}\leq C|||a|||,$$

since we can find a sequence $a_j\in\mathcal{S}$ converging to $a$ narrowly and with $|||a_j|||\leq C|||a|||$, and since $(\mathrm{Op}(a)u|v)=\lim(\mathrm{Op}(a_j)u|v)$ for all $u,v\in\mathcal{S}$.

## 4. Further comments

*a.* The history of exotic symbol spaces and associated pseudors is long ([BF], [B], [H], [CM], [Hw], ... ) and often the Hörmander space $S^0_{0,0}$ serves as a basic building block. Any improvement here may give some improvement in the more elaborate technical theories.

*b.* Hwang [Hw] has obtained $L^2$ boundedness results assuming very little about the derivatives. As communicated to us by N. Lerner, his proof can be extended to the case of symbols of the class $S(1)$.

*c.* The author has the intention to examine some links with Bargman type transforms in the spirit of his old lecture notes [S].

*d.* This work is an outgrowth of a course on spectral asymptotics in Orsay 1993-94, and started with the observation that if $a \in \mathcal{S}'(\mathbb{R}^{2n})$, and

$$\sum_{j\in\Gamma}\int |\widehat{\chi_j a}(\xi)|\, d\xi < \infty,$$

then $\mathrm{Op}(a)$ is of trace class.

*e.* If $U \geq 0$, is a finite measure on $\mathbb{R}^m$, we may introduce the subspaces $S(U,1)$ of all $u \in \mathcal{S}'$ with

$$\sup_j |\widehat{\chi_j u}| \leq C_N(u)(\langle\cdot\rangle^{-N} * U),$$

for every $N > m$. Then the proofs above show that $u \mapsto e^{i\Phi} * u$ maps $S(U,1)$ into itself, and that $u\sharp v \in S(U*V,1)$ if $u \in S(U,1),\ \ v \in S(V,1)$. Notice that $S(\delta_0,1) = S^0_{0,0}$, if $\delta_0$ is the Dirac mass at 0.

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