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ON A CLASS OF SPHERICAL HARMONICS ASSOCIATED WITH RIGID BODY MOTION

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1. Introduction

In the seminal paper [A], Arnold formulated the modern framework for the classical Euler equations. In the case of the generalized n-dimensional rigid body, these may be viewed as Hamilton equations on a coadjoint orbit in $so(n)$ for the kinetic energy form relative to the Kirilov-Kostant symplectic structure. In section 2, we review these constructions. As there are several excellent sources (see [AM], [GS], [R1]) that cover Euler equations in this context, we shall be brief.

Although it was Manakov [Ma] who integrated the Euler equations in n-dimensions, several years earlier, Mishchenko [M] had found a whole series of integrals of 'hydrodynamic type' (ie. quadratic in the Lie algebra coordinates). In section 3, we present a class of quadratic functions, which are integrals in involution for *Legendre transforms* of a class of Mishchenko generalized rigid body Hamiltonians. In the case of $SO(3)$, these are just classical rigid body Hamiltonians. We then reduce left extensions of these integrals with respect to the left action of $SO(n)$ on $SO(n+1)$ at zero momentum. Finally, we show how these integrals also arise via the Moser residue correspondence.

In section 4, we quantize the above integrals, and show that the corresponding differential operators have generalized Lamé harmonics as joint eigenfunctions. To our knowledge, with the exception of the classical case (ie. $S^2 \subset R^3$), these functions have not been studied.

2. The Euler Equations

Let G be a finite-dimensional Lie group with Lie algebra g. Consider T^*G together with the canonical symplectic form ω and a left-invariant Hamiltonian $H : T^*G \to R$. One then easily shows (see [AM]) that the

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corresponding Hamiltonian vector field X_H is left-invariant, and thus, using body coordinates

$$\lambda : T^*G \to G \times g^*$$

given by the formula,

$$\lambda(\alpha_g) = (g, dl_g^t(\alpha_g)), \;\; \alpha_g \in T^*G$$

we get an induced vector field $Y : g^* \to g^*$ satisfying the equation:

$$Y(\xi) \cdot \eta = \xi \cdot [\frac{dx}{dt}|_{t=0}, \eta]$$

Here $\xi \in g^*$, $\eta \in g$, $x(t)$ is the base projection of the Hamiltonian flow on T^*G. Y is called the cotangent Euler vector field. As we already mentioned, Y is a Hamiltonian vector field relative to the Kirilov-Kostant symplectic structure. Turning our attention to the case of the generalized rigid body, we henceforth put $G = SO(n{+}1)$. The Killing form on $so(n{+}1)$ allows us to identify adjoint and coadjoint orbits. Moreover, using the adjoint-invariance of this form, one finds that the Euler equations in adjoint formulation are of Lax type:

$$\frac{d\Omega}{dt} = [\Omega, \psi] \tag{1}$$

where $\psi \in so(n{+}1)$, $\Omega = \psi D{+}D\psi$, $D = \mathrm{diag}(\lambda_0, \dots, \lambda_n)$, and the Hamiltonian is $H(\Omega) = \frac{1}{2}\,\mathrm{Trace}(\Omega\psi)$. Manakov [Ma], managed to integrate this problem, by simply rewriting equation (1) in the form:

$$\frac{d}{dt}(\Omega + D^2 z) = [\Omega + D^2 z, \psi + Dz] \tag{2}$$

where z is any parameter. One then simply reads off the integrals as the coefficients of z in the polynomials:

$$\frac{1}{2k} \cdot \mathrm{Trace}(\Omega + D^2 z)^k \qquad k = 2, \dots, n$$

Moreover, there are precisely $\frac{1}{2}(\frac{n(n-1)}{2} - [\frac{n}{2}])$ independent integrals, which is half the dimension of a generic coadjoint orbit. However, the Manakov integrals do not contain the Hamiltonian $H(\Omega)$. On the other hand, Mishchenko's integrals are all *quadratic*, and are given by the formulas:

$$m_k(\Omega) = -\frac{1}{4} \cdot \mathrm{Trace} \sum_{p=1}^{k} D^{p-1}\Omega D^{k-p}\psi \qquad k = 1, \dots, n-1$$

with $H = m_1$. We note in passing that these functions are only sufficient to integrate the problem in the cases of $so(3)$ and $so(4)$. Nevertheless, it is precisely these integrals that will be of interest to us.

3. The Moser Residue Correspondence

We may write the Mishchenko rigid body Hamiltonians in terms of the standard coordinates ψ_{ij} on $so(n+1)$ as:

$$H(\psi) = m_1(\psi) = \sum_{i<j} (\lambda_i + \lambda_j)\psi_{ij}^2,$$

where $0 < \lambda_0 < \ldots < \lambda_n$. We shall study the Hamiltonians,

$$\mathcal{L}^* H(\Omega) = \sum_{i<j} (\lambda_i + \lambda_j)\Omega_{ij}^2$$

where $\mathcal{L}$ denotes the Legendre transform:

$$\mathcal{L}(\psi_{ij}) = \frac{\partial H}{\partial \psi_{ij}} = \Omega_{ij}$$

Note that we can replace $\mathcal{L}^* H$ by

$$\tilde{H}(\Omega) = \mathcal{L}^* H(\Omega) - (\sum_{i=0}^{n} \lambda_i) \sum_{i<j} \Omega_{ij}^2$$

and get precisely the same flow. This is simply because the Hamiltonian vector field of any multiple of the Casimir is zero. In the case of $so(3)$, we can solve the equations $\bar{\lambda_i} + \bar{\lambda_j} = \frac{1}{\lambda_i + \lambda_j}$, $i < j$ uniquely for the $\bar{\lambda_j}$, and thus the functions $\tilde{H}$ are just classical rigid body energy functions.

Fix $n+1$ numbers $0 < \alpha_0 < \cdots < \alpha_n$, where $\alpha_j = \lambda_j$, and consider the following quadratic functions on $so(n+1) \cong so^*(n+1)$, with $p_1 = -\tilde{H}$:

$$p_k(\Omega) = \sum_{i<j} S_k(\alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_n)\Omega_{ij}^2$$

where S_k, for $k = 0, \ldots, n-1$, denotes the k-th elementary symmetric polynomial on $n-1$ variables.

Theorem 1. *The p_k, for $k = 0, \ldots, n-1$, are independent integrals in involution with respect to the canonical symplectic structure on the coadjoint orbits of $so^*(n+1)$.*

Proof. One can verify directly that the p_k are independent integrals in involution relative to the Kirilov-Kostant symplectic form. Alternatively, this may be proved as follows. Since the intrinsic symplectic structure on a coadjoint orbit is isomorphic to the reduced symplectic structure got by considering the left action of $SO(n+1)$ on itself, it certainly suffices to prove that left-invariant extensions of the p_k are in involution relative to the canonical symplectic structure on $T^*(SO(n+1))$. We thus consider the functions

$$p_k = \sum_{i \neq j} S_k \cdot \sigma\{(dl_g \cdot e_{ij})^2\} \tag{3}$$

where e_{ij} denotes the skew-symmetric matrix with a 1 in the i, j slot, and zeroes everywhere else. The left-invariant vector field generated by e_{ij} is denoted by $dl_g \cdot e_{ij}$. It is easy to see that to prove involution, it suffices to show that:

$$[(dl_g \cdot e_{ij})^2, (dl_g \cdot e_{kl})^2]$$

is independent of cyclic permutations of i, j, k, l. This is indeed so (see, for example [Gur]). $\square$

To understand the functions that arise in the quantization of the p_k, it is important to construct these functions in a different manner. This will be done by what we shall call the *Moser residue correspondence* (see [Mo]).

Consider the following functions on $T^*(R^{n+1})$:

$$G_j(x, \xi) = \sum_{k \neq j}^{n} \frac{(x_j \xi_k - x_k \xi_j)^2}{\alpha_j - \alpha_k} \tag{4}$$

where $0 < \alpha_0 < \cdots < \alpha_n$ are fixed numbers. These *line flow integrals* were introduced by Uhlenbeck, and play a crucial role in the integration of many of the known integrable systems (see [Mo]). Let z be a complex variable. Then, the residue correspondence is given by the formula:

$$\sum_{j=0}^{n} \frac{G_j}{z - \alpha_j} = \frac{\sum_{k=0}^{n-1} \eta_k z^{n-1-k}}{\prod_{j=0}^{n} (z - \alpha_j)}. \tag{5}$$

It is easily verified that $\{G_j, G_k\} = 0$, and moreover the Moser constraining procedure shows that the restrictions of the G_j's to $T^*(S^n)$ are also in involution; hence, so are the η_j's.

Proposition 1. *Let $SO(n)$ act on $SO(n+1)$ by left-multiplication, and let $\Phi : T^*(SO(n+1)) \to so^*(n)$ be the associated momentum mapping. Reduce the mechanical system at the momentum level set $\Phi^{-1}(0)$. Given the standard diffeomorphism $\tilde{f} : SO(n+1)/SO(n) \to S^n$, induced by the map*

$$f : SO(n+1) \to S^n$$
$$f(x_{ij}) = (x_{n+1,\,1}\,, \ldots ,\, x_{n+1,\,n+1})$$

one gets the correspondence,

$$\tilde{\eta}_k \cdot d\tilde{f}^t = \tilde{p}_k$$

where $\tilde{p}_k$ denotes the function on $T^*(SO(n+1)/SO(n))$ induced from p_k, and $\tilde{\eta}_k$ denotes the restriction of η_k to $T^*(S^n)$.

Proof. Consider $SO(n)$ embedded in $SO(n+1)$ in the usual way (ie. as an upper left $n \times n$ block). Since we reduce at $\Phi^{-1}(0)$, we must set all components of spatial angular momentum relative to the left-action equal to zero, that is:

$$(*) \qquad \sigma\{dr_g \cdot e_{ij}\} = 0 \qquad 1 \le i < j \le n.$$

Given $(x_{ij}) \in SO(n+1)$, a point in the quotient space $SO(n+1)/SO(n)$ is identified with the last row vector $(x_{n+1,\,1}, \ldots , x_{n+1,\,n+1}) \in S^n \subset R^{n+1}$ which, for notational simplicity, we shall denote by $(x_0, \ldots , x_n)$. This canonical map establishes the diffeomorphism $SO(n+1)/SO(n) \cong S^n$. Under this map, the left-invariant vector field $dl_g \cdot e_{ij}$, where $g = (x_{ij})$ gets identified with the vector field $-x_j\partial_i + x_i\partial_j$ in R^{n+1}. Moreover, the assertion $(*)$ of spatial momentum conservation becomes $\sum_{k=0}^{n} x_k \cdot \xi_k = 0$. In addition, we note that the induced map between reduced phase space $T^*(SO(n+1)/SO(n))$ and $T^*(S^n)$ is an isomorphism of *canonical* symplectic structures. This is because the Marsden-Weinstein reduced phase space is equipped with the canonical symplectic structure, since we are reducing at $\Phi^{-1}(0)$ and thus, there is no curvature correction in the symplectic form arising from the mechanical connection on the principal reduction bundle. From equations (4) and (5), we get the required formulas:

$$(6) \qquad \eta_k(x,\xi) = \sum_{i<j} S_k \cdot (x_i\xi_j - x_j\xi_i)^2.$$

This completes the proof of the proposition. $\square$

Remark 1. This shows that for the above integrals, Moser constraint and Marsden-Weinstein reduction are really the same thing. One is led to ask whether this is true in other cases. For example, we suspect this to be true for both the C. Neumann problem and geodesic flow on a n-axial ellipsoid. As is well-known ([R2], [GS]), both of these systems are governed by Euler equations on different coadjoint orbits of a semidirect product of Lie algebras.

Thus, we may now think of our reduced rigid body integrals $\tilde{p}_k$ as the functions $\tilde{\eta}_k$ on $T^*(S^n)$. This is important, since the latter have an 'elliptic' symmetry. More precisely, one can show (see [T1], [Mo]) that the $\tilde{\eta}_k$'s are the integrals corresponding to the classical Hamilton-Jacobi equation,

$$H(u, \partial_u \phi) = \tilde{\eta}_1$$

where $u = (u_1, \dots, u_n)$ denote elliptic-spherical variables (see [M], [Gur], [T1]) on S^n, and the generating function is given by the hyperelliptic integral,

$$\phi(u; \eta) = \sum_{j=0}^{n-1} \int_{\alpha_j}^{u_j} \left\{ \frac{\sum_{k=0}^{n-1} \tilde{\eta}_k x^{n-1-k}}{\prod_{l=0}^{n} (x - \alpha_l)} \right\}^{\frac{1}{2}} dx$$

4. Generalized Lamé Harmonics

The quantization of the rigid body integrals (3) is now self-evident:

Theorem 2. *The second order differential operators,*

$$P_k = \sum_{i<j} S_k \cdot (dl_g \cdot e_{ij})^2 \qquad k = 0, \dots, n-1 \tag{7}$$

on $SO(n+1)$ all commute. Moreover, the P_k induce unique operators $\tilde{P}_k$, on $SO(n+1)/SO(n) \cong S^n$. The $\tilde{f}^* \tilde{P}_k$ are all self-adjoint with respect to the constant curvature metric on S^n, and are given by the explicit formulas:

$$\tilde{f}^* \tilde{P}_k = \Delta_k + \nabla_k \log \rho_k \qquad k = 1, \dots, n-1$$
$$\tilde{f}^* \tilde{P}_0 = \Delta_0$$

where Δ_0 denotes the standard Laplacian on S^n, the subscripts k refer to induced positive-definite metrics on S^n, and ρ_k is the 'density' function of Helgason [H]. *Furthermore, the joint eigenfunctions of the $\tilde{f}^* \tilde{P}_k$ are a class of spherical harmonics which are higher-dimensional analogues of the Lamé harmonics. In the case of $SO(3)$, these eigenfunctions are precisely the classical Lamé harmonics* (*see* [Er], [W], [T1], [Gur]).

Proof. By section 3, it suffices to replace the left-invariant vector fields $dl_g \cdot e_{ij}$ by $-x_j\partial_i + x_i\partial_j$ in equation (7). The verification of the formula for the $\tilde{f}^*\tilde{P}_k$ follows by introducing elliptic-spherical variables (see [T1]). The metrics g_k are given by the local formulas:

$$g_k^{jj} = \frac{4(-1)^k \cdot S_k(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_n) \prod_{l=0}^n (u_j - \alpha_j)}{\prod_{l \neq j}(u_l - u_j)}$$

$$g_k^{ij} = 0 \qquad i \neq j$$

and $\rho_k = \{\det g_k^{ij}\}^{1/2}\{\det g_0^{ij}\}^{-1/2}$, which is just the ratio of Riemann measure to Haar measure along an $SO(n)$-orbit. This is precisely the definition of the density function (see [H]). In the case of $SO(3)$, one gets the classical Lamé harmonics as joint eigenfunctions of $\tilde{f}^*\tilde{P}_0 = \Delta_0$ and $\tilde{f}^*\tilde{P}_1$ (see [T1]). That is, $w(u_1, u_2)$ is a joint eigenfunction if and only if $w(u_1, u_2) = \phi_1(u_1) \cdot \phi_2(u_2)$, where $\phi_1(x)$, and $\phi_2(x)$ are certain distinguished solutions of the respective Floquet problems:

$$\frac{d^2\phi_j}{dx^2} = \{k_j^2 n(n+1) sn^2(x; k_j) - \lambda_j\}\phi_j(x) \tag{8}$$

$$\phi_j(x + 4K_j) = \phi_j(x) \qquad j = 1, 2$$

where (8) is called the Lamé equation, $k_1 = \frac{\alpha_0 - \alpha_1}{\alpha_0 - \alpha_2}$, $k_1^2 + k_2^2 = 1$, $K_j = \int_0^1 \{(1 - k_j^2 t^2)(1 - t^2)\}^{-1/2} dt$, and $sn(x; k_j)$ is the Jacobian elliptic function associated to the modulus, k_j (see [Gur], [T1] for details). $\square$

Remark 2. For $SO(n)$, $n > 3$, one encounters complex ODE with automorphic coefficients in determining the joint eigenfunctions of the f^*P_k. This is a separation of variables argument completely analogous to that which takes place in the quantum C. Neumann problem [T2]. We hope to present the analysis of these differential equations in a future paper.

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