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MÖBIUS CONE STRUCTURES ON 3-DIMENSIONAL MANIFOLDS

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Abstract. We show that for any given angle $\alpha \in (0, 2\pi)$, any closed 3-manifold has a Möbius cone structure with cone angle α.

In dimension three, there are eight complete locally homogeneous geometries. Five of them (S^3, H^3, E^3, $H^2 \times R^1$, and $S^2 \times R^1$) are naturally conformally flat. W. Goldman in [Go] showed that manifolds modeled on Sol or Nil geometry cannot have conformally flat structures. As to the last geometry $\widetilde{\mathrm{PSL}}(2, R)$, Gromov-Lawson-Thurston [GLT], Kapovich [Ka], and Kuiper [Ku] showed that some of the closed 3-manifolds modeled on $\widetilde{\mathrm{PSL}}(2, R)$ geometry have conformally flat structures.

In this note, we announce that,

Theorem 1. *For any positive $r < 2$ any closed 3-manifold M has a Riemannian metric ds (singular) of the following form. There are local coordinates (z, t) (z complex and t real) in M so that in the coordinates the metric ds is:*

(a) *conformally flat: $ds = u(z,t)(|dz|^2 + |dt|^2)$ or,*
(b) *conformally flat with cone singularity of angle $r\pi$,*

$$ds = u(z,t)\left(|dz|^2/|z|^{2-r} + |dt|^2\right)$$

where $u(z,t)$ is a smooth positive function of z, t.

Furthermore, if $r = 2/n$ for some positive integer $n > 1$, then the monodromy group is a discrete subgroup of $SO(4,1)$.

We may also state the result in terms of Möbius cone geometry as follows. Given $\alpha \in (0, 2\pi)$, an α-cone 3-sphere S^3_α is the quotient of a Euclidean lens of angle α by a rotation about the edge of the lens which identifies the two boundary half-spheres of the lens. The above result is the same as:

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Theorem 2. *Given any positive $\alpha < 2\pi$, any closed 3-manifold M has a singular conformally flat structure so that each point in M has a neighborhood which is conformal to an open set in S^3_α. Furthermore, if the given cone angle is $2\pi/n$ for some $n \in Z_+$, then the monodromy group is a discrete subgroup of $SO(4,1)$.*

The singularity forms a link in the manifold. We call the singular conformal structure a Möbius cone structure with cone angle α.

The technical details of the proof will appear elsewhere. We will describe below the basic idea of the proof. Essentially, we show that 3-dimensional Dehn surgery can be realized in Möbius cone geometry. The basic idea of the proof comes from Gromov-Lawson-Thurston's construction in [GLT].

Recall that three-dimensional Möbius geometry $(S^3, \mathrm{Mob}(S^3))$ has underlying space the unit 3-sphere or, equivalently, Euclidean 3-space together with infinity, $\bar{R}^3 = R \bigcup \{\infty\}$. The Möbius transformations are the same as those homeomorphisms which are compositions of inversions about 2-spheres. Circles and spheres are invariant under Möbius transformations. Given a circle in S^3, the *half-turn* about the circle is the orientation preserving Möbius involution leaving the circle pointwise fixed.

The following notions introduced by Kuiper [Ku] are crucial to our construction. A *necklace* in R^3 is a union of finitely many closed balls $B_1, B_2, \ldots, B_n$ in R^3 so that only adjacent index balls B_i and B_{i+1} intersect transversely (the index is counted $\mathrm{mod}(n)$). The complement of the interior of a necklace in S^3 is called a *Möbius n-gon* P. The intersection of P with a closed ball B_i is called a *side* of P and the intersection $\partial B_i \cap \partial B_{i+1}$ is called an *edge* of P. A *Möbius annulus* is a topological annulus in a sphere bounded by two disjoint circles. The *middle circle* of a Möbius annulus is the circle in it so that the inversion about the circle leaves the annulus invariant. An *orthogonal arc* in a Möbius annulus is a circular arc orthogonal to the two boundary components.

For simplicity, we will describe the idea of the proof of Theorem 2 in the case that $\alpha = \pi$ and the manifold is orientable.

It is known from the work of Lickorish [Li] that any closed orientable 3-manifold is obtained by doing ± 1-Dehn surgeries on the complement of a closed pure braid in S^3. Our goal is to realize this surgery construction in Möbius cone geometry.

We first cover each component of the braid by small balls so that their union forms a necklace with small exterior angles. These necklaces are all disjoint and form a regular neighborhood of the braid. The edges of the Möbius polygon are the meridian curves of the braids. We will start a sequence of modification in each necklace to achieve the Dehn surgery. Suppose N is such a necklace with cyclically ordered sides $A_1, A_2, \ldots, A_n$, and suppose H_i is the half-turn about the middle circle of A_i for each i. We then introduce an identification on ∂N by these half-turns, i.e., each side A_i is self-identified by H_i. The quotient space will be homeomorphic to the ± 1-Dehn surgeries on the braid if we choose the necklace suitably. To see this, take the edge $E = A_1 \cap A_n$ of the Möbius n-gon $\text{int}(N)^c$. The Möbius transformation $\phi = H_n \circ H_{n-1} \circ \cdots \circ H_1$ sends E to itself. Thus, points x, $\phi(x)$, $\phi^2(x)$, $\ldots$ in E are all identified in the quotient. Thus the quotient is a manifold if and only if ϕ is periodic, i.e., $\phi^k = \text{id}$ in E for some integer k. We require that $k = 1$:

(1) $\phi = \text{id}$ in E.

Since all edges are identified to one edge, we also need

(2) the sum of the exterior angles of N is 2π.

Assume that (1) holds. Then the quotient is homeomorphic to an integer coefficient Dehn surgery on ∂N. To see this, take $t \in E$, let $t_i = H_i \circ \cdots \circ H_1(t)$, and let c_i be the orthogonal arc in the Möbius annulus A_i with end points t_i and t_{i+1}. Then the simple closed curve $C = \bigcup_i c_i$ in ∂N is invariant under the identification and C intersects a meridian of N at one point. The identification in ∂N together with C can be described as follows. Let P be a Euclidean n-sided polygon in the plane, let $m_1, \ldots, m_n$ be the middle points of the sides of P, and let h_i be the Euclidean rotation of angle π about m_i. Then the quotient of P by the identification on ∂P induced by h_i on the i-th side containing m_i is a 2-sphere. The quotient of $P \times S^1$ under the identifications $h_i \times id$ on the $\{i\text{-th side}\} \times S^1$ in $\partial P \times S^1$ is $S^2 \times S^1$. The identification on ∂N is the same as the one on $\partial P \times S^1$ and the invariant curve C is the same as $\partial P \times \{p\}$. This shows that the quotient of $\text{int}(N)^c$ is homeomorphic to the Dehn surgery on ∂N killing C. The coefficient of the Dehn surgery is the linking number between C and the core of the necklace N. We call the linking number the *twist number* of a Möbius polygon satisfying (1).

(3) The twist number of the Möbius polygon is ± 1 (depending on the coefficient of the Dehn surgery).

Having recognized the topology of the quotient, we now modify the necklace to achieve (1), (2), (3). We first construct a necklace N so that (1) holds and the sum of the exterior angles of N is π. Suppose the twist number of $\text{int}(N)^c$ is k. To achieve (3), we will glue a regular Möbius necklace to N along one side. That condition (3) can be realized is morally supported by the Gromov-Lawson-Thurston's construction [GLT] of regular Möbius polygon. We prove that there is a regular Möbius n-gon so that (1) holds; (2) the sum of its inner angles is π ; and (3) the twist number is $k \pm 1$ (any given integer). Having constructed this regular Möbius m-gon, we now modify N so that (1) still holds, the sum of inner angles is again π, the twist number is still k, and one side S of N has the same module as the sides of the regular Möbius m-gon. By attaching this regular polygon to N along the side S, we find a necklace satisfying all the required conditions. Thus the result follows. Note that the group generated by half-turns on all of the middle circles is a discrete group by the Poincaré polyhedron theorem.

Note that if we use the periodic condition $\phi^k = \mathrm{id}$ in E in condition (1), then it corresponds to the rational coefficient Dehn-surgery.

To modify the cone angle to any arbitrary positive number $\alpha < 2\pi$, we replace the Möbius annulus A_i by the Möbius lens which is a union of two Möbius annuli A, B along one of their boundary components L at an angle $2\pi - \alpha$ so that the degree α rotation about L takes A to B.

It seems highly possible due to the solution of the Yamabe problem by Schoen [Sc] that there is a constant scalar curvature metric of the same form (a) and (b) in the conformal class of ds, and the metric is unique if the singular set is non-empty and the manifold pair (M, singular set) is not (S^3, circle). We are informed by G. Tian that such a constant scalar curvature metric exists if the integration of the scalar curvature of a given singular metric in the conformal class is non-positive.

One would hope that under some topological assumptions on M^3 these Möbius cone structures have a limit as the cone angle tends to 2π. This would lead to the existence of conformally flat structures on M^3. Unfortunately, the topological condition is that $\pi_1(M^3)$ is trivial. Indeed, any reasonable convergence of cone structures would imply the algebraic convergence of the monodromy groups. Since the monodromy group of $M^3 - S$ (S denotes the singular set) is trivial and $\pi_1(M^3 - S)$ is mapped surjectively to $\pi_1(M^3)$ under the inclusion map, thus $\pi_1(M^3) = 1$ is a necessary condition for the existence of the limit. On the other hand, one may start with a Dehn surgery description of the given 3-manifold M^3 in any conformally flat manifold N^3. The above construction still works in N^3. This produces cone structures on M^3 with non-trivial monodromy groups on $M^3 - S$.

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