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REMARK ON EXTENSIONS OF THE WATERMELON THEOREM

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In a recent work [H2] L. Hörmander has given some refinements of Kashiwara's Watermelon Theorem (see [H1], [S]). The purpose of this note is to give some slight improvements of one of the results of Hörmander, by extending a proof in [S] (based on ideas of Kashiwara, communicated to the author by P. Schapira and A. Grigis).

Let u be a distribution (or a hyperfunction) on $\mathbb{R}^n$, $n \geq 2$, with compact support K. Let $y_0 \in K$ and let $\eta_0 \in \mathbb{R}^n$, $\|\eta_0\| = 1$, be an exterior normal vector of K at y_0 in the C^2 sense. In other words, there exists a real C^2 function, defined in a neighborhood of y_0 such that $h(y_0) = 0,\ dh(y_0) = \eta_0$, and $h(y) \leq 0$ for all y in K which are in a neighborhood of y_0. After an analytic change of variables (a convexification) and a truncation away from y_0, we may assume that $(y - y_0) \cdot \eta_0|_K \leq 0$. Kashiwara's Watermelon theorem then states that if $\xi_0 \in \mathbb{R}^n \setminus \{0\},\ \lambda \in \mathbb{R},\ (y_0, \xi_0) \notin \mathrm{WF}_a(u)$, then $(y_0, \xi_0 + \lambda\eta_0) \notin \mathrm{WF}_a$. Here WF_a is the analytic wavefront set as defined for instance in [H1] or [S]. The following result is a slight refinement of the first half of Corollary 2.7 of [H2] and we refer to Remark 3 below for the corresponding improvement of the second half. See also Theorem 2.12′ of the same paper which follows from the quoted corollary by geometrical arguments.

Theorem 1. *In the above situation, assume that there is a sequence* $\eta_j \in \mathbb{R}^n$*,* $j = 1, 2, \dots,$ *with* $\eta_j \to \eta_0$*, such that*

$$\frac{\eta_j - \eta_0}{\|\eta_j - \eta_0\|} \to \nu, \qquad j \to \infty, \tag{1}$$

$$(y - y_0) \cdot \eta_j|_K - \delta_j \leq 0, \qquad \textit{where } \delta_j > 0,\ \delta_j \to 0, \tag{2}$$

$$\frac{\delta_j}{\|\eta_j - \eta_0\|^2} \to 0, \qquad j \to \infty. \tag{3}$$

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Then if $\xi_0 \neq 0$, $(y_0, \xi_0) \notin WF_a(u)$, *we have* $(y_0, \xi_0 + \lambda\nu) \notin WF_a(u)$, *for all* $\lambda \in \mathbb{R}$.

We notice that δ_j measures the distance from the point y_0 to the hyperplane $(y - y_0)\eta_j - \delta_j = 0$.

Let $H(\eta) = \sup_{y\in K} y \cdot \eta$, $\eta \in \mathbb{R}^n$, be the support function of K. For $\tau > 0$ small, $\nu \in \mathbb{R}^n$, write the affine linear functionals corresponding to tangent planes of K:

$$(\eta_0 \pm \tau\nu) \cdot y - H(\eta_0 \pm \tau\nu) = (\eta_0 \pm \tau\nu) \cdot (y - y_0 - \delta_\pm \eta_0) = 0, \qquad \delta_\pm \geq 0$$

and keep in mind that $H(\eta_0) = y_0 \cdot \eta_0$. Then,

$$\begin{aligned}\tau^{-2}&(H(\eta_0 + \tau\nu) + H(\eta_0 - \tau\nu) - 2H(\eta_0)) \\ &= \tau^{-2}((\delta_+ + \delta_-) + (\delta_+ - \delta_-)\tau\eta_0\nu) = \tau^{-2}(1 + \mathcal{O}(\tau))(\delta_+ + \delta_-),\end{aligned}$$

so this quantity tends to zero for some (τ_j, ν_j) with $\tau_j \to 0$, $\nu_j \to \nu_0$, if and only if

$$\frac{\delta_{+,j} + \delta_{-,j}}{\tau_j^2} \to 0.$$

The condition (2.9) in the result of Hörmander is the one just mentioned, while the assumption in Theorem 1 amounts to requiring that $\delta_{+,j}/\tau_j^2 \to 0$.

Proof of Theorem 1. (cf. the proof of Theorem 8.3 in [S]). Put

$$Tu(x, \lambda) = \int e^{-\lambda(x-y)^2/2} u(y)\, dy, \qquad \lambda \geq 1,\ x \in \mathbb{C}^n.$$

Then for every $R \geq 1$, $\epsilon > 0$, we have

$$|Tu(x, \lambda)| \leq C(\epsilon, R) e^{\lambda(\Phi_K(x)+\epsilon)}, \qquad |x| \leq R,\ \lambda \geq 1, \tag{4}$$

where

$$\Phi_K(x) = \sup_{y\in K} -\Re\frac{(x-y)^2}{2} = \frac{(\Im x)^2}{2} - \frac{1}{2}d(\Re x, K)^2, \tag{5}$$

and where d denotes the Euclidean distance in $\mathbb{R}^n$. In (4), (5), we can replace K by the larger set K_j: $(y - y_0) \cdot \eta_j - \delta_j \leq 0$ and we shall first assume that (η_j) is a normalized sequence. Writing $x = x_0 + z\eta_j$, $z \in \mathbb{C}$, with $x_0 =$ constant, $\Re x_0 = y_0$, we get $\Phi_j(z) = \Phi_{K_j}(x_0 + z\eta_j)$, given by:

$$\Phi_j(z) = \begin{cases} \frac{1}{2}(\Im x_0 + (\Im z)\eta_j)^2, & \Re z \leq \delta_j \\ \frac{1}{2}((\Im x_0 + (\Im z)\eta_j)^2 - (\Re z - \delta_j)^2), & \Re z \geq \delta_j. \end{cases} \tag{6}$$

Notice that Φ_j is harmonic in the second region.

Now assume that for some $\xi_0 \in \mathbb{R}^n \setminus \{0\}$:

$$(y_0, \xi_0) \notin \mathrm{WF}_a(u). \tag{7}$$

We then choose $x_0 = y_0 - i\xi_0$. By the FBI-characterization of the analytic wavefront set (see [S]), we have

$$|Tu(x,\lambda)| \le \text{Const.}e^{\lambda(\Phi_0(x)-1/C_0)}, \qquad |x-x_0| \le 1/C_0, \tag{8}$$

for some $C_0 > 0$, where $\Phi_0(x) = \frac{1}{2}(\Im x)^2$. Consider the subharmonic function,

$$v_j(z) = \log|Tu(x_0 + z\eta_j, \lambda)| - \lambda\Psi_j(z), \tag{9}$$

where we define $\Psi_j(z)$ to be the harmonic function on $\mathbb{C}$, given by the second expression in (6). With new constants independent of j, we then obtain for $|z|_\infty \le R$, where $|z|_\infty = \max(|\Re z|, |\Im z|)$, and for j large enough:

$$v_j(z) \le -\lambda/C_0, \qquad \text{for } |z|_\infty \le 1/C_0, \tag{10}$$

$$v_j(z) \le \lambda\Big(\epsilon + \frac{(\delta_j - \Re z)^2}{2}\Big) + C(\epsilon, R), \quad \text{for } \Re z \le \delta_j, \tag{11}$$

$$v_j(z) \le \lambda\epsilon + C(\epsilon, R), \qquad \Re z \ge \delta_j. \tag{12}$$

The Poisson integral,

$$I(z) = \frac{1}{C_0\pi}\int_{-1/C_0}^{1/C_0} \frac{y}{y^2 + (x-t)^2}dt, \qquad z = x + iy, \tag{13}$$

is harmonic in the upper half plane with boundary value $\frac{1}{C_0}1_{[-\frac{1}{C_0},\frac{1}{C_0}]}$ on the real axis. Moreover,

$$I(z) = \mathcal{O}(\frac{1}{|z|}), \qquad |z| \to \infty, \tag{14}$$

and

$$I(z) \ge \frac{C_1}{C_0^2}\frac{\Im z}{|z|^2}, \qquad |z|_\infty \ge \frac{1}{2C_0},\ C_1 > 0. \tag{15}$$

For $0 \le t \le \frac{1}{C_0}$, we let $J(z) = I(i(z+t))$, $\Re z \ge -t$. Then by the maximum principle for the subharmonic function $v_j(z) + \lambda J(z)$, we get

$$v_j(z) \le \lambda\Big(- J(z) + \frac{(\delta_j + t)^2}{2} + \epsilon + \mathcal{O}(\frac{1}{R})\Big) + C(\epsilon, R), \qquad \Re z \ge -t,\ |z|_\infty \le R. \tag{16}$$

Combining this with (15) and taking $R \sim \frac{1}{\epsilon}$, we get with a new constant C_1, for $z = iy$, $|y| \geq \frac{1}{C_0}$:

$$v_j(iy) \leq \lambda\big(-\frac{1}{C_1}\frac{t}{|y|^2} + \frac{(\delta_j + t)^2}{2} + C_1\epsilon\big) + C(\epsilon). \tag{17}$$

Here we write,

$$-\frac{1}{C_1}\frac{t}{|y|^2} + \frac{(\delta_j + t)^2}{2} = \frac{1}{2}(\delta_j - t)^2 - (\frac{1}{C_1|y|^2} - 2\delta_j)t,$$

which is negative for $t = \delta_j$, $|y|^2 \leq \frac{1}{3C_1\delta_j}$. Hence,

$$v_j(iy) \leq -\lambda\epsilon_j + C_j, \qquad |y| \leq \frac{1}{C_2\sqrt{\delta_j}} \tag{18}$$

for some $\epsilon_j > 0$. In this argument, we are allowed to vary x_0 slightly within some j-dependent ball, and hence we obtain (for j large enough):

$$|Tu(x - i\tau\eta_j, \lambda)| \leq C_j e^{\lambda(\Phi_0(x - i\tau\eta_j) - \epsilon_j)}, \tag{19}$$

for $|x - x_0| \leq \epsilon_j$, $|\tau| \leq \frac{1}{C_0\sqrt{\delta_j}}$, where $\epsilon_j > 0$. Here C_0 depends on u but is independent of j. (19) shows that $(y_0, \xi_0 + \tau\eta_j) \notin \mathrm{WF}_a$ for the same τ's. In this conclusion we may replace ξ_0 by ξ if $\xi - \xi_0$ is small enough: $|\xi - \xi_0| \leq 1/C_0$. Also, if we now drop the assumption that the η_j are normalized, we get the same conclusion, since we can replace η_j by $\eta_j/|\eta_j|$. Letting $j \to \infty$ we then first recover the ordinary Watermelon theorem. In particular, if ξ_0 is a multiple of η_0, then we get $(y_0, \eta) \notin \mathrm{WF}_a(u)$ for all small η and hence for all η. If ξ_0 is not a multiple of η_0, we have $\xi + \tau_j\eta_j \neq 0$ for j large enough and applying the Watermelon theorem, we get

$$(y_0, \xi + \tau_j(\eta_j - \eta_0)) \notin \mathrm{WF}_a(u), \quad \text{for } |\xi - \xi_0| \leq \frac{1}{C_0},\ |\tau_j| \leq \frac{1}{C_0\sqrt{\delta_j}}, \tag{20}$$

for some sufficiently large $C_0 > 0$ depending on u, ξ_0. Let $\sigma \in \mathbb{R}$. Using the assumptions (1), (3), we get $\sigma\nu = \lim_{j\to\infty}\tau_j(\eta_j - \eta_0)$, where $\tau_j = \frac{\sigma}{\|\eta_j - \eta_0\|}$, so that $|\tau_j| = o(1)\frac{1}{\sqrt{\delta_j}}$, $j \to \infty$. We can then apply (20) to conclude that $(y_0, \xi_0 + \sigma\nu) \notin \mathrm{WF}_a(u)$. $\square$

Remark 2. Let $L \subset \mathbb{R}^n$ be a subspace and assume that $\xi \in \mathbb{R}^n \setminus \{0\}$, $\eta \in L$, $(y_0, \xi) \notin \mathrm{WF}_a(u)$ implies that $(y_0, \xi + \eta) \notin \mathrm{WF}_a(u)$. Consider a sequence

$\eta_j \in \mathbb{R}^n$ with $0 \neq \mathrm{dist}(\eta_j, L) \to 0$, $\|\eta_j\| = 1$ such that (1′), (2), (3′) hold with

$$\frac{\eta_j - \pi_L(\eta_j)}{\|\eta_j - \pi_L(\eta_j)\|} \to \nu, \tag{1′}$$

$$\frac{\delta_j}{\|\eta_j - \pi_L(\eta_j)\|^2} \to 0, \tag{3′}$$

where π_L is a linear projection onto L_0. Then the proof above gives the same conclusion as in Theorem 1.

Remark 3. The proof of Theorem 1 also gives a slight improvement of the second half of Corollary 2.7 of [H2]. We now assume that there are sequences η_j, $\eta_{0,j}$ converging to $\eta_0 \neq 0$, that $\eta_{0,j}$ is an exterior normal of K at $y_{0,j}$ with $y_{0,j} \to y_0$, and that

$$\frac{\eta_j - \eta_{0,j}}{\|\eta_j - \eta_{0,j}\|} \to \nu_0, \qquad (y - y_{0,j}) \cdot \eta_j|_K - \delta_j \leq 0, \qquad \delta_j \|\eta_j - \eta_{0,j}\|^{-2} \to 0,$$

where $\delta_j > 0$. Let $\xi_0 \neq 0$ be non-parallel to η_0 and assume that $(y_0, \xi_0) \notin \mathrm{WF}_a(u)$. Then for j sufficiently large, we have $(y_{0,j}, \xi_0) \notin \mathrm{WF}_a(u)$, and instead of (20) we get

$$\Big(y_{0,j}, \xi + \sigma_j \frac{\eta_j - \eta_{0,j}}{\|\eta_j - \eta_{0,j}\|}\Big) \notin \mathrm{WF}_a(u),$$
$$\text{for } |\xi - \xi_0| \leq \frac{1}{C_0}, \quad |\sigma_j| \leq \frac{1}{C_0} \frac{\|\eta_j - \eta_{0,j}\|}{\sqrt{\delta_j}} \to \infty.$$

In particular, if $\xi_0 = \pm\nu_0$, we deduce that $(y_{0,j}, \widetilde{\xi}) \notin \mathrm{WF}_a(u)$ for all small $\widetilde{\xi}$, and hence for all $\widetilde{\xi}$, in contradiction with the fact that $y_{0,j} \in \partial\, \mathrm{supp}(u)$. In conclusion, $(y_0, \pm\nu_0) \in \mathrm{WF}_a(u)$.

We next discuss an extension of Theorem 1, using some 2-microlocal techniques. For $\mu \in]0, \frac{1}{2}]$, we put $u_\mu(y) = u(\sqrt{\mu}\, y)$ and compute

$$Tu_\mu(x, \lambda) = \int e^{-\lambda(x-y)^2/2} u(\sqrt{\mu}\, y)\, dy$$
$$= C_n \lambda^n \iint e^{-\frac{\lambda}{2}((x-y)^2 - (\sqrt{\mu} y - z)^2)} Tu(z, \lambda)\, dy\, dz,$$

using integration contours as in [S], chapter 3,4 (and 16). Here

$$(x-y)^2 - (\sqrt{\mu}\, y - z)^2 = (1-\mu)(y - \frac{x - \sqrt{\mu}\, z}{1-\mu})^2 + x^2 - z^2 - \frac{(\sqrt{\mu}\, z - x)^2}{1-\mu},$$

so we can eliminate the y-variable and get

$$Tu_\mu(x,\lambda) = \widetilde{C}_n \frac{\lambda^{n/2}}{(1-\mu)^{n/2}} \int e^{-\frac{\lambda}{2}(x^2-z^2-\frac{(\sqrt{\mu}\,z-x)^2}{1-\mu})} Tu(z,\lambda)\,dz. \tag{21}$$

We know that $Tu \mapsto Tu_\mu$: $H_{\Phi_0} \to H_{\Phi_0}$, where

$$H_{\Phi_0} = \{u \in L^2(\mathbb{C}^n; e^{-2\lambda\Phi_0}L(dz)); u \text{ is holomorphic}\}, \quad \Phi_0(x) = \frac{1}{2}(\Im x)^2,$$

where $L(dz)$ denotes the Lebesgue measure on $\mathbb{C}^n$, and in order to make this more explicit, we write

$$\begin{aligned} -\frac{1}{2}(\Im x)^2 + \frac{1}{2}(\Im z)^2 - \frac{1}{2}\Re\left(x^2 - z^2 - \frac{(\sqrt{\mu}\,z-x)^2}{1-\mu}\right) \\ = \frac{1}{2}\left(\frac{(\Re z - \sqrt{\mu}\,\Re x)^2}{1-\mu} - \frac{(\sqrt{\mu}\,\Im z - \Im x)^2}{1-\mu}\right), \end{aligned} \tag{22}$$

so the correct integration contour in (21) is given by $\Re z = \sqrt{\mu}\,\Re x$. Moreover, if (7) holds and $\xi_0 \neq 0$, then for every $\epsilon > 0$:

$$\begin{aligned} |Tu_\mu(x,\lambda)| \le C(\epsilon,\mu) e^{\frac{\lambda}{2}((\Im x)^2 - \frac{\mu}{C} + \epsilon)}, \\ \text{for } \left|\Re x - \frac{1}{\sqrt{\mu}} y_0\right| \le \frac{1}{C\sqrt{\mu}}, \quad |\Im x + \sqrt{\mu}\,\xi_0| \le \frac{\sqrt{\mu}}{C}, \end{aligned} \tag{23}$$

where $C > 0$ is some constant depending on u, ξ_0, but not on μ, λ. (In particular, $(\frac{1}{\sqrt{\mu}} y_0, \sqrt{\mu}\,\xi_0) \notin \mathrm{WF}_a(u_\mu)$.)

Keeping the assumption (7) we assume for simplicity that $y_0 = 0$. Let κ be an analytic diffeomorphism between two neighborhoods of 0 with $\kappa(0) = 0$, $\kappa'(0) = \mathrm{id}$. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be 1 near 0 and have its support close to 0. Put $\widetilde{u}_\mu = \chi(u_\mu \circ \kappa)$ when u is a distribution and in the hyperfunction case, let $\widetilde{u}_\mu$ be an analytic functional with uniformly compact support, $= u_\mu \circ \kappa$ in some fixed neighborhood of 0. It then follows from (23) (as in [S]), that

$$|T\widetilde{u}_\mu(x,\lambda)| \le C(\epsilon,\mu) e^{\frac{\lambda}{2}((\Im x)^2 - \frac{\mu}{C} + \epsilon)}, \quad \text{for } |\Re x| \le \frac{1}{C},\ |\Im x + \sqrt{\mu}\,\xi_0| \le \frac{\sqrt{\mu}}{C}. \tag{24}$$

Assume that $0 \in \operatorname{supp} u$, $\eta_0 \in \mathbb{R}^n$, $\|\eta_0\| = 1$, and that there is a sequence of open balls $B_j \subset \mathbb{R}^n \setminus \operatorname{supp} u$, $j = 1, 2, \dots$ of radius $\sqrt{\mu_j} > 0$ with

$$\mu_j \to 0, \tag{25}$$

(26) The center of B_j is given by $\delta_j\eta_0 + \sqrt{\mu_j}\,\eta_j$,

so that $\|\eta_j\| = 1$,

$$\eta_j \to \eta_0, \qquad \frac{\delta_j}{\mu_j} \to 0. \tag{27}$$

After a Euclidean change of coordinates, we may assume that $\eta_0 = (0, \dots, 1)$. Let κ be the map $\widetilde{y} \mapsto (\widetilde{y}', \widetilde{y}_n + R(\widetilde{y}')^2)$ for some sufficiently large $R > 0$ and put $y = \sqrt{\mu_j}\,\widetilde{y}$. Then we have a ball $\widetilde{B}_j$ of radius 1 in $\mathbb{R}^n \setminus \operatorname{supp} u_{\mu_j}$ of center $\widetilde{\delta}_j\eta_0 + \eta_j$, where $\widetilde{\delta}_j = \frac{\delta_j}{\sqrt{\mu_j}}$ tends to 0 by (25), (27). For $\widetilde{u}_{\mu_j}$ we get (assuming it has its support in a small but fixed neighborhood of 0):

$$(\widetilde{y}\eta_j - \delta_j')|_{\operatorname{supp} u_{\mu_j}} \le 0, \qquad \text{with } \delta_j' = \widetilde{\delta}_j\eta_0\eta_j. \tag{28}$$

Put

$$x_0 = -i\xi_0, \qquad \widetilde{x}_0 = \sqrt{\mu_j}\,x_0, \qquad \widetilde{v}_j(z) = \log(T\widetilde{u}_\mu(\widetilde{x}_0 + z\eta_j, \lambda)) - \lambda\widetilde{\Psi}_j(z),$$

where $\widetilde{\Psi}_j$ is defined as Ψ_j but with x_0, δ_j replaced by $\widetilde{x}_0$, δ_j'. Then $\widetilde{v}_j$ satisfies the following estimates for $|z| \le R$ (suppressing sometimes the parameters λ, μ):

$$\widetilde{v}_j(z) \le -\tfrac{\lambda\mu_j}{C_0} + C(j), \qquad |z|_\infty \le \frac{\sqrt{\mu_j}}{C_0}, \tag{29}$$

$$\widetilde{v}_j(z) \le \lambda(\mu_j\epsilon + (\delta_j' - \Re z)^2) + C(\epsilon, R, j), \qquad \Re z \le \delta_j', \tag{30}$$

$$\widetilde{v}_j(z) \le \lambda\mu_j\epsilon + C(\epsilon, R, j), \qquad \Re z \ge \delta_j'. \tag{31}$$

Put $z = \sqrt{\mu_j}\,\widehat{z}$, $\delta_j' = \sqrt{\mu_j}\,\widehat{\delta}_j$, $(\widehat{\delta}_j = \frac{\delta_j}{\mu_j}\eta_0 \cdot \eta_j)$, $\widehat{v}_j(\widehat{z}) = \widetilde{v}_j(z)$. Then

$$\widehat{v}_j(\widehat{z}) \le -\tfrac{\lambda\mu_j}{C_0} + C(j), \qquad |\widehat{z}|_\infty \le \frac{1}{C_0}, \tag{32}$$

$$\widehat{v}_j(\widehat{z}) \le \lambda\mu_j(\epsilon + (\widehat{\delta}_j - \Re\widehat{z})^2) + C(\epsilon, R, j), \qquad \Re\widehat{z} \le \widehat{\delta}_j, \tag{33}$$

$$\widehat{v}_j(\widehat{z}) \le \lambda\mu_j\epsilon + C(\epsilon, R, j), \qquad \Re\widehat{z} \ge \widehat{\delta}_j. \tag{34}$$

These are just the estimates (10)–(12) with λ, δ_j, z replaced by $\lambda\mu_j$, $\widehat{\delta}_j$, $\widehat{z}$, so as before, we reach the conclusion that $(0, \sqrt{\mu_j}\,\xi_0 + \tau\sqrt{\mu_j}\,\eta_j) \notin \mathrm{WF}_a(u)$, if $|\tau| \le C_2/\sqrt{\widehat{\delta}_j}$, i.e.:

$$(0, \xi_0 + \tau\eta_j) \notin \mathrm{WF}_a(u), \qquad \text{if } |\tau| \le \frac{C_2\sqrt{\mu_j}}{\sqrt{\delta_j}}. \tag{35}$$

In (35) we can replace ξ_0 by ξ with $|\xi-\xi_0| \le \frac{1}{C_0}$ with $C_0 > 0$ independent of j. Since $\eta_j \to \eta_0$, $\mu_j/\delta_j \to \infty$, by (27), we obtain $(0, \xi+\tau\eta_0) \notin \mathrm{WF}_a(u)$, $\tau \in \mathbb{R}$, $|\xi-\xi_0| \le 1/C_0$, which is the conclusion of the ordinary Watermelon theorem, in the case when η_0 is an exterior normal.

Assuming ξ_0, η_0 to be linearly independent (which we may, as in the proof of Theorem 1), we then get from (35)

$$(0, \xi + \tau(\eta_j - \eta_0)) \notin \mathrm{WF}_a(u), \qquad \text{if } |\tau| \le \frac{C_2\sqrt{\mu_j}}{\sqrt{\delta_j}},\ |\xi-\xi_0| \le \frac{1}{C_0}. \tag{36}$$

As for Theorem 1, we get,

Theorem 4. *Let* $0 \in \operatorname{supp} u$, $\eta_0 \in \mathbb{R}^n$, $\|\eta_0\| = 1$, *and let* $B_j \subset \mathbb{R}^n \setminus \operatorname{supp} u$, $j = 1, 2, \dots$ *be a sequence of open balls of radius* $\sqrt{\mu_j}$ *satisfying* (25)–(27). *Then if* $\xi_0 \ne 0$, $(0, \xi_0) \notin WF_a(u)$, *we have* $(0, \xi_0 + \lambda\eta_0) \notin WF_a(u)$ *for every* $\lambda \in \mathbb{R}$. *In particular* (*by a standard argument*), $(0, \pm\eta_0) \in WF_a(u)$.

Assume further that

$$\frac{\eta_j - \eta_0}{\|\eta_j - \eta_0\|} \to \nu_0, \qquad \frac{\sqrt{\mu_j}}{\sqrt{\delta_j}}\|\eta_j - \eta_0\| \to \infty. \tag{37}$$

Then if $(0, \xi_0) \notin WF_a(u)$, *we have* $(0, \xi_0 + \lambda\nu_0) \notin WF_a(u)$ *for every* $\lambda \in \mathbb{R}$.

It is easy to construct examples where Theorem 2 applies and where Theorem 1 cannot be applied. If we assume however that η_0 is an exterior normal of $\operatorname{supp} u$ at 0 in the C^2 sense, then the following discussion shows that Theorem 2 gives nothing more than Theorem 1. For simplicity, we assume that $n = 2$ and that $0 \in \operatorname{supp} u \subset \{(x, y); y \le 0\}$, $\eta_0 = (0, 1)$. Instead of balls, we can consider parabolic regions of the form B_j: $y > \frac{1}{2\sqrt{\mu_j}}(x - x_+)(x - x_-) \overset{\text{def}}{=} f_j(x)$, where $x_\pm = x_\pm(j)$, and $0 < x_- < x_+ \to 0$, $\mu_j \to 0$, in a ball of radius $\sqrt{\mu_j}$. Then $\delta_j = x_+x_-/2\sqrt{\mu_j}$ and $\|\eta_j - \eta_0\|$ is of the order of magnitude $-f_j'(0) = (x_+ + x_-)/2\sqrt{\mu_j} \sim x_+/\sqrt{\mu_j}$. In order to apply Theorem 2, we need

$$\frac{x_+x_-}{\mu_j^{3/2}} \to 0, \qquad \frac{x_+\sqrt{\mu_j}}{\sqrt{\mu_j}\sqrt{\frac{x_+x_-}{\sqrt{\mu_j}}}} = \sqrt{\frac{x_+}{x_-}}\mu_j^{1/4} \to \infty.$$

These conditions remain valid if we replace μ_j by 1, so whenever Theorem 2 applies (in this case), Theorem 1 applies also.

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