Mathematical Research Letters **1**, 481–484 (1994)

ELEMENTARY EXTENSIONS TO THEOREMS OF THE ALTERNATIVE

Irving H. LaValle and Peter C. Fishburn

Abstract. Extensions of Motzkin's and Tucker's Theorems of the Alternative that address multiple sets of relationships in the (I) alternatives and several variously constrained vector variables in the (II) alternatives are derived by elementary means.

Motzkin's (1936) and Tucker's (1956) Theorems of the Alternative are crucial to the derivation of many admissibility-*versus*-dominance alternatives in individual decision theory [see, e.g., Fishburn (1964)] and game theory [see, e.g., Myerson (1991)], and of similar useful alternatives in many areas of application. This note extends these theorems to address multiple sets of relations in the (I) alternative and several vector variables in the (II) alternatives.

In $\mathbb{R}^n$, we write $x \geqq y$, $x = y$, $x > y$, and $x \geq y$ to mean that $(\forall i)\, x_i \geqq y_i$, $(\forall i)\, x_i = y_i$, $(\forall i)\, x_i > y_i$, and $[x \geqq y \;\&\; x \neq y]$ respectively. Let M^T denote the transpose of M, abbreviate "exclusive or" to "xor," and assume as given n-column matrices A, B, C, and D with A and B nonvacuous. Then [see also, e.g., Mangasarian (1969), Panik (1993)]

Motzkin: either (I) $\exists x \ni Ax > \mathbf{0}$, $Cx \geqq \mathbf{0}$, $Dx = \mathbf{0}$

xor (II) $\exists s \geq \mathbf{0}$, $z \geqq \mathbf{0}$, t (unrestricted) $\ni A^T s + C^T z + D^T t = \mathbf{0}$; and

Tucker: either (I) $\exists x \ni Bx \geq \mathbf{0}$, $Cx \geqq \mathbf{0}$, $Dx = \mathbf{0}$

xor (II) $\exists p > \mathbf{0}$, $z \geqq \mathbf{0}$, t (unrestricted) $\ni B^T p + C^T z + D^T t = \mathbf{0}$.

Extension of the relation alternative (I) of Motzkin so that it pertains to several matrices A_i, C_j, and D_k is trivial: form the "tall" matrices A, C, and D by stacking the matrices of the same main symbol on top of each other, and then apply Motzkin's Theorem as stated to obtain the (II)

Key words and phrases. Alternative, theorems of.
Received January 31, 1994 and revised April 10, 1994.

alternative. Similarly, extension of the equation alternative (II) of Tucker is trivial: $\exists p_i > \mathbf{0}, z_j \geqq \mathbf{0}, t_k \ni \sum_i B_i^T p_i + \sum_j C_j^T z_j + \sum_k D_k^T t_k = \mathbf{0} \Leftrightarrow B^T p + C^T z + D^T t = \mathbf{0}$ for $B^T = [B_1^T, \ldots, B_I^T]$, $p^T = [p_1^T, \ldots, p_I^T]$, etc., and the alternative to this is Tucker's (I) as stated for the "stacked" matrices B, C, and D.

Still elementary, but not quite so trivial, are the multivariate extensions to Motzkin's (II) for several semipositive variables $s_i \geq \mathbf{0}$ and summands $A_i^T s_i$, and to Tucker's (I) for several semipositive blocks $B_i x \geq \mathbf{0}$. Since the other terms in Motzkin's (II) and Tucker's (I) are readily dealt with by the "stacking" described in the preceding paragraph, the following extensions concern only the semipositive terms. To make the notation compact, we use a common convention that if $M^T = [M_1^T, \ldots, M_I^T]$, then M_{-i}^T is defined to be $[M_1^T, \ldots, M_{i-1}^T, M_{i+1}^T, \ldots, M_I^T]$.

Proposition 1. [*Extended Motzkin Theorem.*] *Either* (II) $\exists s_1 \geq \mathbf{0}, \ldots, s_I \geq \mathbf{0}$, $z \geqq \mathbf{0}$, t (unrestricted) $\ni \sum_{i=1}^I A_i^T s_i + C^T z + D^T t = \mathbf{0}$ *xor* (I) $\exists i \in \{1, \ldots, I\}\ \exists x \in \mathbb{R}^n \ni A_i x > \mathbf{0}$, $A_{-i} x \geqq \mathbf{0}$, $Cx \geqq \mathbf{0}$, $Dx = \mathbf{0}$.

Proof. We show, first, that (II) is equivalent to
(II′) $\forall i\, \exists y_i^{(i)} \geq \mathbf{0}$, $(\forall h \neq i)\, y_h^{(i)} \geqq \mathbf{0}$, $z^{(i)} \geqq \mathbf{0}$, $t^{(i)} \ni$

$$(*) \qquad \sum_{h=1}^{I} A_h^T y_h^{(i)} + C^T z^{(i)} + D^T t^{(i)} = \mathbf{0}:$$

for "(II) $\Rightarrow$ (II′)," note that any solution to (II) satisfies $(*)$ for every i with $z^{(i)} = z$, $t^{(i)} = t$, and $y_h^{(i)} = s_h (h = 1, \ldots, I)$, since $v \geq \mathbf{0} \Rightarrow v \geqq \mathbf{0}$. For "(II′) $\Rightarrow$ (II)," the sum of solutions to a homogeneous equation is also a solution, so that setting $z = \sum_i z^{(i)}$, $t = \sum_i t^{(i)}$, and $s_h = \sum_i y_h^{(i)}$ produces a solution to (II) from the individual-i solutions to $(*)$, with each $s_h \geq \mathbf{0}$ because $y_h^{(h)} \geq \mathbf{0}$ and $y_h^{(i)} \geqq \mathbf{0}$ for all $i \neq h$.

Given the equivalence of (II) and (II′), failure of (II) is equivalent to existence of some i such that $(*)$ has no solution in appropriately constrained variables. That is, for some i, $\nexists\ s \geq \mathbf{0}$, $y \geqq \mathbf{0}$, $z \geqq \mathbf{0}$, $t \ni A_i^T s + [A_{-i}^T C^T][y^T z^T]^T + D^T t = \mathbf{0}$. By Motzkin's Theorem, this nonexistence entails existence of x such that $A_i x > \mathbf{0}$, $A_{-i} x \geqq \mathbf{0}$, $Cx \geqq \mathbf{0}$, $Dx = \mathbf{0}$. $\square$

Proposition 2. [*Extended Tucker Theorem.*] *Either* (I) $\exists x \in \mathbb{R}^n \ni B_1 x \geq \mathbf{0}, \ldots, B_I x \geq \mathbf{0}$, $Cx \geqq \mathbf{0}$, $Dx = \mathbf{0}$, *xor* (II) $\exists i \in \{1, \ldots, I\}\ \exists p_i > \mathbf{0}$, $z_{-i} \geqq \mathbf{0}$, $z \geqq \mathbf{0}$, t (unrestricted) $\ni B_i^T p_i + B_{-i}^T z_{-i} + C^T z + D^T t = \mathbf{0}$.

Proof. First, we show that (I) is equivalent to
(I′) $\forall i\, \exists x_i \ni B_i x_i \geq \mathbf{0}$, $(\forall h \neq i)\, B_h x_i \geqq \mathbf{0}$, $Cx_i \geqq \mathbf{0}$, $Dx_i = \mathbf{0}$:

for "(I) $\Rightarrow$ (I′)," take $x_i = x$ for every i and note that $B_h x_i \geq \mathbf{0} \Rightarrow B_h x_i \geqq \mathbf{0}$. For "(I′) $\Rightarrow$ (I)," define $x = \sum_{i=1}^{I} x_i$. Then $Cx \geqq \mathbf{0}$ and $Dx = \mathbf{0}$ since $(\forall i)\, Cx_i \geqq \mathbf{0}$ and $Dx_i = \mathbf{0}$, and $B_i x = B_i(x_i + \sum_{h \neq i} x_h) = B_i x_i + \sum_{h \neq i} B_i x_h \geq \mathbf{0}$.

Given (I) $\Leftrightarrow$ (I′), failure of (I) means that, for some i, $[B_i x \geq \mathbf{0}, (\forall h \neq i) B_h x \geqq \mathbf{0}, Cx \geqq \mathbf{0}, Dx = \mathbf{0}]$ has no solution. By Tucker's Theorem, this is the case if and only if $\exists p_i > \mathbf{0}$, $(\forall h \neq i)\, z_h \geqq \mathbf{0}$, $z \geqq \mathbf{0}$, t (unrestricted) $\ni B_i^T p_i + \sum_{h \neq i} B_h^T z_h + C^T z + D^T t = \mathbf{0}$. Rewriting the term involving the z_h's in partitioned-matrix notation concludes the proof. □

At the cost of some clumsiness, these arguments can be applied to extend the more general linear theorems of the alternative; i.e., Slater's (1951) theorem and "Alternative #4" of Mangasarian (1969, p. 34). The basic insights, however, are conveyed by the Motzkin and Tucker Theorems: multiple semipositive relations [in the (I) alternative] are treated as the *joint* satisfaction of all profiles consisting of one semipositive and $I - 1$ nonnegative relations, and multiple semipositive variables [in the (II) alternative] are treated as multiple equations all having solutions, each with one semipositive variable and $I - 1$ nonnegative variables.

Linearity of the transformations $x \to Ax$, etc. is not necessary to our extension arguments, as a referee has shown by obtaining a nonlinear version of Proposition 1 that extends a special case of Jeyakumar's (1985) Theorem 5.1.

Acknowledgements

We thank Michael J. Panik for useful advice and encouragement, and a referee for showing that our arguments pertain to the extension of nonlinear Theorems of the Alternative. Support for LaValle from the Freeman Professorship for Doctoral Studies and Research and the Francis Martin Chair is gratefully acknowledged.

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A. B. Freeman School of Business, Tulane University, New Orleans, LA 70118-5669

E-mail address: lesquerre@office.sob.tulane.edu

AT&T Bell Laboratories, Murray Hill, NJ 07974-0636

E-mail address: jes@research.at.com