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GLOBAL ANALYTIC HYPOELLIPTICITY OF A CLASS OF DEGENERATE ELLIPTIC OPERATORS ON THE TORUS

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0. Introduction

In this paper we prove global analytic hypoellipticity on the torus for certain classes of operators in the form of a sum of squares of vector fields with real-valued and real analytic coefficients. The main conditions are that every point of the torus is of finite type for the vector fields and that the coefficients are independent of some of the variables. We show, for instance, that the well known not locally analytic hypoelliptic operator of Baouendi and Goulaouic [1] is globally analytic hypoelliptic on the torus. Also, we show that the operators that were proved in Hanges-Himonas [11] and in Christ [6], [7] to be not locally analytic hypoelliptic, are globally analytic hypoelliptic.

Global analytic hypoellipticity of sum of squares operators on a compact analytic manifold M is an open problem except for certain cases. From the complex analysis point of view, the most important situation is when M is the boundary of a domain in $\mathbb{C}^n$ and the operator is Kohn's Laplacian, $\Box_b$, for M. In the symplectic case the global analytic hypoellipticity of $\Box_b$ follows from the fundamental result of Tartakoff [24], and Treves [26]. In fact, in this case $\Box_b$ is locally analytic hypoelliptic (see also Metivier [18] and Sjöstrand [22]). Some other, very interesting, partial results on the problem of global analytic hypoellipticity can be found in Chen [4], Derridj [8], Derridj and Tartakoff [9] and Komatsu [17]. It is already apparent from the work of the authors just mentioned above, and in particular the work of Chen [4,5], that $\Box_b$ can be globally analytic hypoelliptic but fail to be locally analytic hypoelliptic.

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1. Statement of results

Let $\mathbb{T}^N = \mathbb{R}^N/2\pi\mathbb{Z}^N$ be the N-dimensional torus. We shall denote by $\mathcal{A}(\mathbb{T}^N)$ the space of all real-analytic functions on $\mathbb{T}^N$. The elements of $\mathcal{A}(\mathbb{T}^N)$ can be identified with the real-analytic functions on $\mathbb{R}^N$ that are 2π-periodic in each variable. The spaces $C^\infty(\mathbb{T}^N)$, $\mathcal{D}'(\mathbb{T}^N)$ are defined in a similar way.

Here we shall consider operators P on $\mathbb{T}^N$ of the following form:

$$P = \sum_{j=1}^{\nu} X_j^2, \tag{1.1}$$

where $X_1, \ldots, X_\nu$ are real vector fields with coefficients in $\mathcal{A}(\mathbb{T}^N)$.

We assume that every point $y \in \mathbb{T}^N$ is of *finite type* for the vector fields $X_1, \ldots, X_\nu$, i.e. at every point $y \in \mathbb{T}^N$ the Lie algebra $\mathcal{L}(X_1, \ldots, X_\nu)$ spans the tangent space $T_y(\mathbb{T}^N)$. We write $\mathbb{T}^N$ as a product of two tori $\mathbb{T}^N = \mathbb{T}^m \times \mathbb{T}^n$ and split the variable y as $y = (x, t)$, where $x = (x_1, \ldots, x_m) \in \mathbb{T}^m$, $t = (t_1, \ldots, t_n) \in \mathbb{T}^n$. If v is a (smooth) function in $\mathbb{T}^{m+n}$ then $\hat{v}(\xi, t)$ will denote its Fourier transform with respect to the variable x,

$$\hat{v}(\xi, t) = \frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} v(x, t) e^{-ix.\xi} dx, \quad \xi \in \mathbb{Z}^m.$$

We will also denote by $(\ ,\ )$ and $\|\ \ \|$ respectively the inner product and the corresponding norm in $L^2(\mathbb{T}^n)$ and by $\|\ \ \|_{(s)}$ the norm in the Sobolev space $H^s(\mathbb{T}^N)$.

We now state our main results. We recall that P is *globally analytic hypoelliptic* on $\mathbb{T}^N$ if the conditions $u \in \mathcal{D}'(\mathbb{T}^N)$, $Pu \in \mathcal{A}(\mathbb{T}^N)$ imply $u \in \mathcal{A}(\mathbb{T}^N)$.

Theorem 1.1. *Let*

$$X_j = \sum_{q=1}^{n} a_{jq}(t) \frac{\partial}{\partial t_q} + \sum_{k=1}^{m} b_{jk}(t) \frac{\partial}{\partial x_k}, \qquad j = 1, \ldots, \nu, \tag{1.2}$$

with a_{jq}, $b_{jk} \in \mathcal{A}(\mathbb{T}^n)$ and real valued. Assume that:

(i) *every point in $\mathbb{T}^{m+n}$ is of finite type for $X_1, \ldots, X_\nu$;*
(ii) *$\sum a_{jq}(t)\partial/\partial t_q$, $j = 1, \ldots, \nu$, span $T_t(\mathbb{T}^n)$ for every t.*

Then P given by (1.1) *is globally analytic hypoelliptic in $\mathbb{T}^{m+n}$.*

Remark. It is not difficult to prove, following the same lines of the proofs presented in Sections 2 and 3, that similar results hold true for more general classes of operators. For instance, on a torus with coordinates (x, t) as above, any linear operator with real analytic coefficients depending only on t, elliptic in the t directions and satisfying an a-priori inequality as (2.2) is necessarily globally analytic hypoelliptic. We leave the details to the interested reader.

It follows from Theorem 1.1 that the operator in $\mathbb{T}^3$ given by

$$P = \partial_{t_1}^2 + [\partial_{t_2} - a(t_1)\partial_x]^2 \tag{1.3}$$

is globally analytic hypoelliptic in $\mathbb{T}^3$ if the function a is in $\mathcal{A}(\mathbb{T}^1)$, real valued and not constant on $\mathbb{T}^1$. We mention that the operator (1.3) is not in general locally analytic hypoelliptic. In [11] it was proved that when $a(t_1) = t_1^{k-1}$, $k = 3, 5, 7, \dots$ then P is not analytic hypoelliptic at 0. In [6] Christ extended this result for $k \geq 3$ and in [7] he extended it for any analytic function $a(t_1)$ with $a(0) = a'(0) = 0$. The case $k = 3$ was first done by Helffer [13] and Pham The Lai-D. Robert [20].

It also follows from Theorem 1.1 that the following generalized Baouendi-Goulaouic operator

$$P = \partial_t^2 + \partial_{x_1}^2 + a^2(t)\partial_{x_2}^2, \tag{1.4}$$

is globally analytic hypoelliptic in $\mathbb{T}^3$, if a is a real valued analytic function which is not identically equal to zero. In fact the following theorem states that this operator is analytic hypoelliptic for "global x" and "local t".

Theorem 1.2. *Let $a \in \mathcal{A}(\mathbb{T}^1)$ be real valued and not identically 0 and let P be as in (1.4). Then given any subinterval $I \subset \mathbb{T}^1$ and given any $u \in \mathcal{D}'(\mathbb{T}^2 \times I)$ the condition $Pu \in \mathcal{A}(\mathbb{T}^2 \times I)$ implies $u \in \mathcal{A}(\mathbb{T}^2 \times I)$.*

2. Proof of Theorem 1.1

We will denote by $C_\mu^\infty(\mathbb{T}^N)$ the space of all $v \in C^\infty(\mathbb{T}^N)$ with $\hat{v}(0) = 0$. The proof of Theorem 1.1 is based on the following proposition, which holds for more general real vector fields on $\mathbb{T}^N$ than the ones given by (1.2):

Proposition 2.1. *Assume that every point in $\mathbb{T}^N$ is of finite type for the vector fields $X_1, \dots, X_\nu$, and let P be given by (1.1). Then there exists a constant $C > 0$ such that*

$$\|u\|_{(0)} \leq C\|Pu\|_{(0)}, \qquad u \in C_\mu^\infty(\mathbb{T}^N). \tag{2.1}$$

Proof. By a theorem of Hörmander [13], given $y_0 \in \mathbb{T}^N$, there are $U_0 \subset \mathbb{T}^N$, an open neighborhood of y_0, and $\sigma_0 > 0$ such that $u \in L^2_{loc}(U_0)$,

$Pu \in L^2_{loc}(U_0)$ imply $u \in H^{\sigma_0}_{loc}(U_0)$. Hence, by compactness, and a simple argument using the closed graph theorem there are $\sigma > 0$ and $C > 0$ such that

$$\|u\|_{(\sigma)} \leq C\left(\|Pu\|_{(0)} + \|u\|_{(0)}\right), \qquad u \in C^\infty(\mathbb{T}^N). \tag{2.2}$$

Note that by Bony's maximum principle [3] the only solutions to the equation $Pu = 0$ are the constants. Using this fact, (2.2), and the compactness of the inclusion $H^\sigma(\mathbb{T}^N) \subset L^2(\mathbb{T}^N)$ (2.1) follows at once (cf. the argument in [14], Vol. IV, p. 64). The proof of Proposition 2.1 is complete. □

We must now show that (2.1) implies that when P is defined by the vector fields (1.3) then it is globally analytic hypoelliptic. Let $u \in \mathcal{D}'(\mathbb{T}^{m+n})$ be a solution to $Pu = f$ with $f \in \mathcal{A}(\mathbb{T}^{m+n})$. We may assume that $u \in C^\infty(\mathbb{T}^{m+n})$, since by Hörmander's Theorem [13] (see also [15], [19], [21]) P is locally C^∞-hypoelliptic. We apply (2.1) replacing u by $\partial_x^\alpha u$, $\alpha \in \mathbb{Z}^m$ and $|\alpha| > 0$. Since the derivatives ∂_{x_j} commute with P and since f is analytic there is a constant $K > 0$ such that

$$\|\partial_x^\alpha u\|_{(0)} \leq C\|P(\partial_x^\alpha u)\|_{(0)} = \|\partial_x^\alpha f\|_{(0)} \leq K^{|\alpha|+1}\alpha!.$$

This last inequality implies that $\hat{u}(\xi, t)$ is exponentially decaying in ξ uniformly in t. Then for some $\varepsilon > 0$ we have

$$|\hat{u}(\xi,\tau)| = \frac{1}{(2\pi)^n}\left|\int_{\mathbb{T}^n} \hat{u}(\xi,t)e^{-it.\tau}\,dt\right| \leq C_2 e^{-\varepsilon|\xi|}, \qquad (\xi,\tau) \in \mathbb{Z}^{m+n}. \tag{2.3}$$

Let now $(x_0, t_0, \xi_0, \tau_0) \in T^*(\mathbb{T}^{m+n}) \smallsetminus 0$ be such that $\xi_0 \neq 0$. Then there is $c > 0$ such that $(\xi_0, \tau_0) \in S \doteq \{(\xi,\tau) \in \mathbb{Z}^{m+n} : |\tau| < c|\xi|\}$. Hence, by (2.3),

$$|\hat{u}(\xi,\tau)| \leq C_2 e^{-\frac{\varepsilon}{2}|\xi| - \frac{\varepsilon}{2c}|\tau|}, \qquad (\xi,\tau) \in S,$$

and thus $(x_0, t_0, \xi_0, \tau_0)$ is not in the analytic wave front set of u. Since this is also true for the points of the form $(x_0, t_0, 0, \tau_0)$ (P is elliptic at those points as a consequence of condition (ii); see [14], Vol. I, Theorem 9.5.1 and also [23]) it follows that $u \in \mathcal{A}(\mathbb{T}^{m+n})$. This completes the proof of Theorem 1.1.

3. On a subclass of operators

Here we will give a self-contained proof of Proposition 2.1 for the case that the vector fields are of the form

$$X_j = \frac{\partial}{\partial t_j} - \sum_{k=1}^{m} a_{jk}(t) \frac{\partial}{\partial x_k}, \qquad j = 1, \dots, n, \tag{3.1}$$

where $a_{jk} \in \mathcal{A}(\mathbb{T}^n)$ and are real valued. The proof is based on the following global inequality, which is stronger than (2.1) and which avoids the use of cutoff functions. We believe that these global arguments may extend to more general situations.

Proposition 3.1. *If the vector fields $X_1, \dots, X_n$ are of the form* (3.1) *then there exists a constant $C > 0$ such that*

$$\|\hat{u}(\xi, \cdot)\| \le C \|\widehat{Pu}(\xi, \cdot)\|, \qquad u \in C^\infty(\mathbb{T}^{m+n}), \quad \xi \in \mathbb{Z}^m, \quad \xi \neq 0. \tag{3.2}$$

Proof. We take $u \in C^\infty(\mathbb{T}^{m+n})$ and write $Pu = f$. We will also write $X_j = \frac{\partial}{\partial t_j} - a_j(t) \cdot \frac{\partial}{\partial x}$, where $a_j = (a_{j1}, \dots, a_{jm})$ and $\partial/\partial x = (\partial/\partial x_1, \dots, \partial/\partial x_m)$, and consider the first order operators in $\mathbb{T}^n$

$$Y_j = \frac{\partial}{\partial t_j} - i a_j(t) \cdot \xi, \qquad \xi \in \mathbb{Z}^m. \tag{3.3}$$

Then, for any function v we have

$$\widehat{X_j v}(\xi, t) = (Y_j \hat{v})(\xi, t), \tag{3.4}$$

so that

$$\sum_{j=1}^{n} \|Y_j \hat{u}(\xi, \cdot)\|^2 = -(\hat{f}(\xi, \cdot), \hat{u}(\xi, \cdot)). \tag{3.5}$$

Let $\mathcal{J} = \bigcup_{\gamma=1}^{\infty} \{1, 2, \dots, n\}^\gamma$. For $J \in \mathcal{J}$, $J = (j_1, \dots, j_p)$ we will set $|J| = p$ and

$$X_J = \left[X_{j_1}, [X_{j_2}, [\dots, [X_{j_{p-1}}, X_{j_p}]] \dots]\right].$$

By using the finite type assumption it can be shown that there exist real functions $c_{k\ell}$ in $\mathcal{A}(\mathbb{T}^n)$, $k = 1, \dots, m$, $\ell = 1, \dots, M$, and $J_1, \dots, J_M \in \mathcal{J}$ with $|J_\ell| \ge 2$, such that

$$\frac{\partial}{\partial x_k} = \sum_{\ell=1}^{M} c_{k\ell}(t) X_{J_\ell}, \qquad \text{in } \mathbb{T}^{m+n}, \quad k = 1, \dots, m. \tag{3.6}$$

By introducing the notation $Y_J = \big[Y_{j_1}, [Y_{j_2}, [\dots, [Y_{j_{p-1}}, Y_{j_p}]] \dots]\big]$, taking the Fourier transform in x in (3.6), and making use of (3.4), we obtain

$$(3.7) \qquad i|\xi|^2 = \sum_{k=1}^{m} \sum_{\ell=1}^{M} \xi_k c_{k\ell}(t) Y_{J_\ell} \qquad \text{in } \mathbb{T}^n, \quad \xi \in \mathbb{Z}^m.$$

The main step in the proof of Proposition 3.1 is the following.

Lemma 3.1. *There exists a constant $C > 0$ such that*

$$(3.8) \qquad \|\varphi\|^2 \le C \sum_{j=1}^{n} \|Y_j \varphi\|^2, \qquad \varphi \in C^\infty(\mathbb{T}^n), \quad \xi \in \mathbb{Z}^m - 0.$$

The proof of this lemma follows the ideas of Kohn ([15], [16]) and will be a consequence of the following lemma:

Lemma 3.2. *Let $b \in C^\infty(\mathbb{T}^n)$ and $J \in \mathfrak{J}$, $|J| \ge 2$. Then there is a constant $C > 0$ such that*

(3.9)

$$|(bY_J\varphi, \varphi)| \le C|\xi| \sum_{j=1}^{n} \left(\|Y_j\varphi\|^2 + \|\varphi\|\, \|Y_j\varphi\| \right), \qquad \varphi \in C^\infty(\mathbb{T}^n), \quad \xi \in \mathbb{Z}^m.$$

Proof. We will prove this result by induction on $|J|$. We start by assuming $|J| = 2$. Write $Y_J = Y_{j_1}Y_{j_2} - Y_{j_2}Y_{j_1}$. By integrating by parts we obtain

$$\begin{aligned}(bY_J\varphi, \varphi) &= -(Y_{j_2}\varphi, Y_{j_1}(\bar{b}\varphi)) + (Y_{j_1}\varphi, Y_{j_2}(\bar{b}\varphi)) \\ &= -(Y_{j_2}\varphi, \bar{b}Y_{j_1}\varphi) - \left(Y_{j_2}\varphi, \frac{\partial \bar{b}}{\partial t_{j_1}}\varphi\right) + (Y_{j_1}\varphi, \bar{b}Y_{j_2}\varphi) \\ &\qquad + \left(Y_{j_1}\varphi, \frac{\partial \bar{b}}{\partial t_{j_2}}\varphi\right).\end{aligned}$$

Hence (3.9) follows from the Cauchy-Schwarz inequality.

Next we assume that the lemma has been proved when $|J| \le p-1$ ($p \ge 3$) and take $J \in \mathfrak{J}$ with $|J| = p$. We have $Y_J = [Y_{j_1}, Y_{J'}]$ with $|J'| = p-1$ and then

$$\begin{aligned}(bY_J\varphi, \varphi) &= (Y_{j_1}Y_{J'}\varphi, \bar{b}\varphi) - (Y_{J'}Y_{j_1}\varphi, \bar{b}\varphi) \\ &= -(Y_{J'}\varphi, \bar{b}Y_{j_1}\varphi) - \left(Y_{J'}\varphi, \frac{\partial \bar{b}}{\partial t_{j_1}}\varphi\right) - (Y_{J'}Y_{j_1}\varphi, \bar{b}\varphi).\end{aligned}$$

Now, by the induction hypothesis, there is $C > 0$ such that

$$\left|\left(Y_{J'}\varphi, \frac{\partial \bar{b}}{\partial t_{j_1}}\,\varphi\right)\right| \leq C|\xi| \sum_{j=1}^{n} \left(\|Y_j\varphi\|^2 + \|\varphi\|\;\|Y_j\varphi\|\right)$$

for all $\varphi \in C^\infty(\mathbb{T}^n)$. On the other hand, the operator $Y_{J'}$ is in fact multiplication by a function $h_{J'}(t,\xi)$ which satisfies $|h_{J'}(t,\xi)| \leq \text{constant } |\xi|$. Thus

$$|(Y_{J'}\varphi, \bar{b}Y_{j_1}\varphi)| + |(Y_{J'}Y_{j_1}\varphi, \bar{b}\varphi)| \leq C|\xi|\,\|\varphi\|\;\|Y_{j_1}\varphi\|$$

for all $\varphi \in C^\infty(\mathbb{T}^n)$ and all $\xi \in \mathbb{Z}^m$. The proof of Lemma 3.2 is complete. $\square$

Proof of Lemma 3.1. If $\xi \in \mathbb{Z}^m$ is not zero we can write, from (3.7),

$$\|\varphi\|^2 = \left(\frac{-i}{|\xi|^2} \sum_{k=1}^{m} \sum_{\ell=1}^{M} \xi_k c_{k\ell}(t) Y_{J_\ell}\varphi, \varphi\right) \leq \frac{1}{|\xi|} \sum_{k=1}^{m} \sum_{\ell=1}^{M} |(c_{k\ell}(t)Y_{J_\ell}\varphi, \varphi)|\,.$$

By Lemma 3.2 there is a constant $C_1 > 0$ such that

$$\begin{aligned}\|\varphi\|^2 &\leq C_1 \sum_{j=1}^{n} \left(\|Y_j\varphi\|^2 + \|\varphi\|\;\|Y_j\varphi\|\right)\\ &\leq C_1 \sum_{j=1}^{n} \left(\|Y_j\varphi\|^2 + \frac{\varepsilon^2}{2}\,\|\varphi\|^2 + \frac{1}{2\varepsilon^2}\|Y_j\varphi\|^2\right)\end{aligned}$$

for all $\varphi \in C^\infty(\mathbb{T}^n)$, $\xi \in \mathbb{Z}^m$, $\xi \neq 0$ and $\varepsilon > 0$. Taking $\varepsilon > 0$ so that $C_1 n\varepsilon^2 \leq 1$ concludes the proof of Lemma 3.1. $\square$

By (3.5), (3.8), and the Cauchy-Schwarz inequality we finally obtain (3.2). The proof of Proposition 3.1 is complete. $\square$

4. Proof of Theorem 1.2

Let u and I be as in Theorem 1.2. As before, we may assume that u is a smooth function. We may also assume that I is an open interval $I_\delta =]-\delta, \delta[$ in the real line. Now, since a is analytic its zero set is finite, and if $a(t) \neq 0$ then P is elliptic, and hence analytic hypoelliptic in $\mathbb{T}^2 \times \{t\}$. Consequently we can further reduce the situation by assuming that

$$a(0) = 0, \qquad a(t) \neq 0, \qquad \forall t \in \bar{I}_\delta \setminus \{0\}. \tag{4.1}$$

Next by the Cauchy-Kovalevsky theorem, if $\delta > 0$ is small enough we can find an analytic function v on $\mathbb{T}^2 \times I_\delta$ such that $Pv = Pu$ there. Therefore our problem is reduced to showing that if $u \in C^\infty(\mathbb{T}^2 \times \bar{I}_\delta)$ and if

$$Pu = 0 \tag{4.2}$$

then u is analytic in $\mathbb{T}^2 \times I_\delta$. Taking partial Fourier transforms we obtain

$$\int_{-\delta}^{\delta} |\hat{u}_t(\xi,t)|^2\,dt + \int_{-\delta}^{\delta} [\xi_1^2 + \xi_2^2 a^2(t)]\,|\hat{u}(\xi,t)|^2\,dt = \hat{u}_t(\xi,t)\bar{\hat{u}}(\xi,t)\Big|_{t=-\delta}^{t=\delta}. \tag{4.3}$$

For $\varphi \in C^\infty(\bar{I}_\delta)$ let

$$\|\varphi\|_a^2 = \int_{-\delta}^{\delta} |\varphi_t(t)|^2\,dt + \int_{-\delta}^{\delta} a^2(t)|\varphi(t)|^2\,dt.$$

We now prove an elementary result:

Lemma 4.1. *There exists a constant $C > 0$ depending on a such that*

$$|\varphi(t)| \le C\|\varphi\|_a, \qquad t \in I_\delta, \quad \varphi \in C^\infty(\bar{I}_\delta). \tag{4.4}$$

Proof. By (4.1) there exists $\alpha > 0$ such that $a^2(t) > \alpha^2$, $t \in J = [\delta/2, \delta]$. This together with the Fundamental Theorem of Calculus, and the Cauchy-Schwarz inequality imply

$$\frac{\delta}{2}|\varphi(t)|^2 \le C_1\left(\frac{1}{\alpha^2}\int_{-\delta}^{\delta} a^2(s)|\varphi(s)|^2\,ds + \frac{\delta}{2}\int_{-\delta}^{\delta} |\varphi_t(\tau)|^2\,d\tau\right),$$

which gives (4.4). □

Now we return to the proof of Theorem 1.2. Since $a^2(t) \le C_2(\xi_1^2 + \xi_2^2 a^2(t))$, $t \in I_\delta$, $|\xi| \ge 1$, by (4.3) and (4.4) we conclude the existence of $C_3 > 0$ such that

$$|\hat{u}(\xi,t)|^2 \le C_3|\hat{u}_t(\xi,\delta)\bar{\hat{u}}(\xi,\delta) - \hat{u}_t(\xi,-\delta)\bar{\hat{u}}(\xi,-\delta)|,\ t \in I_\delta,\ |\xi| \ge 1. \tag{4.5}$$

Now, since $u(\cdot,\delta)$, $u(\cdot,-\delta)$, $u_t(\cdot,\delta)$ and $u_t(\cdot,-\delta)$ belong to $\mathcal{A}(\mathbb{T}^2)$, (4.5) implies the existence of constants $C_4 > 0$, $\varepsilon > 0$ such that

$$|\hat{u}(\xi,t)| \le C_4 e^{-\varepsilon|\xi|}, \qquad t \in I_\delta, \quad \xi \in \mathbb{Z}^2. \tag{4.6}$$

If we finally take $\psi \in C_c^\infty(I_\delta)$ with $\psi = 1$ near 0, define $v(x,t) = \psi(t)u(x,t)$ and consider v as an element of $C^\infty(\mathbb{T}^3)$ we obtain, from (4.6),

$$(4.7) \qquad |\hat{v}(\xi,\tau)| = \frac{1}{2\pi}\left|\int_{\mathbb{T}^1} \hat{v}(\xi,t)e^{-it\tau}\,dt\right| \leq C_4 e^{-\varepsilon|\xi|}, \qquad (\xi,\tau) \in \mathbb{Z}^3.$$

By (4.1) and (4.7) we derive that $(x,t,\xi,\tau) \in T^*(\mathbb{T}^2 \times I_\delta) \setminus 0$ does not belong to the analytic wave front set of u if $\xi \neq 0$. Since this is also true for the points of the form $(x,t,0,\tau) \in T^*(\mathbb{T}^2 \times I_\delta) \setminus 0$ (P is elliptic at those points) the proof is complete.

5. Final remarks

1. It is possible for an operator like (1.1) to be globally analytic hypoelliptic when the vector fields $X_1, \ldots, X_\nu$ fail to satisfy the finite type condition. For example, in $\mathbb{T}^3$, we consider the operator P of the form (1.3), where a is a constant α which is an algebraic number of degree $M > 1$. Then $X_1 = \frac{\partial}{\partial t_1}$, $X_2 = \frac{\partial}{\partial t_2} - \alpha\frac{\partial}{\partial x}$, and no point is of finite type for the vector fields X_1, X_2. Now, by Liouville's theorem, there exists $C > 0$ such that the symbol of P satisfies

$$-P(\xi,\tau) = \tau_1^2 + (\tau_2 - \alpha\xi)^2 \geq (\tau_2 - \alpha\xi)^2 \geq C|\xi|^{-2M+2}, \qquad \tau \in \mathbb{Z}^2,\ \xi \in \mathbb{Z},\ \xi \neq 0.$$

Then by using the Fourier transform and this inequality we obtain that P is globally analytic hypoelliptic in $\mathbb{T}^3$. (In connection with this example, see [10] and also [2]).

2. The methods here can be applied to similar classes of second order degenerate elliptic operators that are of finite type in the more general sense considered by Oleinik and Radkevic [19].

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