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ON THE DIMENSION OF DETERMINISTIC AND RANDOM CANTOR-LIKE SETS

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0. Introduction

In this paper we unify and extend many of the known results on the Hausdorff and box dimension of deterministic and random Cantor-like sets in $\mathbb{R}^d$ determined by geometric constructions (see [PW] for the complete description of results and detailed proofs). Most authors have considered similarity processes which impose a strong restriction on the geometry of the construction. Moreover, these constructions were modeled by either the full shift, or subshifts of finite type. In this paper we weaken these restrictions significantly and consider geometric constructions which need not be self-similar and have more complicated geometry. Our constructions are also modeled by *arbitrary* symbolic dynamical systems. Symbolic dynamics and the thermodynamic formalism thus become essential tools in our analysis. We also introduce two new fundamental classes of geometric constructions: *asymptotic constructions and random constructions determined by an arbitrary ergodic stationary process.*

Another innovation in this paper is the use of geometric constructions to build dynamical systems exhibiting pathological behaviors. For instance, in Section 5 we construct a homeomorphism having an ergodic invariant Gibbs measure with positive entropy that has different upper and lower pointwise dimensions almost everywhere.

Our basic construction, which we call a *symbolic geometric construction*, defines a Cantor-like set of $\mathbb{R}^d$ by using a symbolic description in the space of all one-sided infinite sequences $(i_1 i_2 \cdots)$ on p symbols. We denote this space by Σ_p^+ and endow it with its usual topology.

Key words and phrases. Hausdorff dimension, box dimension, Cantor-like set, geometric construction, random geometric construction, gauge function, Eckmann-Ruelle Conjecture.

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Preliminary Definitions.

(1) A *symbolic geometric construction* is defined by

a) a compact set $Q \subset \Sigma_p^+$ invariant under the shift σ (i.e., $\sigma(Q) = Q$) such that $\sigma \,|\, Q$ is topologically transitive;

b) a family of closed sets called *basic sets* $\{\Delta_{i_1 \cdots i_n}\} \subset \mathbb{R}^d$ for $i_j = 1, 2, \ldots, p$ and $n \in \mathbb{N}$ where the n-tuples $(i_1 \cdots i_n)$ are admissible with respect to Q (i.e., there exists $\omega = (i_1', i_2', \cdots) \in Q$ such that $i_1' = i_1, i_2' = i_2, \cdots, i_n' = i_n$) and these sets satisfy

$$\lim_{n \to \infty} \max_{(i_1 \cdots i_n)} \operatorname{diam}(\Delta_{i_1 \cdots i_n}) = 0.$$

For any admissible sequence $(i_1 \cdots i_{n+1}) \in \{1, \cdots, p\}^n$, we require that

a) $\Delta_{i_1 \cdots i_n i_{n+1}} \subset \Delta_{i_1 \cdots i_n}$;

b) $\Delta_{i_1 \cdots i_n} \cap \Delta_{i_1' \cdots i_n'} = \emptyset$ if $(i_1 \cdots i_n) \neq (i_1', \cdots, i_n')$.

(2) The *limit set* F for this construction is defined by

$$F = \bigcap_{n=1}^{\infty} \bigcup_{\substack{(i_1 \cdots i_n) \\ \text{admissible}}} \Delta_{i_1 \cdots i_n}.$$

The set F is a Cantor-like set, i.e., it is a perfect, nowhere dense, and totally disconnected set. The placement of the sets $\{\Delta_{i_1 \cdots i_n}\}$ is completely arbitrary, and we make no assumptions on the regularity of the boundaries of the sets $\{\Delta_{i_1 \cdots i_n}\}$ which can be fractal. The basic sets need not be connected.

(3) We classify symbolic geometric constructions according to the symbolic dynamics determined by the set Q and the geometry of the construction. We call a symbolic geometric construction that is modeled on the Bernoulli shift on p symbols (i.e. $Q = \Sigma_p^+$) a *simple construction* and a geometric construction that is modeled on a subshift of finite type a *Markov construction*.

Our main results provide lower and upper estimates for the Hausdorff dimension and the box dimension of Cantor-like limit sets in $\mathbb{R}^d$ for large classes of symbolic geometric constructions defined by *arbitrary* subsets Q which are shift-invariant. These constructions, most of which are being studied for the first time in this paper, include:

(1) one-dimensional constructions ($d = 1$);
(2) Moran constructions where the basic sets are *essentially* balls (see (1));

(3) constructions with rectangles where the basic sets are multi-dimensional rectangles;
(4) constructions where the gaps between the basic sets can decrease exponentially;
(5) constructions with aligned basic sets;
(6) asymptotic Moran and one-dimensional constructions where the the ratio coefficients for the basic sets depend on the step; of the construction and admit a *good* asymptotic behavior;
(7) random Moran and one-dimensional constructions which are essentially Moran constructions with ratio coefficients chosen randomly from an arbitrary ergodic stationary process.

One particular case is a simple construction where the p^n basic sets at the n^{th} step are strictly geometrically similar to the corresponding sets at the $(n-1)^{\text{th}}$ step. This means that there is a collection of similarity maps (affine contractions) $h_1, \cdots, h_p$ such that

$$\Delta_{i_1 \cdots i_n} = h_{i_1} \circ \cdots \circ h_{i_n}(\Delta),$$

where Δ denotes a ball in $\mathbb{R}^d$. This construction is called a *similarity construction* since it exposes a self-similar character of the geometric process. These very rigid constructions have been the main object of study in dimension theory for many years. The dimensions of limit sets for similarity constructions defined by subshifts of finite types have been computed by [MW1].

About 50 years ago, Moran [Mo] computed the Hausdorff dimension of simple geometric constructions in $\mathbb{R}^d$ given by p^n non-overlapping balls $\Delta_{i_1 \ldots i_n}$ satisfying $\operatorname{diam}(\Delta_{i_1 \ldots i_n j}) = \lambda_j \operatorname{diam}(\Delta_{i_1 \ldots i_n})$ where $0 < \lambda_j < 1$ for $j = 1, \ldots, p$ are the *ratio coefficients*. We call this *a Moran construction*. It *need not* be a similarity construction. Moran discovered the formula for the Hausdorff dimension of the set limit F, $\dim_H F = s$, where s is the unique root of the equation

$$\sum_{i=1}^{p} \lambda_i^t = 1.$$

Moran's great insight was to realize that the similarity maps, or even the spacing of the balls $\{\Delta_{i_1 \cdots i_n}\}$ are not important in the calculation of the Hausdorff dimension of the limit set; the dimension depends only on the ratio coefficients. Moran proved this using the uniform mass distribution principle applied to the s-dimensional Hausdorff measure.

Moran only studied constructions modeled on the full shift on p symbols, where p is the number of basic sets on the first step of the construction.

Stella [St] considered Moran constructions determined by a subshift of finite type with some additional strong assumptions.

Since the basic sets in Moran constructions are balls, these constructions are also quite rigid. We are aware of very few papers in the literature on (deterministic) geometric constructions which are more general that (Markov) Moran constructions. In this paper we introduce and study several new classes of constructions which have significantly more complex geometry than Moran constructions.

For the simple Moran process, the crucial observation is the existence of a measure m on the set F such that the Hausdorff dimension of F is equal to the Hausdorff dimension of m. This measure is the pullback of the equilibrium state for the function $s\phi(x) = s \log \lambda_{i_1}$ where x is associated with a sequence $(i_1 i_2 \cdots)$ and s is the unique root of the equation

$$P(s\phi) = 0.$$

For the Moran constructions, s is the Hausdorff dimension of the limit set as well as its box dimension. We show this is true for a Moran construction which is modeled by an arbitrary symbolic process.

The equation $P(s\phi) = 0$ was discovered by Bowen [B] and seems to be universal. We will show that all known equations previously used to compute the Hausdorff dimension can be deduced from this equation.

For the general symbolic geometric constructions the measure m is an equilibrium measure and admits the non-uniform mass distribution principle. If the measure is Gibbs, then we can also prove strict positivity and boundedness of the Hausdorff measure of the limit set. There is a crucial difference between Gibbs measures and equilibrium measures in Statistical Physics (see [R]). These notions coincide for subshifts of finite type, but need not coincide for general symbolic systems.

One can not expect to obtain any refined estimates for the Hausdorff and box dimensions of the limit set F of a construction with arbitrary shape and spacing of the basic sets. We control the geometry of the construction by either restricting the shapes or sizes of the basic sets, the spacing of the basics sets, or both. If one has strong control over the sizes of the basic sets, then the spacing can be fairly arbitrary, and vice-versa. In [PW], we introduce a new approach to studying geometric constructions having complicated geometry. Our approach is based on the notions of regularity and boundedness of the construction. *Regular and bounded constructions* are those where the control over the geometry is effected in the spirit of Moran processes by *generalized ratio coefficients* that encode the information about *both* the shape and spacing of the basic sets. For some constructions these coefficients are determined by the largest inscribed balls and smallest circumscribed balls for the basic sets. However, in [PW] we construct an example where these numbers are completely independent of these balls.

Given $x \in F$ and $n > 0$, there exists a unique set $\Delta_{i_1 \cdots i_k}(x)$ that contains x and hence $x = \bigcap_{k=1}^{\infty} \Delta_{i_1 \cdots i_k}(x)$. This gives a unique one-sided infinite sequence $(i_1 i_2 \cdots)$ such that the mapping $\chi : F \to Q$ defined by $x \mapsto (i_1 i_2 \cdots)$ is a homeomorphism from F onto Q. The associated symbolic dynamics is one of our main tools to compute the Hausdorff dimension of the limit set F.

Consider the symbolic dynamical system (Q, σ), where $Q \subset \Sigma_p^+$. Given a p-tuple $\alpha = (\alpha_1, \cdots, \alpha_p)$ such that $0 < \alpha_i < 1$, there exists a uniquely defined number s_α such that $P(s_\alpha \log \alpha_{i_1}) = 0$, where P denotes the topological pressure . Let μ_α denote an equilibrium measure for the function $(i_1 i_2 \cdots) \mapsto s_\alpha \log \alpha_{i_1}$ on Q, and let m_α be the pull back measure on F under the coding map χ.

1. Moran constructions

The simpliest case is when the basic sets are essentially balls. Such constructions modeled by the full shift were first considered by Moran [Mo].

Definition. A *Moran symbolic construction* is a symbolic construction such that each basic set $\Delta_{i_1 \cdots i_n}$ satisfies

$$D(C_1 \prod_{j=1}^{n} \lambda_{i_j}) \subset \Delta_{i_1 \cdots i_n} \subset D(C_2 \prod_{j=1}^{n} \lambda_{i_j}) \tag{1}$$

where $0 < \lambda_i < 1$, $i = 1, \cdots, p$ and C_1, C_2 are positive constants. We emphasize once again that the placement of the basic sets is arbitrary.

Proposition 1. *Let F be the limit set for a Moran symbolic construction. Then*

$$s = s_\lambda = dim_H F = \underline{dim}_B F = \overline{dim}_B F,$$

where $\lambda = (\lambda_1, \ldots, \lambda_p)$ and $dim_H F, \underline{dim}_B F, \overline{dim}_B F$ denote the Hausdorff dimension, lower box dimension, and upper box dimension (respectively) of the limit set F. Moreover, if the measure m_λ is Gibbs, then the Hausdorff measure $m_H(s, *)$ is equivalent to m_λ and $0 < m_H(s, F) < \infty$.

We now consider new and more general types of geometric constructions where the basic sets are *asymptotically* balls.

2. Asymptotic Moran constructions

We introduce a new class of geometric constructions called *asymptotic geometric constructions* where the ratio coefficients depend on steps of the construction and admit a *good* asymptotic behavior.

Definition. An *asymptotic Moran symbolic construction* is a symbolic construction for which there exist two sequences of numbers

$$\underline{\lambda}_{i,n} = \lambda_i \exp(\underline{a}_{i,n}), \quad \overline{\lambda}_{i,n} = \lambda_i \exp(\overline{a}_{i,n})$$

where $0 < \lambda_i < 1, i = 1, \cdots, p$ such that

a) for m_λ-almost every $x \in F$ with $\chi(x) = (i_1 i_2 \cdots)$ and $\lambda = (\lambda_1, \ldots, \lambda_p)$,

$$\frac{1}{n}\sum_{j=1}^{n} \underline{a}_{i_j,j} \to 0 \quad \text{and} \quad \frac{1}{n}\sum_{j=1}^{n} \overline{a}_{i_j,j} \to 0 \quad \text{as } n \to \infty;$$

b) each basic set $\Delta_{i_1,\cdots,i_n}$ satisfies

$$D(C_1 \prod_{j=1}^{n} \underline{\lambda}_{i_j,j}) \subset \Delta_{i_1 \cdots i_n} \subset D(C_2 \prod_{j=1}^{n} \overline{\lambda}_{i_j,j})$$

where C_1, C_2 are positive constants.

We use the non-uniform mass distribution principle to estimate the Hausdorff dimension of the limit set. For asymptotic constructions, the uniform mass distribution principle does not hold.

Proposition 2. *Let F be the limit set for an asymptotic Moran symbolic construction. Then $s_\lambda \le dim_H F$.*

Condition (a) in the definition of asymptotic Moran symbolic construction is quite weak; one can obtain more information about the Hausdorff and box dimension of the limit set if the construction satisfies the following uniform versions of (a):

a1) $$\sup_{(i_1 \ldots i_n)} \frac{1}{n}\sum_{j=1}^{n} \underline{a}_{i_j,j} \to 0 \quad \text{as } n \to \infty;$$

a2) $$\sup_{(i_1 \ldots i_n)} \frac{1}{n}\sum_{j=1}^{n} \overline{a}_{i_j,j} \to 0 \quad \text{as } n \to \infty.$$

Proposition 3. *Let F be the limit set for an asymptotic Moran symbolic construction. Assume that the construction satisfies condition (a2). Then* $s = s_\lambda = dim_H F = \underline{dim}_B F = \overline{dim}_B F$.

Unlike the Moran symbolic construction, the limit set for an asymptotic Moran symbolic construction (even one satisfying the strong asymptotic condition (a1) and (a2)) with Gibbs measure μ_λ may have zero Hausdorff measure (see [PW]). We also prove a version of Proposition 2 for one-dimensional asymptotic constructions.

3. Moran random symbolic constructions

We apply our study of asymptotic constructions to analyze a random version of the Moran geometric construction. These constructions are essentially Moran constructions with ratio coefficients chosen randomly from an arbitrary ergodic stationary process.

Geometric constructions of random type have been considered by Falconer [F], Graf [G], Kahane [K], Graff, Mauldin and Williams [GMW], and Mauldin and Williams [MW2]. These authors studied special types of branching processes that correspond to the full shift on p symbols with p^n ratio coefficients at step n chosen randomly, essentially independently and with the same distribution on $(0, 1)$. They also assume some independence conditions over n. In this paper, we consider branching processes that are associated with arbitrary compact shift-invariant subsets. We generate the ratio coefficients by chosing p random numbers on the interval (a, b) where $0 < a \le b < 1$. We do not require that these numbers be independent nor be identically distributed.

In the literature, random constructions are usually considered as a distinct class of geometric constructions and always as constructions modeled by the full shift. Our main idea is to reduce the study of random Moran constructions to asymptotic Moran constructions.

Definition. A *Moran random symbolic construction* is defined by

- **a)** a stochastic process $(\Lambda, \mathfrak{F}, \nu)$ with $\Lambda = \{\vec{\lambda} = (\lambda_{i,n}),\ i = 1, \cdots, p$ and $n \in \mathbb{N}\}$ where $0 < \alpha \le \lambda_{j,n} \le \beta < 1$, $\mathfrak{F}$ denotes the σ-algebra of Borel sets in Λ, and ν is an arbitrary stationary, shift-invariant ergodic Borel probability measure on Λ;
- **b)** a compact set $Q \subset \Sigma_p^+$ invariant under the shift (i.e., $\sigma(Q) = Q$) such that $\sigma \,|\, Q$ is topologically transitive;
- **c)** for ν-almost every $\vec{\lambda} \in \Lambda$, a family of sets $\{\Delta_{i_1 \cdots i_n}\}(\vec{\lambda}) \subset \mathbb{R}^n$ for $i_j = 1, 2, \ldots, p$, where the n-tuple $(i_1 \cdots i_n)$ is admissible with re-

spect to Q and satisfies

$$D(C_1 \prod_{j=1}^{n} \lambda_{i_j,j}(\vec{\lambda})) \subset \Delta_{i_1\cdots i_n}(\vec{\lambda}) \subset D(C_2 \prod_{j=1}^{n} \lambda_{i_j,j}(\vec{\lambda}))$$

where C_1, C_2 are positive constants;

d) for any sequence $(i_1 \cdots i_n) \in \{1, \cdots p\}^n$, we require that

(1) $\Delta_{i_1\cdots i_{n+1}}(\vec{\lambda}) \subset \Delta_{i_1\cdots i_n}(\vec{\lambda})$

(2) $\Delta_{i_1\cdots i_n}(\vec{\lambda}) \cap \Delta_{i'_1\cdots i'_n}(\vec{\lambda}) = \emptyset$, if $(i_1 \cdots i_n) \neq (i'_1 \cdots i'_n)$.

For every $\vec{\lambda} \in \Lambda$, the limit set

$$F(\vec{\lambda}) = \bigcap_{n=1}^{\infty} \bigcup_{\substack{(i_1\cdots i_n) \\ \text{admissible}}} \Delta_{i_1\cdots i_n}(\vec{\lambda})$$

is a perfect, nowhere dense, totally disconnected set.

The following proposition, which uses an ergodic theorem in [BFKO], describes the limiting behavior of the numbers $\lambda_{i,n}$ in the Moran random symbolic construction:

Proposition 4. *Let F be the limit set specified by a random symbolic geometric construction. Then there are numbers λ_i, $0 < \lambda_i < 1$, $i = 1, \cdots, p$ such that for ν-almost every $\vec{\lambda} \in \Lambda$ the following limit exists for μ_{λ}-almost every sequence $(i_1 i_2 \cdots) \in Q(\vec{\lambda})$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_{i_k,k} = 0$$

where $a_{i,j} = \log\left(\frac{\lambda_{i,j}}{\underline{\lambda}_i}\right)$, $\lambda = (\lambda_1, \ldots, \lambda_p)$.

The next statement immediately follows from Propositions 3 and 4..

Proposition 5. *Let F be the limit set specified by a Moran random symbolic construction. Then for ν-almost every $\vec{\lambda} \in \Lambda$,*

$$s_{\underline{\lambda}} \leq dim_H F(\vec{\lambda}).$$

We also prove a version of proposition 5 for one-dimensional random constructions.

4. Constructions with rectangles

We now consider geometric constructions where the basic sets are (multi-dimensional) rectangles. These constructions are obviously quite different from any type of Moran construction.

Definition. We call a symbolic construction a *construction with rectangles* if there exist $2p$ numbers $\underline{\lambda}_i, \overline{\lambda}_i$, $i = 1, \cdots, p$, $0 < \underline{\lambda}_i \leq \overline{\lambda}_i < 1$ such that the basic set $\Delta_{i_1 \cdots i_n} \subset \mathbb{R}^d$ is a rectangle (the direct product of intervals, called sides, lying on n orthogonal lines) with the largest side equal to $C_1 \prod_{j=1}^{n} \overline{\lambda}_{i_j}$ and the smallest side equal to $C_2 \prod_{j=1}^{n} \underline{\lambda}_{i_j}$ where C_1, C_2 are positive constants.

Proposition 6. *Let F be the limit set for a symbolic construction with rectangles. Then*

$$s_{\underline{\lambda}} \leq dim_H F \leq \underline{dim}_B F \leq \overline{dim}_B F \leq s_{\overline{\lambda}},$$

where $\underline{\lambda} = (\underline{\lambda}_1, \ldots, \underline{\lambda}_p)$ and $\overline{\lambda} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_p)$.

5. The Eckman-Ruelle Conjecture

For a Borel measure μ, we define the lower and upper *pointwise dimensions of* μ, $\underline{d}_\mu(x) = \liminf_{\epsilon \to 0} \frac{\log \mu(B(x,\epsilon))}{\log \epsilon}$ and $\overline{d}_\mu(x) = \limsup_{\epsilon \to 0} \frac{\log \mu(B(x,\epsilon))}{\log \epsilon}$. Eckmann and Ruelle conjectured that for an ergodic measure invariant under a $C^{1+\alpha}$ diffeomorphism with non-zero Lyapunov exponents, the pointwise dimension exists, i.e., $\underline{d}_\mu(x) = \overline{d}_\mu(x)$ and is constant almost everywhere, see [ER]. This was proved in [Y] for two-dimensional maps and in [L, PY] for some measures including Gibbs measures for Axiom A diffeomorphisms. Below we construct a Hölder homeomorphism having an ergodic invariant Gibbs measure with positive entropy which has different upper and lower pointwise dimensions almost everywhere.

We first exhibit a simple construction with horizontally and vertically stacked rectangles in $\mathbb{R}^2$ with $p = 2$, $\underline{\lambda}_1 = \underline{\lambda}_2 = \underline{\lambda}$, $\overline{\lambda}_1 = \overline{\lambda}_2 = \overline{\lambda}$, $0 < \underline{\lambda} < \overline{\lambda} < \frac{1}{3}$ such that the limit set F satisfies

$$\dim_H F = \frac{\log 2}{-\log \underline{\lambda}}, \qquad \underline{\dim}_B F = \gamma \frac{\log 2}{-\log \underline{\lambda}}, \qquad \overline{\dim}_B F = \frac{\log 2}{-\log \overline{\lambda}},$$

where $\gamma \in (1, \alpha)$ is an arbitrary number and $\alpha = \left[\frac{\log \underline{\lambda}}{\log \overline{\lambda}}\right] > 1$. We employ several propositions including Proposition 6 in the calculation.

Let F be the limit set of the simple geometric construction above, $G : F \to F$ the map $G = \chi^{-1} \circ \sigma \circ \chi$, where χ denotes the coding map and

$\sigma : \Sigma_p^+ \to: \Sigma_p^+$ is the full shift. Consider the set $\tilde{F} = F \times F$ endowed with the metric

$$\tilde{\rho}((x_1, y_1), (x_2, y_2)) = \rho(x_1, x_2) + \rho(y_1, y_2), \qquad x_1, x_2, y_1, y_2 \in F$$

and the coding map $\tilde{\chi} : \tilde{F} \to \Sigma_p$ defined by

$$\tilde{\chi}(x, y) = (\cdots i_{-1} i_0 i_1 \cdots)$$

where Σ_p denotes the space of two-sided sequences $(\cdots i_{-1} i_0 i_1 \cdots)$, $i_j = 1 \cdots p$ and $\chi(x) = (\cdots i_{-1})$, $\chi(y) = (i_0 i_1 \cdots)$. Set $\tilde{G} = \tilde{\chi}^{-1} \circ \sigma \circ \tilde{\chi}$. One can show that $\tilde{G}$ is a Hölder homeomorphism and that for any $(x, y) \in \tilde{F}$,

$$\pi_1 \tilde{G}(x, y) = G(x), \quad \pi_2 \tilde{G}^{-1}(x, y) = G(y)$$

where π_1, π_2 are the projections $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. Consider the measure $\underline{\tilde{m}} = \tilde{\chi}^* \underline{\tilde{\mu}}$ where $\underline{\tilde{\mu}}$ is the measure on Σ_p^+ defined by

$$\underline{\tilde{\mu}}(\Delta_{i_k \cdots i_n}) = \underline{\lambda}^{(n-k)\underline{s}}.$$

The measure $\underline{\tilde{\mu}}$ is invariant under σ and hence $\underline{\tilde{m}}$ is invariant under $\tilde{G}$. It is easy to see that $\underline{\tilde{m}} = \underline{m} \times \underline{m}$. It follows that for $\underline{\tilde{m}}$-almost every (x, y),

$$\underline{d}_{\underline{\tilde{m}}}(x, y) = \underline{d}_{\underline{m}}(x) + \underline{d}_{\underline{m}}(y) = \underline{s} \quad \text{and} \quad \overline{d}_{\underline{\tilde{m}}}(x, y) = \overline{d}_{\underline{m}}(x) + \overline{d}_{\underline{m}}(y) = \overline{s}.$$

The measure $\underline{\tilde{m}}$ is the Gibbs measure corresponding to the function $\underline{\phi}(x, y) = 2\underline{s} \log \underline{\lambda}$. Thus the map $\tilde{G}$ provides an example of a homeomorphism that is not a smooth map but possesses a Gibbs measure with different lower and upper pointwise dimension almost everywhere. The same is also true with respect to the measure $\overline{\tilde{m}} = \tilde{\chi}^* \overline{\tilde{\mu}}$ where $\overline{\tilde{\mu}}$ is defined by $\overline{\tilde{\mu}}(\Delta_{i_k \cdots i_n}) = \overline{\lambda}^{(n-k)\overline{s}}$.

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