GLOBAL ANALYTIC HYPOELLIPTICITY
IN THE PRESENCE OF SYMMETRY

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A linear partial differential operator, L, is said to be globally analytic hypoelliptic on some real analytic manifold M without boundary if, for any $u \in \mathcal{D}'(M)$ such that $Lu \in C^\omega(M)$, one has $u \in C^\omega(M)$. It is of some interest to determine under what circumstances this property holds, especially for sums of squares of vector fields satisfying the bracket hypothesis of Hörmander, and for related operators such as $\Box_b$ arising in complex analysis in several variables. Examples are known (and discussed at the end of this note) in which global analytic hypoellipticity holds [C1], [C2], [CoH], [DT1], [DT2], yet analytic hypoellipticity in the more usually studied local sense fails [Ch1], [Ch2], [HH]. Those examples all possess certain global symmetries. We have recently shown [Ch3] that global analytic hypoellipticity does not always hold for certain of these operators, so that symmetry hypotheses are no mere crutch eventually to be discarded, as might have been hoped. Our aim here is to show how global analytic hypoellipticity follows quite directly from the presence of certain symmetry, emphasizing the general nature of the result and structure of the argument.

Throughout this note, "analytic" means "real analytic", and is denoted by C^ω. The symbols L^2, C^∞, C^ω refer to sections of whichever vector bundle is relevant. Our results are formulated for sections of vector bundles, rather than for functions, because that situation arises for $\Box_b$, but the reader will lose no ideas by assuming all functions to be scalar-valued.

Let there be given an analytic manifold Ω without boundary, not necessarily compact, a compact subset $K \subset \Omega$, and a linear partial differential operator L mapping sections of one analytic vector bundle over Ω to sections of a second such bundle. We consider the following notion of regularity.

Definition. *L is globally analytic hypoelliptic relative to the pair* (Ω, K)
if, whenever $u \in \mathcal{D}'(\Omega)$*,* $Lu \in C^\omega(\Omega)$ *and* $u \in C^\omega(\Omega \backslash K)$*, then necessarily*
$u \in C^\omega(\Omega)$*.*

When Ω is a compact manifold and $K = \Omega$, this reduces to the usual notion
of global regularity. An interesting example in which this type of regularity
fails, for K a single point, is in [M1], while other examples are (implicit) in
[Ch3].

Let G be a compact, connected Lie group. Let M be an analytic mani-
fold, without boundary, equipped with the structure of an analytic principal
fiber bundle with structure group G. Thus G acts on M, the quotient is
an analytic manifold, and the fibers of the quotient map may be identified
with G. Let E_1, E_2 be analytic vector bundles over M, to which the G-
action is assumed to lift as invertible (analytic) endomorphisms. Let L be
a linear partial differential operator on M, with C^ω coefficients, mapping
sections of E_1 to sections of E_2.

Let $TG \subset TM$ be the subbundle tangent to the G-orbits. Denote by
$\Sigma \subset T^*M \otimes E_1$ the characteristic variety on which the principal symbol
of L fails to be invertible, and by $G^\perp \subset T^*M \otimes E_1$ the annihilator of the
subbundle $TG \otimes E_1^*$. Given any compact subset of M, we write $\|f\|_{H^t}$ for
some fixed Sobolev-type norm of order $t \geq 0$ on all C^∞ functions supported
in that subset.

Our hypotheses are:

(1) L commutes with the action of G.
(2) L is C^∞ hypoelliptic.
(3) L is transversally elliptic, that is, $\Sigma \cap G^\perp = M \times \{0\}$.
(4) There exist ε, $s \in \mathbb{R}^+$ such that for any compactly supported func-
    tions $\eta, \tilde{\eta} \in C^\infty$ such that $\tilde{\eta} \cdot \eta \equiv \eta$, there exists $C < \infty$ such that
    for any smooth function u,

$$\|\eta u\|_{H^\varepsilon} \leq C \|\tilde{\eta} Lu\|_{H^s} + C \|\tilde{\eta} u\|_{L^2}.$$

Theorem. *Under the above hypotheses, L is globally analytic hypoelliptic
relative to* (M, K)*, for any G-invariant compact subset K of M.*

Variations on this set of hypotheses are possible, as will be apparent from
the proof. For instance, (3) may be replaced by the weaker assumption that
L is microlocally analytic hypoelliptic in some conic neighborhood of $G^\perp$.
When $K = M$, (4) may be replaced by the assumption that there exists
$s \in \mathbb{R}^+$ such that, for all smooth u, $\|u\|_{L^2} \leq C \|Lu\|_{H^s}$.

A device useful in the proof will be a bi-invariant orthogonal projection
$P : L^2(G) \mapsto L^2(G)$, such that for any function f belonging to the range

of P, $\|f\|_{L^2(G)} \leq \delta\|f\|_{H^\varepsilon(G)}$. For any $\varepsilon, \delta > 0$, there exists such a projection. Indeed, fixing a large constant λ and a bi-invariant Laplace-Beltrami operator A on G, define $I - P$ to be the orthogonal projection of $L^2(G)$ onto the direct sum of all eigenspaces associated to eigenvalues of A not exceeding λ. Then for any function, $f = (-A + I)^{-\varepsilon/2} \circ (-A + I)^{\varepsilon/2}f$, so that $\|f\|_{H^\varepsilon} \sim \|(-A + I)^{\varepsilon/2}f\|_{L^2}$; if f belongs to the range of P, then this last quantity is $\geq \lambda^{\varepsilon/2}\|f\|_{L^2}$.

Since A is elliptic, its eigenfunctions are all real analytic, and hence $I - P$ maps $L^2(G)$ continuously to $C^\omega(G)$. P is bi-invariant since A is, and hence (or because $I - P$ is C^ω-regularizing) preserves $C^\omega(G)$. The G-action transfers P to an operator Q acting on sections of any G-invariant vector bundle over M, which preserves C^∞ and C^ω. Since $I - P$ may be realized as convolution, on either the left or the right, with a function on G, and since L commutes with the action of G, Q commutes with L.

Fix a basis $\{X_j\}$ for the Lie algebra of left-invariant vector fields on G; these will be regarded also as operators acting on functions defined on M, via the action of G. Denote by X^α any finite composition, with $|\alpha|$ factors, of any of the X_j.

For any $t > 0$, denote by $\tilde{H}^t$ the Sobolev space of functions in $L^2(M)$ which possess t derivatives along the fibers.

Proof of Theorem. Suppose that $Lu \in C^\omega(M)$. u is C^∞ since L is assumed to be C^∞ hypoelliptic, so the restriction of $(I - Q)u$ to each fiber is C^ω, uniformly in the fiber, and consequently the analytic wave front set of $(I - Q)u$ is contained in $G^\perp$. On the other hand, since $L((I - Q)u) = (I - Q)Lu \in C^\omega$, the analytic wave front set must be contained in the characteristic variety Σ of L. We have assumed $G^\perp \cap \Sigma$ to be empty, so $(I - Q)u$ is analytic.

Set $g = Qu$. Then $Lg = QLu \in C^\omega(M)$, and moreover, $g \in C^\omega(M\backslash K)$ since u is C^ω there, and since Q preserves C^ω in any open G-invariant set. Consider any α. Since L commutes with the action of G, it commutes with X^α. Then for any fixed N, and any relatively compact $U \subset M$, $\|L(X^\alpha g)\|_{C^N(U)} = \|X^\alpha Lg\|_{C^N(U)} \leq C^{1+|\alpha|}|\alpha|^{|\alpha|}$, uniformly in α.

Fix compactly supported, G-invariant C^∞ functions $\eta, \tilde{\eta}$, such that $\tilde{\eta} \cdot \eta \equiv \eta$ and $\eta \equiv 1$ in a neighborhood of K. By (4), then,

$$\|\eta X^\alpha g\|_{H^\varepsilon} \leq C^{1+|\alpha|}|\alpha|^{|\alpha|} + C'\|\tilde{\eta} X^\alpha g\|_{L^2},$$

where C' is independent of the choices made in the construction of Q. Now

$$\|\tilde{\eta} X^\alpha g\|_{L^2} \leq C^{1+|\alpha|}|\alpha|^{|\alpha|} + \|\eta X^\alpha g\|_{L^2},$$

since $g \in C^\omega(M \backslash K)$. Since P is bi-invariant and g is in the range of Q, $X^\alpha g$ is also in the range of Q, and consequently $\|\eta X^\alpha g\|_{L^2} \leq \delta \|\eta X^\alpha g\|_{\tilde{H}^\varepsilon}$, where Q may be constructed so that δ is as small as we please. Putting this all together,

$$\|\eta X^\alpha g\|_{H^\epsilon} \leq C^{1+|\alpha|}|\alpha|^{|\alpha|} + \delta C' \|\eta X^\alpha g\|_{\tilde{H}^\epsilon}.$$

Choosing δ to be sufficiently small and then absorbing the second term from the right-hand side into the left, we conclude that $X^\alpha g = O(C^{|\alpha|}|\alpha|^{|\alpha|})$, in the $\tilde{H}^\varepsilon$ norm and hence in the L^2 norm, in a neighborhood of K.

The analytic wave front set of g is therefore contained in $G^\perp$. As in the discussion of $(I - Q)u$, this implies that g, hence u, is analytic. $\square$

Our results are by no means definitive; there should for instance be generalizations to boundary value problems, and generalizations which are simultaneously microlocal and global in character.

Here are some examples satisfying our hypotheses. In all of these, M is a compact manifold without boundary, and $K = \emptyset$. In all except example (4), L maps scalar-valued functions to scalar-valued functions.

(1) Let M be the torus $\mathbb{T}^3$ with coordinates (x, y, t), and take $L = \partial_x^2 + (\partial_y - \theta(x)\partial_t)^2$, where θ is periodic and real-valued, and has the property that for each x there exists $k > 0$ such that $d^k \theta(x)/dx^k \neq 0$. Regarding $\mathbb{T}^3$ as a principal fiber bundle with structure group $\mathbb{T}$, acting in the t variable, all hypotheses of the Theorem are satisfied. However, if there exists x_0 such that $d\theta(x_0)/dx = 0$, then L fails to be analytic hypoelliptic in the local sense [Ch2], [Ch4].

(2) Let G be any compact Lie group and $\{X_j\}$ a collection of left-invariant vector fields on G which generate its Lie algebra. Take $M = G$, and $L = \sum_j X_j^2$. Again, all hypotheses of the Theorem are fulfilled. But if the X_j are independent, if the number of X_j is one less than the dimension of G, and if that dimension is even, then L fails to be analytic hypoelliptic in the local sense [M2].

(3) Fix a compact analytic manifold M' without boundary and a compact Lie group G, and set $M = M' \times G$. Fix left-invariant vector fields $\{\tilde{Y}_j\}$ on G which generate its Lie algebra. Letting G act on M by left multiplication in the second factor of the product, these induce corresponding vector fields Y_j on M. Fix analytic vector fields $\tilde{Z}_j$ on M' which span its tangent space at every point, and denote by Z_j the corresponding vector fields on M. Fix analytic, real-valued functions θ_j on M', regarding these also as functions defined on M and independent of the second factor, and finally set $X_j = Z_j + \varepsilon \theta_j Y_j$ and $L_\varepsilon = \sum_j X_j^2$, where $\varepsilon \in \mathbb{R}$ is a parameter.

All hypotheses of our Theorem are satisfied, provided that the Lie algebra generated by the X_j spans the tangent space to M at every point. Many examples can be constructed in which this holds for generic $\varepsilon \in \mathbb{R}$.

(4) Let $\Omega \subset \mathbb{C}^n$ be a bounded, real analytic, circular domain (that is, if $z \in \Omega$ and $\theta \in \mathbb{R}$ then $(e^{i\theta}z_1, \ldots e^{i\theta}z_n) \in \Omega$). Let $M = \partial\Omega$ and take G to be the circle group $\mathbb{T} = \{e^{i\theta}\}$. Assume that the orbits of G are transverse to the complex tangent space of M. Let $L = \Box_b$ be the Kohn Laplacian, mapping smooth (p, q) forms on M to (p, q) forms on M. Then the hypotheses of our Theorem are satisfied, provided that L is subelliptic, which holds under a standard hypothesis of finite type.

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