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THE HOMOTOPY TYPE OF ARTIN GROUPS

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Abstract. Let $\mathbf{W}$ be a group generated by reflections in $\mathbb{R}^n$. $\mathbf{W}$ acts on the complement $\mathbf{Y} \subset \mathbb{C}^n$ of the complexification of the reflection hyperplanes of $\mathbf{W}$. The fundamental group of the orbit space $\mathbf{Y}/\mathbf{W}$ is the so called *Artin group* of type $\mathbf{W}$. Here we give a new description of the homotopy type of $\mathbf{Y}/\mathbf{W}$ in terms of a convex polyhedrum in $\mathbb{R}^n$ with identifications on the faces. Such identifications are quite easy to describe and are naturally connected to the combinatorics of $\mathbf{W}$. We derive an associated algebraic complex which computes the cohomology of local systems on $\mathbf{Y}/\mathbf{W}$: its k^{th}-module is freely generated by the *k-subsets* of $\{1, \dots, n\}$ and the coboundary is explicitly given by a formula involving the Poincaré series of the group. In particular, we are able to compute the cohomology of the Artin group associated to $\mathbf{W}$ for all the exceptional groups.

Introduction

Let $\mathbf{W}$ be a group generated by reflections in $\mathbb{R}^n$; then $\mathbf{W}$ acts freely on $\mathbf{Y} = \mathbf{Y}(\mathbf{A}) = \mathbb{C}^n \setminus \bigcup_{H \in \mathbf{A}} H$, where $\mathbf{A}$ is the *arrangement* in $\mathbb{C}^n$ obtained by complexifying the reflection hyperplanes of $\mathbf{W}$. The fundamental group $\mathbf{G_W}$ of $\mathbf{Y}/\mathbf{W}$ is the so called *Artin group* associated to $\mathbf{W}$; a presentation of it is given in terms of generalized braid relations (see [3], [4]).

In this paper we give a very simple picture for $K(\mathbf{G_W}, 1)$, as the quotient space obtained by glueing faces of a convex polyhedrum. More precisely, one starts from the complex in $\mathbb{R}^n$ which is *dual* to the stratification induced by the hyperplanes: this can be realized in $\mathbb{R}^n$ as a $\mathbf{W}$-invariant convex polyhedrum $\mathbf{Q}$ by choosing any point v_0 inside a chamber C_0 and taking the convex hull of its $\mathbf{W}$-orbit. For each face e of $\mathbf{Q}$, let $\gamma_{(e)} \in \mathbf{W}$ be the unique element of minimal length such that $\gamma_{(e)}^{-1}(e) \cap C_0 \neq \emptyset$ ($\Leftrightarrow \gamma_{(e)}^{-1}(F) \subset closure(C_0)$, where F is the facet dual to e). Here the length is measured in the Coxeter system associated to C_0. Identify two cells e,e' of $\mathbf{Q}$ iff they are in the same $\mathbf{W}$-orbit by using the homeomorphism induced by $\gamma_{(e')}(\gamma_{(e)})^{-1}$: then $\mathbf{Y}/\mathbf{W}$ contracts onto the so obtained complex $\mathbf{X_W}$

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(Theorem 1.4). The construction also works for infinite groups; in that case, however, it is not known whether **Y** is a $K(\Pi, 1)$

The complex $\mathbf{X_W}$ appears as a sort of "deformed" simplex: if **W** is finite and essential in $\mathbb{R}^n$, $\mathbf{X_W}$ has exactly $\binom{n}{i}$ *i-cells*, $(i = 0, \ldots, n)$.

By giving a suitable orientation to the cells of $\mathbf{X_W}$, an explicit formula for the coboundary operators is obtained (Theorem 1.9). By exploiting a well known product formula for the generating function of the weak Bruhat ordering in **W**. This produces an algorithm for computing the cohomology with integer coefficients of $\mathbf{X_W}$. Actually, the coboundary formula contains a parameter q which allows to compute the cohomology over an arbitrary rank-1 local system. (For local systems on **Y**, see [7], [11].)

The integer cohomology of **Y** is well known (see [3], [8], [1],[5]) but that of $\mathbf{Y_W}$ has been studied only for the classical cases ([13], [6]). Here we give calculations of the integer cohomology for all exceptional cases; this concludes calculations of the integer cohomology for Artin groups associated to finite Coxeter groups. Our approach also allows calculations of local systems in more complicated rings, and we give tables at the end (the interesting consequences of similar computations will be investigated in future papers). The calculations of part two were done by using a personal computer.

1.

1.1. Let **W** be a group generated by reflections in the affine space $\mathbb{A}^n(\mathbb{R})$. Let $\mathbf{A} = \{H_i\}_{i \in \mathcal{I}}$ be the arrangement in $\mathbb{A}^n$ of the reflection hyperplanes of **W**, which induces a stratification $\mathcal{S}$ of $\mathbb{A}^n$ into *facets* ([2; ch. 5]). The set $\mathcal{S}$ is partially ordered by: $F > F'$ iff $closure(F) \supset F'$. We shall indicate by **Q** the cellular complex which is *dual* to $\mathcal{S}$. Note that **Q** is isomorphic to the Coxeter complex of **W**. In a standard way, this can be realized inside $\mathbb{A}^n$ by baricentrical subdivision of the facets (see [9]): inside each codimensional-j facet F^j of $\mathcal{S}$ choose one point $v(F^j)$, and consider the simplexes

$$s(F^{i_0}, \ldots, F^{i_j}) \ = \ \{\textstyle\sum_{k=0}^{j} \lambda_k \ v(F^{i_k}) \ \ : \ \sum \lambda_k = 1, \ \lambda_k \in [0,1]\}$$

where $F^{i_k} \ > \ F^{i_{k+1}}$, $k = 0, \ldots, j-1$. The dimensional-j cell $e^j(\bar{F}^j)$ which is dual to $\bar{F}^j$ is obtained by taking the union

$$\bigcup \ s(F^0, \ldots, F^{j-1}, \bar{F}^j)$$

over all chains $F^0 > \cdots > F^{j-1} > \bar{F}^j$. So $\mathbf{Q} = \bigcup e^j(F^j)$, the union being over all facets of $\mathcal{S}$.

One can think of the 1-*skeleton* $\mathbf{Q}_1$ as a graph (with vertex-set the 0-*skeleton* $\mathbf{Q}_0$) and define the combinatorial distance between two vertices v,v' as the minimum number of edges in an edge-path connecting v and v'.

For each cell e^j of $\mathbf{Q}$, indicate by $V(e^j) = \mathbf{Q}_0 \cap e^j$, the 0-*skeleton* of e^j. Recall from [9, 10] the easily proved property of $\mathbf{Q}$:

Proposition 1.1. *Given a vertex* $v \in \mathbf{Q}_0$ *and a cell* $e^i \in \mathbf{Q}$*, there is a unique vertex* $\underline{w}(v, e^i) \in V(e^i)$ *with the lowest combinatorial distance from* v*, i.e.:*

$$d(v, \underline{w}(v, e^i)) < d(v, v') \text{ if } v' \in V(e^i) \setminus \{\underline{w}(v, e^i)\}.$$

If $e^j \subset e^i$ *then* $\underline{w}(v, e^j) = \underline{w}(\underline{w}(v, e^i), e^j)$. □

The first claim is the geometric translation of the existence of a unique element of minimal length in each coset of a parabolic subgroup.

Fix a chamber C_0 of $\mathcal{S}$, and let v_0 be the vertex of $\mathbf{Q}$ which is contained in C_0. Let also $\mathcal{F}_0$ be the system of facets of C_0 (i.e., of facets contained in the closure of C_0) and let $\mathcal{Q}_0$ be the set of cells in $\mathbf{Q}$ which are dual to a facet in $\mathcal{F}_0$, i.e., the cells which intersect C_0 (note: $\mathcal{Q}_0$ is not a cellular complex).

Given a cell $e \in \mathbf{Q}$, we shall indicate for brevity: $w_0(e) := \underline{w}(v_0, e)$.

The group $\mathbf{W}$ acts also on the complex $\mathbf{Q}$, by taking the cell $e^j(F^j)$ into $e^j(\gamma(F^j))$, $\gamma \in \mathbf{W}$. More precisely, choose one point $v(F)$ inside each facet $F \in \mathcal{F}_0$; inside the facet $\gamma(F)$, $\gamma \in \mathbf{W}$, choose the point $\gamma(v(F))$. Since the stabilizer of a facet fixes pointwise that facet, this point does not depend on the choice of the element $\gamma \in \mathbf{W}$. Then construct $\mathbf{Q}$ as above, by using these points; clearly $\mathbf{Q}$ and its $i-skeletons$ $\mathbf{Q}_i$, $i = 0, 1, \dots$ are invariant for $\mathbf{W}$.

The following proposition comes easily from standard facts.

Proposition 1.2. *For every facet* F *of the stratification* $\mathcal{S}$*, there is exactly one facet* $F_0 \in \mathcal{F}_0$ *in the same orbit as* F *with respect to* $\mathbf{W}$*.*

For each cell e *of* $\mathbf{Q}$*, there is exactly one cell* $e_0 \in \mathcal{Q}_0$ *in the same orbit with respect to* $\mathbf{W}$*. The elements* $\gamma \in \mathbf{W}$ *such that* $\gamma(e_0) = e$ *describe a left-coset of the stabilizer* $\mathbf{W}_{F_0}$ *of* F_0*, where here* F_0 *is the dual of* e_0*; so they are as many as the vertices of* e_0*.* □

Recall also that $\mathbf{W}$ acts simply transitively on the set of chambers and therefore on $\mathbf{Q}_0$. Then the following corollary immediately comes from proposition 1.2.

Corollary 1.3. *Let* $e \in \mathbf{Q}$ *be a cell,* $e_0 \in \mathcal{Q}_0$ *the unique cell which is in the same orbit as* e *with respect to* $\mathbf{W}$ *(according to prop. 1.2). Then the unique element* $\gamma_{(e)} \in \mathbf{W}$ *such that*

$$\gamma_{(e)}(v_0) = w_0(e)$$

satisfies also

$$\gamma_{(e)}(e_0) = e . \quad \square$$

Remark. The element $\gamma_{(e)}$ is the unique element of minimal length (in the Coxeter system given by the reflections with respect to the walls of C_0) in the left-coset $\gamma_{(e)}.\mathbf{W}_{F_0}$. It preserves the weak Bruhat ordering of the vertices of e_0 and e (see the introduction).

Now we come to the main result of this paper. Recall that $\mathbf{W}$ acts on the space $\mathbf{Y} = \mathbb{C}^n \setminus \bigcup_{i \in \mathcal{I}} H_{i,\mathbb{C}}$, the complement of the complexified arrangement. The action on $\mathbf{Y}$ is free.

Theorem 1.4. *Let* $\mathbf{Y}$ *be as above. Then the orbit space* $\mathbf{Y}/\mathbf{W}$ *has the same homotopy type as the following cellular complex* $\mathbf{X_W}$:

identify two cells $e, e' \in \mathbf{Q}$ *iff they are in the same* $\mathbf{W}$*-orbit by using the homeomorphism induced by the element*

$$\gamma_{(e')}(\gamma_{(e)})^{-1}.$$

Corollary 1.5. *If* $\mathbf{W}$ *is finite and essential then the number of* i*-cells of* $\mathbf{X_W}$ *is* $\binom{n}{i}$, $i = 0, \ldots, n$.

In the affine case if $\mathbf{W}$ *is irreducible and essential then the number of* i*-cells of* $\mathbf{X_W}$ *is* $\binom{n+1}{i}$, $i = 0, \ldots, n$.

Note. For general Coxeter groups $\mathbf{W}$, one can use the geometric realization of $\mathbf{W}$ and, instead of the whole space $\mathbb{R}^n$, the cone $\mathbf{W}.C_0$; theorem 1.4 remains true.

Proof of corollary 1.4. In the first case, each chamber is a simplicial cone of dimension n; in the second case each chamber is an n-simplex: so the number of codimensional-i facets of C_0 is easy to calculate. $\square$

Proof of theorem 1.4. The proof is based on the construction given in [9]. Recall from [9] that $\mathbf{Y}$ is homotopy equivalent to the complex $\mathbf{X}$ which is constructed as follows.

Take a cell $e^j = e^j(F^j) = \bigcup s(F^0, \ldots, F^{j-1}, F^j)$ of $\mathbf{Q}$ as defined above and let $v \in V(e^j)$. Embed each simplex $s(F^0, \ldots, F^j)$ into $\mathbb{C}^n$ by the formula

$\phi_{v,e_j}(\sum_{k=0}^{j} \lambda_k \ v(F^k)) \ =$

$$\sum_{k=0}^{j} \lambda_k \ v(F^k) \ + \ i \ \sum_{k=0}^{j} \ \lambda_k \ (\underline{w}(v, e^k) \ - \ v(F^k)) \qquad (*).$$

It is shown in [9]:

(i) the preceding formulas define an embedding of e^j into $\mathbf{Y}$;

(ii) if $E^j \ = \ E^j(e^j, v)$ is its image, then varying e^j and v one obtains a cellular complex

$$\mathbf{X} \ = \ \bigcup E^j$$

which is homotopy equivalent to $\mathbf{Y}$.

The boundary of the cell $E^i(e^i, v)$ is given by all cells of the kind

$$E^j(e^j, \underline{w}(v, e^j))\,,$$

where $e^j \in \partial(e^i)$. The formula of proposition 1.1 implies compatibility on the boundaries.

It is clear that $\mathbf{X}$ is invariant for the action of $\mathbf{W}$ on $\mathbf{Y}$: $\gamma \in \mathbf{W}$ takes the cell $E^j(e^j, v)$ of $\mathbf{X}$ into $E^j(\gamma(e^j), \gamma(v))$.

The fundamental remark is that the explicit homotopies constructed in [9] are $\mathbf{W}$-equivariant: points belonging to the same orbit are taken into points belonging to the same orbit. In fact, all the homotopies defined in [9] have the form

$$x + iy \ \mapsto \ (1-t)(x+iy) + tP(x, y)$$

where $P(x, y) \in \mathbf{Y}$ is defined only in terms of the positions of x and y with respect to the hyperplanes and is equivariant.

It follows that the quotient space $\mathbf{Y}/\mathbf{W}$ is homotopy equivalent to $\mathbf{X}/\mathbf{W}$. So now we have to understand how the cells of $\mathbf{X}$ identify under $\mathbf{W}$.

By looking at formula $(*)$, one sees immediately that all cells $E^j(e^j, v)$, $v \in V(e^j)$, identify each other under the action of the stabilizer $\mathbf{W}_{F^j}$; more precisely, the element $\gamma \in \mathbf{W}_{F^j}$ such that $\gamma(v) \ = \ v'$ takes homeomorphically $E^j(e^j, v)$ onto $E^j(e^j, v')$. In particular, $\mathbf{W}$ acts freely over $\mathbf{X}$.

Let $\mathcal{X}$ be the set of cells $E^j(e^j, w_0(e^j))$, $e^j \in \mathbf{Q}$, of $\mathbf{X}$: there is an incidence preserving bijection between $\mathbf{Q}$ and $\mathcal{X}$; each map

$\phi_{w_0(e^j),e^j}$, $e^j \in \mathbf{Q}$, is a homeomorphism between e^j and $E^j(e^j, w_0(e^j))$, and by part two of proposition 1.1, such homeomorphisms glue into a homeomorphism of $\mathbf{Q}$ and $\mathcal{X}$ ([10; prop. 11]). Since $\mathcal{X}$ contains cells from any orbit of $\mathbf{W}$ acting on $\mathbf{X}$, theorem follows from the shape of this action and from corollary 1.3. In fact, each cell $E^j(e^j, w_0(e^j))$ is $\mathbf{W}$-equivalent to a unique cell of the type $E^j(e_0^j, v_0), e_0^j \in \mathcal{Q}_0$, through the element $\gamma_{(e^j)}$. □

Corollary 1.6. *When* $\mathbf{W}$ *is finite, the cellular complex* $\mathbf{X_W}$ *is a space of type* $K(\mathbf{G_W}, 1)$, *where* $\mathbf{G_W} = \pi_1(\mathbf{Y}/\mathbf{W})$.

Proof. In fact, it is known ([4]) that $\mathbf{Y}$ is a space of type $K(\pi, 1)$. Since there is a regular covering $\mathbf{Y} \to \mathbf{Y}/\mathbf{W}$ the corollary follows. □

By choosing the point $v(F)$, $F \in \mathcal{F}_0$, as the orthogonal projection of v_0 into F, all cells of $\mathbf{Q}$ become convex polyhedra in $\mathbb{A}^n$. $\mathbf{Q}$ itself is a finite convex polyhedrum if $\mathbf{W}$ is finite, and in general, $\mathbf{Q}$ is the convex hull of the $\mathbf{W}$-orbit of v_0. So theorem 1.4 gives the homotopy type of $\mathbf{Y}/\mathbf{W}$ in terms of a complex which is obtained by a convex polyhedrum by identifications over its faces.

1.2. We assume from now on $\mathbf{W}$ is finite and essential; all we say remains true with very few changes in case $\mathbf{W}$ is infinite. Let the hyperplanes of C_0 be $H_1, \ldots, H_n$. Let also $v_i \in H_i \cap closure(C_0)$ be the chosen points in $\mathbf{Q}$, $i = 1, \ldots, n$, say the orthogonal projections of v_0 onto each H_i. Each facet $F \in \mathcal{F}_0$ corresponds to an unique intersection $H_{i_1} \cap \cdots \cap H_{i_k}$, $k = codim(F)$, where the H_{i_j} are the hyperplanes of C_0 which contain F, and $i_1 < \cdots < i_k$. Let us give to the dual cell $e(F)$ the orientation induced by the ordering $v_0, v_{i_1}, \ldots, v_{i_k}$. Next, give an orientation to each cell $e \in \mathbf{Q}$ by requiring that the element $\gamma_{(e)}$ above is orientation preserving. Then the incidence number $[e : e'] \in \{0, 1, -1\}$ among the cells of $\mathbf{Q}$ is defined, and it passes to the quotient in $\mathbf{X_W}$.

The cells of $\mathbf{X_W}$ correspond to the cells in $\mathcal{Q}_0$, therefore to the facets $\mathcal{F}_0$ of C_0. Each facet $F \in \mathcal{F}_0$ corresponds (as said before) to a unique intersection of hyperplanes of C_0, hence to a unique subset $\Gamma = \Gamma(F) \subset I_n = \{1, \ldots, n\}$, where $|\Gamma| = codim(F) = dim\ e(F)$.

Lemma 1.7. *Let* $F \in \mathcal{F}_0$ *correspond to the subset* Γ. *Let* G' *be a facet which is* $\mathbf{W}$*-equivalent to the facet* $F' \in \mathcal{F}_0$ *corresponding to* $\Gamma' \subset \Gamma$, *with* $|\Gamma'| = |\Gamma| - 1$, *and such that* $closure(G') \supset F$ *(so* G' *is* $\mathbf{W}_F$*-equivalent to* F'*). Then* $e(F) \supset e(G')$ *and their incidence number holds*:

$$[e(f) : e(G')] \ = \ (-1)^{l(G')}[e(F) : e(F')]$$

where $l(G')$ *is the shortest length* $l(g)$ *of an element* $g \in\mathbf{W}$ *which takes* F' *into* G'.

Remark. $l(G') = d(v_0, w_0(e(G')))$.

Proof. Let $g \in \mathbf{W}_F$ be of minimal length such that $g(F') = G'$. Then g is the element $\gamma_{(e(G'))}$ of corollary 1.3. Then

$$[e(F) : e(G')] = [g^{-1}(e(F)) : g^{-1}(e(G'))] = [g^{-1}(e(F)) : e(F')].$$

Since $g^{-1}(e(F)) = e(F)$ and g^{-1} preserves the orientation iff $l(g^{-1}) = l(g)$ is even, one obtains:

$$[e(F) : e(G')] = (-1)^{l(g)}[e(F) : e(F')].$$

But $l(g) = l(G')$, so lemma follows. □

We can now construct in a standard way an algebraic complex which computes the cohomology of $\mathbf{X_W}$.

Let $\mathcal{C}^k$ be the free $\mathbb{Z}$-module generated by the k-cells of $\mathbf{X_W}$, $k = 0, \ldots, n$. By the above discussion,

$$\mathcal{C}^k \cong \{\sum \nu_\Gamma \Gamma \ : \ \nu_\Gamma \in \mathbb{Z}, \Gamma \subset I_n, |\Gamma| = k\}.$$

Let $\pi_{\mathbf{W}} : \mathbf{Q}\to \mathbf{X_W}$ be the quotient map.

Theorem 1.8. *One has:*

$$H^*(\mathbf{X_W}; \mathbb{Z}) \cong H^*(\mathcal{C}^*) ,$$

where the coboundary map $\delta^k : \mathcal{C}^k \to \mathcal{C}^{k+1}$ *is given by:*

$$\delta^k(\Gamma) = \sum_{j\in I_n\setminus\Gamma} (-1)^{\sigma(j,\Gamma)+1} \Big(\sum_{\underline{h}\in\mathbf{W}_{\Gamma\cup\{j\}}/\mathbf{W}_\Gamma} (-1)^{l(\underline{h})}\Big) (\Gamma \cup \{j\}) \qquad (1)$$

Here we set $\sigma(j, \Gamma) = |\{i \in \Gamma : i < j\}|$ *and* $\mathbf{W}_{\Gamma(F)} = \mathbf{W}_F$. $l(\underline{h})$ *is the minimal length of an element* $h \in \underline{h}$.

Proof. Let $\mathcal{F}^k$ be the set of facets of codimension k, $\mathcal{F}_0^k = \mathcal{F}^k \cap \mathcal{F}_0$. Then, by the proof of theorem 1.4, one has:

$$\delta^k(\Gamma(F)) = \sum_{G\in\mathcal{F}_0^{k+1}} \Big(\sum_{H\in\mathcal{F}^k,\ \pi_{\mathbf{W}}(e(H))=\Gamma(F)} [e(G) : e(H)]\Big) \Gamma(G) =$$

$$\sum_{G\in\mathcal{F}_0^{k+1}} \Big(\sum_{H\in\mathcal{F}^k,\ H\equiv_{\mathbf{W}} F} [e(G) : e(H)]\Big) \Gamma(G)$$

(since $\pi_{\mathbf{W}}(e(H)) = \Gamma(F)$ iff H and F are in the same $\mathbf{W}$-orbit).

The incidence number $[e(G) : e(H)]$ does not vanish iff G is a facet of H. Then there remain only those G which are facets also of F, since H is taken into F by an element of the stabilizer of G. Then

$$\delta^k(\Gamma(F)) \;=\; \sum_{G\subset cl(F)} (\sum_{G\subset cl(H),\; H\equiv_{\mathbf{W}} F} [e(G) : e(H)])\; \Gamma(G) \;=$$

$$= \sum_{G\subset cl(F)} (\sum_{h\in \mathbf{W}_G,\; h=\gamma_{(e(h(F)))}} [e(G) : e(h(F))])\; \Gamma(G) =$$

$$= \sum_{G\subset cl(F)} (\sum_{\underline{h}\in \mathbf{W}_G/\mathbf{W}_F} [e(G) : e(h(F))])\; \Gamma(G)$$

(by the remark after corollary 1.3: here $h \in \underline{h}$ is the unique element of minimal length)

$$= \sum_{G\subset cl(F)} (\sum_{\underline{h}\in \mathbf{W}_G/\mathbf{W}_F} (-1)^{l(h)}[e(G) : e(F)])\; \Gamma(G)$$

(by lemma 1.7). If G corresponds to $\Gamma \cup \{j\}$ then

$$[e(G) : e(F)] \;=\; (-1)^{\sigma(j,\Gamma)+1}$$

by the choice of orientations, so theorem follows. □

Corollary 1.9. *The mod* 2 *coboundary operator is given by*

$$\delta^k(\Gamma) \;=\; \sum_{j\in I_n\setminus\Gamma} [\mathbf{W}_{\Gamma\cup\{j\}} : \mathbf{W}_\Gamma]\; \Gamma\cup\{j\}$$

(Γ *as in theorem 1.8).* □

1.3. For any part $\mathbf{H}$ of $\mathbf{W}$ let us denote as usual

$$\mathbf{H}(q) \;=\; \textstyle\sum_{w\in\mathbf{H}} q^{l(w)}.$$

Recall that for every $\Gamma \subset I_n$, $\mathbf{W}_\Gamma(q)$ is a rational function of q. Moreover, if $m_1, \dots, m_n$ are the exponents of the group $\mathbf{W}$, then the following product formula holds ([12]):

$$\mathbf{W}(q) \;=\; \textstyle\prod_{i=1}^n (1 + q + \cdots + q^{m_i}).$$

Since one has

$$\sum_{\underline{h}\in \mathbf{W}_{\Gamma\cup\{j\}}/\mathbf{W}_\Gamma} q^{l(\underline{h})} \;=\; \mathbf{W}_{\Gamma\cup\{j\}}(q)/\mathbf{W}_\Gamma(q)\,,$$

one can use the product formula to compute the coboundary in (1), setting $q=-1$.

More generally, to the parameter q can be given a geometrical meaning as follows.

Theorem 1.10. (*i*) *Let* $\mathbf{W}$, $\mathbf{X_W}$ *be as above, let* R *be a commutative ring with* 1 *and let* q *be a unit of* R. *Let* $\mathcal{L}_q = \mathcal{L}_q(\mathbf{X_W};R)$ *be the local system on* $\mathbf{X_W}$ *with coefficients in* R *which corresponds to the map taking the generator of* $\pi_1(\mathbf{X_W})$ *represented by the oriented* 1*-cell* $\pi_\mathbf{W}(e)$, $e\in \mathcal{Q}_0\cap \mathbf{Q}_1$, *into the automorphism of* R *given by the multiplication by* q. *Then*

$$H^*(\mathbf{X_W};\mathcal{L}_q) \;\cong\; H^*(\mathcal{C}^*)$$

where

$$\mathcal{C}^k(q) \;\cong\; \{\sum \nu_\Gamma\Gamma \;:\nu_\Gamma\in R, \Gamma\subset I_n, |\Gamma|=k\}$$

and the coboundary map is:

$$\delta^k(q)(\Gamma) \;=\; \sum_{j\in I_n\setminus\Gamma} (-1)^{\sigma(j,\Gamma)+1}\;(\frac{\mathbf{W}_{\Gamma\cup\{j\}}(-q)}{\mathbf{W}_\Gamma(-q)})\;(\Gamma\cup\{j\}) \qquad (2)$$

(*here* $\sigma(j,\Gamma)$ *is as in theorem 1.9*).

(*ii*) *Let* $\mathbf{W}$ *be the Weyl group of an irreducible root system. In cases where there are two different lengths for the roots, one can consider the local system* $\mathcal{L}_{q',q''}(\mathbf{X_W};R)$ *which associates to the generator of* $\pi_1(\mathbf{X_W})$ *corresponding to the short* (*long*) *root the multiplication by* q' (q''), *where* q' *and* q'' *are units of* R. *The cohomology is computed analogously by using the same* R*-modules as in part* (*i*), *but with coboundary*

$$\delta^k(q',q'')(\Gamma) \;=\; \sum_{j\in I_n\setminus\Gamma} (-1)^{\sigma(j,\Gamma)+1}\;(\frac{\mathbf{W}_{\Gamma\cup\{j\}}(-q',-q'')}{\mathbf{W}_\Gamma(-q',-q'')})\;(\Gamma\cup\{j\}) \quad (3)$$

where

$$\mathbf{W}_\Gamma(q',q'') \;=\; \textstyle\sum_{h\in\mathbf{W}_\Gamma}(-q')^{l'(h)}(-q'')^{l''(h)}$$

	H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7	H^8
$\mathbf{I}_2(2s)$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$						
$\mathbf{I}_2(2s+1)$	$\mathbb{Z}$	$\mathbb{Z}$							
$\mathbf{H}_3$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$					
$\mathbf{H}_4$	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z}$				
$\mathbf{F}_4$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}$				
$\mathbf{E}_6$	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_3$		
$\mathbf{E}_7$	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_6 \times \mathbb{Z}_6$	$\mathbb{Z}_3 \times \mathbb{Z}_6 \times \mathbb{Z}$	$\mathbb{Z}$	
$\mathbf{E}_8$	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_3 \times \mathbb{Z}_6$	$\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}$	$\mathbb{Z}$

Table 1. $R = \mathbb{Z}$, $q = 1$ (trivial coefficients)

	H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7	H^8
$\mathbf{I}_2(p)$	0	$\mathbb{Z}_2$	$\mathbb{Z}_p$						
$\mathbf{H}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$					
$\mathbf{H}_4$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$				
$\mathbf{F}_4$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_{24}$				
$\mathbf{E}_6$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_9$		
$\mathbf{E}_7$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	
$\mathbf{E}_8$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{240}$

Table 2. $R = \mathbb{Z}$, $q = -1$ ($q' = q'' = -1$ in the case of two parameters).

	H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7	H^8
$\mathbf{I}_2(p)$	0	$R/([2])$	$R/([p])$						
$\mathbf{H}_3$	0	$R/([2])$	0	$R/(\frac{[6][10]}{[3][5]})$					
$\mathbf{H}_4$	0	$R/([2])$	0	$R/(\frac{[4]}{[2]})$	$R/(\frac{[20][30]}{[10]})$				
$\mathbf{F}_4$	0	$R/([2])$	$R/([2])$	$R/([6])$	$R/(\frac{[8][12]}{[4]})$				
$\mathbf{E}_6$	0	$R/([2])$	0	0	0	$R/(\frac{[6][8]}{[2][3][4]})$	$R/(\frac{[8][12]}{[4]})$		
$\mathbf{E}_7$	0	$R/([2])$	0	0	0	$R/(\frac{[6]}{[2][3]})$	$R/(\frac{[6]}{[2][3]})$	$R/(\frac{[14][18]}{[2][7][9]})$	
$\mathbf{E}_8$	0	$R/([2])$	0	0	0	$R/(\frac{[4]}{[2]})$	0	$R/(\frac{[2][8][12]}{[4][4][6]})$	$R/(\frac{(q-1)[20][24][30]}{[2][3][3][4][10]})$

Table 3. $R = \mathbb{Q}[q, q^{-1}]$.

$l'(h)$ $[l''(h)]$ *being the number of reflections in short (long) roots in a decomposition of* h *(so* $l'(h)+l''(h)=l(h)$ *).*

Remark. In case (ii) a product formula still holds.

Proof. (i) Clearly the map described in the statement defines a homomorphism of $\pi_1(\mathbf{X_W})$ in $Aut(R)$. So the proof will be completely analogous to that of theorem 1.8. In fact, the cohomology of a local system over a cellular complex can be computed similar to the case of constant coefficients; in the computation of the coboundary of a cell $\pi_{\mathbf{W}}(e)$, $e \in \mathcal{Q}_0$, one has to multiply each cell $\pi_{\mathbf{W}}(e') \subset \partial\pi_{\mathbf{W}}(e)$, $e' \subset \partial(e)$, by the (signed) coefficient $\pi_{\mathbf{W}}(\gamma)_*(1) \in R$, where γ is a path in e which connects v_0 with $w_0(e')$.

Thus we have only to modify the proof of theorem 1.8 by observing that if γ is a path in $e(G)$ connecting v_0 with $w_0(e(h(F)))$ (in the notations there used) then $\pi_{\mathbf{W}}(\gamma) \subset \pi_{\mathbf{W}}(e(G))$ crosses $l(h)$ 1-cells of $\mathbf{X_W}$ in the positive sense.

(ii) Here the map takes a generator of $\pi_1(\mathbf{X_W})$ represented by the oriented 1-*cell* $\pi_{\mathbf{W}}(e(F))$, $F \in \mathcal{F}_0$, such that the hyperplane generated by F is orthogonal to a short (long) root, into the multiplication by q' (by q''). It is easy to see that this defines a homomorphism of $\pi_1(\mathbf{X_W})$ into $Aut(R)$. Then one proceeds as for (i). $\square$

Remark. If the Dynkin diagram of $\mathbf{W}$ is disconnected, one can apply the above theorem to each component, using different parameters q.

2.

By using the preceding formulas we obtained the cohomology over $\mathbb{Z}$ and over some local system $\mathcal{L}_q$ for all the exceptional groups. Since the spaces $\mathbf{X_W}$ are in these cases $K(\pi,1)$-*spaces*, their cohomology is also the cohomology of the Artin group $\mathbf{G_W}$. The integer cohomology is reported in Table 1 (the case $\mathbf{G}_2$ coincides with $\mathbf{I}_2(6)$).

The computations of this section were done by the use of a personal computer.

With regard to local systems, we give some cases in Tables 2 and 3. It is convenient to introduce the *q-analog*

$$[n] \;=\; \frac{q^n-1}{q-1}$$

and to apply formulas of part 1 to the local system corresponding to the multiplication by $-q$.

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