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BOGOMOLOV UNSTABILITY ON ARITHMETIC SURFACES

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Abstract. In this paper, we will consider an arithmetic analogue of Bogomolov unstability theorem, i.e. if (E, h) is a torsion free Hermitian sheaf on an arithmetic surface X and $\widehat{\deg}\left((\operatorname{rk} E - 1)\widehat{c}_1(E,h)^2 - (2\operatorname{rk} E)\widehat{c}_2(E,h)\right) > 0$, then there is a non-zero saturated subsheaf F of E such that the point $\widehat{c}_1(F, h|_F)/\operatorname{rk} F - \widehat{c}_1(E,h)/\operatorname{rk} E$ lies in the positive cone of X.

0. Introduction

In [Bo], Bogomolov proved an unstability theorem, namely, if a vector bundle E on a complex projective surface S satisfies an inequality $(\operatorname{rk} E - 1)c_1(E)^2 - (2\operatorname{rk} E)c_2(E) > 0$, then there is a saturated subsheaf F of E such that $c_1(F)/\operatorname{rk} F - c_1(E)/\operatorname{rk} E$ belongs to the positive cone of S. In this paper, we would like to consider an arithmetic analogue of Bogomolov unstability theorem.

Let $f : X \to \operatorname{Spec}(O_K)$ be a regular arithmetic surface over the ring of integers of a number field K with $f_*\mathcal{O}_X = O_K$, and $\deg_K : \widehat{\operatorname{CH}}^1(X)_{\mathbb{R}} \to \mathbb{R}$ the natural homomorphism given by

$$\deg_K : \widehat{\operatorname{CH}}^1(X)_{\mathbb{R}} \xrightarrow{z} \operatorname{CH}^1(X)_{\mathbb{R}} \xrightarrow{\otimes K} \operatorname{CH}^1(X_K)_{\mathbb{R}} \xrightarrow{\deg} \mathbb{R}.$$

The positive cone $\widehat{C}_{++}(X)$ of X is defined by the set of all elements $x \in \widehat{\operatorname{CH}}^1(X)_{\mathbb{R}}$ with $\widehat{\deg}(x^2) > 0$ and $\deg_K(x) > 0$. A torsion free Hermitian sheaf (E, h) on X is said to be *arithmetically unstable* if there is a non-zero saturated subsheaf F of E with

$$\frac{\widehat{c}_1(F, h|_F)}{\operatorname{rk} F} - \frac{\widehat{c}_1(E,h)}{\operatorname{rk} E} \in \widehat{C}_{++}(X),$$

where $h|_F$ is the Hermitian metric of F given by the restriction of h to F. The main theorem of this paper is the following.

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Theorem A. *If (E, h) is a torsion free Hermitian sheaf on X and*

$$\widehat{\deg}\left((\operatorname{rk} E - 1)\widehat{c}_1(E,h)^2 - (2\operatorname{rk} E)\widehat{c}_2(E,h)\right) > 0,$$

then (E, h) is arithmetically unstable.

Let A be an element of $\widehat{\mathrm{CH}}^1(X)_{\mathbb{R}}$ with $\widehat{\deg}(A\cdot x) > 0$ for all $x \in \widehat{C}_{++}(X)$, i.e. according to terminology in §1.1, A is an element of the weak positive cone $\widehat{C}_{+}(X)$. A torsion free Hermitian sheaf (E, h) on X is said to be *arithmetically μ-semistable with respect to A* if, for all non-zero saturated subsheaves F of E,

$$\frac{\widehat{\deg}(\widehat{c}_1(F, h|_F)\cdot A)}{\operatorname{rk} F} \leq \frac{\widehat{\deg}(\widehat{c}_1(E,h)\cdot A)}{\operatorname{rk} E}.$$

Under the above terminology, we have the following corollary of Theorem A.

Corollary B. *If (E, h) is arithmetically μ-semistable with respect to A, then*

$$\widehat{\deg}\left((2\operatorname{rk} E)\widehat{c}_2(E,h) - (\operatorname{rk} E - 1)\widehat{c}_1(E,h)^2\right) \geq 0.$$

If $A = \left(0, \sum_{\sigma\in K(\mathbb{C})} 1/[K:\mathbb{Q}]\right)$, then arithmetic μ-semistability of (E,h) with respect to A is nothing more than μ-semistability of $E_{\overline{\mathbb{Q}}}$. So this corollary gives a generalization of [Mo1], [Mo2] and [So].

In §1, we will prepare several basic facts of the positive cone and Hermitian vector spaces. In §2, we will consider finiteness of saturated subsheaves in a Hermitian vector bundle, which will be crucial for the proof of Theorem A. §3 is devoted to the proof of Theorem A and Corollary B.

1. Preliminaries

1.1. Positive cone of arithmetic Chow group. Here, we consider basic properties of the positive cone of the arithmetic Chow group of codimension 1.

Let K be a number field and O_K the ring of integers of K. Let $f : X \to \operatorname{Spec}(O_K)$ be a regular arithmetic surface with $f_*\mathcal{O}_X = O_K$, and $\deg_K : \widehat{\mathrm{CH}}^1(X)_{\mathbb{R}} \to \mathbb{R}$ the natural homomorphism defined by

$$\deg_K : \widehat{\mathrm{CH}}^1(X)_{\mathbb{R}} \xrightarrow{z} \mathrm{CH}^1(X)_{\mathbb{R}} \xrightarrow{\otimes K} \mathrm{CH}^1(X_K)_{\mathbb{R}} \xrightarrow{\deg} \mathbb{R}.$$

We set

$$\widehat{C}_{++}(X) = \left\{ x \in \widehat{\mathrm{CH}}^1(X)_{\mathbb{R}} \mid \widehat{\deg}(x^2) > 0 \text{ and } \deg_K(x) > 0 \right\} \quad \text{and}$$

$$\widehat{C}_{+}(X) = \left\{ x \in \widehat{\mathrm{CH}}^1(X)_{\mathbb{R}} \mid \widehat{\deg}(x\cdot y) > 0 \text{ for all } y \in \widehat{C}_{++}(X) \right\}.$$

$\widehat{C}_{++}(X)$ (resp. $\widehat{C}_{+}(X)$) is called *the positive cone of X* (resp. *the weak positive cone of X*). First of all, we have the following lemma.

Lemma 1.1.1. (1.1.1.1) *If* $h \in \widehat{C}_{++}(X)$, $x \in \widehat{\mathrm{CH}}^1(X)_{\mathbb{R}}$ *and* $\widehat{\deg}(x \cdot h) = 0$, *then* $\widehat{\deg}(x^2) \leq 0$.

(1.1.1.2) *If* $x \in \widehat{\mathrm{CH}}^1(X)_{\mathbb{R}}$, $\widehat{\deg}(x^2) \geq 0$ *and* $\deg_K(x) > 0$, *then* $x \in \widehat{C}_{+}(X)$.

(1.1.1.3) *If* $h \in \widehat{C}_{++}(X)$, $x \in \widehat{\mathrm{CH}}^1(X)_{\mathbb{R}}$, $\widehat{\deg}(x^2) \geq 0$ *and* $\widehat{\deg}(x \cdot h) > 0$, *then* $x \in \widehat{C}_{+}(X)$.

Proof. (1.1.1.1) Let t be a real number with $\deg_K(x - th) = 0$. Then, by Hodge index theorem (cf. [Fa], [Hr] or [Mo3]), $\widehat{\deg}((x-th)^2) \leq 0$. Thus, $\widehat{\deg}(x^2) + t^2\widehat{\deg}(h^2) \leq 0$. Therefore, $\widehat{\deg}(x^2) \leq 0$.

(1.1.1.2) Let $y \in \widehat{C}_{++}(X)$ and t a real number with $\deg_K(y - tx) = 0$. Then, $t > 0$. By Hodge index theorem, $\widehat{\deg}((y-tx)^2) \leq 0$. Thus, we have

$$\widehat{\deg}(y^2) + t^2\widehat{\deg}(x^2) \leq 2t\widehat{\deg}(x \cdot y).$$

Therefore, $\widehat{\deg}(x \cdot y) > 0$. Hence, $x \in \widehat{C}_{+}(X)$.

(1.1.1.3) Let $y \in \widehat{C}_{++}(X)$ and t a real number with $\widehat{\deg}(y - tx \cdot h) = 0$. Then, $t > 0$ by (1.1.1.2). (1.1.1.1) implies $\widehat{\deg}((y-tx)^2) \leq 0$. Thus, by the same way as above, we have $\widehat{\deg}(x \cdot y) > 0$, which says $x \in \widehat{C}_{+}(X)$. □

$\widehat{C}_{++}(X)$ and $\widehat{C}_{+}(X)$ have the following properties.

Proposition 1.1.2. (1.1.2.1) $\widehat{C}_{++}(X) \subset \widehat{C}_{+}(X)$.

(1.1.2.2) *If* $x, y \in \widehat{C}_{++}(X)$ *and* $t > 0$, *then* $x + y, tx \in \widehat{C}_{++}(X)$.

(1.1.2.3) *If* $x, y \in \widehat{C}_{+}(X)$ *and* $t > 0$, *then* $x + y, tx \in \widehat{C}_{+}(X)$.

(1.1.2.4) $\widehat{C}_{++}(X) = \left\{ x \in \widehat{\mathrm{CH}}^1(X)_{\mathbb{R}} \mid \widehat{\deg}(x \cdot y) > 0 \ \forall\, y \in \widehat{C}_{+}(X) \right\}$.

Proof. (1.1.2.1) and (1.1.2.2) are straightforward from (1.1.1.2). (1.1.2.3) is obvious.

(1.1.2.4) Clearly, we have

$$\widehat{C}_{++}(X) \subseteq \left\{ x \in \widehat{\mathrm{CH}}^1(X)_{\mathbb{R}} \mid \widehat{\deg}(x \cdot y) > 0 \text{ for all } y \in \widehat{C}_{+}(X) \right\}.$$

We assume that $\widehat{\deg}(x \cdot y) > 0$ for all $y \in \widehat{C}_{+}(X)$. If we set $B = \left(0, \sum_{\sigma \in K(\mathbb{C})} 1/[K:\mathbb{Q}]\right)$, then $B \in \widehat{C}_{+}(X)$. Thus, $\deg_K(x) = \widehat{\deg}(x \cdot B) > 0$. Hence, it is sufficient to show $\widehat{\deg}(x^2) > 0$. Here, we fix $h \in \widehat{C}_{++}(X)$ with $\widehat{\deg}(h^2) = 1$. We set $t = \widehat{\deg}(x \cdot h) > 0$. Since $\widehat{\deg}(x - th \cdot h) = 0$, by (1.1.1.1), $\widehat{\deg}((x-th)^2) \leq 0$. If $\widehat{\deg}((x-th)^2) = 0$, then

$$\widehat{\deg}(x^2) = \widehat{\deg}((th + (x - th))^2) = t^2 > 0.$$

Thus, we may assume that $\widehat{\deg}((x - th)^2)$ is negative. If we set $s = \left(-\widehat{\deg}((x - th)^2)\right)^{1/2}$ and $l = (x - th)/s$, then $x = th + sl$, $\widehat{\deg}(l^2) = -1$ and $\widehat{\deg}(h \cdot l) = 0$. Let us consider $y = h + l$. Since $\widehat{\deg}(y^2) = 0$ and $\widehat{\deg}(y \cdot h) = 1$, by (1.1.1.3), $y \in \widehat{C}_+(X)$. Thus, $\widehat{\deg}(x \cdot y) = t - s > 0$. Therefore

$$\widehat{\deg}(x^2) = t^2 - s^2 = (t + s)(t - s) > 0.$$

Hence, $x \in \widehat{C}_{++}(X)$. □

Finally we consider the following proposition.

Proposition 1.1.3. *For $z \in \widehat{\mathrm{CH}}^1(X)_{\mathbb{R}}$, we set*

$$W(z) = \left\{u \in \widehat{C}_+(X) \mid \widehat{\deg}(z \cdot u) > 0\right\}. \quad (1.1.3.1)$$

If $x \notin \widehat{C}_{++}(X)$, $y \in \widehat{C}_{++}(X)$ and $W(x) \neq \emptyset$, then $W(x) \subsetneqq W(x + y)$.

Proof. Clearly, $W(x) \subseteq W(x + y)$. By virtue of (1.1.2.4), $W(x) \subsetneqq \widehat{C}_+(X)$. Let $u_1 \in W(x)$, $u_2 \in \widehat{C}_+(X) \setminus W(x)$ and $t = -\widehat{\deg}(x \cdot u_2)/\widehat{\deg}(x \cdot u_1)$. Then, $t \geq 0$ and $\widehat{\deg}(x \cdot u_2 + tu_1) = 0$. Hence, $u_2 + tu_1 \notin W(x)$. On the other hand, by (1.1.2.3), $u_2 + tu_1 \in \widehat{C}_+(X)$. Moreover, $\widehat{\deg}(y \cdot u_2 + tu_1) > 0$. Thus, $\widehat{\deg}(x + y \cdot u_2 + tu_1) > 0$. Therefore $u_2 + tu_1 \in W(x + y)$. □

1.2. Hermitian vector space. Let V be a $\mathbb{C}$-vector space, h_V a Hermitian metric on V and W a subvector space of V. Considering the restriction of h to W, the metric h_V induces a metric h_W of W, which is called *the submetric of W induced by h_V*. Let $W^\perp$ be the orthogonal complement of W. Then the natural homomorphism $W^\perp \longrightarrow V/W$ is isomorphic. Thus we have a metric $h_{V/W}$ of V/W given by $h|_{W^\perp}$. This metric is called *the quotient metric of V/W induced by h_V*.

Proposition 1.2.1. *Let (V, h_V) be a Hermitian vector space over $\mathbb{C}$ and U, W subspaces of V with $U \subset W$. Let $h_W = (h_V)|_W$ and $h_{V/U}$ the quotient metric of V/U induced by h_V. We consider two Hermitian metrics of W/U. Let $h_s = (h_{V/U})\big|_{W/U}$ and h_q the quotient metric induced by h_W. Then, we have $h_s = h_q$.*

Proof. Let $V = U \oplus U^\perp$ be the orthogonal decomposition of V and $f : U^\perp \to V/U$ the natural isomorphism. Then, for $x, y \in W/U$, $h_s(x, y) = h_V(f^{-1}(x), f^{-1}(y))$.

Let $W = U \oplus U^\perp_W$ be the orthogonal decomposition of W, i.e. $U^\perp_W$ is the orthogonal complement of U in W, and $g : U^\perp_W \to W/U$ the natural isomorphism. Then, $h_q(x, y) = h_V(g^{-1}(x), g^{-1}(y))$ for $x, y \in W/U$.

On the other hand, $U_W^\perp \subset U^\perp$ and the following diagram is commutative.

$$\begin{array}{ccc} U_W^\perp & \hookrightarrow & U^\perp \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ W/U & \hookrightarrow & V/U \end{array}$$

Thus, we have $h_s = h_q$. □

2. Finiteness of saturated subsheaves

In this section, we will consider finiteness of saturated subsheaves in a Hermitian vector bundle, which will be crucial for the proof of Theorem A.

Theorem 2.1. *Let K be a number field, and O_K the ring of integers of K. Let $f : X \to \operatorname{Spec}(O_K)$ be a regular arithmetic surface with $f_*\mathcal{O}_X = O_K$, (E, h) a Hermitian vector bundle on X, and (H, k) a Hermitian line bundle on X. If H_K is ample, then, for constants C_1 and C_2, the set*

$$\mathcal{F} = \left\{ L \;\middle|\; \begin{array}{l} L \text{ is a rank-1 saturated subsheaf of } E \text{ with} \\ \widehat{\deg}(\widehat{c}_1(L, h|_L) \cdot \widehat{c}_1(H, k)) \geq C_1 \text{ and } \deg(L_K) \geq C_2 \end{array} \right\}$$

is finite.

Proof. First of all, we need the following Lemma.

Lemma 2.1.1. *Let K be an infinite field, C a smooth projective curve over K, and E a vector bundle on C of rank $r \geq 2$. Then, for every real number M, there is a rank-1 saturated subsheaf L of E with $\deg(L) < M$.*

Proof. Let H be an ample line bundle on X, and let n be a positive integer such that $n\deg(H) > -M$ and $E \otimes H^n$ is generated by global sections. Let us consider the following closed set Σ in $C \times \mathbb{P}(H^0(C, E \otimes H^n))$;

$$\Sigma = \left\{ (x, s) \in C \times \mathbb{P}(H^0(C, E \otimes H^n)) \mid s(x) = 0 \right\}.$$

Let $p : \Sigma \to C$ be the natural projection. Since $E \otimes H^n$ is generated by global sections, $\operatorname{codim}(p^{-1}(x), \mathbb{P}(H^0(C, E \otimes H^n))) = r$. Therefore, $\operatorname{codim}(\Sigma, C \times \mathbb{P}(H^0(C, E \otimes H^n))) = r$ and $\dim \Sigma < \dim \mathbb{P}(H^0(C, E \otimes H^n))$. Thus, the natural projection $q : \Sigma \to \mathbb{P}(H^0(C, E \otimes H^n))$ is not surjective. Hence, since $\#(K)$ is infinite,

$$(p(\Sigma))(K) \subsetneqq \mathbb{P}(H^0(C, E \otimes H^n))(K).$$

Thus, there is a section $s \in H^0(C, E \otimes H^n)$ with $\operatorname{div}(s) = \emptyset$, which induces an injective homomorphism $H^{-n} \to E$. Since $\operatorname{div}(s) = \emptyset$, the image of $H^{-n} \to E$ is saturated. $\square$

Let us start the proof of Theorem 2.1. Clearly we may assume $r = \operatorname{rk} E \geq 2$. By Lemma 2.1.1, there is a filtration : $\{0\} = F_0 \subset F_1 \subset \cdots \subset F_{r-1} \subset F_r = E$ such that

(2.1.2) F_i/F_{i-1} is a rank-1 torsion free sheaf for every $1 \leq i \leq r$.

(2.1.3) $\deg((F_i/F_{i-1})_K) < C_2$ for $1 \leq i \leq r-1$.

Here we claim

Claim 2.1.4. *If L is a line bundle on X with $\deg(L_K) \geq C_2$, and $\varphi : L \to F_{r-1}$ is a homomorphism, then $\varphi = 0$.*

Proof. We assume that $\varphi \neq 0$. Choose i in such a way that $\varphi(L) \subseteq F_i$ and $\varphi(L) \not\subseteq F_{i-1}$. Then, we have an injective homomorphism $L \to F_i/F_{i-1}$. Since $i \leq r-1$, $\deg(L_K) \geq C_2 > \deg((F_i/F_{i-1})_K)$. This is a contradiction. $\square$

Let Q be the double dual of E/F_{r-1} and h_Q the quotient metric of Q induced by h via $E \to E/F_{r-1}$, i.e. $h_Q = h|_{F^\perp}$. Pick up $L \in \mathcal{F}$. By Claim 2.1.4, we have the natural injection $L \to Q$. So there is an effective divisor D_L on X such that $L \otimes \mathcal{O}_X(D_L) \simeq Q$. Let D_L^h (resp. D_L^v) be the horizontal part of D_L (resp. the vertical part of D_L). Then, we have

Claim 2.1.5. *If $D_L^h = D_{L'}^h$ for $L, L' \in \mathcal{F}$, then $L = L'$.*

Proof. Let us consider $M = L \otimes \mathcal{O}_X(-D_{L'}^v)$ and $M' = L' \otimes \mathcal{O}_X(-D_L^v)$. Since $D_L^h = D_{L'}^h$, M and M' have the same image in Q via $E \to Q$. Therefore, we have $M + F_{r-1} = M' + F_{r-1}$. Moreover, since $M \to Q$ and $M' \to Q$ are injective, $M \cap F_{r-1} = M' \cap F_{r-1} = \{0\}$. Hence, we have a homomorphism $M \to M' \oplus F_{r-1} \to F_{r-1}$, which must be zero by Claim 2.1.4. Thus $M \subseteq M'$. By the same way, $M' \subseteq M$. Hence, $M = M'$. So $L = L'$ because L is the saturation of M in E and L' is the saturation of M' in E. $\square$

We set $C_3 = \widehat{\deg}(\widehat{c}_1(Q, h_Q) \cdot \widehat{c}_1(H, k)) - C_1$ and $C_4 = \deg(Q_K) - C_2$. Let $D_L^h = \sum_i a_i \Gamma_i$ be the irreducible decomposition of D_L^h. Then, we have

Claim 2.1.6. $\sum_i a_i \widehat{\deg}(\widehat{c}_1((H,k)|_{\Gamma_i})) \leq C_3$ *and* $\sum_i a_i [K(\Gamma_i) : K] \leq C_4$.

Proof. Since $L \otimes \mathcal{O}_X(D_L) \simeq Q$, we have $\deg(L_K) + \deg((D_L)_K) = \deg(Q_K)$. Thus we get $\sum_i a_i [K(\Gamma_i) : K] \leq C_4$.

We choose a Hermitian metric h_{D_L} of $\mathcal{O}_X(D_L)$ in such a way that $(L, h|_L) \otimes (\mathcal{O}_X(D_L), h_{D_L})$ is isometric to (Q, h_Q). Let 1 be the canonical section of $H^0(X, \mathcal{O}_X(D_L))$ with $\operatorname{div}(1) = D_L$, and $D_L^v = \sum_j b_j l_j$ the irreducible decomposition of the vertical part of D_L. Then, we have

$$\widehat{\deg}(\widehat{c}_1(\mathcal{O}_X(D_L), h_{D_L}) \cdot \widehat{c}_1(H,k)) = \sum_i a_i \widehat{\deg}(\widehat{c}_1((H,k)|_{\Gamma_i})) + \sum_j b_j \deg(H|_{l_j}) - \frac{1}{2} \sum_{\sigma \in K(\mathbb{C})} \int_{X_\sigma} \log(h_{D_L}(1,1)) c_1(H_\sigma, k_\sigma).$$

Since h_Q is a quotient metric of h, we can see $h_{D_L}(1,1) \leq 1$ for all points of each infinite fiber X_σ. Therefore, we get

$$\sum_i a_i \widehat{\deg}(\widehat{c}_1((H,k)|_{\Gamma_i})) \leq \widehat{\deg}(\widehat{c}_1(\mathcal{O}_X(D_L), h_{D_L}) \cdot \widehat{c}_1(H,k)) = \widehat{\deg}(\widehat{c}_1(Q, h_Q) \cdot \widehat{c}_1(H,k)) - \widehat{\deg}(\widehat{c}_1(L, h|_L) \cdot \widehat{c}_1(H,k)) \leq C_3.$$

Thus, we obtain our claim. □

To complete our theorem, by Claim 2.1.5, it is sufficient to see that $\{D_L^h \mid L \in \mathcal{F}\}$ is finite. Since $\sum_i a_i [K(\Gamma_i) : K] \leq C_4$, we have $a_i \leq C_4$ and $[\mathbb{Q}(\Gamma_i) : \mathbb{Q}] \leq C_4 [K : \mathbb{Q}]$. Let $D_L^h = D_L^h(+) + D_L^h(-)$ be the decomposition of D_L^h such that

$$D_L^h(+) = \sum_{\widehat{\deg}(\widehat{c}_1((H,k)|_{\Gamma_i}))>0} a_i \Gamma_i \quad \text{and} \quad D_L^h(-) = \sum_{\widehat{\deg}(\widehat{c}_1((H,k)|_{\Gamma_i}))\leq 0} a_i \Gamma_i.$$

By Northcott's theorem (cf. Theorem 2.6 in Chapter 2 of [La]), the set $\{D_L^h(-) \mid L \in \mathcal{F}\}$ is finite. Hence, there is a constant C_5 depending only on $\mathcal{F}$ with

$$\sum_{\widehat{\deg}(\widehat{c}_1((H,k)|_{\Gamma_i}))>0} a_i \widehat{\deg}(\widehat{c}_1((H,k)|_{\Gamma_i})) \leq C_5.$$

Thus, $\{D_L^h(+) \mid L \in \mathcal{F}\}$ is finite. Therefore, $\{D_L^h \mid L \in \mathcal{F}\}$ is finite. □

Corollary 2.2. *Let K be a number field, and O_K the ring of integers of K. Let $f : X \to \operatorname{Spec}(O_K)$ be a regular arithmetic surface with $f_*\mathcal{O}_X = O_K$, (E,h) a Hermitian vector bundle on X, and (H,k) a Hermitian line bundle on X. If H_K is ample, then, for constants C_1 and C_2, the set*

$$\left\{ \widehat{c}_1(F, h|_F) \in \widehat{\mathrm{CH}}^1(X) \;\middle|\; \begin{array}{l} F \text{ is a non-zero saturated subsheaf of } E \text{ with} \\ \widehat{\deg}(\widehat{c}_1(F, h|_F) \cdot \widehat{c}_1(H,k)) \geq C_1 \\ \text{and } \deg(F_K) \geq C_2 \end{array} \right\}$$

is finite.

Proof. Since $\det F$ is a saturated subsheaf of $\bigwedge^{\operatorname{rk} F} E$, our corollary is an immediate consequence of Theorem 2.1. □

3. Proof of Bogomolov unstability (Theorem A)

Before the proof of Theorem A, we will fix notations. Let K be a number field and O_K the ring of integers of K. Let $f : X \to \operatorname{Spec}(O_K)$ be a regular arithmetic surface with $f_* \mathcal{O}_X = O_K$. Let (F, h_F) and (E, h_E) be torsion free Hermitian sheaves on X. We set

$$\widehat{\delta}(E, h_E) = \widehat{\deg}\left(\frac{\operatorname{rk} E - 1}{2 \operatorname{rk} E} \widehat{c}_1(E, h_E)^2 - \widehat{c}_2(E, h_E) \right).$$

and

$$\widehat{d}((F, h_F), (E, h_E)) = \frac{\widehat{c}_1(F, h_F)}{\operatorname{rk} F} - \frac{\widehat{c}_1(E, h_E)}{\operatorname{rk} E}.$$

Then, we have the following formula.

Lemma 3.1. *Let $0 \to (S, h_S) \to (E, h_E) \to (Q, h_Q) \to 0$ be an exact sequence of torsion free Hermitian sheaves on X such that h_S and h_Q are the induced metric by h_E. Then, we have*

$$\widehat{\delta}(E, h_E) \leq \widehat{\delta}(S, h_S) + \widehat{\delta}(Q, h_Q) + \frac{(\operatorname{rk} E)(\operatorname{rk} S)}{2 \operatorname{rk} Q} \widehat{\deg}(\widehat{d}((S, h_S), (E, h_E))^2).$$

Proof. First of all, $\widehat{c}_1(E, h_E) = \widehat{c}_1(S, h_S) + \widehat{c}_1(Q, h_Q)$. Moreover, by Proposition 7.3 of [Mo1],

$$\widehat{\deg}\left(\widehat{c}_2(E, h_E) - \widehat{c}_2((S, h_S) \oplus (Q, h_Q))\right) \geq 0.$$

Thus, by an easy calculation, we have our lemma. □

Let us start the proof of Theorem A. Let $E^{\vee\vee}$ be the double dual of E. Then, $\widehat{c}_2(E^{\vee\vee}, h) = \widehat{c}_2(E, h) - \log\left(\#\left(E^{\vee\vee}/E\right)\right)$. So we may assume that E is locally free. First we claim

Claim 3.2. *There is a non-zero saturated subsheaf F of E such that*

$$\frac{\deg(F_K)}{\operatorname{rk} F} - \frac{\deg(E_K)}{\operatorname{rk} E} > 0.$$

Proof. Since $\widehat{\deg}\left((\operatorname{rk} E - 1)\widehat{c}_1(E, h)^2 - (2 \operatorname{rk} E)\widehat{c}_2(E, h)\right) > 0$, by the main theorem of [Mo1], $E_{\overline{\mathbb{Q}}}$ is not semistable. Let F' be the maximal destabilizing sheaf of $E_{\overline{\mathbb{Q}}}$. For $\tau \in \operatorname{Gal}(\overline{\mathbb{Q}}/K)$, let us consider $\tau(F')$. Then, $\tau(F') \subset E_{\overline{\mathbb{Q}}}$, $\deg(\tau(F')) = \deg(F')$ and $\operatorname{rk} \tau(F') = \operatorname{rk} F'$, which means that $\tau(F')$ is also a maximal destabilizing sheaf of $E_{\overline{\mathbb{Q}}}$. Thus, by the uniqueness of the maximal destabilizing sheaf, we have $\tau(F') = F'$. Therefore, F' is defined over K. Hence, there is a saturated subsheaf F of E with $F_K = F'$. Thus, we have our claim. $\square$

Let (H, k) be a Hermitian line bundle on X such that H_K is ample. Since

$$\widehat{\deg}(\widehat{d}((F, h|_F), (E, h)) \cdot \widehat{c}_1(H, ck)) = \widehat{\deg}(\widehat{d}((F, h|_F), (E, h)) \cdot \widehat{c}_1(H, k)) - \frac{\log(c)[K : \mathbb{Q}]}{2} \left(\frac{\deg(F_K)}{\operatorname{rk} F} - \frac{\deg(E_K)}{\operatorname{rk} E} \right)$$

and $\widehat{\deg}(\widehat{c}_1(H, ck)^2) = \widehat{\deg}(\widehat{c}_1(H, k)^2) - \log(c)[K : \mathbb{Q}] \deg(H_K)$, we may assume that $\widehat{\deg}(\widehat{d}((F, h|_F), (E, h))) \cdot \widehat{c}_1(H, k)) > 0$ and $\widehat{c}_1(H, k) \in \widehat{C}_{++}(X)$ if we replace k by ck with sufficiently small positive number c. Here we consider the following set.

$$\mathcal{G} = \left\{ G \;\middle|\; \begin{array}{l} G \text{ is a non-zero saturated subsheaf of } E \text{ with} \\ \widehat{\deg}(\widehat{d}((G, h|_G), (E, h)) \cdot \widehat{c}_1(H, k)) > 0 \text{ and} \\ \deg_K(\widehat{d}((G, h|_G), (E, h))) > 0 \end{array} \right\}.$$

Then, $F \in \mathcal{G}$. Moreover, by Corollary 2.2,

$$\left\{ \widehat{d}((G, h|_G), (E, h)) \in \widehat{\operatorname{CH}}^1(X)_{\mathbb{Q}} \mid G \in \mathcal{G} \right\}$$

is finite.

We will prove Theorem A by induction on $\operatorname{rk} E$.

Claim 3.3. *If* $\operatorname{rk} E = 2$*, then* $\widehat{d}((F, h|_F), (E, h)) \in \widehat{C}_{++}(X)$.

Proof. Let $h_{E/F}$ be the quotient metric of E/F, i.e. $h_{E/F} = h|_{F^\perp}$. By Lemma 3.1,

$$\widehat{\delta}(E, h) \leq \widehat{\delta}(F, h|_F) + \widehat{\delta}(E/F, h_{E/F}) + \frac{(\operatorname{rk} E)(\operatorname{rk} F)}{2 \operatorname{rk} E/F} \widehat{\deg}(\widehat{d}((F, h|_F), (E, h))^2).$$

Since $\operatorname{rk} F = \operatorname{rk} E/F = 1$, $\widehat{\delta}(F, h|_F) \leq 0$ and $\widehat{\delta}(E/F, h_{E/F}) \leq 0$. Therefore,

$$\widehat{\deg}(\widehat{d}((F, h|_F), (E, h))^2) > 0.$$

Thus, $\widehat{d}((F, h|_F), (E, h)) \in \widehat{C}_{++}(X)$. $\square$

From now on, we assume $\operatorname{rk} E \geq 3$. As in (1.1.3.1), for $x \in \widehat{\operatorname{CH}}^1(X)_{\mathbb{R}}$, we set

$$W(x) = \left\{ u \in \widehat{C}_{+}(X) \mid \widehat{\deg}(x \cdot u) > 0 \right\}.$$

Here we claim

Claim 3.4. *Under the hypothesis of induction, if* $\widehat{\deg}(\widehat{d}((G, h|_G), (E, h))^2) \leq 0$ *for* $G \in \mathcal{G}$*, then there is* $G_1 \in \mathcal{G}$ *with*

$$W(\widehat{d}((G, h|_G), (E, h))) \subsetneqq W(\widehat{d}((G_1, h|_{G_1}), (E, h))).$$

Proof. We set $h_{E/G} = h|_{G^\perp}$. First of all, by Lemma 3.1,

$$\widehat{\delta}(E, h) \leq \widehat{\delta}(G, h|_G) + \widehat{\delta}(E/G, h_{E/G}) + \frac{(\operatorname{rk} E)(\operatorname{rk} G)}{2 \operatorname{rk} E/G} \widehat{\deg}(\widehat{d}((G, h|_G), (E, h))^2).$$

Since $\widehat{\delta}(E, h) > 0$ and $\widehat{\deg}(\widehat{d}((G, h|_G), (E, h))^2) \leq 0$, we have either that $\widehat{\delta}(G, h|_G) > 0$ or that $\widehat{\delta}(E/G, h_{E/G}) > 0$.

If $\widehat{\delta}(G, h|_G) > 0$, then by hypothesis of induction there is a non-zero saturated subsheaf G_1 of G with $\widehat{d}((G_1, h|_{G_1}), (G, h|_G)) \in \widehat{C}_{++}(X)$. Here since

$$\widehat{d}((G_1, h|_{G_1}), (E, h)) = \widehat{d}((G_1, h|_{G_1}), (G, h_G)) + \widehat{d}((G, h|_G), (E, h)),$$

we have $G_1 \in \mathcal{G}$. Moreover, by Proposition 1.1.3, we get

$$W(\widehat{d}((G, h|_G), (E, h))) \subsetneqq W(\widehat{d}((G_1, h|_{G_1}), (E, h))).$$

If $\widehat{\delta}(E/G, h_{E/G}) > 0$, then by hypothesis of induction there is a non-zero saturated subsheaf T of E/G such that $\widehat{d}((T, h_{E/G}|_T), (E/G, h_{E/G})) \in \widehat{C}_{++}(X)$. Take a saturated subsheaf G_1 in E with $G \subset G_1$ and $G_1/G = T$. Let $h_{G_1/G}$ be the induced quotient metric by h_{G_1}. By Proposition 1.2.1, we have $h_{E/G}|_T = h_{G_1/G}$. Thus, by an easy calculation, we get

$$\widehat{d}((G_1, h|_{G_1}), (E, h)) = \frac{\operatorname{rk}(G_1/G)}{\operatorname{rk} G_1} \widehat{d}((T, h_{E/G}|_T), (E/G, h_{E/G})) + \frac{\operatorname{rk} G \operatorname{rk}(E/G_1)}{\operatorname{rk} G_1 \operatorname{rk}(E/G)} \widehat{d}((G, h|_G), (E, h_E)).$$

Therefore, $G_1 \in \mathcal{G}$, and by Proposition 1.1.3,

$$W(\widehat{d}((G, h|_G), (E, h))) \subsetneqq W(\widehat{d}((G_1, h|_{G_1}), (E, h))).$$

Hence we get Claim 3.4. □

Here we assume that $\widehat{\deg}(\widehat{d}((G, h|_G), (E, h))^2) \leq 0$ for all $G \in \mathcal{G}$. Then, since $F \in \mathcal{G}$, by Claim 3.4, $\exists$ a sequence $\{G_0 = F, G_1, G_2, \dots, G_n, \dots\}$ in $\mathcal{G}$ such that

$$W(\widehat{d}((G_i, h|_{G_i}), (E, h))) \subsetneqq W(\widehat{d}((G_j, h|_{G_j}), (E, h)))$$

for all $i < j$. In particular, $\widehat{d}((G_i, h|_{G_i}), (E, h))$ gives distinct elements in $\widehat{\mathrm{CH}}^1(X)_{\mathbb{Q}}$. On the other hand,

$$\left\{ \widehat{d}((G, h|_G), (E, h)) \in \widehat{\mathrm{CH}}^1(X)_{\mathbb{Q}} \mid G \in \mathcal{G} \right\}$$

is finite. This is a contradiction. Thus, to get our theorem, we note that there is $G \in \mathcal{G}$ with $\widehat{\deg}(\widehat{d}((G, h|_G), (E, h))^2) > 0$. □

3.5 Proof of Corollary B. Finally, we give the proof of Corollary B. We assume that

$$\widehat{\deg}\left((2 \operatorname{rk} E)\widehat{c}_2(E, h) - (\operatorname{rk} E - 1)\widehat{c}_1(E, h)^2\right) < 0.$$

Then, by Theorem A, there is a non-zero saturated subsheaf F of E with

$$\frac{\widehat{c}_1(F, h|_F)}{\operatorname{rk} F} - \frac{\widehat{c}_1(E, h)}{\operatorname{rk} E} \in \widehat{C}_{++}(X).$$

Thus,

$$\frac{\widehat{\deg}(\widehat{c}_1(F, h|_F) \cdot A)}{\operatorname{rk} F} - \frac{\widehat{\deg}(\widehat{c}_1(E, h) \cdot A)}{\operatorname{rk} E} > 0.$$

This is a contradiction. □

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